NuHAG Numerical Harmonic Analysis Group

From Linear Algebra to Continuous Variables: A Mild View on Continuous Dirac-like Bases

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Abstract submitted for the talk I

It is widespread common perception that Quantum Mechanics is about *operators on abstract Hilbert spaces* $\mathcal{H}$, states which are normalized trace-class operator and observables which can be general linear operators on $\mathcal{H}$. The pairing comes from the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$ via the trace operation: $\langle T, S \rangle_{HS} = trace(TS^*)$. By complex interpolation one obtains the Schatten S^p-classes of compact operators, with $HS = S^2$. This is quite analogous to scale of Lebesgue spaces L^p, say, with $1 \leq p \leq \infty$, which play a significant role in functional analysis in general and Fourier Analysis in particular, although sometimes wavelet theory appears to be more suitable for questions involving such spaces.

Abstract submitted for the talk II

In this talk the author would like to challenge the audience with an alternative model, based on the Banach Gelfand Triple (S_0, L_2, S_0^*), based on the Feichtinger algebra and the dual space, called the space of mild distributions, with the corresponding concept of "mild convergence" (namely the w^*-convergence). It also allows to give a proper meaning to Dirac BRAs and KETs, which in its simple form allows to describe the basic operations of Linear Algebra in a compact form, which is also taken as an inspiration for the general case (and in addition almost exactly what can be done by using the MATLAB mathematical software).

Abstract submitted for the talk III

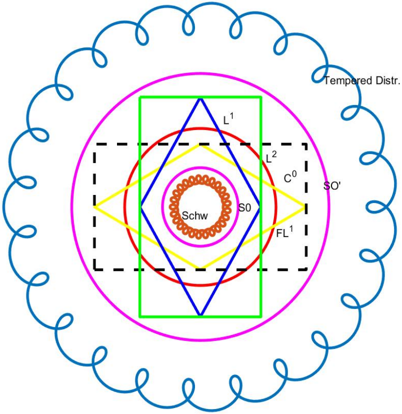
We would like to announce a couple of bold claims:

1. Typical signals is very well modeled as mild distributions;
2. Signals are measured by applying them on test functions;
3. Hence signals are very similar if they agree up to a small error for a large number of measurements;
4. Operators from $S_0(R^d)$ to $S_0^*(R^d)$ are characterized by their "kernels": signals of $2d$ variables, spreading functions, or KNS-symbols or Wigner(-Weyl)transforms;
5. They appear to be a suitable model for observables, larger than $\mathcal{L}(L^2)$, allowing some additional important operators;
6. The setting allows to describe Dirac-like bases properly;
7. Invariance under the metaplectic group gives extra insight.

Abstract submitted for the talk IV

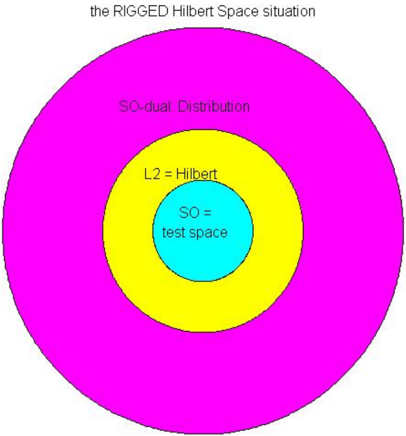
As time permits we would like to point out that so-called *unitary Banach Gelfand Triple isomorphism* are the correct analogue of unitary matrices in the finite dimensional case. The enrichment of the structure from the abstract Hilbert space to the ambient triple (or rigged Hilbert space) provides a lot of extra information. It can be used as an inspiration for the approximation of continuous models by finite-dimensional ones of increasing size. Roughly speaking it corresponds to learn a signal up to some *resolution*, like an audio-signal recorded on a CD (providing the relevant information up to $20kHz$)

A schematic description: all the spaces



A schematic description: the simplified setting

Testfunctions $\subset$ Hilbert space $\subset$ Distributions, like $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$!



Goals of this lecture

- Recall the view on Dirac bases.
- Discuss how problematic they are.
- Remind the concept of Banach Gelfand Triples.
- Specifically $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.
- The Fourier transform as a basic
  unitary Banach Gelfand Triple automorphism.
- Alternative Gaussian perspective
- Definition of *mild* (Dirac-like) bases.
- Examples and illustrations, e.g.
  Fractional Fourier Transforms as a change of bases!
- Signals $\sigma \in \boldsymbol{S}_0'$ are measured by evaluation $\mapsto \sigma(f)$!

MATHEMATICAL SIGNALS I

Asking a mathematician or even a mathematically well-trained engineer what a "signal is" most likely the answer be: a function $f \in \boldsymbol{L}^2(\mathbb{R})$. As a justification there is the comment that "every real-life signal has finite energy" and $\int_{-\infty}^{\infty} |f(t)|^2 dt$ is considered the "energy in the signal f". Plancherel's Theorem (stating that the Fourier transform is a unitary automorphism of the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$) is interpreted as "energy preservation".
Of course, when it comes to do actual computations the engineer will say: Since the computer only allows to deal with finite length vectors one has to replace the integral transform by the FFT, and it is natural to obtain such a finite sequence by sampling the function (regularly and) sufficiently "fine" over a sufficiently "long interval".
Discussions about the connections between the discrete and the continuous setting are still rare (would be important)!

Different Settings of Signals I

Engineering Books

Engineers are use to have signals of different types:

- Signals are either continuous of discrete;
- Signals are either periodic or non-periodic;
- or combinations: finite = discrete + periodic;

For each setting we have a corresponding Fourier transform.

20th Century Mathematical Analysis

In Abstract Harmonic (Fourier) Analysis we view such signals as functions (or distributions) over suitable LCA groups: e.g., $\mathbb{R}^d$ versus $\mathbb{Z}^d$, or $\mathbb{R}$ versus $\mathbb{T}$ (torus: Fourier series!).
In this general setting one finds that compact groups have discrete dual groups ("pure frequencies") and vice versa. $\mathbb{R}^d$ is non-compact and non-discrete. Periodic and discrete signals live on finite Abelian groups (such as the unit roots of order $N \in \mathbb{N}$).

Different Settings of Signals II

In the classical setting $\left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$, the space of absolutely convergent Fourier series, which is dense in $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$. Together with its dual space, known as the space $\boldsymbol{PM}(\mathbb{T})$ of so-called pseudo-measures we have the "prototypical Banach Gelfand triple" $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})$, and the mapping from functions (on the torus) to their Fourier coefficients can be considered the first BGT-isomorphism. It identifies (via distribution theory) the space of pseudo-measures with the set of (periodic, tempered) distributions which have bounded Fourier. Thus the target is $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.
We observe, that the unitary mapping at the Hilbert space level restricts to an identification of $\boldsymbol{A}(\mathbb{T})$ with $\ell^1(\mathbb{Z})$, and extends (in a "unique way") by duality to the "outer spaces".

Integration Theory: Why Lebesgues Spaces? I

It is clear that integration theory plays a significant role in the development of Fourier Analysis, first through the Riemann integral and later via the Lebesgue integral, which is also a cornerstone of functional analysis.
After all, it APPEARS as obvious that one needs the (perfect) integration methods if one wants to study the Fourier transforms (and its inversion), as it IS (?) an integral transform, given by

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t}\, dt \quad \text{and} \quad f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega}\, d\omega, \tag{1}$$

Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$, but it is not natural, because $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ does NOT imply $\widehat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$, hence summability methods had to be used!

Integration Theory: Why Lebesgues Spaces? II

Later on the Fourier transform (over $\mathbb{R}^d$, for this discussion) was extended to $\boldsymbol{L}^p(\mathbb{R}^d)$, but with the restriction to $1 \leq p \leq 2$, known as the *Hausdorff-Young* Theorem. Nowadays it is derived using the "complex interpolation method" (going back to Riesz Thorin). Boundedness of quite general operators (Fourier multipliers, pseudo-differential operators, etc.) are still considered over $\boldsymbol{L}^p$-spaces (with the Hilbert space case $p = 2$ as the cornerstones). Many of the modern function spaces, such as Besov-Triebel-Lizorkin spaces are obtained using Fourier methods, but basically using norms defined naturally on $\boldsymbol{L}^p(\mathbb{R}^d)$, and using *dyadic partitions of unity* (Paley-Littlewood Theory).

But $\boldsymbol{L}^p$-spaces and Fourier transforms do not go along too well!

Only $\boldsymbol{L}^2(\mathbb{R}^d)$ is invariant under the Fourier transform!

REAL LIFE signals I

Let us look at different *real-life signals*. Is it really true that we have (almost everywhere) well-defined functions if we consider a signal? In a few cases we could consider a signal as a bounded and continuous functions, say the temperature as measured by a classical thermometer. BUT in reality this is already an average of the *fluctuating temperature* that we might sense! (due to an open window, or hot air from the hair dryer).
A more typical case is an audio signal, which can be recorded, stored, compressed nowadays without problems (MP3). The HiFi standard used for the reproduction of music stored in a digital format on a CD (or transmitted for streaming) makes use of a sampling rate of 44100 samples per second, which (based on the Shannon Sampling Theorem) allows a (*quasi real-time*) reconstruction of the underlying "function" up to the audible range (resolution) up to $20kHz$ (using spectrograms).

REAL LIFE signals II

In summary: an acoustic signal (of time) is "something" that can be recorded (using a microphone) which (with the help of an A/D-converter) allows to compute a spectrogram (or Short-Time Fourier Transform, or Sliding-Window Fourier Transform).
Two-dimensional signal analysis is seen as the basis for digital image processing. Here we have a similar situation. While the standard mathematical approach talks about the Hilbert spaces $\boldsymbol{L}^2(\mathbb{R}^2)$ it is not realistic to define the (gray or color) value at a physical point but only at local average values as obtained with the help of photonic sensors. Depending on the quality of such a digital camera one has different resolution levels.
Thus in a way an IMAGE can be seen as a collection of possible pixel images of given scene, again up to some resolution.

REAL LIFE signals III

As a final example, emphasizing the idea that a "signal" is something which can be recorded or measured, let us think of the gravitational force acting on mass (i.e. materials of different types). Here is is clear that we cannot measure the infinitesimal action of a force, say on an individual atom. But we can observe the acceleration of a piece a falling stone on earth or the moon, and deduce, using physical laws and some mathematics (going back to Newton) the strength of the gravitational force at this location. On the other hand it is common in physics to talk about point-charge or in our case of point mass, i.e. the whole mass of body concentrated in one point. For long distances this simplifies a lot computations and is good enough to explain the action of the moon on the ocean (tides), or the path of a stone thrown into the air, following a parabolic curve, or "dithering" (print).

Changing the Perspective I

Without spending too much time on the technical details I would like to claim that the outer layer, called the space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ of *mild distributions* provides a good setup of the description of what real-life signals are.
It can be defined as the subspace of all tempered distributions (quasi: most general objects) which have a **bounded STFT**, endowed with the supnorm. Since one can show that this boundedness is independent of the (say) non-zero Schwartz window $g \in \mathcal{S}(\mathbb{R}^d)$ we can assume that $V_g(\sigma) = \sigma(\overline{\pi(\lambda)g})$ is using the Gaussian window g, normalized in the $\boldsymbol{L}^2$-norm.

Alternatively there is a sequential approach to construct $\boldsymbol{S}_0'(\mathbb{R}^d)$ from specific Cauchy-sequences in $\boldsymbol{C}_b(\mathbb{R}^d)$, comparable to the construction of $\mathbb{R}$ from $\mathbb{Q}$. Also in this case the resulting "limit" has a bounded spectrogram (STFT).

Changing the Perspective II

In contrast to a naive approach to signals which (for the continuous domain $\mathbb{R}^d$) models them as continuous functions signals are much more vague than true (continuous) functions which have an exact, pointwise value for each $t \in \mathbb{R}^d$. This is however in reality quite rare! (try to think of a concrete case). It is rather more plausible that signals are "things" which allow a measurement, which should be linear. Also, we can shift signals in time and multiply them with pure frequencies (assumption!) and then find that they turn out to be a dual Banach spaces, where the predual has an atomic structure, in fact another characterization of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (using atoms which are TF-shifted Gaussians of the form $\pi(\lambda)g$, with ℓ^1-coefficients.)
In short: $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ is a signal, and the allowed measurements are the mappings $\sigma \to (\sigma(f_k))_{k=1}^K$, with $f_k \in \boldsymbol{S}_0(\mathbb{R}^d)$.

Convergence I

With this view-point it is clear that one can express similarity with respect to measurements: Given a finite set $F = \{f_k, 1 \leq k \leq K\}$ and $\varepsilon > 0$ we can call two signals σ_1, σ_2 (F, ε)-close of the difference satisfies

$$|(\sigma_1 - \sigma_2)(f_k)| < \varepsilon, \quad for 1 \leq k \leq K$$

This is of course w^*-convergence for sequences or nets in the dual space $\boldsymbol{S}_0'(\mathbb{R}^d)$. In fact, since $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a separable Banach space it suffices to discuss convergence of sequences. On the other hand one can consider Riemannian sums (over the real line) as a convergent net, which converges to $\int_{\mathbb{R}^d} f(t)dt$, for any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$. The approximating functionals are discrete measures (for $d = 1$) $\mu = \sum_{k \in \mathbb{Z}} l_k \delta_{\xi_k}$, with $l_k =$ length of the interval $I_k \ni \xi_k$.

Practical convergence using spectrograms I

Note that we have $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, and thus any finite set $F \subset \boldsymbol{S}_0(\mathbb{R}^d)$ defines a finite dimensional subspace $\boldsymbol{V}_F$. For such subspaces one can define (abstractly, oblique) projection operators. They can be used to describe the "mild convergence" (by measurements) as the convergence of all the projected variants of the given sequence $(\sigma_n)_{n\geq 1}$ in $\boldsymbol{S}_0(\mathbb{R}^d)$. More practically one can consider good Gabor frames, say a Gabor frame using the fact that for any TF-lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, with $ab < 1$ (redundancy $1/ab$) and observe for such a mildly convergent sequence convergence of all the Gabor coefficients. Thus the finite partial sums of the Gabor expansion converge

$$\sum_{\lambda \in F} V_g f(\lambda) \widetilde{g}_\lambda \to_{w^*} f, \quad f \in \boldsymbol{S}_0'(\mathbb{R}^d)$$

Operations on mild distributions I

Except for the lack of the possibility of evaluating a mild distributions (think of δ_0!) most of the concepts and operations that can be performed on ordinary functions can be extended to mild distributions, i.e. to signals.
We can extend (always in unique way if we want to ensure that the extension of a given operator also preserves mild convergence!) the following operations:

1. translate, dilate, rotate, flip (parity);
2. take a (fractional) Fourier transform: $\widehat{\sigma}(f) := \sigma(\widehat{f})$;
3. define the product of a test function (or pointwise multiplier thereof) with a mild distribution: $\sigma h(f) := \sigma(hf)$.
4. define the support of a mild distribution.

The CONSISTENCY PRINCIPLE I

As for any extension (such as $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ for the extension of fields) it is of utmost importance to ensure that the concept which are valid at different levels are not in conflict with each other. Thus we can multiple rational numbers first and then convert the result to decimal description, or we convert them to decimal numbers and multiply them as real numbers. The RESULT IS THE SAME. The same idea continuous to be valid for the Banach Gelfand Triple situation. All the concepts which are valid for ordinary functions (the one mentioned above) are the *unique* extensions of the corresponding concepts for test functions, using the fact that they are compatible with mild/distributional/w^*-convergence. As such they are even uniquely determined, and the way to describe by duality is just an elegant way to define them.

sectionThe Dirac Family

Towards the Dirac family I

The Dirac family $(\delta_x)_{x\in\mathbb{R}^d}$ or the family of pure frequencies $(\chi_s)_{s\in\mathbb{R}^d}$ (which are considered the Dirac basis for the frequency domain) have all similar properties respectively problems.
First of all they do not belong to the Hilbert space, which is $\mathcal{H} = \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ for our purposes. This is highly problematic because it does not allow to naively take the scalar products with this family, unless the one has extra conditions, such as continuity for the Dirac family or integrability for the definition of the FT via integrals. Still, as a Banach space of *equivalence classes of measurable functions* $\boldsymbol{L}^2(\mathbb{R}^d)$ (and also any $\boldsymbol{L}^p(\mathbb{R}^d), 1 \leq p \leq \infty$) there is still some "trace of pointwise values", in contrast to general measures (where one likes to take histograms) or even more general (mild) distributions.

Continuous Orthonormal Bases? I

For the world of Hilbert spaces many aspects (mostly geometric ones) are well understood by the analogy to the Euclidean space $\mathbb{R}^d$ or $\mathbb{C}^d$ with the standard scalar product and Euclidean norm. For example: any (closed) subspace has an orthonormal basis which can be used to describe the orthogonal projection onto this subspace. The Pythagorean principle guarantees that the projection also minimizes the distance of a given point (outside of that subspace) to the subspace in a unique fashion.
The situation is less obvious for bases. It is the finite dimensionality allows to derive the following well known theorem:
A BASIS $(\mathbf{b}_k)_{k=1}^d$ is an orthonormal basis if and only if the norm of a given vector $\mathbf{x} \in \mathbb{C}^d$ equals the Euclidean length of the vector of coefficients $(\langle \mathbf{x}, \mathbf{b}_k \rangle)$.

Continuous Orthonormal Bases? II

!But there are many redundant families (tight frames) with the same property, and they can have arbitrary redundancy. Just think of the arrows pointing from 0 to the unit roots of order $N \geq 3$ in the complex plane!
The same is true for (say) typical Gaussian Gabor frames, using the lattices $a\mathbb{Z}^d \times b\mathbb{Z}^d$ with redundancy $1/(ab) > 1$, but arbitrarily large redundancy.
One can also apply dilation to a given Gabor frame, i.e. make the windows more narrow by replacing g by the compressed version of g (rescaled isometrically in $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$) by a factor $\rho > 0$, the same with the time lattice $a \to \rho a$, but correspondingly rescale b by b/ρ.

Continuous Orthonormal Bases? III

If we normalize (for a moment) g in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, i.e. put $\|g\|_1 = 1$ the $\boldsymbol{L}^1$-normalized compression of a Gaussian window will converge to a Dirac delta, the lattice $\rho a\mathbb{Z}^d$ get finer and finer and one can this of the Dirac family as the limit of those frames (or tight frames).
As we said in this case we have observed that these elements are kind a "continuous tight frame" but not of test functions, but now of mild distributions.

Recalling Linear Algebra Concepts I

Let us take the Euclidean space $\mathbb{R}^d$ with the usual scalar product as a starting points. For a comprehensive discussion of the basic linear algebra concepts it is better to allow complex coefficients, so let us start with $\mathbb{C}^n$ as the prototypical finite dimensional vector space, typically realized as collection of column vectors $\mathbf{x} = [x_1, \ldots, x_d]$. Endowed with the usual scalar product it is also the key example of a (finite-dimensional complex) Hilbert space.

For NON-FINITE dimensional vector space (does not mean that there is an infinite basis). Typical examples of *Hilbert spaces* $\mathcal{H}$ are of course ℓ^2 and $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, again with the usual scalar products.

Even more generally there are the (quasi-)Banach spaces, or Frechet spaces (such as $\mathcal{S}(\mathbb{R}^d)$) and their duals, also interesting for the so-called *kernel-theorem*.

Recalling Linear Algebra Concepts II

For $\boldsymbol{V} = \mathbb{R}^d$ let us discuss the terminology.
According to the vector space axioms we can define *linear combinations* of n vectors with a coefficient sequence $\mathbf{c} \in \mathbb{C}^n$. If stored as columns of a $d \times n$ matrix $\boldsymbol{A}$ the resulting linear combination is just $\mathbf{u} = \boldsymbol{A} * \mathbf{c}$. If this mapping from $\mathbb{C}^n$ to $\mathbb{C}^d$ is surjective we have a *generating system* and if the representation of $\mathbf{u}$ is unique we call the family $(\mathbf{a}_k)_{k=1}^n$ a *linear independent* family of (column) vectors. The columns of $\boldsymbol{A}$ form a *basis* of both properties are given.
In addition, any linear operator T from $\mathbb{C}^n$ to $\mathbb{C}^m$ is given by a unique matrix, describing T using two basis (in the domain and in the target space). The columns of $\boldsymbol{A}$ describe the image of the basis vectors in $\mathbb{C}^n$ under T.

Recalling Linear Algebra Concepts III

Matrix multiplication is defined in order to ensure that composition of linear operators, whenever it makes sense, can be realized at the matrix representation level.
Row vectors describe functionals ($m = 1$). Often we say that the *dual space* of $\mathbb{C}^n$ is just $\mathbb{C}^n$.
Matrices of maximal rank $r = \min(m, n)$ are exactly those which define surjective mappings ($n \geq m$) or (their transpose) injective linear mappings $n \leq m$.

We can also view a matrix $\boldsymbol{A}$ as a description of T as a sum of $m \cdot n$ rank-one operators, where the rows of $\boldsymbol{A}$ define the linear functionals acting on the input vector and the unit vectors $\mathbf{e}_k, k = 1, ..., m$ are the target vectors, so (using Dirac notation) we talk about $|r_j\rangle\langle\mathbf{e}_l|$.

Important Facts

1. The columns (or rows) of a matrix form a basis if and only if $\boldsymbol{A}$ is *invertible* (thus $m = n$), i.e., $\boldsymbol{A} \in GL(n, \mathbb{C})$.
2. Given any two bases there is exactly one matrix mapping a given basis into another given basis. This corresponds to the group property of $GL(n, \mathbb{C})$.
3. Inside of this group the group of *unitary operators* plays a specific role. It describes exactly the invertible, linear operators which map ONBs onto ONBs, and it is a subgroup of $GL(n, \mathbb{C})$, called $SL(n, \mathbb{C})$.
4. All the ONBs for $\mathbb{C}^n$ can be obtained by applying the full group $SL(n, \mathbb{C})$ on any given ONB for $\mathbb{C}^n$, typically the unit matrix (collection of unit vectors).
5. Base change matrices describe the identity map in different bases and thus contain basis vectors.

Hilbert Space Context I

In the Hilbert space context stability estimates do not come automatically, and thus extend the finite dimensional concepts in a correct by to the setting of (separable) Hilbert spaces.

generating system	frame
linear independence	Riesz sequence
basis	Riesz basis
isometric embedding	tight frame
matrix	integral kernel
unitary matrix	unitary operator

All these concepts can be characterized via diagrams, which are also valid for *Banach frames* and Riesz projection bases.

BGT and Unitary Isomorphisms I

We do not deal with the (restricted) setting of (abstract or concrete) Hilbert spaces, but rather go for the corresponding concepts which make sense in the context of Banach Gelfand Triples.
This we replace the involved vector space by a *rigged Hilbert space*, based on the Feichtinger algebra $\boldsymbol{S}_0(\mathbb{R}^d)$, with the dual space, denoted by $\boldsymbol{S}_0'(\mathbb{R}^d)$ being the space of mild distributions. The natural duality is the one coming from $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$.
Similarly, be can start with the Hilbert space of Hilbert Schmidt operators with the scalar product

$$\langle\langle T, S\rangle\rangle_{\mathcal{HS}} = \operatorname{trace} TS^*, \quad T, S \in \mathcal{HS}.$$

Banach Gelfand Triples appear to be the correct structure in order to imitate situations like those encountered by the inclusion of the number systems $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

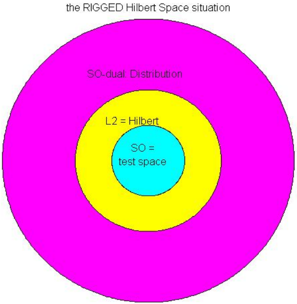


Abbildung: Three layers

The "inner layer" is where the actual computations are done, the focus in mathematical analysis is all to often with the (yellow) Hilbert spaces (taking the role of $\mathbb{R}$, more complete with respect to a scalar product, more symmetric, because it allows to be identify the dual, via the Riesz representation Theorem, very much like matrix theory is working, with row and column vectors), and the outside world where things sometimes can be explained, and with completeness in an even more general sense (distributional convergence). In other words, we do not assume anymore that $\sigma_n(f)$ is convergent for all $f \in \mathcal{H}$ (the completion of the test functions in $\mathcal{H}$), but *only for* elements f in the core space!
What we are going to suggest/present is THE Banach Gelfand Triple

$$(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$$

consisting of the *Feichtinger algebra* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, known as space of *mild distributions*. Note that these spaces can be defined without great difficulties on any LCA group G and that it satisfies many desirable *functorial properties*, see the early work of V. Losert (**lo83-1 :[3]**).

For $\mathbb{R}^d$ the most elegant way (which is describe in **gr01 :[1]** or **ja18 :[2]**) is to define it by the integrability (actually in the sense of an infinite Riemann integral over $\mathbb{R}^{2d}$ if you want) of the STFT

$$V_{g_0}(f)(x,y) := \int_{\mathbb{R}^d} f(y)g(y-x)e^{-2\pi i s y}\,dy$$

and the corresponding norm

$$\|f\|_{\boldsymbol{S}_0} := \int_{\mathbb{R}^{2d}} |V_{g_0}(f)(x,y)|dxdy < \infty.$$

From a practical point of view one can argue that one has the following list of good properties of $\boldsymbol{S}_0(\mathbb{R}^d)$.

Theorem

1. $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right) \hookrightarrow \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d)$;
2. $\mathcal{F}(\boldsymbol{S}_0(\mathbb{R}^d)) = \boldsymbol{S}_0(\mathbb{R}^d)$ *(isometrically)*;
3. *Isometrically invariant under TF-shifts*
$$\|\pi(\lambda)(f)\|_{\boldsymbol{S}_0} = \|M_s T_t f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}, \quad \forall (t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$
4. $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ *is an essential double module (convolution and multiplication)*
$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d) \quad \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d),$$
in fact a Banach ideal and hence a double Banach algebra.
5. *Tensor product property* $\boldsymbol{S}_0(\mathbb{R}^d)\widehat{\otimes}\boldsymbol{S}_0(\mathbb{R}^d) \approx \boldsymbol{S}_0(\mathbb{R}^{2d})$ *which implies the Kernel Theorem.*
6. *Restriction property: For* $H \lhd G$: $R_H(\boldsymbol{S}_0(G)) = \boldsymbol{S}_0(H)$.

Theorem

1. $\mathcal{F}(\boldsymbol{S}_0'(\mathbb{R}^d)) = \boldsymbol{S}_0'(\mathbb{R}^d)$ *via* $\widehat{\sigma}(f) := \sigma(\widehat{f}), f \in \boldsymbol{S}_0'$.
2. *Identification of TLIS:* $\boldsymbol{H}_G(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(G)$ *(as convolutions of the form)* $T(f) = \sigma * f$;
3. *Kernel Theorem:* $\mathcal{B} := \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(\mathbb{R}^{2d})$ *Inner Kernel Theorem reads:* $\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0) \approx \boldsymbol{S}_0(\mathbb{R}^{2d})$.
4. *Regularization via product-convolution or convolution-product operators:* $(\boldsymbol{S}_0' * \boldsymbol{S}_0) \cdot \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0$, $(\boldsymbol{S}_0' \cdot \boldsymbol{S}_0) * \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0$
5. *The finite, discrete measures or trig. pols. are* w^*-*dense.*
6. $H \lhd G \rightarrow \boldsymbol{S}_0(H) \hookrightarrow \boldsymbol{S}_0(G)$ *via* $\iota_H(\sigma)(f) = \sigma(R_H f), f \in \boldsymbol{S}_0(G)$. *Moreover the range characterizes* $\{\tau \in \boldsymbol{S}_0(G) \mid \mathrm{supp}(\tau) \subset H\}$.

Identifications

The well-known kernel theorem which identifies the Hilbert space of Hilbert-Schmidt operators on the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ with $\left(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\right)$, thus associating (in a unitary way) to each $K(x,y) \in \boldsymbol{L}^2(\mathbb{R}^{2d})$ the linear (HS)-operator

$$T_k(f)(x) := \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d)$$

can be extended to a unitary Banach Gelfand Triple isomorphism. The inner layer describes the operators with kernel $K \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ as w^*-nuclear operators from $\boldsymbol{S}_0'(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\mathbb{R}^d)$, hence regularizing operators, while the dual space can be identified with all the bounded linear operators
from $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ into $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$.

Dirac-like Bases

There are plenty of examples of Dirac-like bases, understanding them as generalizations of "unit vectors", now with continuous parameters.

1. Dirac basis $(\delta_x)_{x \in \mathbb{R}^d}$
2. Fourier basis $(\chi_s)_{s \in \mathbb{R}^d}$
3. Chirp bases arising from Fractional Fourier transform
4. Kernel representation with basis $\delta_x \otimes \delta_y$
5. spreading representation with unitary TF-shifts as basis
6. Kohn-Nirenberg representation
7. Wigner-Weyl calculus
8. Kirkwood-Rihaczek distributions

Definition

We call a family $(\tau_x)_{x\in\mathbb{R}^d}$ a *mild basis*[a] for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ if there exists some BGT-automorphism α such that $\tau_x = \alpha(\delta_x)$, $x \in \mathbb{R}^d$. We call it a *unitary mild basis* (for short, a **UMB**) for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ if there exists some unitary BGT-automorphism.

[a] This terminology relates well to the use of the Dirac basis in physics and other application areas. Strictly speaking it would be more correct to call it a *tight, mild frame*, because a basis is supposed to be minimal, while a frame is a generating system which allows some redundancy. All the *relevant* properties of the $(\delta_x)_{x\in\mathbb{R}^d}$ would be valid also for the family (for illustration only) $(\delta_q)_{q\in\mathbb{Q}}$ as well, since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. In a way, one can consider the *tight Dirac frame* $(\delta_x)_{x\in\mathbb{R}^d}$ as the limit of Gabor frames of the form $(\mathrm{St}_\rho g_0, \rho a, b/\rho)$, using compressed Gaussians as atoms, for $\rho \to 0$, starting from any pair (a, b) with $ab < 1$. Thus the Dirac basis consists of in infinitely fine grid in time and an infinitely sparse grid in frequency, and thus the concept of redundancy does not make much sense anymore for this limiting case.

Fractional Fourier Transform I

The FrFT (*Fractional Fourier Transforms*) $\mathcal{F}_\alpha$, $\alpha \in \mathbb{R}$ define a group of unitary operators on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ which correspond to rotations in phase space. They form a compact subgroup of the *metaplectic group*, isomorphic to the torus $d = 1$.
Although going back to the time of Wiener's description of the Fourier transform in terms of the Hermite basis it was re-invented a couple of times in the last 50 years. The study of the metaplectic group initiated by Andre Weil is largly unknown in the engineering community.
Namias ([4]) and others describe it as an integral transform with a smooth kernel $K_\alpha(x, y)$ of absolute value one (like the ordinary Fourier transform) except for the cases of $\alpha = k\pi$, $k \in \mathbb{Z}$ (identity or parity/flip operator).

Fractional Fourier Transform II

But $\mathcal{F}_\alpha$ (like any metaplectic operator) leaves $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ invariant, and by duality also $\boldsymbol{S}_0'(\mathbb{R}^d)$ and thus it is not difficult to verify that any such metaplectic transformation represents a unitary BGT-automorphism for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$. The "columns" of the kernel K_α, which are a mild basis for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

It is then also clear that the commutative group of translations then goes over to a commutative group of "generalized translations" as it is discussed in the context of the SAFT.

The spreading representation of operators is another such example. One can say that the spreading distribution of a general operator is the description of T in the mild basis $\pi(\lambda)$, $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. It is (by a simple transformation) equivalent to the Weyl-Wigner transform, replacing the TF-shifts by the Weyl-Heisenberg operators $e^{\pi its} M_s T_t$, $(t,s) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

Realizing BGT-isomorphisms

We can suggest a common recipe for the realization that a given unitary mapping between to $\boldsymbol{L}^2$-spaces over LCA groups is a BGT-isomorphism, modeled along the Fourier transform, which maps $\boldsymbol{L}^2(\mathbb{R}^d)$ onto $\boldsymbol{L}^2(\widehat{\mathbb{R}^d})$ in a unitary way:

1. First verify that the integral transform makes sense in the setting of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (typically it suffices to use Riemann integrals!)
2. Verify that the mapping is isometric (then it extends to all of $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, due to a simple density argument)
3. Then extend it by duality (like the usual rule) $\widehat{\sigma}(f) = \sigma(\widehat{f}), f \in \boldsymbol{S}_0(\mathbb{R}^d)$. In fact, this defines the only extension to $\boldsymbol{S}_0'(\mathbb{R}^d)$ which preserves mild convergence.

The Group of Unitary BGT Automorphism I

In the same way as the group structure of the unitary group of $n \times n$-matrices is very helpful, and thus allows to identify the columns (or rows) of a unitary matrix an (general) ONB for $\mathbb{C}^n$ the group structure of the set of all unitary BGT-isomorphisms allows to generate all the mild bases (by definition now). On the other hand, the "integral kernels" of such transformations belong to $\boldsymbol{S}_0'(\mathbb{R}^{2d})$ (and thus cover even the exceptional cases of the Fractional Fourier transform). The kernel description of the identity operator as a "Dirac along the main diagonal" (the continuous version of a unit matrix) is thus not really an exceptional case!

Generalizations I

All the methods described above are meaningful in the general setting allowing Abelian Harmonic Analysis, thus according to A.Weil (and H.Reiter, etc.) can be carried out naturally in the context of LCA groups!
The BGT setting should allow approximation (at the level of test functions) by finite models, using the idea of structure preserving approximation.
The "flat torus idea" allows to view the collection of discrete and periodic signals as a finite dimensional subspace of $\boldsymbol{S}_0'(\mathbb{R}^d)$, and the fact that the collection of all those subspaces allows for a mild approximation of general elements of $\boldsymbol{S}_0'(\mathbb{R}^d)$ (the same at the operator level) should open the way to efficient numerical methods together with quantitative error estimates.

K. Gröchenig.
Foundations of Time-Frequency Analysis.
Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.

M. S. Jakobsen.
On a (no longer) New Segal Algebra: a review of the Feichtinger algebra.
J. Fourier Anal. Appl., 24(6):1579–1660, 2018.

V. Losert.
Segal algebras with functorial properties.
Monatsh. Math., 96:209–231, 1983.

V. Namias.
The fractional order Fourier transform and its application to quantum mechanics.
J. Inst. Math. Appl., 25:241–265, 1980.

Links and Refs I

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hans.feichtinger@univie.ac.at.

This talk provides some thoughts related to the idea of

Conceptual harmonic analysis.