Gabor Analysis, Mild Distributions and Rigged Hilbert Spaces: Concepts, Tools and Applications

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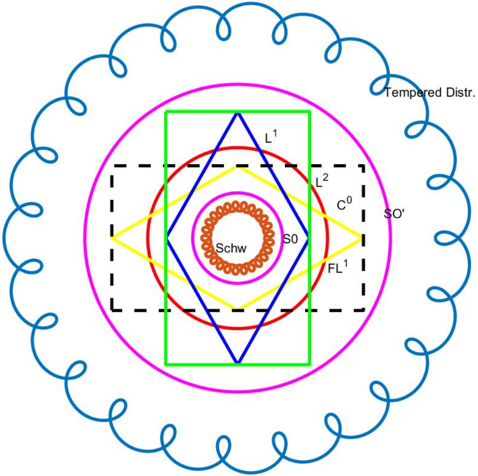
Outline

A Zoo of Banach Spaces for Fourier Analysis



Rigged Hilbert Spaces in Quantum Mechanics I

Historical background

The mathematical concept of a Rigged Hilbert Space (or Gelfand Triple) originated in the work of Gelfand and his school during the 1950s, providing a rigorous setting for generalized eigenvectors and spectral expansions.

Rigged Hilbert Spaces in Quantum Mechanics II

The contribution of Arno Böhm

Arno Böhm (1936–2024) recognized that the Rigged Hilbert Space provides the natural mathematical language for quantum mechanics whenever one has to deal with

- continuous spectra,
- Dirac's generalized eigenvectors,
- scattering theory,
- resonances and Gamow vectors,
- irreversible quantum evolution.

Together with collaborators (notably Manuel Gadella), he developed the Rigged Hilbert Space approach into a systematic framework for mathematical quantum mechanics.

Rigged Hilbert Spaces in Quantum Mechanics III

Relation to the present talk

The present lecture advocates a Banach Gelfand Triple $(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))$, which may be viewed as a Banach-space analogue of the classical Rigged Hilbert Space $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$
It is designed primarily for time-frequency analysis, signal processing, sampling, Banach frames, and operator theory.

For the physics viewpoint see, e.g., A. Böhm, *The Rigged Hilbert Space and Quantum Mechanics*, Lecture Notes in Physics, Springer, 1978,
and
A. Böhm and M. Gadella, *Dirac Kets, Gamow Vectors and Gel'fand Triplets*, Lecture Notes in Physics 348, Springer, 1989.

From ℓ^2 to L^2 and the Role of Trace Class Operators I

- In the discrete setting, the natural Banach Gelfand Triple is

$$(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty), \quad \text{with} \quad \ell^1(\mathbb{Z}^d) \subset \ell^2(\mathbb{Z}^d) \subset \ell^\infty(\mathbb{Z}^d),$$

with the dual pairing extending the scalar product on ℓ^2.

- Passing to continuous variables suggests to consider the NON-VALID! spaces: $L^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, with the scalar product for $\boldsymbol{L}^2(\mathbb{R}^d)$ given by $\langle f, g \rangle = \int_{\mathbb{R}^d} f(x)\, \overline{g(x)}\, dx$, one better uses the scale, increasing for $p \in [1, \infty]$ and closed under duality, with $1/p + 1/q = 1$ for $p < \infty$:

$$\boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{M}^1(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}^p(\mathbb{R}^d) \hookrightarrow \boldsymbol{M}^\infty(\mathbb{R}^d) = \boldsymbol{S}_0'(\mathbb{R}^d).$$

From ℓ^2 to L^2 and the Role of Trace Class Operators II

- In analogy, one considers spaces of operators:
  - Hilbert–Schmidt operators $\mathcal{HS}$ as the analogue of L^2,
  - Trace class operators $\mathcal{L}^1$ as the analogue of L^1.
- The Hilbert–Schmidt class carries a natural inner product:

$$\langle T, S\rangle_{\mathcal{HS}} := \mathrm{trace}(S^* T),$$

  which is the operator analogue of the L^2 scalar product.
- Thus:
  - vectors $\leftrightarrow$ functions, or (mild) distributions;
  - matrices $\leftrightarrow$ operators (not only bounded operators on $\boldsymbol{L}^2(\mathbb{R}^d)$)
  - matrix trace $\leftrightarrow$ integral.
- This analogy provides the bridge from finite-dimensional linear algebra to operator theory in harmonic analysis.

Slight Variant of the Submitted Abstracts I

Starting from the finite dimensional situation described very well by the linear algebra interpretation of Paul Dirac's Bra-Ket notation it is natural to first move to the setting of Hilbert spaces such as $\boldsymbol{L}^2(\mathbb{R}^d)$. But already the analysis of pointwise multipliers shows that decent bounded operators may not have true eigenvectors (one expects Dirac measures in this situation). This problem has lead to the introduction of "rigged Hilbert spaces", such as $(\mathcal{S}, \boldsymbol{L}^2, \mathcal{S}')(\mathbb{R}^d)$, making use of Schwartz tempered distributions. They are also used in advanced engineering courses to deal with sampling (Dirac combs) or in order to introduce the description of linear operators by their distributional kernel, allowing also alternative descriptions, such as the Wigner/Weyl or Kohn-Nirenberg or Kirkwood-Dirac calculus.

It is the **goal of this talk to demonstrate that a technically much less sophisticated and often more efficient description**

Slight Variant of the Submitted Abstracts II

of the situation can be given using tools which have first shown to be useful in the context of Time-Frequency or Gabor Analysis. It avoids the use of topological vector space but is based on simple principles of linear functional analysis.

The key tool is the Banach Gelfand Triple based on the Feichtinger algebra $\boldsymbol{S}_0(\mathbb{R}^d)$, the dual space, the space of mild distributions, together with the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$. There is a natural description using Gabor expansions of distributions. With the help of Wilson bases one can establish a unitary isomorphism to the triple $(\ell^1, \ell^2, \ell^\infty)$. Among others a number of concepts appearing in physics and engineering (Dirac bases, impulse response, transfer function, etc.) can be formulated properly in this context, and the analogy to the finite dimensional situation can be realized in a rather intuitive way, also paving the way for numerical work.

The key-players for time-frequency analysis I

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x}\hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle,\ \ \lambda = (t,\omega);$$

Some important properties of the STFT I

Starting from the Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ with scalar product

$$\langle f, g \rangle_{\boldsymbol{L}^2} = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$$

one can find that for fixed g the linear mapping $f \mapsto V_g f$ is isometric from $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ into $(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2)$. But also

$$\|V_g f\|_\infty = \max_{\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(\lambda)| \leq \|f\|_2 \|g\|_2$$

by the Cauchy-Schwarz inequality, since $\|\pi(\lambda)g\|_2 = \|g\|_2$, for any $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

There is also orthogonality for the windows, since we have

$$V_g f(\lambda) = \langle f, \pi(\lambda)g \rangle = \langle \pi(\lambda)^* f, g \rangle,$$

Some important properties of the STFT II

which gives essentially symmetry between f and g.
For the Fourier transform we have:

$$V_g f(t,s) = e^{2\pi i t s} V_{\widehat{g}} \widehat{f}(s,-t).$$

Next comes the so-called *covariance property* of the STFT.

Lemma (3.1.3)

Whenever $V_g f$ is defined, we have

$$V_g(T_u M_\eta f)(x,\omega) = e^{-2\pi i u\cdot\omega} V_g f(x-u,\omega-\eta) \tag{3.14}$$

for $x,u,\omega,\eta \in \mathbb{R}^d$. In particular,

$$|V_g(T_u M_\eta f)(x,\omega)| = |V_g f(x-u,\omega-\eta)|.$$

Reconstruction Formula

The isometric property of the embedding $f \mapsto V_g f$ of $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ into $(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2)$ implies that the adjoint linear mapping, given by

$$V_g^* : F \mapsto \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g\delta_\lambda$$

is the inverse on the range, and we recover f back from $V_g f$ via

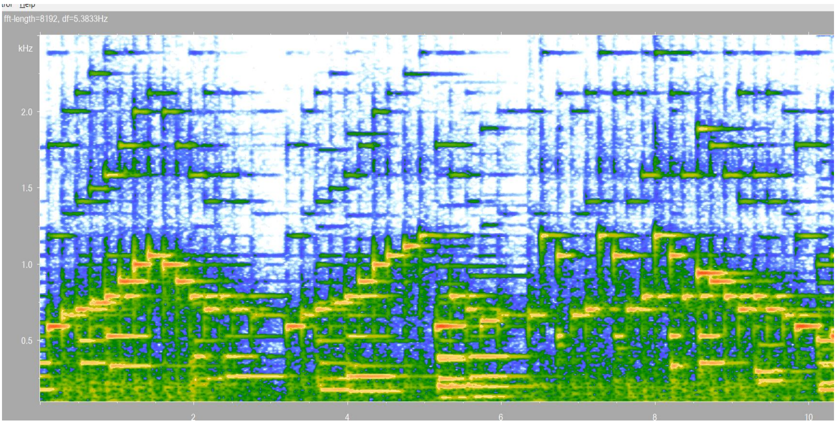
$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f \pi(\lambda) g d\lambda = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda \rangle g_\lambda d\lambda. \qquad (1)$$

Compare with the Fourier inversion theorem:

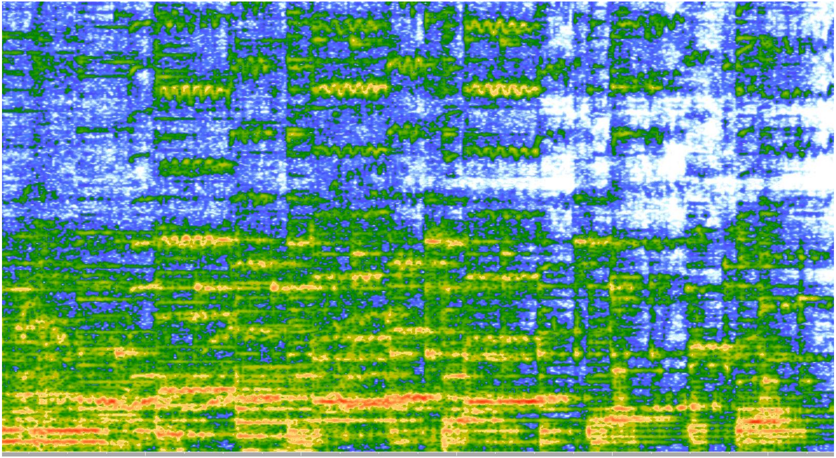
$$f = \int_{\widehat{\mathbb{R}}^d} \langle f, \chi_s \rangle \chi_s ds.$$

A Typical Musical STFT

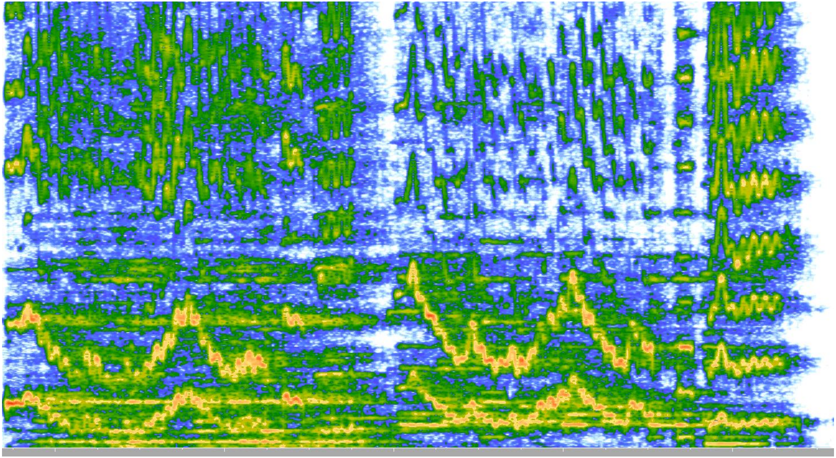
A typical piano spectrogram (Mozart), from digital recording (realized with the help of the STX software, free download)



A Musical STFT: Brahms, Cello

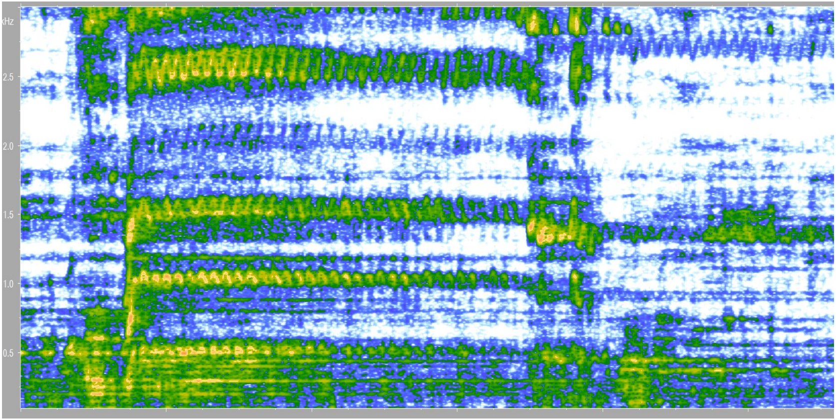


A Musical STFT: Maria Callas



A Musical STFT: Tenor: VINCERO!

Obtained via STX Software from ARI (Austrian Acad. Sci.)



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{\boldsymbol{S}_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $\boldsymbol{M}^1 = \boldsymbol{S}_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}, \quad \forall f \in \boldsymbol{S}_0$.

In fact, $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

Various Function Spaces

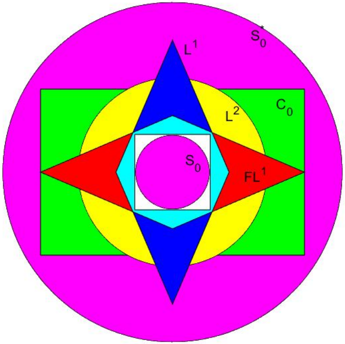


Figure: The usual Lebesgues space, the Fourier algebra, and the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ inside all these spaces

Motivation and Background I

Given my *personal background* (study of basic physics, education in Abstract Harmonic Analysis (AHA), and a record of application oriented European projects) I observed over the years an increasing gap between they way how Fourier Analysis is performed in different disciplines, notably in **Mathematics, Physics** and **Engineering**. The differences are **by notation, by conventions** (level of precision) and **by tradition**.

Over the years it become more and more that somewhat different tools (e.g. more suitable Banach spaces of distributions) are needed, and that a new approach, called **Conceptual Harmonic Analysis** (CHA) is required.

I was first thinking of unifying abstract and computational Fourier Analysis with a twist towards real world applications (e.g. in mobile communication of digital signal processing) Now I think more of reaching out to physics and engineering as well.

Motivation and Background II

Nowadays I see it more as a kind of **INFRA-STRUCTURE PROJECT** in the scientific domain. We have to establish common goals, common terminology and tools to identify best practice examples and figures of merit.

Of course this is a *long term task*, requiring to develop many aspects: provide examples and arguments, convince steak-holders, demonstrate effective advantages over traditional methods, take into account psychological aspects (traditions, etc.) and encourage the necessary investments (in the form doing ground-work, learning things, preparing the ground for future developments).

In fact, such things occur in may areas and might be worthwhile being considered from this perspective, such as

- mobility (automobiles, airplanes, trains);
- travel platforms (customer opinion, filters, personal settings)
- need for integrated and versatile systems.

Motivation and Background III

In order to connect to the topic of this conference let me mention that for me '**Harmonic Analysis** is the branch of Functional Analysis (dealing with linear space of functions, distributions, linear operators, spaces of such operators and their spectral properties, etc.) which satisfy some invariance properties. Needless to say that the natural setting for HA are LCA groups. The same is true for Quantum Harmonic Analysis. For signal processing tasks we are

happy to use the FFT as part of a fast algorithm, and in Time-Frequency Analysis (relevant e.g. for mobile communication) the Heisenberg group is a fundamental object, also relevant for Quantum Information Theory.

Finally some of the most well-known examples of applied TFA are the MP3 compression scheme and the detection of gravitational waves by the LIGO team (using Wilson basis).

Motivation and Background IV

The central object of my approach is the so-called **Banach Gelfand Triple** $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, a kind of rigged Hilbert spaces, but with the difference (compared to the usual examples) that it consists of three Banach spaces, with a lot of good (invariance) properties (among others Fourier invariance). The space of distributions arising in this context is nowadays called the space of *mild distributions*

Let me express right away that these objects (like the sound arising when a piece of music is played) appear to be a convenient model for what is usually called a **signal**, while the corresponding test functions from the space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ (known meanwhile as Feichtinger algebra) describe the valid measurements, so that obviously the pairing $(\sigma, f) \to \sigma(f)$ defines a continuous, bilinear pairing.

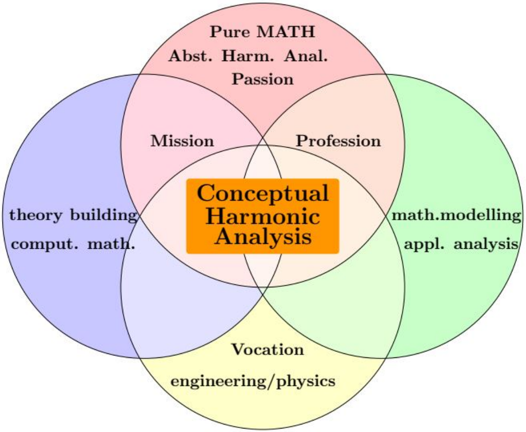
Motivation and Background V

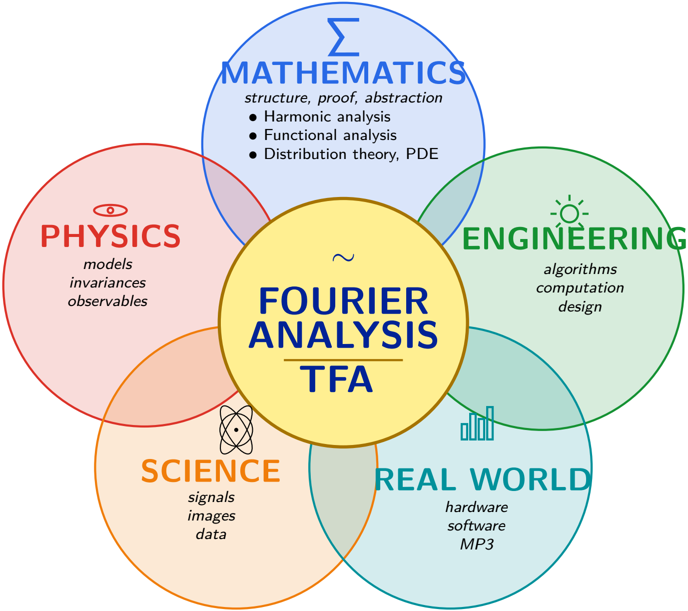
So let me also shortly mention the aspects which one might address in a course on this topic, namely

1. How are the spaces defined and their properties;
2. How do they compare to the standard tools;
3. What is the "reason d'etre" for these tools;
4. Which limitations do they bring along;
5. Where and how can one use the new tools;
6. **Which classical settings in mathematics, physics (including optics) and engineering can be treated better?**
7. What are good analogies for explaining the situation and for teaching (in various disciplines).

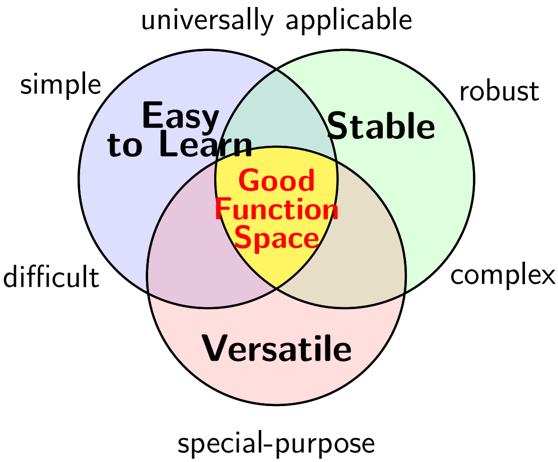
IKIGAI25HA

IKIGAI on Fourier Analysis




MATHEMATICS
structure, proof, abstraction
• Harmonic analysis
• Functional analysis
• Distribution theory, PDE
PHYSICS
models
invariances
observables
ENGINEERING
algorithms
computation
design
FOURIER ANALYSIS
TFA
SCIENCE
signals
images
data
REAL WORLD
hardware
software
MP3

How to Evaluate a Function Space I



The Setting for this Talk

The setting for this talk is Fourier Analysis and the more recent branch of Time-Frequency Analysis (**TFA**) and Gabor Analysis. For me important components for modern linear analysis (signals, systems, operators, representations) are built on methods from Functional Analysis (FA). The unavoidable ingredients are NOT-finite dimensional vector spaces and suitable concepts of convergence, allowing to extend concepts from Linear Algebra.

These concepts are centered about linear combinations of elements $\mathbf{x}_k$ from a vector space $\boldsymbol{V}$, with real or complex coefficients , which are unique in the case of linear independence, and allow to represent any element we have a generating system, and a (Riesz) basis if both properties coincide. Linear Mappings respect linear combinations and thus they can be completely described using matrix representations with respect to given bases on both sides.

Fourier Analysis as the Theme of Choice I

Aside from the topic and the tools there is s strong motivation *to promote an exchange of ideas between different disciplines*, notably mathematics, physics and engineering.

From personal experience through interaction with scientists from the different disciplines I have realized how different their use of Fourier Analysis is. Physicists generously write (sometimes divergent) integrals supporting well-founded intuition, Mathematicians tend to be seen as pedantic persons, but effectively do not care too much about real-world applications or computational issues (if they do modern analysis). Finally Engineers argue that one has to use the FFT instead of the integral transform, because, after all, the computer only allows to deal with finite vectors.

!Fourier Analysis has a different meaning in those fields!

Fourier Analysis as the Theme of Choice II

Fourier Analysis is also a good example of a scientific area with a long history of 200 years (starting 1822). It has widespread applications in many disciplines, quite often leading to independent developments and diverging terminologies.

In mathematics it has contributed enormously to the development of the current concepts of functions and integrals and finally to the development of Functional Analysis (FA) and distribution theory, most importantly in the version of Laurent Schwartz (using $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ and the space $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ of tempered distributions).

Expressions such as the Dirac Delta (?function) are used both in physics and engineering, often with the hint to mathematics, where a "rigorous but complicated theory" underpinning of its use can be found (?where, and in a suitable format???).

The Goal of this Presentation

Although there is abundance of material concerning a rigged Hilbert space setting, called THE Banach Gelfand Triple, $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, consisting of the Feichtinger algebra $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$, the Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ and the ambient space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ of *mild distributions*, the main emphasize in this presentation is on mathematical tools.

This is a somewhat unusual task, as it requires not only a reflection of existing tools, understood based on the historical development, and an analysis of what they are good for, and what are their possible limitations. And as so often in life people are getting used to a setting, even if it a but cumbersome, once it has become “classical”.

Let us summarize the task: We have to model and understand signals, vector spaces of signals, and, primarily, linear operators between such signal spaces.

THE Banach Gelfand Triple

Feichtinger Algebra and Mild Distributions

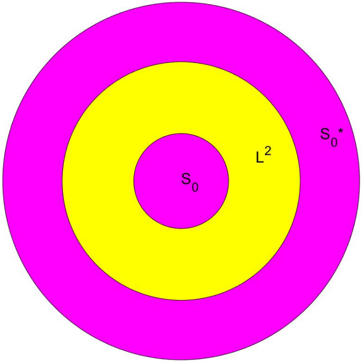


Figure: Feichtinger Algebra and Mild Distributions

Outline of the talk

During the talk I will shortly address the following topics:

- What are the current tools for Fourier Analysis;
- Functions spaces, Lebesgue integration; convolution;
- Hilbert spaces and (un)bounded linear operators;
- Modern Applications: MP3, digital cameras, etc.;
- TFA (Time-Frequency Analysis) and Gabor Analysis;
- Suitability of function/distribution spaces;
- From Linear Algebra to Rigged Hilbert spaces;
- Definition and basic properties of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$;
- The model $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ (computational, completion);
- Indication of numerous applications, TFA and Classical Fourier Analysis, up to Generalized Stochastic Processes.

Riemann versus Lebesgue Integration

Riemann integral	Lebesgue integral
Easy to learn • based on partitions of an interval; • geometrically intuitive; • immediately suitable for continuous functions; • close to elementary approximation by step functions.	Stable and versatile • integrates a much larger class of functions; • compatible with the Banach spaces $\left(\boldsymbol{L}^p, \|\cdot\|_p\right)$; • well adapted to limits of functions; • concept extends to measure spaces.

The underlying point of view

Riemann Integration: Simple and Concrete

Riemann sums

For a bounded function $f : [a, b] \to \mathbb{R}$, one considers partitions $a = x_0 < x_1 < \cdots < x_N = b$ and approximates the area under the graph by sums of the form

$$\sum_{j=1}^{N} f(\xi_j)(x_j - x_{j-1}) = \left[\sum_{j=1}^{N} w_j \delta_{\xi_j}\right](f) \qquad \xi_j \in [x_{j-1}, x_j].$$

This is highly effective for continuous functions and many piecewise regular functions.

Limitation

Riemann integration is less robust under limiting procedures. A sequence of Riemann integrable functions can converge pointwise to a function which is no longer Riemann integrable.

Lebesgue Integration: the Functional-Analytic Setting

Measure and function spaces

Lebesgue integration is built on measurable sets and measure. It gives normed spaces

$$\boldsymbol{L}^p(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} |f(x)|^p \, dx < \infty \right\}, \qquad 1 \leq p < \infty.$$

For $p = \infty$, one obtains the space of essentially bounded functions. These spaces have a rich and stable structure. For $\frac{1}{p} + \frac{1}{q} = 1$:

$$(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p) \text{ is complete}, \quad (\boldsymbol{L}^p(\mathbb{R}^d))^* = \boldsymbol{L}^q(\mathbb{R}^d), \quad 1 < p < \infty.$$

Why Lebesgue Integration is More Robust

Reliable passage to limits

Convergence theorems provide tools for passing to limits:

- monotone convergence, Fatou's lemma;
- dominated convergence;
- Tonelli's theorem, Fubini's theorem.

They make Lebesgue integration particularly suitable for Fourier analysis, probability theory, partial differential equations, and functional analysis.

The price

Lebesgue theory requires measurable sets, equivalence classes modulo null sets, and a more abstract language. Thus it is less immediate than the Riemann integral, although more powerful.

Riemann versus Lebesgue Integration: Summary

Criterion	Riemann	Lebesgue
Easy to learn	moderate	measure theory
Stable under limits	limited	very strong
Universally applicable	mainly cubes in $\mathbb{R}^d$	general measure spaces
Function-space framework	less known Wiener amalgams	$\boldsymbol{L}^p$-spaces and duality
Best role	elementary intuition	modern analysis

Conclusion

The Riemann integral is a good elementary tool; the Lebesgue integral is the established, more stable and versatile analytical tool.

The "natural role" of $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ for Fourier Analysis

There are many good reasons for using the Lebesgue integral in a course on Fourier Analysis. First of all of course the definition of the FT as an integral transform with

$$\widehat{f}(s) = \int_{\mathbb{R}^d} f(t)\overline{\chi_s(t)}dt = \int_{\mathbb{R}^d} f(t)exp(-2\pi ist)dt,$$

with $\chi_s(t) = \exp(2\pi ist)$, $t, s \in \mathbb{R}^d$.
It turns $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ into a Banach algebra, with convolution as multiplication, given (a.e.) by

$$g * f(x) := \int_{\mathbb{R}^d} f(x-y)g(y)dy, \quad x, y \in \mathbb{R}^d.$$

But already the Fourier Inversion Theorem poses problems if $\widehat{f} \notin \boldsymbol{L}^1(\mathbb{R}^d)$, such as $\mathrm{SINC} = \mathcal{F}(\mathbf{1}_{[-1/2,1/2]})$.

The Price paid and the limitations which still exist

On the other hand the "members" of $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ are (*strictly speaking*) not functions anymore, but only *equivalence classes of measurable functions*, based on the identification of two measurable functions (whatever this means, in the sense of non-pathological) f, g if $f(x) = g(x)$ almost everywhere (a.e.), i.e. with the exception of a null-set, a set whose measure is zero! Still, the Lebesgue integral does not allow to deal with the Dirac delta (?function?), because it is zero "almost everywhere" but still $\int_{-\infty}^{\infty} \delta(x)dx = 1$ (according to physics or engineering books). It does not justify the frequently used formulas

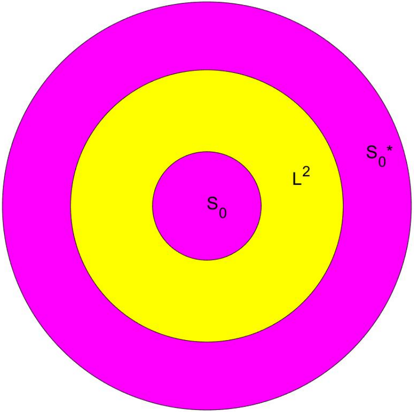
$$\int_{-\infty}^{\infty} \exp(2\pi ist)ds = \delta(t) \quad \text{or} \quad f(t) = \int_{-\infty}^{\infty} f(y)\delta(x-y)dy,$$

called the *sifting property* of the Dirac Delta!.

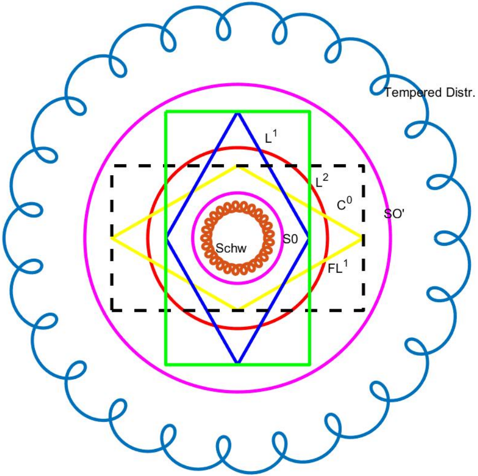
The problems are more serious than joining $+\infty$ to $\mathbb{R}$!!

THE Banach Gelfand Triple

Feichtinger Algebra and Mild Distributions



A Zoo of Banach Spaces for Fourier Analysis



The central theme of this survey talk should be the description of a mind-set (and of course corresponding mathematical tools) which allows move from discrete variables to continuous variable models, as they are relevant for physics and engineering. Methods which are often well understood in the linear algebra setting (also computationally) are typically transferred to the more general setting using methods from linear functional analysis. Thus Hilbert spaces, Lebesgue spaces such as $L^2(R^d)$ or $L^1(R^d)$ or tempered distributions are accepted as appropriate tools and widely used in all branches dealing with Fourier Analysis or Time-Frequency Analysis, but also in Abstract Harmonic Analysis and Engineering. The aim of this survey talk will be to open up the mind for a slightly different mindset, making use of concepts arising from time-frequency and Gabor analysis. The current viewpoint is certainly inspired by the ideas of Paul Dirac, who suggested to use Bras and Kets. The theory of mild distributions, i.e. of tempered distributions with a uniformly bounded spectrogram appears to capture the naive ideas about "signals" and "operators" better than the established theory. They form a Banach space, more precisely they have been introduced as the dual space to the Segal Algebra $S_0(G)$ (G some LCA group), known as the Feichtinger Algebra. Together with the intermediate Hilbert space $L^2(G)$ they form a so-called Banach Gelfand Triple, with can be compared with the triple of numbers: $Q > R > C$ (of rational, real and complex numbers).

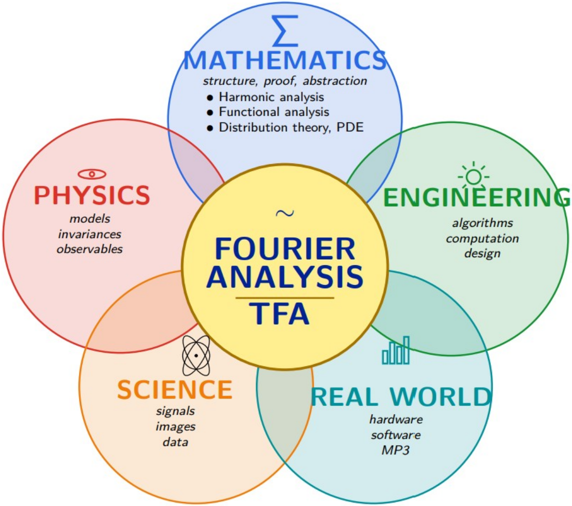
This setup is not only useful for the treatment of questions arising in Gabor Analysis or the study of pseudo-differential operators, but can be used to explain classical questions (e.g. summability), or the basic theory of generalized stochastic processes. On of the key facts is the existence of a so-called "kernel theorem" (the analogue of a matrix representation of a linear mapping on R^n).

Overall the talk will try to provide a glimpse into the world of Conceptual Harmonic Analysis, which can be seen as an integration of abstract and computational harmonic analysis, with a strong relation to the needs of applied scientists in engineering and physics.

Links with talks or publications of HGFei are found at www.nuhag.eu/feitalks and www.nuhag.eu/feipub

Material for a course in this spirit, starting from scratch, with links to YouTube recordings (with some MATLAB) are found at `www.nuhag.eu/ETH20`

Fourier analysis as a connecting discipline



Basic Principles of Linear Algebra I

- We start from the theory of finite-dimensional vector spaces, typically over $\mathbb{C}$ (sometimes over $\mathbb{R}$).
- A vector space allows:
  - formation of linear combinations using scalar coefficients,
  - definition of linear mappings: respecting linear combinations.
- Fundamental concepts:
  - **Generating systems** (frames): spanning families of vectors,
  - **Linear independence**: independent families (Riesz sequence).
- Together, these yield:
  - the existence of a **basis**,
  - the notion of **dimension** as a well-defined invariant.
  - For any such space various **scalar products** can be defined using the isomorphism with $\mathbb{C}^n$.

Basic Principles of Linear Algebra II

- Given appropriate bases:
  - linear mappings are represented by matrices,
  - composition corresponds to matrix multiplication;
  - invertibility corresponds to matrix inversion ($n \times n$!)
- This framework provides the computational and conceptual model for more general settings.

Matrices multiplication follows what we call the Domino-Principle. The speaker prefers to a kind of "Polish notation", which means: vectors are primarily row vectors and matrix multiplication is done from the right, using the coordinates of the row vector as coefficients, producing a linear combination of the row vectors. Thus the MATLAB realization of a scalar product becomes quite simple: $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{C}^n} = \sum_{k=1}^{n} x_k \overline{y_k} = \mathbf{x} * \mathbf{y}'$.

From Linear Algebra to Bra–Ket Notation I

- Linear algebra already contains the basic pattern:

$$\text{vectors} \quad \longleftrightarrow \quad \text{coordinates} \quad \longleftrightarrow \quad \text{matrices.}$$

- In Dirac's notation one distinguishes between:
  - **kets** $|v\rangle$: vectors, discrete finite signals;
  - **bras** $\langle w|$: linear functionals, evaluations
  - **operators** A: linear mappings.

- The basic scalar quantity is the pairing (linear in w) $\langle w|v\rangle$, while an operator is tested through matrix elements

$$\langle w|A|v\rangle.$$

- In finite dimensions this is simply matrix algebra:

$$\langle w|A|v\rangle \quad \leftrightarrow \quad w * A * v'.$$

From Linear Algebra to Bra–Ket Notation II

- Thus Dirac's notation provides a compact symbolic language:
  - vectors and dual vectors,
  - scalar products and duality pairings,
  - operators and their matrix elements.

- A similar pattern appears with Banach Gelfand Triples:

$$S_0(G) \subset L^2(G) \subset S_0'(G),$$

where test functions, Hilbert space vectors, and mild distributions play the roles of increasingly flexible objects.

While row/column vectors are "the same" we prefer to think of "mild distributions" as signals (BRAs) and test functions as measurements (KETs). Obviously (up to conjugation) the heart is the bilinear pairing $(\sigma, \varphi) \mapsto \sigma(\varphi), \varphi \in \boldsymbol{S}_0(\mathbb{R}^d), \sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$.

From ℓ^2 to L^2 and the Role of Trace Class Operators I

- In the discrete setting, the natural Banach Gelfand Triple is

$$(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty), \quad \text{with} \quad \ell^1(\mathbb{Z}^d) \subset \ell^2(\mathbb{Z}^d) \subset \ell^\infty(\mathbb{Z}^d),$$

with the dual pairing extending the scalar product on ℓ^2.

- Passing to continuous variables suggests to consider

$$L^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d),$$

and the resulting scalar product for $L^2(\mathbb{R}^d)$ being given by

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f(x)\,\overline{g(x)}\,dx.$$

From ℓ^2 to L^2 and the Role of Trace Class Operators II

- In analogy, one considers spaces of operators:
  - Hilbert–Schmidt operators $\mathcal{HS}$ as the analogue of L^2,
  - Trace class operators $\mathcal{L}^1$ as the analogue of L^1.
- The Hilbert–Schmidt class carries a natural inner product:

$$\langle T, S\rangle_{\mathcal{HS}} := \mathrm{trace}(S^* T),$$

  which is the operator analogue of the L^2 scalar product.
- Thus:
  - vectors $\leftrightarrow$ functions, or (mild) distributions;
  - matrices $\leftrightarrow$ operators (not only bounded operators on $\boldsymbol{L}^2(\mathbb{R}^d)$)
  - matrix trace $\leftrightarrow$ integral.
- This analogy provides the bridge from finite-dimensional linear algebra to operator theory in harmonic analysis.

Basic Inclusions for $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$ I

- The Feichtinger algebra $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is continuously embedded into $L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$.
- In fact we have the continuous embeddings

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (S_0(\mathbb{R}^d), \|\cdot\|_{S_0}) \hookrightarrow (L^p(\mathbb{R}^d), \|\cdot\|_p) \qquad (1 \le p \le \infty),$$

and in particular $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}) \hookrightarrow (L^2(\mathbb{R}^d), \|\cdot\|_2)$.

- In turn, mild distributions are a subset of tempered distributions, or $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'}) \qquad (1 \le p \le \infty)$.
- Hence we obtain the chain of continuous embeddings

$$(L^p(\mathbb{R}^d), \|\cdot\|_p) \hookrightarrow (S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

Basic Inclusions for $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$ II

- As a consequence, every bounded linear operator

$$T : (\mathbf{L}^p(\mathbb{R}^d), \|\cdot\|_p) \to (\mathbf{L}^q(\mathbb{R}^d), \|\cdot\|_q)$$

can also be viewed as a continuous operator

$$T : (S_0(\mathbb{R}^d), \|\cdot\|_{S_0}) \to (S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'}),$$

whenever the restriction to $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is considered and the output is interpreted in $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$.

- On the other hand many operators (like periodization or sampling, or a combination of them) are not bounded on $(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2)$.

Function Spaces and Operator Classes (Quick Reference) I

Symbol	Name	Role / Interpretation
$(\boldsymbol{S_0}(\mathbb{R}^d), \lVert\cdot\rVert_{\boldsymbol{S_0}})$	Feichtinger algebra	well-localized test functions
$(\boldsymbol{S_0'}(\mathbb{R}^d), \lVert\cdot\rVert_{\boldsymbol{S_0'}})$	Dual of $(\boldsymbol{S_0}(\mathbb{R}^d), \lVert\cdot\rVert_{\boldsymbol{S_0}})$	mild distributions (generalized sig
$(\boldsymbol{L^2}(\mathbb{R}^d),\ \lVert\cdot\rVert_2)$	$L^2(\mathbb{R}^d)$	Hilbert space (energy space)
$(\boldsymbol{L^p}(\mathbb{R}^d),\ \lVert\cdot\rVert_p)$	$L^p(\mathbb{R}^d)$	classical signal spaces
$\boldsymbol{L^1}(\mathbb{R}^d)$	$L^1(\mathbb{R}^d)$	integrable functions
$\boldsymbol{C_0}(\mathbb{R}^d)$	$C_0(\mathbb{R}^d)$	continuous, vanishing at infinity
$\ell^2(\mathbb{Z}^d)$	sequence space	discrete Hilbert space
$\ell^1(\mathbb{Z}^d)$	sequence space	summable sequences
$\ell^\infty(\mathbb{Z}^d)$	sequence space	bounded sequences

Operator Class	Notation	Analogy
Bounded operators	$\mathcal{B}(L^2)$	matrices on $\mathbb{C}^n$
Hilbert–Schmidt	$\mathcal{HS}$	L^2-kernels
Trace class	$\mathcal{L}^1$ or $\mathcal{T}^1$	L^1-kernels
Rank-one operators	$\lvert g\rangle\langle f\rvert$	outer products

Function Spaces and Operator Classes (Quick Reference) II

- Banach Gelfand Triple:

$$\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right) \subset \left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right) \subset \left(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'}\right)$$

- Discrete analogue:

$$\ell^1 \subset \ell^2 \subset \ell^\infty$$

- Principle:
  - test functions $\to$ Hilbert space $\to$ distributions
  - vectors $\to$ operators $\to$ kernels

Physics language versus time-frequency language I

Physics	**TF-Analysis**	**Math. operation**
position q	time t	translation
momentum p	frequency ω	modulation
phase space (q, p)	TF-plane (t, ω)	$\mathbb{R}^d \times \widehat{\mathbb{R}}^d$
state / wave fct.	signal	$f \in \boldsymbol{L}^1(\mathbb{R}^d)$
position repres.	time representation	$f(t)$
momentum repres.	frequency repr.	$\widehat{f}(\omega)$
position operator	multiplication by t	$Qf(t) = tf(t)$
momentum op.	differentiation	$Pf(t) = \frac{1}{2\pi i} f'(t)$
uncertainty principle	TF-uncertainty	localization conflict
coherent states	Gabor atoms	$M_\omega T_x g$
Wigner function	Wigner distribution	quadratic TF repr.

Physics language versus time-frequency language II

A useful dictionary translates the terminology of quantum mechanics into the terminology of time-frequency analysis. The formal correspondence is based on the analogy

phase space in physics $\leftrightarrow$ time-frequency plane in signal analysis.

In time-frequency analysis one usually writes the elementary time-frequency shifts as

$$T_x f(t) = f(t-x), \qquad M_\omega f(t) = e^{2\pi i \omega t} f(t).$$

Thus the analogue of a point in physical phase space is the parameter pair

$$z = (x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d,$$

Physics language versus time-frequency language III

and the corresponding shifted atom is

$$\pi(z)g = M_\omega T_x g.$$

The physical terminology emphasizes position and momentum. The engineering and signal-analysis terminology emphasizes time and frequency. Mathematically these are two versions of the same structural idea: one variable describes location, the other describes oscillation.

The bridge is not just linguistic, but operator-theoretic.

The commutation relation behind the analogy is

$$M_\omega T_x = e^{2\pi i \omega x} T_x M_\omega,$$

which is the time-frequency version of the non-commutativity of position and momentum observables.

From L^2-functions to equivalence classes I

The classical Hilbert space setting is based on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ But, strictly speaking, its "elements" are not pointwise defined functions. They are equivalence classes of measurable functions:

$$f \sim g \quad \Longleftrightarrow \quad f(x) = g(x) \quad \text{for almost every } x \in \mathbb{R}^d.$$

Hence values on null sets are invisible in $L^2(\mathbb{R}^d)$. This is a powerful feature for analysis, but it is also a warning sign: pointwise objects cannot always be represented faithfully inside $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$.

Lebesgue integration solves many problems,
but it does not turn every useful object into a function.

From L^2-functions to equivalence classes II

The difficulty becomes apparent when one tries to interpret the Dirac delta as a function. Formally one wants

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1,$$

although the intuitive picture says that $\delta(x) = 0$ for all $x \neq 0$. This cannot be a Lebesgue integrable function. If δ were an ordinary measurable function which is zero almost everywhere, then its Lebesgue integral would have to be zero.

Thus the problem is not just that the value at one point is "infinite". The real problem is structural:

The Dirac delta is not a function in the Lebesgue sense.

The traditional remedy: Schwartz distributions I

Schwartz' solution was to change the question. Instead of asking whether δ is a function, one asks how it acts on test functions. The classical test function space is the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ consisting of smooth functions with all derivatives decaying rapidly. The space of $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ of tempered distributions is its continuous dual:

$$\boldsymbol{\mathcal{S}}'(\mathbb{R}^d) = (\boldsymbol{\mathcal{S}}(\mathbb{R}^d))'.$$

In this setting the Dirac delta is perfectly well-defined:

$$\delta(\varphi) = \varphi(0), \qquad \varphi \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d).$$

Thus δ is not a function, but a continuous linear functional on a suitable space of test functions.

The traditional remedy: Schwartz distributions II

This also gives a rigorous meaning to the sifting property. The formal expression

$$\int_{-\infty}^{\infty} f(y)\delta(x-y)\,dy = f(x)$$

is not interpreted as an ordinary Lebesgue integral involving a function δ.
Instead, for fixed x, the shifted delta acts by

$$\delta_x(\varphi) = \varphi(x).$$

Symbolically one writes

$$\langle \delta_x, \varphi \rangle = \varphi(x),$$

which is the rigorous content behind the physicists' notation.

The traditional remedy: Schwartz distributions III

The slogan is:

The delta is not integrated against test functions;
it evaluates them.

For "ordinary functions" $g(t)$ (e.g. locally integrable functions of polynomial growth) one can define a regular tempered distributions via integration against test functions

$$\sigma_g(\varphi) = \int_{\mathbb{R}^d} \varphi(t) g(t) dt, \quad \varphi \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d).$$

Thus tempered distributions are "generalized functions", which still can be shifted, dilated, transformed, or have support.

Fourier transform and the delta I

The distributional setting also explains the formal identity

$$\int_{-\infty}^{\infty} e^{2\pi ist}\, ds = \delta(t).$$

This is not a statement about ordinary functions. It is a statement inside $\mathcal{S}'(\mathbb{R}^d)$, after testing against $\varphi \in \mathcal{S}(\mathbb{R}^d)$.
In one common normalization, the constant function 1 has Fourier transform equal to the Dirac delta:

$$\mathcal{F}(1) = \delta_0, \quad \text{and conversely} \quad \mathcal{F}(\delta_0) = 1.$$

Thus the plane waves $e^{2\pi ist}$ are not being integrated in the Lebesgue sense over s. The formula is a shorthand for a distributional limiting principle.

Fourier transform and the delta II

The traditional chain of spaces is therefore

$$\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \subseteq (\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1) \quad \text{or} \quad (\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2) \subseteq \boldsymbol{\mathcal{S}}'(\mathbb{R}^d),$$

where the first inclusion gives very good test functions, the middle space gives the Hilbert space of finite energy signals, and the last space contains generalized objects such as

$$\delta_0, \quad \delta_x, \quad 1, \quad e^{2\pi i\omega x}, \quad \text{and many oscillatory limits.}$$

This is the classical Schwartz theory of tempered distributions; see the classical books of Schwartz (1950), Gelfand-Shilov (1964), Treves (1967) and Rudin (1973).
There is also a so-called *kernel theorem*, due to the fact that $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is a nuclear Frechet space. This is the analogue of the matrix representation of linear operators for continuous variables.

Fourier transform and the delta III

A further advantage of the Schwartz setting is its excellent behavior under regularization. If $\Phi \in \mathcal{S}(\mathbb{R}^d)$ is a good smoothing kernel and $T \in \mathcal{S}'(\mathbb{R}^d)$ is a tempered distribution, then convolution with Φ produces a smooth function,

$$\Phi * T \in C^\infty(\mathbb{R}^d),$$

with at most polynomial growth. More generally, product-convolution operators of the form

$$T \mapsto \Phi * (\Psi T), \qquad \Phi, \Psi \in \mathcal{S}(\mathbb{R}^d),$$

provide a very flexible way of localizing and smoothing distributions. They are compatible with the basic operations of harmonic analysis and allow one to pass from singular objects to regular functions without leaving the distributional framework.

Fourier transform and the delta IV

Equally important is the Fourier invariance of the setting:

$$\mathcal{F} : \boldsymbol{\mathcal{S}}(\mathbb{R}^d) \to \boldsymbol{\mathcal{S}}(\mathbb{R}^d) \quad \text{is an isomorphism,}$$

and by duality

$$\mathcal{F} : \boldsymbol{\mathcal{S}}'(\mathbb{R}^d) \to \boldsymbol{\mathcal{S}}'(\mathbb{R}^d) \quad \text{is again an isomorphism.}$$

Thus differentiation, multiplication by polynomials, convolution, regularization, and Fourier transformation all remain within one stable and coherent framework.
Overall we have a Fourier invariant rigged Hilbert space situation' with

$$\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$$

Related references I

Books describing Modulation Spaces:

A. Benyi and K. A. Okoudjou.
Modulation Spaces. With Applications to Pseudodifferential Operators and Nonlinear Schrödinger Equations.
Appl. Num. Harm. Anal. (ANHA). Springer (Birkhäuser), New York, 2020.

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Time-Frequency Analysis of Operators and Applications.
De Gruyter Studies in Mathematics, Berlin, 2020.

H. G. Feichtinger.
Modulation spaces on locally compact Abelian groups.
In R. Radha, M. Krishna, and S. Thangavelu, editors, *Proc. Internat. Conf. on Wavelets and Applications (Chennai)*, pages 1–56, Chennai, January 2002, 2003. New Delhi Allied Publishers.

K. Gröchenig.
Foundations of Time-Frequency Analysis.
Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.

Gabor Systems

- Time-frequency shifts:

$$\pi(x, \omega) g(t) = e^{2\pi i \omega t} g(t - x)$$

- Gaussian window as canonical choice
- Leads to stable expansions and frames

Mild Distributions

- The dual space $S_0'(\mathbb{R}^d)$ is interpreted as a space of *mild distributions*.
- Interpretation via the short-time Fourier transform:

$$V_g f(x, \omega)$$

- Weak-* convergence corresponds to convergence tested against elements of $S_0(\mathbb{R}^d)$.

Extended Fourier Transform

- $\mathcal{F} : S_0 \to S_0$, hence also on S_0^*
- Compatible with Gabor representation
- Natural framework for operator theory

Matrix-representation and kernels

We know also from linear algebra, that any linear mapping can be expressed by a matrix (once two bases are fixed). We have a similar situation through the so-called kernel theorem.
Naively the operator has a representation as an integral operator: $f \mapsto Tf$, with

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy, \;\; x,y \in \mathbb{R}^d.$$

But clearly no multiplication operator can be represented in this way (not even identity), for any locally integrable function $K(x,y)$.
But we can reformulate the connection distributionally, as

$$\langle Tf, g\rangle = \langle K, f \otimes g\rangle,$$

and still call K (the uniquely determined) distributional kernel (on $\mathbb{R}^{2d}$) corresponding to T (and vice versa).

The Kernel Theorem in the S_0-setting

We will use $\boldsymbol{B} = \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0)$ and observe and $\boldsymbol{B}'$ coincides with $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ (correctly: the linear operators which are w^* to norm continuous!), using the scalar product of Hilbert Schmidt operators: $\langle T, S \rangle_{\mathcal{HS}} := \mathrm{trace}(T \circ S^*), \ T, S \in \mathcal{HS}$.

Theorem

There is a natural BGT-isomorphism between $(\boldsymbol{B}, \mathcal{HS}, \boldsymbol{B}')$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$.
This in turn is isomorphic via the spreading and the Kohn-Nirenberg symbol to $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.
Moreover, the spreading mapping is uniquely determined as the BGT-isomorphism, which established a correspondence between TF-shift operators $\pi(\lambda)$ and the corresponding point masses δ_λ.

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $\boldsymbol{M}^1 = \boldsymbol{S}_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

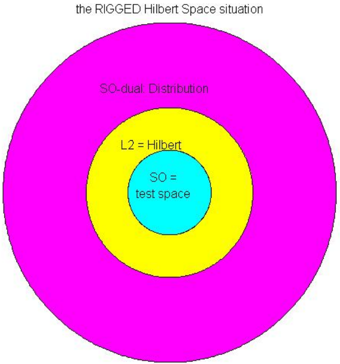
(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

A schematic description: the simplified setting

In our picture this simply means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{2}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Atomic decompositions

One can characterize $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ as the smallest Banach space containing at least one non-zero Schwartz function $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, which is isometrically invariant under TF-shifts (e.g. like $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p))$, in fact for any such $g \neq 0$ one has

$$\boldsymbol{S}_0(\mathbb{R}^d) = \{f = \sum_{n \geq 1} c_n \pi(\lambda_n) g \mid \sum_{n \geq 1} |c_n| < \infty\}.$$

The natural inf-norm is then an equivalent norm on $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ for any such g. Consequently the w^*-convergence in $\boldsymbol{S}_0'(\mathbb{R}^d)$ can be characterized via pointwise convergence (uniformly over compact regions) over compact subsets.

Retracts in Interpolation Theory

A linear mapping $\mathcal{C} : \boldsymbol{X} \to \boldsymbol{Y}$ defines a retract from $\boldsymbol{X}$ into $\boldsymbol{Y}$ if there exists a bounded linear mapping $\mathcal{R} : \boldsymbol{Y} \to \boldsymbol{X}$ such that

$$\mathcal{R} \circ \mathcal{C} = \mathrm{Id}_{\boldsymbol{X}} .$$

Set $\boldsymbol{Y}_0 := \mathrm{range}(\mathcal{C})$. Then $P := \mathcal{C} \circ \mathcal{R}$ is an idempotent operator on $\boldsymbol{Y}$, and therefore $\boldsymbol{Y}_0$ is a closed complemented subspace of $\boldsymbol{Y}$.

Moreover, $\mathcal{C}$ establishes an isomorphism $\mathcal{C} : \boldsymbol{X} \to \boldsymbol{Y}_0$.
Thus $\boldsymbol{X}$ may be identified with $\boldsymbol{Y}_0$, and also with the quotient

$$\boldsymbol{Y} / \ker(\mathcal{R}).$$

Retract diagram

$$
\begin{tikzcd}
 & \boldsymbol{Y} \arrow[dl, "\mathcal{R}"'] \arrow[d, "P"] \\
\boldsymbol{X} \arrow[r, shift right, "\mathcal{C}"'] & \boldsymbol{Y}_0 \arrow[l, shift right, "\mathcal{R}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \ell^2(I) \arrow[dl, "\mathcal{R}"'] \arrow[d, "\mathbf{P}"] \\
\mathcal{H} \arrow[r, shift right, "\mathcal{C}"'] & C(\mathcal{H}) \arrow[l, shift right, "\mathcal{R}"']
\end{tikzcd}
$$

Banach Frames for the Banach Gelfand Triples

$$\begin{tikzcd}
 & (\ell^1, \ell^2, \ell^\infty)(\Lambda) \arrow[dl, "\mathcal{R}"'] \arrow[d, "\mathbf{P}"] \\
(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0') \arrow[r, shift right, "\mathcal{C}"'] & C((\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')) \arrow[l, shift right, "\mathcal{R}"']
\end{tikzcd}$$

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

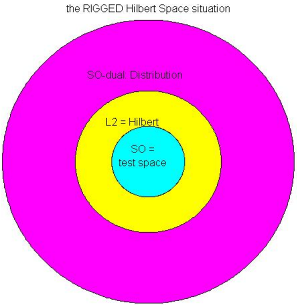
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{3}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

THE Banach Gelfand Triple Situation

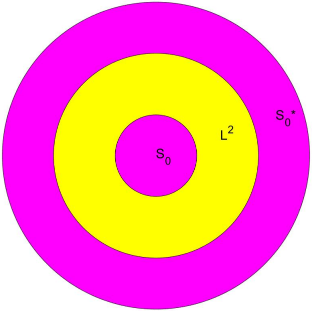


Figure: Three layers of Banach spaces, with the Hilbert space in the middle

BGT Morphism: Four Layers

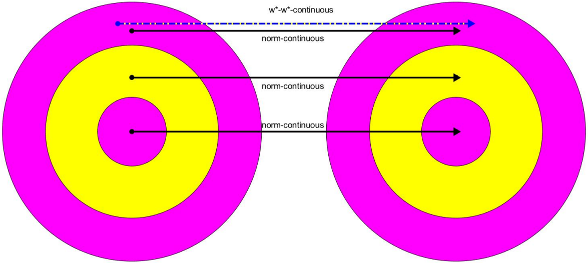


Figure: BGT3morph1.jpg: Note that there are four continuity requirements!

Regularizing operators on BGTs

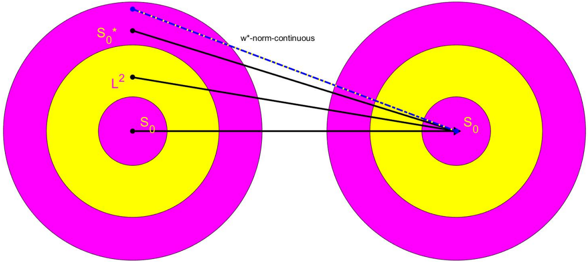


Figure: Regularizing operators: w^*-convergent nets to norm convergent ones.

Good operators compared to Trace-Class Operators

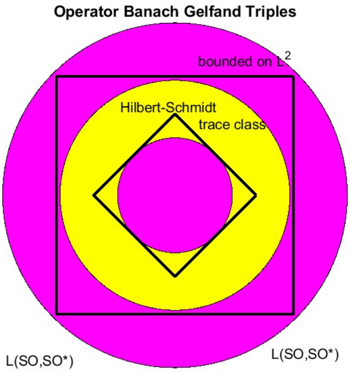


Figure: QRCtrace26a.jpg: Larger predual (trace class) implies smaller dual space: only operators bounded on L2!

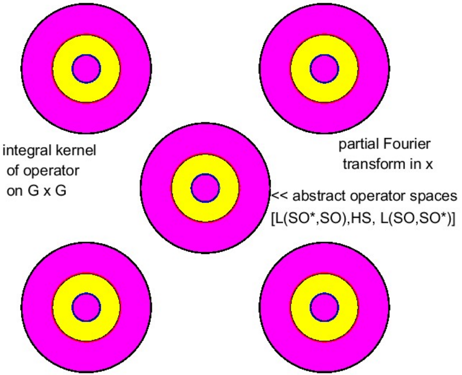


Figure: The spaces of operators $(\boldsymbol{B}, \mathcal{HS}, \boldsymbol{B}')$ can be identified with four different representations of operators.

Conclusion

- Unified framework for signals and distributions
- Strong compatibility with numerical methods
- Basis for approximation via finite structures

Links and references I

Thanks for your attention!

Let us provide a few links.

ETH Course Notes are found at:

www.nuhag.eu/ETH20

At the page

www.nuhag.eu/feitalks

you find my talks and also recorded talks from some courses, for example

From linear algebra to Gelfand triples.

Links and references II

Various Lecture Notes can be downloaded from

https://www.univie.ac.at/nuhag-php/home/skripten.php

The public version of my publication list can be provided upon request via

hans.feichtinger@univie.ac.at.

This talk provides some thoughts related to the idea of

Conceptual harmonic analysis.

Applications I

- Impulse response and transfer function
- Easy proof of the Shannon Sampling Theorem
- MP3
- Graviational Waves (LIGO)
- a theory of Continuous ONBs (Dirac bases etc.)
- Sifting property of the Dirac Delta
- Approximation of the FT through FFT

Why should we care about convolutions?

In engineering, especially in communication theory, the *theory of time-invariant channels*, i.e. operators which are linear and commute with translations (motivated by the time-invariance of physical laws!), also called TILS (time-invariant linear systems) is one of the cornerstones. The collection of all these operators form a commutative algebra, with the pure frequencies as common eigenvalues.

Still in an engineering terminology: Every such system T is a *moving average* or (equivalently) a *convolution operator*, described by the *impulse response* of the system, or alternatively, it can be described as a *Fourier multiplier*, being understood as a pointwise multiplier on the Fourier transform side, by the so-called *transfer function*.

The concrete case of BIBOs systems

From my point of view the most natural setting is that of so-called *BIBOS*, bounded-input-bounded-output systems, or more precisely the bounded linear operators on $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ which commute with translations.
It is not hard to show that they are exactly the *convolution operators* which are arising from a given linear functional $\mu \in \boldsymbol{M}_b(\mathbb{R}^d) := (\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'})$:

$$\mu * f(x) = \mu(T_x f^{\checkmark}), \quad x \in \mathbb{R}^d, \tag{4}$$

where $f^{\checkmark}(x) = f(-x), x \in \mathbb{R}^d$.

Any TILS on $\boldsymbol{C}_0(\mathbb{R}^d)$ is if this form!

Given T choose $\mu(f) := [T(f^{\checkmark})](0)$, $f \in \boldsymbol{C}_0(\mathbb{R}^d)$.

More general "multipliers"

According to R. Larsen (his book on the multiplier problem appeared in 1972) it is meaningful to ask for a characterization of the space of "multipliers", i.e. all linear, bounded operators from one translation invariant Banach space into another one, which commute with translations. You may think of such operators from $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ to $(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q)$, for two values $p, q \in [1, \infty]$. Using the concept of *quasi-measures* introduced by G. Gaudry he demonstrates that each such operator is a convolution operator by some quasi-measure, which also has an alternative description as a Fourier multiplier. The drawback of the concept of quasi-measures is the fact, that they do not involve any global condition and thus a general quasi-measure does NOT have a well defined Fourier transform. So the expected claim that the transfer function is the FT of the impulse response (and vice versa) cannot be formulated in that context.

Multipliers between L^p-spaces

The key result for multipliers be summarized as follows:

Theorem

Any bounded linear operator from $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ into $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ which commutes with translations can be described by the convolution with a uniquely $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$, i.e. via

$$Tf(x) = \sigma(T_x[f^{\checkmark}]), \quad f \in \boldsymbol{S}_0(\mathbb{R}^d), x \in \mathbb{R}^d.$$

Thus in fact, T maps $\boldsymbol{S}_0(\mathbb{R}^d)$ even into $(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty)$, with norm equivalence between the operator norm of the operator T and the functional norm (in $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$) of σ.
If σ is a regular distribution, induced by some test function $h \in \boldsymbol{S}_0(\mathbb{R}^d)$, then we even have that $f \mapsto h * f$ (equivalently given in a pointwise sense) maps $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ into $(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty)$.

How to construct a unitary BGT homomorphism I

Let $(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))$ be the standard Banach Gelfand triple. A unitary BGT homomorphism is typically obtained by one of the following two procedures.

(a) Start with a unitary operator

$$U : \boldsymbol{L}^2(\mathbb{R}^d) \longrightarrow \boldsymbol{L}^2(\mathbb{R}^d)$$

and verify that its restriction preserves the test-function space:

$$U(\boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d), \qquad U^{-1}(\boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d).$$

Thus the restriction

$$U|_{\boldsymbol{S}_0(\mathbb{R}^d)} : \boldsymbol{S}_0(\mathbb{R}^d) \longrightarrow \boldsymbol{S}_0(\mathbb{R}^d)$$

How to construct a unitary BGT homomorphism II

is a bounded Banach-space isomorphism. The action of U is then extended to mild distributions by duality:

$$\langle U\sigma, \varphi\rangle = \langle \sigma, U^{-1}\varphi\rangle, \qquad \sigma \in \boldsymbol{S}_0'(\mathbb{R}^d), \quad \varphi \in \boldsymbol{S}_0(\mathbb{R}^d).$$

In concrete situations, this extension may also be described by regularizing a mild distribution, applying U to the regularized functions, and then passing to the weak-$*$ limit.

Typical example: the Fourier transform, after Plancherel's theorem has established its unitary action on $\boldsymbol{L}^2(\mathbb{R}^d)$.

How to construct a unitary BGT homomorphism III

(b) A conceptually preferable approach starts directly on the test-function space $\boldsymbol{S}_0(\mathbb{R}^d)$.
One first defines

$$T : \boldsymbol{S}_0(\mathbb{R}^d) \longrightarrow \boldsymbol{S}_0(\mathbb{R}^d)$$

and verifies that both T and its formal adjoint T^* act isometrically with respect to the two relevant norms:

$$\|Tf\|_{\boldsymbol{S}_0(\mathbb{R}^d)} = \|f\|_{\boldsymbol{S}_0(\mathbb{R}^d)}, \qquad \|Tf\|_{\boldsymbol{L}^2(\mathbb{R}^d)} = \|f\|_{\boldsymbol{L}^2(\mathbb{R}^d)},$$

and similarly

$$\|T^*f\|_{\boldsymbol{S}_0(\mathbb{R}^d)} = \|f\|_{\boldsymbol{S}_0(\mathbb{R}^d)}, \qquad \|T^*f\|_{\boldsymbol{L}^2(\mathbb{R}^d)} = \|f\|_{\boldsymbol{L}^2(\mathbb{R}^d)}.$$

How to construct a unitary BGT homomorphism IV

The $\boldsymbol{L}^2(\mathbb{R}^d)$-isometry then extends uniquely, by density of $\boldsymbol{S}_0(\mathbb{R}^d)$ in $\boldsymbol{L}^2(\mathbb{R}^d)$, to an isometry

$$\widetilde{T} : \boldsymbol{L}^2(\mathbb{R}^d) \longrightarrow \boldsymbol{L}^2(\mathbb{R}^d).$$

The corresponding extension of T^* shows that

$$\widetilde{T}^* \widetilde{T} = I, \qquad \widetilde{T} \widetilde{T}^* = I,$$

so that $\widetilde{T}$ is unitary on $\boldsymbol{L}^2(\mathbb{R}^d)$.
Finally, the operator is extended to $\boldsymbol{S}_0'(\mathbb{R}^d)$ by duality:

$$\langle T\sigma, \varphi \rangle = \langle \sigma, T^*\varphi \rangle, \qquad \sigma \in \boldsymbol{S}_0'(\mathbb{R}^d), \quad \varphi \in \boldsymbol{S}_0(\mathbb{R}^d).$$

How to construct a unitary BGT homomorphism V

Guiding principle: the test-function space carries both the Banach-space structure and the dense Hilbert-space core. Hence it is the natural level on which one should verify simultaneously the $\boldsymbol{S}_0(\mathbb{R}^d)$- and $\boldsymbol{L}^2(\mathbb{R}^d)$-behaviour before passing to the Hilbert-space completion and finally to the dual space of mild distributions.

This is the standard pattern behind the Fourier transform, the fractional Fourier transform, and many metaplectic operators.

Definition

We call a family $(\tau_x)_{x\in\mathbb{R}^d}$ a *mild basis*[a] for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ if there exists some BGT-automorphism α such that $\tau_x = \alpha(\delta_x)$, $x \in \mathbb{R}^d$. We call it a *unitary mild basis* (for short, a **UMB**) for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ if there exists some unitary BGT-automorphism.

---

[a] This terminology relates well to the use of the Dirac basis in physics and other application areas. Strictly speaking it would be more correct to call it a *tight, mild frame*, because a basis is supposed to be minimal, while a frame is a generating system which allows some redundancy. All the *relevant* properties of the $(\delta_x)_{x\in\mathbb{R}^d}$ would be valid also for the family (for illustration only) $(\delta_q)_{q\in\mathbb{Q}}$ as well, since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. In a way, one can consider the *tight Dirac frame* $(\delta_x)_{x\in\mathbb{R}^d}$ as the limit of Gabor frames of the form $(\mathrm{St}_\rho g_0, \rho a, b/\rho)$, using compressed Gaussians as atoms, for $\rho \to 0$, starting from any pair (a, b) with $ab < 1$. Thus the Dirac basis consists of in infinitely fine grid in time and an infinitely sparse grid in frequency, and thus the concept of redundancy does not make much sense anymore for this limiting case.

Mild and Tempered Distributions I

- The classical setting of distribution theory in harmonic analysis is the Schwartz triple

$$\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^2(\mathbb{R}^d) \hookrightarrow \boldsymbol{\mathcal{S}}(\mathbb{R}^d)'.$$

- The Banach Gelfand triple associated with the Feichtinger algebra is

$$\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^2(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d).$$

- Since

$$\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0(\mathbb{R}^d)$$

continuously and densely, restriction of functionals gives the continuous inclusion

$$\boldsymbol{S}_0'(\mathbb{R}^d) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d).$$

Mild and Tempered Distributions II

- Thus every tempered distribution defines a mild distribution, but not every mild distribution must be tempered.
- The difference comes from the choice of test functions:
  - $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is a Fréchet space controlled by infinitely many seminorms;
  - $\boldsymbol{S}_0(\mathbb{R}^d)$ is a Banach space controlled by one natural time-frequency norm.

Advantages of the Mild-Distribution Setting I

- The test space $\boldsymbol{S}_0(\mathbb{R}^d)$ is a Fourier-invariant Banach space.
- It is invariant under translations, modulations, and hence under time-frequency shifts.
- Both pointwise multiplication and convolution act naturally on $\boldsymbol{S}_0(\mathbb{R}^d)$.
- The dual space $\boldsymbol{S}_0'(\mathbb{R}^d)$ contains important signals such as
  - Dirac measures and finite measures;
  - bounded functions;
  - periodic distributions and Dirac combs;
  - tempered distributions.
- The Banach-space structure permits direct use of
  - bounded-operator theory;
  - Banach frames and atomic decompositions;
  - Gabor analysis and modulation-space techniques;
  - the kernel theorem for operators between Feichtinger algebras.

Advantages of the Mild-Distribution Setting II

- Convergence and boundedness can often be expressed through one norm and its dual norm, rather than through infinitely many seminorms.

Benefits of the Tempered-Distribution Setting I

- The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is adapted to differentiation and multiplication by polynomials.
- The space $\mathcal{S}(\mathbb{R}^d)'$ is therefore especially natural for
  - partial differential equations;
  - differential operators;
  - polynomial growth conditions;
  - classical Fourier-transform identities involving derivatives.
- The Schwartz topology contains detailed information about simultaneous smoothness and rapid decay.
- Tempered distributions form a well-established framework with a large classical literature.
- The price is that $\mathcal{S}(\mathbb{R}^d)$ is not a Banach space:
  - continuity must be checked with respect to infinitely many seminorms;

Benefits of the Tempered-Distribution Setting II

- standard Banach-space arguments cannot always be applied directly;
- frame-theoretic and operator-norm formulations may be less immediate.

Comparison and Practical Choice I

	Tempered distributions	**Mild distributions**
Test space	$\mathcal{S}(\mathbb{R}^d)$, a nuclear Fréchet space	$\boldsymbol{S}_0(\mathbb{R}^d)$, a Fourier-inva nach space
Underlying dual	$\mathcal{S}(\mathbb{R}^d)'$	$\boldsymbol{S}_0'(\mathbb{R}^d)$
Main strength	Smoothness, derivatives, polynomial weights	Time-frequency shifts bounded operators
Topology	Infinitely many seminorms	One Banach norm
Typical applications	PDE, classical distribution theory	Gabor analysis, signal operator kernels
Size of the dual	Smaller	Larger

- Use tempered distributions when differentiation, polynomial growth, or PDE structure is central.
- Use mild distributions when time-frequency localization, Banach frames, sampling, or operator theory is central.

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