Banach Gelfand triples, mild distributions and Gabor frames: New perspectives on Fourier analysis

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Outline

Fourier transform and inverse Fourier transform

The forward Fourier transform (Fourier Analysis) reads

$$\hat{f}(s) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i s \cdot t}\, dt = \langle f, \chi_s \rangle_{\boldsymbol{L}^2} \qquad (1)$$

The inverse Fourier transform (synthesis) then has the form

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(s) \cdot e^{2\pi i t \cdot s}\, ds = \int_{\mathbb{R}^d} \langle f, \chi_s \rangle \chi_s ds, \qquad (2)$$

providing a continuous resolution of the identity.
According to Plancherel's Theorem it extends to a unitary automorphism of $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$. Physicists see it as a change of basis from the Dirac ONB (?) to the Fourier basis $(\chi_s)_{s \in \mathbb{R}^d}$.

D. Gabor's elementary signals

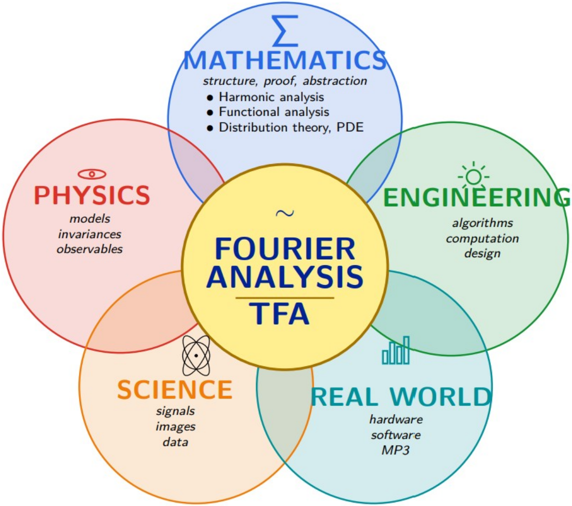
Gabor's 1946 paper suggests to replace the pure frequences χ_s by time-frequency shifted Gaussians. For $g(t) = \mathrm{e}^{-\pi|t|^2}$, the time-frequency shifted Gabor atoms are

$$g_{m,n}(t) = [M_{bn} T_{am} g](t) = \mathrm{e}^{2\pi i bn\cdot t}\, g(t - am), \qquad m, n \in \mathbb{Z}^d.$$

He argues for $a = 1 = b$, the von Neumann lattice in phase space, or critical density 1. Nowadays Gabor families $\{ M_{bn} T_{am} g : (m, n) \in \mathbb{Z}^d \times \mathbb{Z}^d \}$, with density $1/ab > 1$ is preferred. The family then forms a frame in $(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2)$, i.e. a *stable generating system*.

- each atom is localized near m in time;
- its oscillation is centered near frequency n;

Fourier analysis as a connecting discipline



Fourier analysis: 200 years in a nutshell I

1822: J. B. Fourier

- Heat equation and trigonometric series expansion
- Representation of functions by oscillatory building blocks
- Beginning of harmonic analysis

19th century: rigor and convergence

- Dirichlet, Riemann: convergence of Fourier series
- Weierstrass: uniform approximation phenomena
- Growing awareness of pathological examples

1900–1930: H. Lebesgue and modern integration

- Lebesgue integral and L^p-spaces
- Fourier transform defined on L^1 and extended to L^2
- Plancherel and Parseval identities

Fourier analysis: 200 years in a nutshell II

1930–1950: abstract harmonic analysis

- Pontryagin duality for LCA groups
- Fourier transform as representation of characters
- Unified treatment of $\mathbb{R}^d$, $\mathbb{Z}^d$, $\mathbb{T}^d$, finite groups

1950: L. Schwartz

- Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and tempered distributions $\mathcal{S}'(\mathbb{R}^d)$
- Fourier transform as automorphism on $\mathcal{S}'(\mathbb{R}^d)$
- Rigorous treatment of generalized signals (e.g. Dirac delta)

1965: Cooley–Tukey (FFT)

- Fast computation of discrete Fourier transform
- Reduction from $O(N^2)$ to $O(N \log N)$
- Revolution in numerical signal processing

Fourier analysis: 200 years in a nutshell III

Modern developments

- Time-frequency analysis (Gabor, STFT, modulation spaces)
- Wavelets and multiresolution analysis
- Pseudodifferential operators and phase-space methods

Key message: Fourier analysis evolved from formal expansions to a unified theory of signals, operators and representations.

For me the transition from classical and abstract harmonic analysis is comparable to the *transition* from the study of linear functions to *differentiable* functions, which behave locally very similar.
Compare: Geometry $>$ Differential Geometry?
Slowly varying channels like audio-mixers do not commute, but have Gabor atoms as common collection of (non-orthogonal) *approximate eigenvectors*.

Fourier analysis and functional analysis I

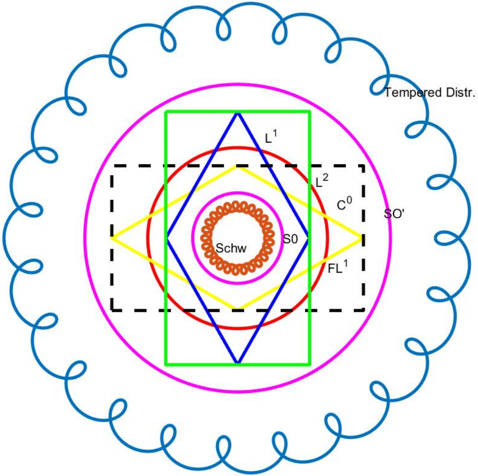
Fourier Analysis pushes Function Analysis

- Long before the language of distributions became standard, Fourier analysis was already a major driving force behind functional analysis.
- Questions about Fourier series led to fundamental problems about convergence, completeness, approximation, summability and the precise meaning of a “function”.
- The Lebesgue integral and the spaces $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ were strongly motivated by the need to control limiting processes and convergence in mean for Fourier expansions and integral transforms.

Fourier analysis and functional analysis II

- The Hilbert space viewpoint was shaped by orthogonality, Parseval identities, the Riesz–Fischer theorem and the idea that functions can be treated like infinite-dimensional vectors.
- Linear operators entered naturally through integral operators, Fourier multipliers, projections and summability methods. This prepared the systematic study of bounded operators.
- Banach algebra ideas are already present in convolution algebras such as $L^1(G)$ and in the Wiener algebra of absolutely convergent Fourier series.
- Thus the classical development moved from concrete Fourier formulas toward abstract spaces, norms, duality, operators and algebras: precisely the basic language of functional analysis.

A Zoo of Banach Spaces for Fourier Analysis



ROMAN NUMBER SYSTEM (diffuclties!)

A guiding analogy: $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ I

- The chain $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ represents a natural hierarchy of number systems, technically speaking of fields.
- Each level has its own strength:
  - **Rational numbers** $\mathbb{Q}$: good for computation, involving finite arithmetic and multiplicative inversion.
  - **Real numbers** $\mathbb{R}$: complete; limits exist, allowing the definition of numbers such as π, $\sqrt{2}$, or exponential functions.
  - **Complex numbers** $\mathbb{C}$: algebraically closed and structurally richer; provide conceptual clarity, e.g. via Euler's formula and addition theorems for trigonometric functions.
- In short:
  - $\mathbb{Q} \rightarrow$ computational viewpoint,
  - $\mathbb{R} \rightarrow$ analytic completeness,
  - $\mathbb{C} \rightarrow$ structural and conceptual clarity.
- This hierarchy serves as a guiding analogy for Banach Gelfand triples, specifically $(\ell^1, \ell^2, \ell^\infty)$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

A guiding analogy: $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ II

Another useful observation is the fact, that the DFT (the Discrete Fourier Transform), the *workhorse of modern digital signal processing*, as realized by the FFT (the Fast Fourier Transform) is nothing else but the evaluation of a polynomial $p_a(z)$ with coefficients $[a_0, \cdots, a_{N-1}]$ over the unit roots of order N in the complex domain, taken in the clockwise sense.

This interpretation of the DFT-matrix as a Vandermode matrix paves the way to fast realization of multiplication of integers, or the verification that the DFT of a Dirac comb of step size $R|N$ is a Dirac comb of step size N/R under the DFT, simply by combining the exponential law with the formula for geometric series.

The observation that $111111^2 = 12345654321$ can also used as a challenge to students, connecting it with probability theory.

THE Banach Gelfand Triple

Feichtinger Algebra and Mild Distributions

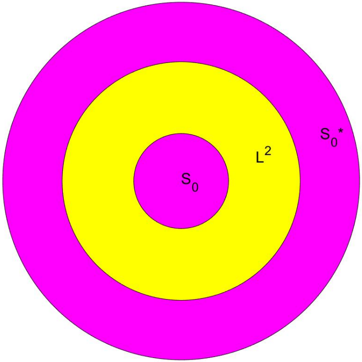


Figure: Feichtinger Algebra and Mild Distributions

Fractions and quotients in Narvik

The key-players for time-frequency analysis

Time-shifts and Frequency shifts (II)

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \quad \lambda = (t, \omega).$$

Demonstration of “Happy Birthday” in TF-plane

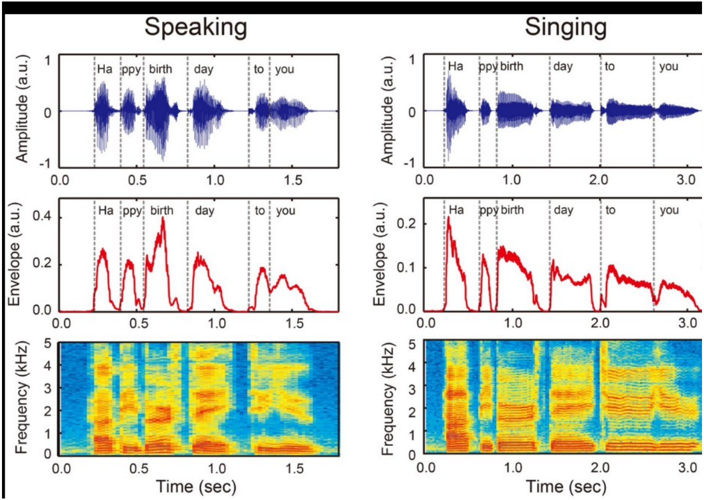


Figure: Speaking and singing “Happy Birthday”, shown by waveform, amplitude envelope and spectrogram.

The Gaborator as a modern example I

- The Gaborator is a practical software realization of time-frequency ideas, designed for audio visualization and analysis. See www.gaborator.com
- It produces constant-Q spectrograms, well adapted to musical perception: frequency resolution is organized logarithmically, closer to octaves and semitones than to a linear frequency scale.
- It is close in spirit to Gabor analysis, but not simply the standard uniform-lattice Gabor frame transform.
- Such examples show that time-frequency analysis provides concrete tools for visualizing and processing real-world signals.
- From the mBGT point of view, this illustrates the usefulness of a framework where test functions, signals, distributions, kernels and operators can be treated coherently.

Demonstration of the possible output of the GABORATOR

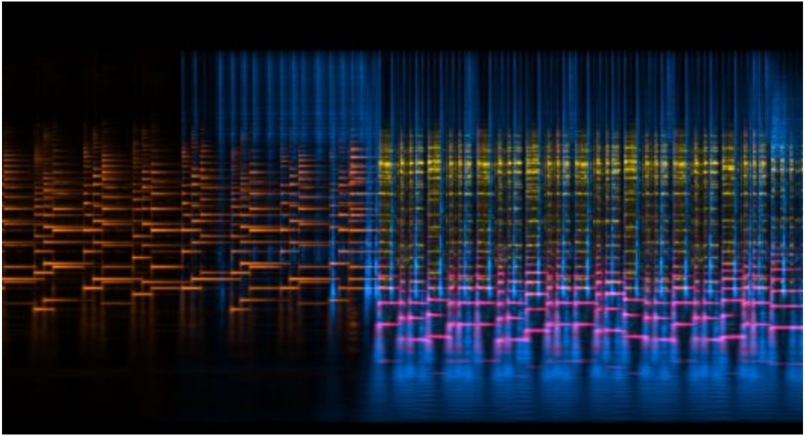


Figure: A spectrogram-like time-frequency visualization, illustrating the type of output produced by the Gaborator.

Spectrogram apps on Android I

- Modern smartphones make elementary time-frequency analysis directly visible through the microphone.
- A quick search in the Google Play Store already gives a number of examples, listed the next page.

- This shows the pedagogical value of time-frequency analysis: everyone can immediately see harmonic patterns, transients, pitch changes and noise components.
- Such apps are informal demonstrations, but they point toward the same conceptual framework: signals are best understood jointly in time and frequency.

Spectrogram apps on Android II

Spectroid	Real-time audio spectrum analyzer.
Spectral Audio Analyzer	Colored spectrogram from microphone input; useful for voices, instruments and environmental noise.
Spectrogram	FFT-based real-time spectrogram, including frequency and musical note detection.
SpecStream Audio Spectrum Plot	Scrolling real-time spectrum image with adjustable sample rate and FFT buffer.
FNam FFT Spectrogram Analyzer	Real-time FFT and spectrogram analysis, including data export and post-analysis features.
Sound Analyzer Basic	Spectrum, waterfall/waveform display.
Audio Spectrum Monitor	Real-time spectrum display on a musical scale, with pitch detection.
Spectrolizer	Music player and microphone-based audio spectrum visualization.

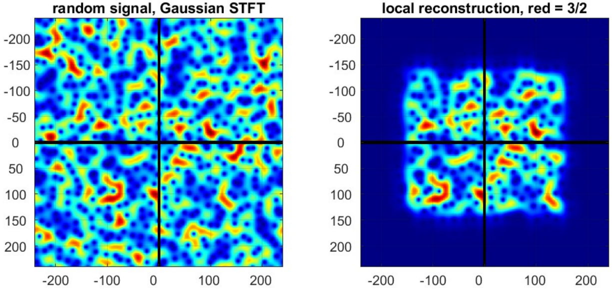


Figure: Local reconstruction of a signal via localization: up to the use of the so-called masking effect, a psycho-acoustic phenomenon, this is what **MP3** coding is doing, using pieces of length 512 samples, or 11.6 ms. Thus 132300 samples, cover up to 20kHz for a 3 minute piece of music.

Gabor atoms and the corresponding TF-lattice

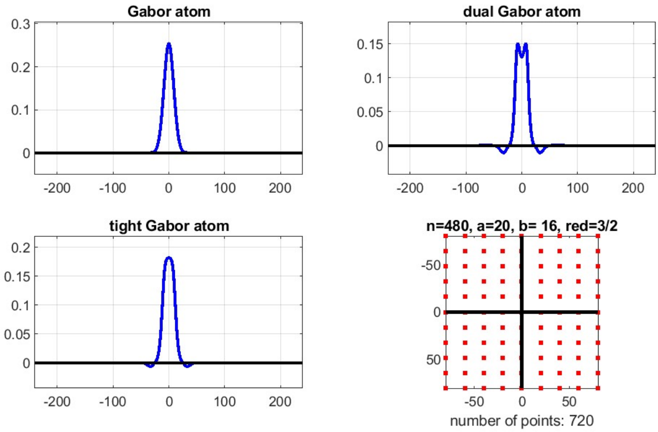


Figure: A (discrete) Gaussian atom, its dual (upper right) and the corresponding tight Gabor atom, together with TF-lattice used.

Fractional Fourier Transform I

The α-order fractional Fourier transform of $f \in \boldsymbol{L}^2(\mathbb{R})$ (!rotation in phase space!) is defined as:

$$\mathcal{F}_\alpha\{f(t)\} = F_\alpha(u) = \int_{-\infty}^{\infty} f(t) K_\alpha(t,u)\, dt,$$

where the kernel $K_\alpha(t,u)$ is given by:

$$K_\alpha(t,u) = \begin{cases} \sqrt{1 - i\cot\alpha}\; e^{i\pi(t^2+u^2)\cot\alpha - 2i\pi tu\csc\alpha}, & \alpha \notin n\pi, \\ \delta(t-u), & \alpha = 0, \\ \delta(t+u), & \alpha = \pi. \end{cases}$$

Each Hermite function (h_0 = Gaussian!!) is invariant under the group formed by these FrFTs! It is interesting to observe that the BGT $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ is invariant under the fractional FT, hence the "round symbol" for the involved spaces.

Fractional Fourier Transform II

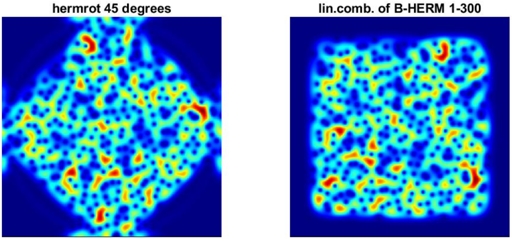
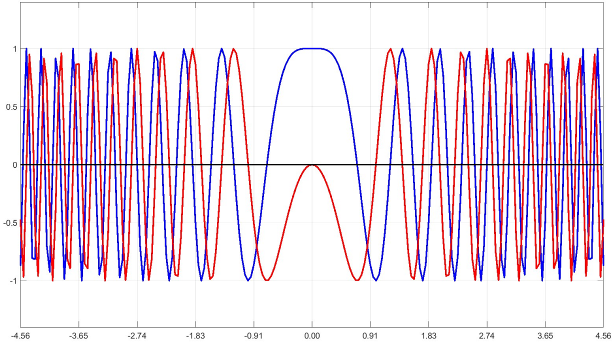


Figure: The spectrogram after FrFT. It can be implemented as Hermite multiplier or an ordinary FT conjugated with a poitwise multiplier with a chirp. It also maps chirps to chirps, changing the angle.

A sampled chirp signal

A chirp, typically of the form $t \mapsto exp(2\pi i \alpha t^2)$, or a linear frequency sweep, look like this. Algebraically it is a *character of second degree*. It is also used for geophysical exploration. Its FT is another (complementary) chirp, a complex Gaussian!



Basic principles of linear algebra I

We recall some elementary principles of finite-dimensional linear algebra, mostly for vector spaces over $\mathbb{C}$, or over $\mathbb{R}$ if real scalars are sufficient.

- A vector space consists of objects which can be added and multiplied by scalars. Thus one can form finite linear combinations
$$c_1 v_1 + \cdots + c_n v_n.$$
- A linear mapping $T : V \to W$ is a mapping which respects linear combinations:
$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 T(v_1) + \cdots + c_n T(v_n).$$
- A family $(v_i)_{i \in I}$ is a generating system if every vector in the space can be written as a linear combination of the v_i.

Basic principles of linear algebra II

- A family is linearly independent if no non-trivial finite linear combination gives zero.
- A basis is a family which is both generating and linearly independent. In finite dimension every vector then has a unique coordinate representation.
- Once bases are fixed in V and W, every linear mapping $T : V \to W$ is represented by a matrix. Composition of linear mappings corresponds to matrix multiplication.
- Invertibility of a linear mapping corresponds to invertibility of the representing matrix.

Thus finite-dimensional linear algebra provides the dictionary:

$$\text{vectors} \leftrightarrow \text{coordinates}, \qquad \text{linear maps} \leftrightarrow \text{matrices}.$$

Kernel Theorem I

The Kernel Theorem for linear operators is (or SHOULD BE) the analogue of matrix representations of a linear mapping T between finite-dimensional vector spaces, using bases (on the domain and the target space). Typically the columns of the matrix $\boldsymbol{A}$ which represents T describe the images of the basis vectors under T.
If the operator is already given and we only want to know its mapping properties it is enough to do the same for frames (with some extra information). A typical situation is the study of Gabor multipliers or even pseudo-differential operators using Gabor frames arising from Gabor atoms which are very well localized in the time-frequency sense.
What is lacking in the literature (both in pure mathematics, but also in the applied sciences) is a clear concept of continuous bases which would allow to transfer such ideas which are well understood in the finite-dimensional case to the case of continuous variables.

Kernel Theorem II

Discussion of the representation theorem of linear operators T by matrices $\boldsymbol{A}$. Choosing a fixed basis (say the unit vectors) in a finite dimensional vector space (say $\mathbb{R}^3$) we can represent any matrix by a 3x3 real matrix. The columns of $\boldsymbol{A}$ are just the images of the basis vectors, e.g. $T(\mathbf{e}_k)$ under T.
Equivalently the matrix is a superposition of the rank one operators of the form

$$\mathbf{x} \mapsto P_{j,k}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{e}_k \rangle \mathbf{e}_j.$$

This is also equivalent to the claim that s

$$\boldsymbol{A} = \sum_{j,k} a_{j,k} P_{j,k}.$$

Kernel Theorem III

A continuous analogue should look like

$$K(x,y) = \int_{\mathbb{R}^{2d}} K(u,v) P_{u,v}(x,y) dudv.$$

While it is obvious that this (double indexed family of rank-one operators) forms an ONB for the space $\mathcal{M}_{n,n}(\mathbb{C})$ of all complex-valued $n \times n$-matrices it is not so clear how to interpret mathematically the claim the these rank one operators are supposed to for an ONB for (?) the space of all Hilbert Schmidt operators (?), forming a resolution of the identity

$$\mathrm{Id} = \int_{u,v} P_{u,v} \, dudv.$$

Kernel Theorem IV

Given the identification of a separable Hilbert space one can characterize the space of all Hilbert-Schmidt operators exactly as the linear span of such projection operators, using the scalar product obtained by

$$\langle T, S\rangle_{\mathcal{HS}} := \mathrm{trace}(TS^*), \quad T, S \in \mathcal{HS}.$$

But for general (even bounded) linear operators on the Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ such characterizations are not available! In contrast, using the embedding

$$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$$

we can view any operator $T \in \mathcal{L}(\boldsymbol{L}^2(\mathbb{R}^d))$ as an operator from $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ to $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (even isometrically).

Kernel Theorem V

And the kernel theorem for Mild distributions allows to identify each operator in $\mathcal{L}\left(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d)\right)$ with a uniquely determined mild distribution σ_T such that

$$Tf(g) = \sigma(f \otimes g), \quad f, g \in \boldsymbol{S}_0(\mathbb{R}^d).$$

This opens up the way to many other representations via mild distributions!

In the theory of *pseudo-differential operators* the description of such an operator can be given via the Kohn-Nirenberg distributions which belongs to $\boldsymbol{S}_0'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

Since the symplectic Fourier transform is another unitary Banach Gelfand triple automorphism of $\boldsymbol{S}_0'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ we can look into the spreading representation of operators.

Kernel Theorem VI

Let us mention on the side that the theory of *Quantum Harmonic Analysis* is closely connected with convolution at the level of the Kohn-Nirenberg symbols.
The convolution of operators (actually their KNS-symbols) gives a function (in fact a mild distribution) over phase space $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, while the convolution of a function (on phase space) with an operator, rather with the KNS-symbol of an operator, gives and operator (the operator corresponding to the convolution product just obtained).
There are meanwhile quite a few interesting recent papers on this subject.

Rigged Hilbert Space Theory I

Definition 2.3 Given a rigged Hilbert space $\mathcal{D} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times$, a weakly measurable function $\omega : X \to \mathcal{D}^\times$ is called a *distribution basis* for $\mathcal{D}$ if, for every $f \in \mathcal{D}$, there exists a *unique* measurable function ξ_f such that:

$$\langle f|g\rangle = \int_X \xi_f(x) \langle \omega_x|g\rangle \, d\mu, \qquad \forall g \in \mathcal{D}$$

and, for every $x \in X$, the linear functional $f \in \mathcal{D} \mapsto \xi_f(x) \in \mathbb{C}$ is continuous on $\mathcal{D}[t]$.

Remark 2.4 If ω is a distribution basis, then it is μ-independent. However, a μ-independent function ω need not be, in general, a distribution basis, but only uniqueness of the coefficient function $\xi_f(x)$ is guaranteed.

Rigged Hilbert Space Theory II

If ω is a distribution basis, there exists a *unique* weakly measurable map $\theta : X \to \mathcal{D}^\times$ such that $\xi_f(x) = \langle f|\theta_x\rangle$, for every $f \in \mathcal{D}$. Hence the following identity holds:

$$\langle f|g\rangle = \int_X \langle f|\theta_x\rangle \, \langle \omega_x|g\rangle \, d\mu, \qquad \forall f, g \in \mathcal{D}. \tag{7}$$

Definition 3.13

A weakly measurable distribution function ω is called a *Parseval distribution frame* if

$$\int_X |\langle f|\omega_x\rangle|^2 \, d\mu = \|f\|^2, \quad f \in \mathcal{D}.$$

It is clear that a Parseval distribution frame is a bounded Bessel distribution map and a frame in the sense of Definition 3.6 with $S_\omega = I_\mathcal{D}$, the identity operator of $\mathcal{D}$.

Rigged Hilbert Space Theory III

The book by Antoine and Trapani on Partial Inner product spaces describes the following situation:

Definition 2. Let $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ be a rigged Hilbert space and $\zeta : x \in X \mapsto \zeta_x \in \mathcal{D}^\times$ be a weakly measurable map from $\mathcal{D}$ into $\mathcal{D}^\times$. We say that ζ is a *Gelfand distribution basis* if ζ is μ-independent and satisfies the Parseval identity; i.e.,

$$\int_X |\langle f|\zeta_x\rangle|^2 d\mu = \|f\|^2, \qquad f \in \mathcal{D}. \tag{3}$$

From linear algebra to bra–ket notation I

In finite-dimensional Hilbert spaces, the language of vectors, coordinates, matrices and scalar products can also be expressed in Dirac's bra–ket notation.

- A vector v is written as a ket $|v\rangle$.
- A continuous linear functional, in finite dimension just a row vector or adjoint vector, is written as a bra $\langle w|$.
- The scalar product is written as

$$\langle w|v\rangle.$$

- A linear operator A is inserted between bra and ket:

$$\langle w|A|v\rangle.$$

From linear algebra to bra–ket notation II

- In matrix notation this is simply

$$\langle w|A|v\rangle = w * A * v'.$$

  Here we use *row vectors* and v' is the transpose conjugat (column) vector s
- Thus bra–ket notation is a compact notation for the ordinary finite-dimensional matrix formalism, but it also prepares the transition to Hilbert spaces and to distributional coordinate systems.

The important point is that the notation should not obscure the underlying linear-algebraic structure: vectors, functionals, operators and pairings must still live in compatible spaces.

Generating systems and finite frames I

Let $\mathcal{H} = \mathbb{C}^d$ be a finite-dimensional Hilbert space, and let

$$\Phi = (\varphi_1, \ldots, \varphi_N)$$

be a finite family of vectors in $\mathcal{H}$.

- Φ is a generating system if

$$\operatorname{span}\{\varphi_1, \ldots, \varphi_N\} = \mathcal{H}.$$

- Equivalently, every $x \in \mathcal{H}$ has at least one representation

$$x = \sum_{k=1}^{N} c_k \varphi_k.$$

- If $N > d$, such a representation is usually not unique.

Generating systems and finite frames II

- Writing the vectors φ_k as columns of a matrix

$$A = (\varphi_1 \cdots \varphi_N),$$

the representation problem becomes

$$Ac = x.$$

- Thus Φ is generating if and only if

$$\mathrm{rank}(A) = d.$$

Generating systems and finite frames III

In finite dimension, every generating system is a frame. More explicitly, there exist constants $0 < A_0 \leq B_0 < \infty$ such that

$$A_0\|x\|^2 \leq \sum_{k=1}^{N} |\langle x, \varphi_k \rangle|^2 \leq B_0\|x\|^2, \qquad x \in \mathcal{H}.$$

The synthesis operator is

$$D_\Phi c = \sum_{k=1}^{N} c_k \varphi_k = Ac,$$

and the analysis operator is

$$C_\Phi x = (\langle x, \varphi_k \rangle)_{k=1}^{N} = A^* x.$$

Generating systems and finite frames IV

The frame operator is

$$S_\Phi = D_\Phi C_\Phi = AA^*, \qquad S_\Phi x = \sum_{k=1}^{N} \langle x, \varphi_k \rangle \varphi_k.$$

Thus the finite frame condition is exactly the invertibility of S_Φ.

The pseudo-inverse and the MNLSQ solution I

Let

$$A = U\Sigma V^*$$

be the singular value decomposition of A. The Moore–Penrose pseudo-inverse is defined by

$$A^\dagger = V\Sigma^\dagger U^*,$$

where $\Sigma^\dagger$ is obtained by replacing every non-zero singular value σ_j by $1/\sigma_j$ and leaving the zero singular values at zero.

The pseudo-inverse provides the minimal-norm least-squares solution, briefly the MNLSQ solution:

$$c^\dagger = A^\dagger x = \arg\min \left\{ \|c\|_2 : c \in \arg\min_{d \in \mathbb{C}^N} \|Ad - x\|_2 \right\}.$$

The pseudo-inverse and the MNLSQ solution II

Thus $A^{\dagger}x$ first minimizes the residual

$$\|Ac - x\|_2,$$

and, among all least-squares solutions, chooses the one with minimal Euclidean norm.
For a generating system

$$\Phi = (\varphi_1, \ldots, \varphi_N)$$

with synthesis matrix

$$A = (\varphi_1 \ \cdots \ \varphi_N) : \mathbb{C}^N \to \mathbb{C}^d,$$

every vector $x \in \mathbb{C}^d$ has representations

$$x = Ac.$$

The pseudo-inverse and the MNLSQ solution III

If $N > d$, the coefficient vector c is usually not unique. Since A has full row rank in the generating case, the least-squares residual can be made zero. Hence the MNLSQ solution is simply the exact coefficient vector of minimal Euclidean norm:

$$c^{\dagger} = A^{\dagger}x.$$

If A has full row rank, equivalently if Φ is generating, then

$$A^{\dagger} = A^{*}(AA^{*})^{-1}.$$

Since

$$S_{\Phi} = AA^{*},$$

this can be written as

$$A^{\dagger} = A^{*}S_{\Phi}^{-1}.$$

The pseudo-inverse and the MNLSQ solution IV

Thus the MNLSQ coefficient vector is

$$c^{\dagger} = A^* S_{\Phi}^{-1} x = \left(\langle S_{\Phi}^{-1} x, \varphi_k \rangle\right)_{k=1}^{N}.$$

Equivalently, with the canonical dual frame

$$\widetilde{\varphi}_k = S_{\Phi}^{-1} \varphi_k,$$

one obtains the reconstruction formula

$$x = \sum_{k=1}^{N} \langle x, \widetilde{\varphi}_k \rangle \varphi_k.$$

Thus the canonical dual frame coefficients are exactly the MNLSQ coefficients provided by the Moore–Penrose pseudo-inverse.

Retracts in Interpolation Theory

A linear mapping $\mathcal{C} : \boldsymbol{X} \to \boldsymbol{Y}$ defines a retract from $\boldsymbol{X}$ into $\boldsymbol{Y}$ if there exists a bounded linear mapping $\mathcal{R} : \boldsymbol{Y} \to \boldsymbol{X}$ such that

$$\mathcal{R} \circ \mathcal{C} = \mathrm{Id}_{\boldsymbol{X}} \,.$$

Set $\boldsymbol{Y}_0 := \mathrm{range}(\mathcal{C})$. Then $P := \mathcal{C} \circ \mathcal{R}$ is an idempotent operator on $\boldsymbol{Y}$, and therefore $\boldsymbol{Y}_0$ is a closed complemented subspace of $\boldsymbol{Y}$.

Moreover, $\mathcal{C}$ establishes an isomorphism $\mathcal{C} : \boldsymbol{X} \to \boldsymbol{Y}_0$.
Thus $\boldsymbol{X}$ may be identified with $\boldsymbol{Y}_0$, and also with the quotient

$$\boldsymbol{Y} / \ker(\mathcal{R}).$$

Retract diagram

$$
\begin{tikzcd}
 & & \boldsymbol{Y} \arrow[dll, "\mathcal{R}"'] \arrow[d, "P"] \\
\boldsymbol{X} \arrow[rr, shift right, "\mathcal{C}"'] & & \boldsymbol{Y}_0 \arrow[ll, shift right, "\mathcal{R}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \ell^2(I) \arrow[dl, "\mathcal{R}"'] \arrow[d, "\mathbf{P}"] \\
\mathcal{H} \arrow[r, shift right, "\mathcal{C}"'] & C(\mathcal{H}) \arrow[l, shift right, "\mathcal{R}"']
\end{tikzcd}
$$

Banach Frames for the Banach Gelfand Triples

$$
\begin{tikzcd}
 & (\ell^1, \ell^2, \ell^\infty)(\Lambda) \arrow[dl, "\mathcal{R}"'] \arrow[d, "\mathbf{P}"] \\
(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0') \arrow[r, shift right, "\mathcal{C}"'] & C((\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')) \arrow[l, shift right, "\mathcal{R}"']
\end{tikzcd}
$$

Gabor frames generated by the Gaussian I

Let

$$\mathcal{H} = \boldsymbol{L}^2(\mathbb{R}^d), \qquad g(x) = \mathrm{e}^{-\pi|x|^2}, \quad x \in \mathbb{R}^d.$$

For $z = (x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$ we write

$$\pi(z)g = M_\omega T_x g,$$

where

$$T_x f(t) = f(t - x), \qquad M_\omega f(t) = \mathrm{e}^{2\pi i\, \omega \cdot t} f(t).$$

Choose the rectangular lattice

$$\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d, \qquad 0 < a, b < 1,$$

for instance $a = b < 1$, and assume

$$ab < 1.$$

Gabor frames generated by the Gaussian II

Then the Gaussian Gabor system

$$\mathcal{G}(g,\Lambda) = \{M_{bn}T_{ak}g : k,n \in \mathbb{Z}^d\}$$

is a Gabor frame for $\boldsymbol{L}^2(\mathbb{R}^d)$. Hence every $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is described by its Gabor coefficients

$$\langle f, M_{bn}T_{ak}g\rangle.$$

The Gabor frame operator is

$$S_g f = \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{bn}T_{ak}g\rangle M_{bn}T_{ak}g.$$

It is bounded, positive and invertible on $\boldsymbol{L}^2(\mathbb{R}^d)$.

Gabor frames generated by the Gaussian III

The canonical dual Gabor atom is

$$\widetilde{g} = S_g^{-1} g.$$

It gives the reconstruction formula

$$f = \sum_{k,n \in \mathbb{Z}^d} \langle f, M_{bn} T_{ak} \widetilde{g} \rangle M_{bn} T_{ak} g,$$

and also

$$f = \sum_{k,n \in \mathbb{Z}^d} \langle f, M_{bn} T_{ak} g \rangle M_{bn} T_{ak} \widetilde{g}.$$

The canonical tight Gabor atom is obtained from the inverse square root:

$$h = S_g^{-1/2} g.$$

Gabor frames generated by the Gaussian IV

Then

$$\mathcal{G}(h, \Lambda) = \{M_{bn}T_{ak}h : k, n \in \mathbb{Z}^d\}$$

is a tight, in fact Parseval, Gabor frame:

$$f = \sum_{k,n \in \mathbb{Z}^d} \langle f, M_{bn}T_{ak}h \rangle M_{bn}T_{ak}h,$$

and

$$\|f\|_2^2 = \sum_{k,n \in \mathbb{Z}^d} |\langle f, M_{bn}T_{ak}h \rangle|^2.$$

Thus g is the original analysis atom, $\widetilde{g}$ gives stable dual reconstruction, while h provides the best balanced tight version of the same Gabor system.

The role of time-frequency analysis in different areas

Let us just give a short list of examples connecting the mathematics of time-frequency and Gabor analysis to real-world applications:

- The Physics Nobel Prize 2017 was given for the detection of *gravitational waves* using time-frequency methods;
- early contact to colleagues from communication theory at TU Wien exposed us to slowly time-variant channels, Kohn–Nirenberg calculus and Gabor expansions;
- MP3 performs a localized DFT in order to discard perceptually irrelevant information;
- wavelet theory is used for digital image compression;
- Quantum Harmonic Analysis is of current interest;
- reconstruction of band-limited signals from irregular samples.

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

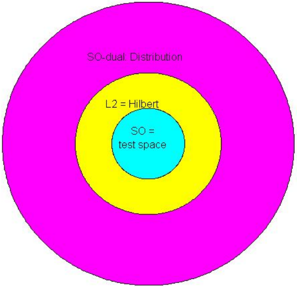
If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!

the RIGGED Hilbert Space situation



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{4}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

THE Banach Gelfand Triple Situation

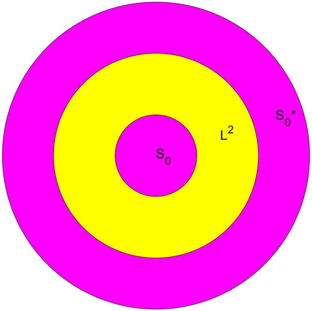


Figure: Three layers of Banach spaces, with the Hilbert space in the middle

BGT Morphism: Four Layers

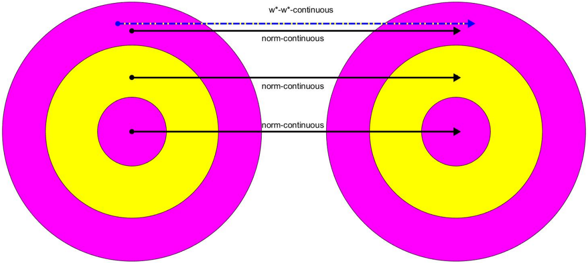


Figure: BGT3morph1.jpg: Note that there are four continuity requirements!

Regularizing operators on BGTs

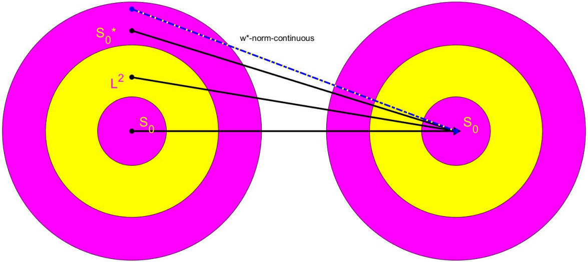


Figure: Regularizing operators: w^*-convergent nets to norm convergent ones.

Good operators compared to Trace-Class Operators

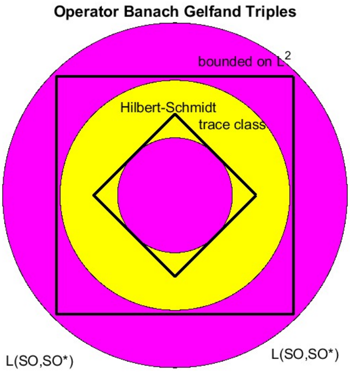


Figure: QRCtrace26a.jpg: Larger predual (trace class) implies smaller dual space: only operators bounded on L2!

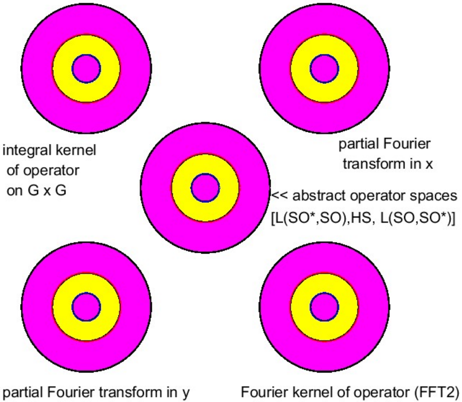


Figure: The spaces of operators $(\boldsymbol{B}, \mathcal{H}S, \boldsymbol{B}')$ can be identified with four different representations of operators.

Why should we care about convolutions?

In engineering, especially in communication theory, the *theory of time-invariant channels*, i.e. operators which are linear and commute with translations (motivated by the time-invariance of physical laws!), also called TILS (time-invariant linear systems) is one of the cornerstones. The collection of all these operators form a commutative algebra, with the pure frequencies as common eigenvalues.

Still in an engineering terminology: Every such system T is a *moving average* or (equivalently) a *convolution operator*, described by the *impulse response* of the system, or alternatively, it can be described as a *Fourier multiplier*, being understood as a pointwise multiplier on the Fourier transform side, by the so-called *transfer function*.

The concrete case of BIBOs systems

From my point of view the most natural setting is that of so-called *BIBOS*, bounded-input-bounded-output systems, or more precisely the bounded linear operators on $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ which commute with translations.
It is not hard to show that they are exactly the *convolution operators* which are arising from a given linear functional $\mu \in \boldsymbol{M}_b(\mathbb{R}^d) := (\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'})$:

$$\mu * f(x) = \mu(T_x f^\checkmark), \quad x \in \mathbb{R}^d, \tag{5}$$

where $f^\checkmark(x) = f(-x), x \in \mathbb{R}^d$.

Any TILS on $C_0(\mathbb{R}^d)$ is if this form!
Given T choose $\mu(f) := [T(f^\checkmark)](0)$, $f \in \boldsymbol{C}_0(\mathbb{R}^d)$.

More general "multipliers"

According to R. Larsen (his book on the multiplier problem appeared in 1972) it is meaningful to ask for a characterization of the space of "multipliers", i.e. all linear, bounded operators from one translation invariant Banach space into another one, which commute with translations. You may think of such operators from $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ to $(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q)$, for two values $p, q \in [1, \infty]$. Using the concept of *quasi-measures* introduced by G. Gaudry he demonstrates that each such operator is a convolution operator by some quasi-measure, which also has an alternative description as a Fourier multiplier. The drawback of the concept of quasi-measures is the fact, that they do not involve any global condition and thus a general quasi-measure does NOT have a well defined Fourier transform. So the expected claim that the transfer function is the FT of the impulse response (and vice versa) cannot be formulated in that context.

Multipliers between L^p-spaces

The key result for multipliers be summarized as follows:

Theorem

Any bounded linear operator from $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ *into* $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ *which commutes with translations can be described by the convolution with a uniquely* $\sigma \in \mathbf{S}_0'(\mathbb{R}^d)$, *i.e. via*

$$Tf(x) = \sigma(T_x[f^{\checkmark}]), \quad f \in \mathbf{S}_0(\mathbb{R}^d), x \in \mathbb{R}^d.$$

Thus in fact, T *maps* $\mathbf{S}_0(\mathbb{R}^d)$ *even into* $(\mathbf{C}_b(\mathbb{R}^d), \|\cdot\|_\infty)$, *with norm equivalence between the operator norm of the operator* T *and the functional norm (in* $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$*) of* σ.
If σ *is a regular distribution, induced by some test function* $h \in \mathbf{S}_0(\mathbb{R}^d)$, *then we even have that* $f \mapsto h * f$ *(equivalently given in a pointwise sense) maps* $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ *into* $(\mathbf{C}_b(\mathbb{R}^d), \|\cdot\|_\infty)$.

Banach Gelfand triples as a general framework I

Banach Gelfand triples appear naturally in many different situations. The basic model is $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(G)$, where $\boldsymbol{S}_0(G)$ is Feichtinger's algebra on the LCA group G, and $\boldsymbol{S}_0'(G)$ is its dual space, the space of mild distributions.

The concrete form depends on the group. For compact groups, $\boldsymbol{S}_0(G)$ is essentially the Wiener algebra, i.e. the space of functions with absolutely summable Fourier coefficients. For discrete groups one obtains $\boldsymbol{S}_0(G) = \ell^1(G)$. Thus the same concept covers continuous, discrete, compact and finite situations within one common language.

The space of mild distributions allows us to relate objects on a given group, for example on $\mathbb{R}^d$, to corresponding objects on finite subquotients. This is important because finite models should not be viewed as unrelated discretizations, but as finite-dimensional shadows of the same underlying phase-space structure.

Banach Gelfand triples as a general framework II

This gives a natural perspective on numerical approximation. One can pass from continuous objects to finite models, perform computations there, and interpret the result again in the ambient Banach Gelfand triple. In this sense the triple

$$\boldsymbol{S}_0(G) \subset L^2(G) \subset \boldsymbol{S}_0'(G)$$

acts as a bridge between abstract harmonic analysis and computational harmonic analysis.
Moreover, Banach Gelfand triples form a category with natural morphisms: many relevant operators act simultaneously on the test-function space, on the Hilbert space, and on the distribution space. Typical examples are the Fourier transform, automorphisms of the group, time-frequency shifts, and metaplectic operators in the Euclidean case.

Banach Gelfand triples as a general framework III

Finally, the operator theory associated with these triples is especially rich. The kernel theorem identifies operators with mild distributions on product groups. In phase-space language, the spreading representation and the Kohn–Nirenberg symbol provide complementary descriptions of operators. These tools are central for understanding Gabor multipliers, pseudodifferential operators and structure-preserving approximations.

Main message

Banach Gelfand triples provide a common language for analysis of signals or operators and their representations, using discrete non-orthogonal expansions such as Gabor expansions or Wilson basis, but it alos opens the way to numerical computations.

Take-Home Message

- The Fourier transform is not only a formula, but a change of coordinates adapted to translation, oscillation, and symmetry.
- Basic principles appear in finite-dimensional linear algebra: bases, matrices, scalar products, duality, diagonalization.
- Passing from finite to continuous models requires more than Hilbert spaces: Dirac objects, sampling, kernels, and operators naturally lead to distributional and Banach-space viewpoints.
- The Banach Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ provides a robust framework for test functions, signals, and mild distributions.
- Gabor analysis give localized Fourier tools which connect rigorous analysis with signal processing, physics, engineering, and numerical approximation.

Fourier analysis is a conceptual bridge from linear algebra to modern applied mathematics.

Conceptual Harmonic Analysis (CHA) I

The guiding philosophy of this survey talk is the transition from finite-dimensional and discrete models towards continuous structures relevant for physics and engineering.

Classical linear algebra provides the basic intuition: vectors, matrices, orthogonality, spectral decompositions, and finite Fourier transforms. The passage to continuous-variable models is usually achieved through functional analysis, replacing finite-dimensional spaces by objects such as $\boldsymbol{L}^1(\mathbb{R}^d)$, $\boldsymbol{L}^2(\mathbb{R}^d)$ via Plancherel theory, or $\boldsymbol{S}_0(\mathbb{R}^d)$, together with the corresponding operator-theoretic framework.

Conceptual Harmonic Analysis (CHA) II

CHA proposes a slightly different perspective, inspired by modern time-frequency analysis and Gabor analysis. It emphasizes localization in phase space and the use of Banach-space methods adapted to signal analysis.

A central role is played by the Banach Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, generated from $\boldsymbol{S}_0(\mathbb{R}^d)$, the Feichtinger algebra. This is a Fourier-invariant Segal algebra, together with its Hilbert-space middle term and its dual space $\boldsymbol{S}_0'(\mathbb{R}^d)$, the space of *mild distributions*.

Mild distributions may be characterized through uniformly bounded spectrograms and often reflect intuitive notions of 'signals", such as audio or optical signals, and of 'operators", more faithfully than the traditional distributional framework.

Conceptual Harmonic Analysis (CHA) III

Overall, the CHA approach allows one to integrate ideas from abstract harmonic analysis over LCA groups, computational methods, and mathematically rigorous models for a variety of applications in engineering and physics.

This framework is appropriate for topics such as:

- summability methods and modulation spaces;
- impulse response and transfer functions;
- Gabor expansions and frame theory;
- pseudo-differential operators and Quantum Harmonic Analysis;
- kernel theorems in analogy to matrix representations;
- discrete approximation of test functions;
- structure-preserving discretization and norm estimates;
- generalized stochastic processes.

From ℓ^2 to L^2 and the Role of Trace Class Operators I

- In the discrete setting, the natural Banach Gelfand Triple is

$$(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty), \quad \text{with} \quad \ell^1(\mathbb{Z}^d) \subset \ell^2(\mathbb{Z}^d) \subset \ell^\infty(\mathbb{Z}^d),$$

  with the dual pairing extending the scalar product on ℓ^2.

- Passing to continuous variables suggests to consider the NON-VALID! spaces:

$$L^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d),$$

  and the resulting scalar product for $L^2(\mathbb{R}^d)$ being given by

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f(x)\, \overline{g(x)}\, dx.$$

From ℓ^2 to L^2 and the Role of Trace Class Operators II

- In analogy, one considers spaces of operators:
  - Hilbert–Schmidt operators $\mathcal{HS}$ as the analogue of L^2,
  - Trace class operators $\mathcal{L}^1$ as the analogue of L^1.
- The Hilbert–Schmidt class carries a natural inner product:

$$\langle T, S \rangle_{\mathcal{HS}} := \mathrm{trace}(S^* T),$$

  which is the operator analogue of the L^2 scalar product.
- Thus:
  - vectors $\leftrightarrow$ functions, or (mild) distributions;
  - matrices $\leftrightarrow$ operators (not only bounded operators on $\boldsymbol{L}^2(\mathbb{R}^d)$)
  - matrix trace $\leftrightarrow$ integral.
- This analogy provides the bridge from finite-dimensional linear algebra to operator theory in harmonic analysis.

Topics not covered in this talk

- the relationship between $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, or $\boldsymbol{S}_0'(\mathbb{R}^d)$ and $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, the tempered distributions;
- the connection between the finite-dimensional case, FFT algorithms, and the $\boldsymbol{S}_0$-setting on the real line, for example via sampling, periodization, and piecewise linear interpolation;
- the many variants of the kernel theorem for mild distributions, including Kohn–Nirenberg symbols and the spreading representation;
- modulation spaces as time-frequency analogues of Besov spaces, in particular for time-frequency operators;
- the theory of Gabor multipliers and their symbolic calculus;
- connections to PDEs, especially the nonlinear Schrödinger equations.

These topics indicate the broader scope of Conceptual Harmonic Analysis.

Links and references I

Thanks for your attention!

Let us provide a few links.

ETH Course Notes are found at:

www.nuhag.eu/ETH20

At the page

www.nuhag.eu/feitalks

you find my talks and also recorded talks from some courses, for example

From linear algebra to Gelfand triples.

Links and references II

Various Lecture Notes can be downloaded from

https://www.univie.ac.at/nuhag-php/home/skripten.php

The public version of my publication list can be provided upon request via

hans.feichtinger@univie.ac.at.

This talk provides some thoughts related to the idea of

Conceptual harmonic analysis.