

Tools for Modern Fourier and Gabor Analysis

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Abstract I

There are several paths leading to a common set of ideas in modern Fourier and time-frequency analysis. From the mathematical side, one needs suitable notions of “generalized functions” in order to describe objects such as Dirac measures and Dirac combs, which occur naturally in engineering and physics. Convolution describes linear time-invariant systems, while on the Fourier transform side the same systems are represented by multiplication with a transfer function. Gabor analysis provides another point of view: functions and distributions are represented by time-frequency shifted building blocks, typically well-localized windows such as Gaussians, resulting in a description which may be compared heuristically with a musical score.

From the applied side, signals arise through measurements. An audio signal is recorded by a microphone, while an image represents a scene captured by a digital camera. In both cases only a finite region of interest can be observed, and only up to a certain resolution. At the same time, the uncertainty principle places an intrinsic limitation on simultaneous localization in time and frequency.



Abstract II

This suggests the use of function and distribution spaces which are flexible enough for applications while retaining a robust mathematical structure.

Against this background, the presentation will focus on the Banach Gelfand Triple $(S_0(\mathbb{R}^d), L^2(\mathbb{R}^d), S_0^*(\mathbb{R}^d))$, consisting of the Feichtinger algebra $S_0(\mathbb{R}^d)$, the Hilbert space $L^2(\mathbb{R}^d)$, and its surrounding dual space $S_0^*(\mathbb{R}^d)$ of mild distributions. The continuous inclusions $S_0(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset S_0^*(\mathbb{R}^d)$ provide a setting in which well-localized functions, finite-energy signals, measures, and more general distributions can be treated within one framework.

A central feature is the close connection with Gabor analysis. For suitable Gabor systems the three spaces are characterized by coefficient sequences in ℓ^1 , ℓ^2 , and ℓ^∞ , respectively. Using Wilson bases, this correspondence becomes particularly transparent and provides natural sequence-space models for the three levels of the triple.



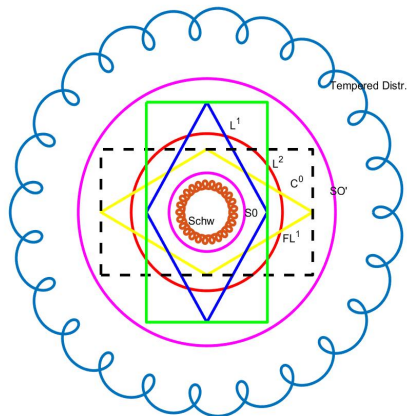
Abstract III

The natural convergence notion in S_0^* is weak-* convergence, also called mild convergence. It provides a convenient framework for approximation and limiting procedures and allows many familiar operators to be treated consistently throughout the triple. The Fourier transform is a basic example, and the same viewpoint also applies to fractional Fourier transforms and related transformations.

As time permits, applications in modern Gabor and time-frequency analysis, as well as connections with classical Fourier analysis, will be presented. The aim is to illustrate how the Banach Gelfand Triple provides a flexible bridge between classical harmonic analysis, generalized functions, and signal analysis.



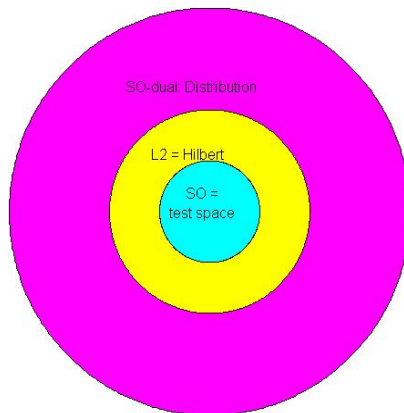
A schematic description: all the spaces



A schematic description: the simplified setting

Testfunctions $\subset$ Hilbert space $\subset$ Distributions, like $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$!

the RIGGED Hilbert Space situation



What are numbers?

We are used to work with spaces of signals which are vector spaces over the real or complex numbers. Usually these spaces carry some topology, allowing us to speak about convergence.

In many important situations this topology comes from a norm, and completeness is a major advantage.

Before discussing such spaces, let treat an elementary question:

What are numbers, and what does it mean to compute with them?

Numbers arise naturally from

counting, measuring, adding, multiplying.

The rational numbers $\mathbb{Q}$ already form a remarkably rich system: with addition, subtraction, multiplication and division by nonzero elements. This described the mathematical notion of a *field*.



Rational numbers are not enough

But not every “natural length” can be described by rational numbers. Consider a square of side length 1. Its diagonal d satisfies $d^2 = 2$. Obviously there is no rational number $d \in \mathbb{Q}$ with this property.

Nevertheless, there are arbitrarily good rational approximations:

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad 1.41421, \dots$$

leading to the concept of “infinite decimal representations”. Once their existence is established one can assign a symbol. Thus

$\mathbb{Q}$ contains arbitrarily good approximations to $\sqrt{2}$, but not $\sqrt{2}$ itself.

This is the first hint that a *completion* is needed.



From approximation to real numbers

The rational numbers come with their natural distance

$$d(x, y) = |x - y|.$$

There are **Cauchy sequences** of rational numbers which do not converge inside $\mathbb{Q}$. The real numbers can therefore be viewed as the completion:

$$\mathbb{R} = \text{completion of } \mathbb{Q}.$$

A familiar practical realization is given by infinite decimal expressions. For example, $r = 3.14159265358979 \dots$ is represented (among others, by the specific sequence)

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad \dots$$

Thus an “infinite” object is accessed through finite approximations.

A minor technical point: decimal representations are not unique, since $1.0000 \dots = 0.9999 \dots$, but only one is truly infinite!



Computing by approximation

Suppose

$$x_n \longrightarrow x, \quad y_n \longrightarrow y,$$

with $x_n, y_n \in \mathbb{Q}$ and $x, y \in \mathbb{R}$. Then (one can show that)

$$x_n + y_n \longrightarrow x + y, \quad x_n y_n \longrightarrow xy,$$

and both operations on $\mathbb{Q}$ extend continuously to the completion:

$$x + y = \lim_{n \rightarrow \infty} (x_n + y_n), \quad xy = \lim_{n \rightarrow \infty} x_n y_n.$$

The relevant estimates guarantee that the result is independent of the chosen approximating sequences.

approximate + compute + pass to the limit.



Infinite objects and finite computation

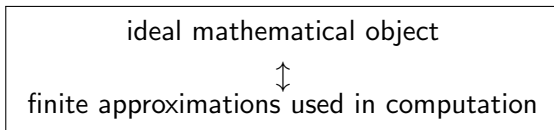
A real number is mathematically an infinite object, but every actual computation uses only finite information.

For example, $\pi = 3.1415926535 \dots$ is replaced in a computer by some finite approximation π_N .

Likewise, $\sqrt{2}$ is replaced by $(\sqrt{2})_N$, and the computer evaluates

$$\pi_N + (\sqrt{2})_N, \quad \pi_N \cdot (\sqrt{2})_N.$$

Mathematical analysis tells us how close these finite computations are to the ideal quantities.



The same distinction will reappear when we speak about signals.



From real to complex numbers

We now pass from $\mathbb{R}$ to $\mathbb{C}$. This extension is primarily algebraic. We introduce an element i satisfying $i^2 = -1$ and write

$$z = x + iy, \quad x, y \in \mathbb{R}.$$

The old arithmetic is preserved:

$$(x + iy) + (u + iv) = (x + u) + i(y + v),$$

$$(x + iy)(u + iv) = (xu - yv) + i(xv + yu).$$

Moreover,

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}, \quad x + iy \neq 0.$$

Thus equations which have no real solution may become solvable after enlarging the number system.



Compatible extensions

Altogether we have the chain

$$\mathbb{Q} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C}.$$

An element of a smaller system is not discarded when we pass to a larger one:

$$q \in \mathbb{Q} \rightsquigarrow q \in \mathbb{R} \rightsquigarrow q + i0 \in \mathbb{C}.$$

The old operations agree with the new ones. In this way we enlarge the universe while preserving the already established structure.

Extend the objects, preserve the structure.



A first lesson for analysis

The passage

$$\mathbb{Q} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C}$$

already illustrates several principles which will reappear later:

- start with simple objects on which the operations are clear;
- introduce a notion of approximation and convergence;
- complete or enlarge the class of objects;
- extend the important operations continuously or algebraically;
- retain compatibility with the original objects.

compute on good approximations and pass to the limit.

We proceed in a similar way with spaces of functions and signals!



Towards spaces of signals

For modern Fourier and Gabor analysis we encounter exactly this kind of situation.

We want spaces which are large enough to contain the objects occurring in applications, but structured enough that the basic operations remain well behaved.

The guiding pattern will be

nice signals $\hookrightarrow$ finite-energy signals $\hookrightarrow$ generalized signals.

Later this leads to the Banach Gelfand Triple

$$S_0(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow S_0^*(\mathbb{R}^d).$$

The philosophy is already familiar:

approximation, completion, extension, compatibility.



The Key-players for Time-Frequency Analysis (TFA)

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

—

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t),$$

with $x, \omega, t \in \mathbb{R}^d$, compatible with the Fourier transform

$$(T_x f)^\wedge = M_{-x} \hat{f} \quad (M_\omega f)^\wedge = T_\omega \hat{f}$$

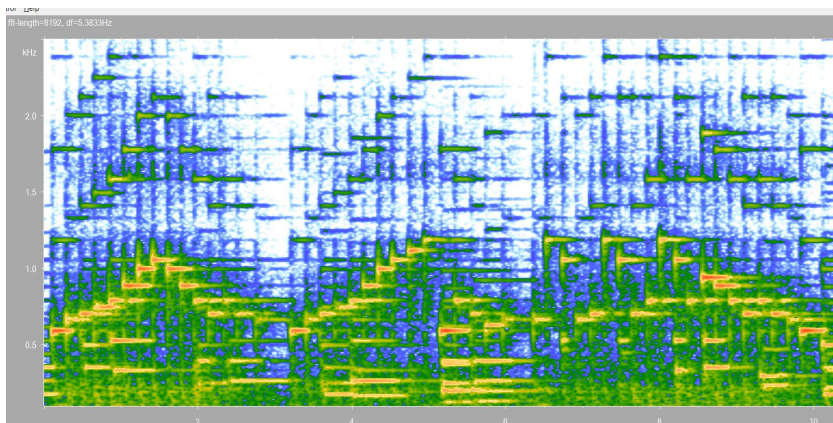
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \quad \lambda = (t, \omega);$$

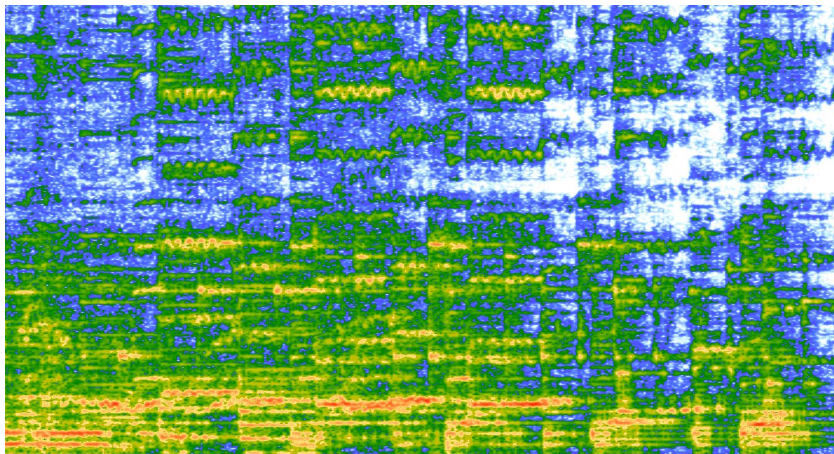


A Typical Musical STFT

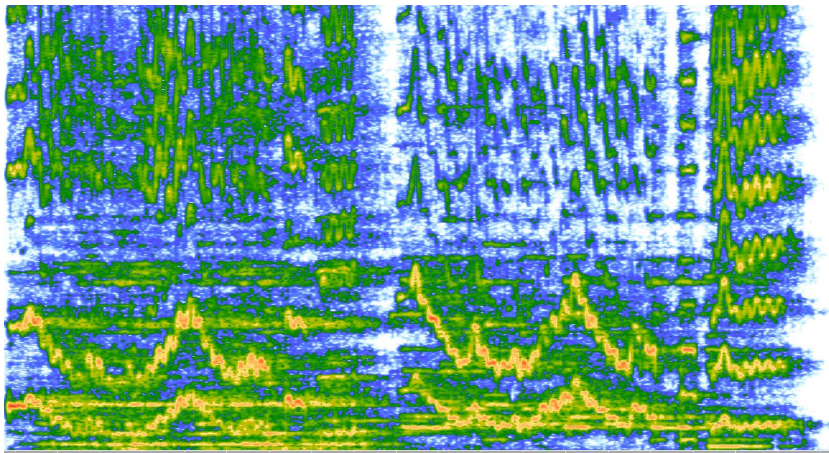
A typical piano spectrogram (Mozart), from recording



A Musical STFT: Brahms, Cello



A Musical STFT: Maria Callas



A Banach Space of Test Functions (Fei 1979)

A function in $f \in L^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the “window”) in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{\mathbf{S}_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.



Basic properties of $M^1(\mathbb{R}^d) = \mathcal{S}_0(\mathbb{R}^d)$

Lemma

Let $f \in \mathcal{S}_0(\mathbb{R}^d)$, then the following holds:

- ① $\mathcal{S}_0(\mathbb{R}^d) \hookrightarrow L^1 \cap C_0(\mathbb{R}^d)$ (dense embedding);
- ② $(\mathcal{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathcal{S}_0})$ is a Banach algebra under both pointwise multiplication and convolution;
- (1) $\pi(u, \eta)f \in \mathcal{S}_0(\mathbb{R}^d)$ for $(u, \eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and $\|\pi(u, \eta)f\|_{\mathcal{S}_0} = \|f\|_{\mathcal{S}_0}$.
- (2) $\hat{f} \in \mathcal{S}_0(\mathbb{R}^d)$, and $\|\hat{f}\|_{\mathcal{S}_0} = \|f\|_{\mathcal{S}_0}$.

In fact, $(\mathcal{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathcal{S}_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p -spaces (and their Fourier images).



From finite expressions to generalized signals

For real numbers we started with finite decimal expressions and passed to infinite ones by approximation and completion.

A closely related idea appears in time-frequency analysis.

Fix a lattice $\Lambda \triangleleft \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and a window $g \in \mathbf{S}_0(\mathbb{R}^d)$ such that

$$\mathcal{G}(g, \Lambda) = \{ \pi(\lambda)g : \lambda \in \Lambda \}$$

is a Gabor frame for $L^2(\mathbb{R}^d)$.

We first understand finite linear combinations

$$\sum_{\lambda \in F} c_\lambda \pi(\lambda)g, \quad F \subset \Lambda \text{ finite,}$$

and then with infinitely many coefficients (bounded case).

finite Gabor sums $\longrightarrow$ infinite Gabor expansions.





Finite Gabor sums

If $F \subset \Lambda$ is finite and $c_\lambda \in \mathbb{C}$, then $f_F = \sum_{\lambda \in F} c_\lambda \pi(\lambda)g$ belongs to $\mathbf{S}_0(\mathbb{R}^d)$, because the Feichtinger algebra is isometrically invariant under time-frequency shifts:

$$g \in \mathbf{S}_0(\mathbb{R}^d) \implies \pi(\lambda)g \in \mathbf{S}_0(\mathbb{R}^d).$$

Hence every finite Gabor sum is again a “good” signal.

In coefficient language, $c = (c_\lambda)_{\lambda \in \Lambda} \in c_{00}(\Lambda)$ gives

$$D_g c = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g \in \mathbf{S}_0(\mathbb{R}^d).$$

So $c_{00}(\Lambda)$ plays the role of the finite expressions.



Passing from finite sums to S_0

For absolutely summable coefficients $c = (c_\lambda)_{\lambda \in \Lambda} \in \ell^1(\Lambda)$
 $\sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g$ still converges absolutely in $\mathbf{S}_0(\mathbb{R}^d)$. Thus the
 synthesis operator is bounded:

$$D_g : \ell^1(\Lambda) \longrightarrow \mathbf{S}_0(\mathbb{R}^d).$$

Moreover, if $\gamma \in \mathbf{S}_0(\mathbb{R}^d)$ is a dual Gabor window, then every
 $f \in \mathbf{S}_0(\mathbb{R}^d)$ has an absolutely convergent Gabor expansion

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g, \quad \text{with} \quad (\langle f, \pi(\lambda)\gamma \rangle)_{\lambda \in \Lambda} \in \ell^1(\Lambda).$$

$\mathbf{S}_0(\mathbb{R}^d) \quad \leftrightarrow \quad \ell^1\text{-type Gabor coefficients.}$



What should an “infinite signal” be?

For a general signal we may want to allow coefficients which no longer decay summably. Suppose only that $c = (c_\lambda)_{\lambda \in \Lambda} \in \ell^\infty(\Lambda)$. Formally we would like to write

$$\sigma = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g.$$

This series need not converge in L^2 , and certainly not in $\mathbf{S}_0(\mathbb{R}^d)$.

But for every test signal $h \in \mathbf{S}_0(\mathbb{R}^d)$ we may consider

$$\sum_{\lambda \in \Lambda} c_\lambda \langle \pi(\lambda)g, h \rangle.$$

Because $g, h \in \mathbf{S}_0(\mathbb{R}^d)$, the sampled Gabor coefficients

$$(\langle \pi(\lambda)g, h \rangle)_{\lambda \in \Lambda}$$

belong to $\ell^1(\Lambda)$. Hence the pairing is absolutely convergent.



Bounded coefficients define mild distributions

For $c \in \ell^\infty(\Lambda)$ define, for $h \in \mathbf{S}_0(\mathbb{R}^d)$,

$$\langle D_g c, h \rangle := \sum_{\lambda \in \Lambda} c_\lambda \langle \pi(\lambda)g, h \rangle.$$

The estimate

$$|\langle D_g c, h \rangle| \leq \|c\|_{\ell^\infty} \sum_{\lambda \in \Lambda} |\langle \pi(\lambda)g, h \rangle|$$

shows that this defines a continuous linear functional on $\mathbf{S}_0(\mathbb{R}^d)$. Therefore $D_g c \in \mathbf{S}'_0(\mathbb{R}^d)$. Every bounded coefficient sequence generates a mild distribution:

$$\ell^\infty(\Lambda) \xrightarrow{D_g} \mathbf{S}'_0(\mathbb{R}^d).$$

This is the analogue of allowing an infinite expansion rather than only finite expressions (but note: no uniqueness claim here!)



Every mild distribution has such a Gabor representation

Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame with $g \in \mathbf{S}_0(\mathbb{R}^d)$.

Its canonical dual window γ also belongs to $\mathbf{S}_0(\mathbb{R}^d)$.

For every mild distribution $\sigma \in \mathbf{S}'_0(\mathbb{R}^d)$, the coefficient sequence

$$c_\lambda = \langle \sigma, \pi(\lambda)\gamma \rangle$$

is bounded:

$$(c_\lambda)_{\lambda \in \Lambda} \in \ell^\infty(\Lambda).$$

Moreover,

$$\sigma = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g$$

in the weak-* or mild sense. Equivalently,

$$\langle \sigma, h \rangle = \sum_{\lambda \in \Lambda} c_\lambda \langle \pi(\lambda)g, h \rangle, \quad h \in \mathbf{S}_0(\mathbb{R}^d).$$



What does convergence mean here?

For a finite set $F \subset \Lambda$, put

$$\sigma_F = \sum_{\lambda \in F} c_\lambda \pi(\lambda) g.$$

Each σ_F belongs to $\mathbf{S}_0(\mathbb{R}^d)$. For a mild distribution $\sigma \in \mathbf{S}'_0(\mathbb{R}^d)$, the corresponding finite Gabor sums satisfy

$$\langle \sigma_F, h \rangle \longrightarrow \langle \sigma, h \rangle, \quad h \in \mathbf{S}_0(\mathbb{R}^d).$$

Thus

$$\sigma_F \xrightarrow{w^*} \sigma \quad \text{in } \mathbf{S}'_0(\mathbb{R}^d).$$

The passage from finite sums to a mild distribution is again:

approximate by finite objects + pass to the appropriate limit.

Because the scalar series obtained after testing against $h \in \mathbf{S}_0(\mathbb{R}^d)$ is absolutely convergent, the resulting functional does not depend on the order in which the lattice points are enumerated.



The coefficient-space picture

The basic coefficient spaces form the familiar chain

$$c_{00}(\Lambda) \subset \ell^1(\Lambda) \subset \ell^2(\Lambda) \subset \ell^\infty(\Lambda).$$

Under Gabor synthesis this corresponds heuristically to

$$\begin{array}{ccccccc} c_{00}(\Lambda) & \subset & \ell^1(\Lambda) & \subset & \ell^2(\Lambda) & \subset & \ell^\infty(\Lambda) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{finite sums} & \subset & \mathbf{S}_0(\mathbb{R}^d) & \hookrightarrow & L^2(\mathbb{R}^d) & \hookrightarrow & \mathbf{S}'_0(\mathbb{R}^d). \end{array}$$

For a redundant Gabor frame the coefficient representation is generally not unique.

What is important is the existence of bounded analysis and synthesis maps which are compatible across the whole scale.



A “decent” Gabor system is enough

No special orthonormal basis is required. It is enough to take $g \in \mathbf{S}_0(\mathbb{R}^d)$ and a lattice $\Lambda \triangleleft \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ such that $\mathcal{G}(g, \Lambda)$ is a Gabor frame. Then the canonical dual window satisfies $\gamma \in \mathbf{S}_0(\mathbb{R}^d)$, and the same Gabor system provides compatible expansions on $\mathbf{S}_0(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$, and $\mathbf{S}'_0(\mathbb{R}^d)$.

Thus one and the same time-frequency coordinate system allows us to move from nice signals to finite-energy signals and further to mild distributions.

$$\mathbf{S}_0(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathbf{S}'_0(\mathbb{R}^d)$$

becomes, the functional-analytic counterpart of

$$\ell^1(\Lambda) \subset \ell^2(\Lambda) \subset \ell^\infty(\Lambda).$$

The analogy with decimal expansions is structural rather than literal: $\mathbf{S}'_0(\mathbb{R}^d)$ is not the norm completion of $\mathbf{S}_0(\mathbb{R}^d)$, but finite Gabor sums approximate mild distributions in the natural weak-* sense.





Typical Wilson functions I

Using the Gaussian window

$$g(x) = 2^{1/4} e^{-\pi x^2},$$

we display typical Wilson-system atoms, with the standard convention

$$w_{\ell,0}(x) = g(x - \ell),$$

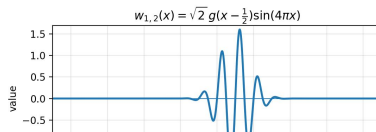
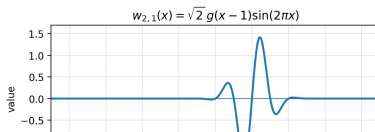
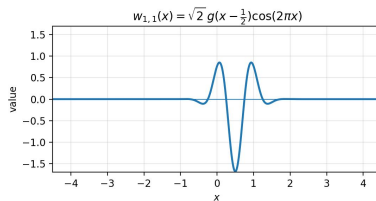
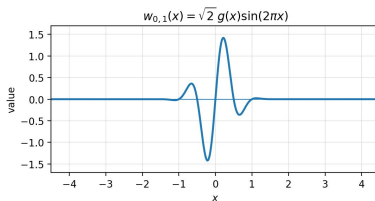
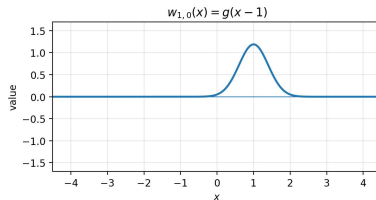
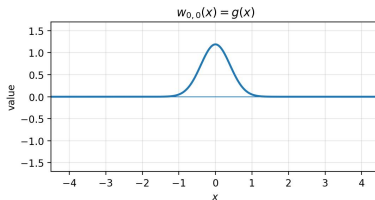
and for $n \geq 1$,

$$w_{\ell,n}(x) = \sqrt{2} g\left(x - \frac{\ell}{2}\right) \begin{cases} \cos(2\pi nx), & \ell + n \text{ even,} \\ \sin(2\pi nx), & \ell + n \text{ odd.} \end{cases}$$



Typical Wilson functions II

Typical Wilson functions for the Gaussian window $g(x) = \exp(-x^2)$
 Standard 60 dBm system parameters:
 $w_0,0(x) = g(x)$, $w_{0,1}(x) = \sqrt{2} g(x) \sin(2\pi x)$, $w_{1,1}(x) = \sqrt{2} g(x - \frac{1}{2}) \cos(2\pi x)$
 and $w_{2,1}(x) = \sqrt{2} g(x - 1) \sin(4\pi x)$

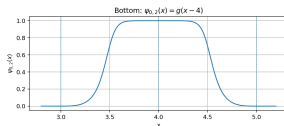
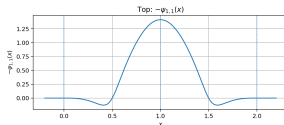


Two Functions from the Cubic-Spline Wilson Basis

Cubic spline Wilson family: two sample functions

Top panel: $-\psi_{0,1}(x) = -\sqrt{3}/2 \, g(x-1) \cos(\pi x)$.

Bottom panel: $\psi_{0,2}(x) = g(x-4)$, i.e. the zero-frequency branch at translation $2\pi = 4$.



Window: normalized compactly supported cubic spline generator $g(x)=0, \frac{3}{2}\Delta(x)/\sqrt{3}\pi(g(x))$.



Operators: the BGT changes the perspective

$$\mathbf{S}_0(\mathbb{R}^d) \hookrightarrow \mathbf{L}^2 \hookrightarrow \mathbf{S}'_0(\mathbb{R}^d).$$

In a pure Hilbert-space setting the basic question is often

$$T : \mathbf{L}^2 \longrightarrow \mathbf{L}^2 : \quad \text{bounded? unitary? compact? self-adjoint?}$$

For the Banach Gelfand Triple there are several natural levels:

- (A) T preserves the BGT: it acts on $\mathbf{S}_0(\mathbb{R}^d)$, $\mathbf{L}^2$, and (by duality) on $\mathbf{S}'_0(\mathbb{R}^d)$.
- (B) T is bounded on $\mathbf{L}^2$, but does not preserve $\mathbf{S}_0(\mathbb{R}^d)$.
- (C) T need not act on all of $\mathbf{L}^2$, but is a natural bounded map

$$T : \mathbf{S}_0(\mathbb{R}^d) \longrightarrow \mathbf{S}'_0(\mathbb{R}^d).$$

Sampling and periodization belong naturally to class (C).



Operators preserving the whole BGT

Suppose that

$$T : \mathbf{S}_0(\mathbb{R}^d) \rightarrow \mathbf{S}_0(\mathbb{R}^d) \quad \text{and} \quad T^* : \mathbf{S}_0(\mathbb{R}^d) \rightarrow \mathbf{S}_0(\mathbb{R}^d)$$

are bounded. Then the dual action is defined by

$$\langle T^\times \sigma, g \rangle := \langle \sigma, T^* g \rangle, \quad \sigma \in \mathbf{S}'_0(\mathbb{R}^d), \quad g \in \mathbf{S}_0(\mathbb{R}^d).$$

Thus one obtains (since L^2 can be obtained by complex interpolation):

$$\begin{array}{ccccc} \mathbf{S}_0(\mathbb{R}^d) & \hookrightarrow & L^2 & \hookrightarrow & \mathbf{S}'_0(\mathbb{R}^d) \\ \downarrow T & & \downarrow T & & \downarrow T^\times \\ \mathbf{S}_0(\mathbb{R}^d) & \hookrightarrow & L^2 & \hookrightarrow & \mathbf{S}'_0(\mathbb{R}^d). \end{array}$$

Typical examples are

$\mathcal{F}$, $\mathcal{F}_\alpha$, T_x , M_ω , dilations and metaplectic operators.

In particular, the ordinary and fractional Fourier transforms are **BGT automorphisms**. See Cordero–Feichtinger–Luef (2008), de Gesson–Luef (2014), and Jakobsen (2018).



L^2 -bounded does not imply BGT-preserving: BOXCAR

Let

$$b(x) = \mathbf{1}_{[-1/2, 1/2]}(x), \quad M_b f = b f.$$

Then

$$\|M_b f\|_2 \leq \|f\|_2,$$

so M_b is a perfectly harmless bounded operator on L^2 .

But

$$\mathcal{S}_0(\mathbb{R}^d) \subset C_0(\mathbb{R}^d).$$

Choose $g \in \mathcal{S}_0(\mathbb{R}^d)$ with $g(1/2) \neq 0$. Then bg has a jump at $1/2$, hence

$$bg \notin \mathcal{S}_0(\mathbb{R}^d).$$

Therefore

$$M_b : L^2 \rightarrow L^2 \quad \text{but} \quad M_b : \mathcal{S}_0(\mathbb{R}^d) \not\rightarrow \mathcal{S}_0(\mathbb{R}^d).$$

More generally, a nontrivial indicator function cannot be a multiplier of $\mathcal{S}_0(\mathbb{R}^d)$ if it creates a jump at a boundary point.



Why the BOXCAR obstruction matters for mild distributions

For a good multiplier m one extends multiplication to $\mathbf{S}'_0(\mathbb{R}^d)$ by

$$\langle m\sigma, g \rangle := \langle \sigma, mg \rangle, \quad g \in \mathbf{S}_0(\mathbb{R}^d).$$

For the BOXCAR b this prescription fails in general:

$$bg \notin \mathbf{S}_0(\mathbb{R}^d).$$

Hence the expression $\langle \sigma, bg \rangle$ is not defined for an arbitrary $\sigma \in \mathbf{S}'_0(\mathbb{R}^d)$ and the Hilbert-space statement

$$b \in L^\infty \implies M_b \in \mathcal{L}(L^2)$$

does **not** give a canonical multiplication operator on all mild distributions.

This is a first example where the BGT viewpoint is genuinely different from the L^2 viewpoint.



Fourier–Stieltjes functions are good multipliers

Let

$$B(\mathbb{R}^d) = \widehat{M(\mathbb{R}^d)}$$

denote the Fourier–Stieltjes algebra. If

$$m = \widehat{\mu}, \quad \mu \in M(\mathbb{R}^d),$$

then m acts boundedly by pointwise multiplication on $\mathbf{S}_0(\mathbb{R}^d)$. Indeed, $\mathbf{S}_0(\mathbb{R}^d)$ is Fourier invariant and is a Banach convolution module over bounded measures. Thus, with the usual harmless convention-dependent reflection,

$$\widehat{mf} = \mu * \widehat{f}$$

and consequently

$$\|mf\|_{S_0} \leq C \|\mu\|_M \|f\|_{S_0}.$$

Therefore every $m \in B(\mathbb{R}^d)$ induces bounded operators on

$$\mathbf{S}_0(\mathbb{R}^d), \quad L^2, \quad \mathbf{S}'_0(\mathbb{R}^d).$$



There are more multipliers: quadratic chirps

For a real symmetric matrix A , put

$$c_A(x) = e^{\pi i x \cdot A x}.$$

Multiplication by c_A is a metaplectic shear, hence

$$f \mapsto c_A f, \quad c_A \mathbf{S}_0(\mathbb{R}^d) = \mathbf{S}_0(\mathbb{R}^d),$$

and it extends canonically to the whole BGT.

For $A \neq 0$, c_A is not uniformly continuous, while

$$B(\mathbb{R}^d) \subset BUC(\mathbb{R}^d).$$

Consequently $c_A \notin B(\mathbb{R}^d)$, although it is an S_0 -multiplier. After Fourier conjugation,

$$\mathcal{F}^{-1} M_{c_A} \mathcal{F},$$

one obtains a [Fresnel/chirp convolution operator](#); for nonsingular A its distribution kernel is again a quadratic chirp, up to constants and a linear change of variables.

See de Gosson–Luef (2014).



The large natural class: $\mathcal{S}_0(\mathbb{R}^d) \longrightarrow \mathcal{S}'_0(\mathbb{R}^d)$!

The BGT analogue for operators $T : \mathcal{S}(\mathbb{R}^{d_1}) \longrightarrow \mathcal{S}'(\mathbb{R}^{d_2})$ is

$$T : \mathcal{S}_0(\mathbb{R}^{d_1}) \longrightarrow \mathcal{S}'_0(\mathbb{R}^{d_2}).$$

The outer kernel theorem for the Feichtinger algebra gives the correspondence

$$\mathcal{L}\left(\mathcal{S}_0(\mathbb{R}^{d_1}), \mathcal{S}'_0(\mathbb{R}^{d_2})\right) \simeq \mathcal{S}'_0(\mathbb{R}^{d_1+d_2}),$$

$$T \longleftrightarrow K_T.$$

Every bounded L^2 -operator is contained in this class through

$$\mathcal{S}_0(\mathbb{R}^d) \hookrightarrow L^2 \xrightarrow{T} L^2 \hookrightarrow \mathcal{S}'_0(\mathbb{R}^d),$$

but the class is much larger. Its kernels may describe outputs which are discrete, periodic, or genuinely distributional.



The large natural class: $\mathbf{S}_0(\mathbb{R}^d) \longrightarrow \mathbf{S}'_0(\mathbb{R}^d) \parallel$

See Feichtinger–Jakobsen (2022), especially the discussion of the [inner and outer kernel theorems](#).

Physicists may write the operator, using the “continuous ONB of Dirac measures $(\delta_x)_{x \in \mathbb{R}^d}$ as

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y)f(y)dy,$$

in complete analogy to matrix multiplication, just with continuous variables. One can discuss the terminology.

But for the case that one has $f \in \mathbf{S}_0(\mathbb{R}^d)$ and $K_T \in \mathbf{S}_0(\mathbb{R}^{2d})$ (the so-called inner kernel theorem) one can discretize these integrals and proof convergence (using e.g. piecewise linear interpolation). For the $\mathbf{L}^2$ -case one has a unitary identification with the space of (compact) Hilbert-Schmidt operators, with $K_T \in \mathbf{L}^2(\mathbb{R}^{2d})$.



The Dirac comb is a mild distribution

For $a > 0$ define the lattice Dirac comb $\Delta_a := \sum_{k \in \mathbb{Z}^d} \delta_{ak}$. belongs to $\mathcal{S}'_0(\mathbb{R}^d)$, with $\Delta_a(f) = \sum_{k \in \mathbb{Z}^d} f(ak)$. In fact, for $f \in \mathcal{S}_0(\mathbb{R}^d)$, lattice samples have sufficient summability, and Δ_a defines a bounded linear functional.

$$f \Delta_a = \sum_{k \in \mathbb{Z}^d} f(ak) \delta_{ak},$$

and

$$f * \Delta_a = \sum_{k \in \mathbb{Z}^d} T_{ak} f.$$

They represent two fundamental signal-processing operations:

sampling = multiplication by a Dirac comb

periodization = convolution with a Dirac comb

Feichtinger (2024), *Sampling via the Banach Gelfand Triple*.



Sampling and periodization are usually not L^2 operators

Define

$$\mathcal{S}_a f := f \Delta_a = \sum_{k \in \mathbb{Z}^d} f(ak) \delta_{ak}.$$

Its output is a **discrete signal** and is not an L^2 -function.

Likewise,

$$\mathcal{P}_a f := f * \Delta_a = \sum_{k \in \mathbb{Z}^d} T_{ak} f$$

is an $a\mathbb{Z}^d$ -periodic continuous function. Unless it vanishes, it is not in $L^2(\mathbb{R}^d)$.

Thus

$$\mathcal{S}_a, \mathcal{P}_a : \mathbf{S}_0(\mathbb{R}^d) \longrightarrow \mathbf{S}'_0(\mathbb{R}^d)$$

are natural bounded BGT operators, although neither operation is naturally an operator

$$L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d).$$

Periodic and discrete signals fit naturally into the dual space



Fourier transform interchanges sampling and periodization

With

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx,$$

Poisson's formula takes the compact form

$$\mathcal{F}\Delta_a = a^{-d} \Delta_{1/a}.$$

Hence

$$\begin{aligned} \mathcal{F}(\mathcal{S}_a f) &= a^{-d} \widehat{f} * \Delta_{1/a} \\ &= a^{-d} \sum_{k \in \mathbb{Z}^d} \widehat{f}\left(\cdot - \frac{k}{a}\right), \end{aligned}$$

while

$$\begin{aligned} \mathcal{F}(\mathcal{P}_a f) &= a^{-d} \widehat{f} \Delta_{1/a} \\ &= a^{-d} \sum_{k \in \mathbb{Z}^d} \widehat{f}(k/a) \delta_{k/a}. \end{aligned}$$

sampling $\xleftrightarrow{\mathcal{F}}$ periodization.



Sampling + periodization gives a finite cyclic signal

Take

$$L = Na, \quad N \in \mathbb{N},$$

and combine the two operations:

$$\mathcal{D}_{a,N}f := \mathcal{P}_{Na}\mathcal{S}_af = \mathcal{S}_a\mathcal{P}_{Na}f.$$

The result is simultaneously **discrete** and **Na -periodic**:

$$\mathcal{D}_{a,N}f = \sum_{r=0}^{N-1} c_r \sum_{k \in \mathbb{Z}} \delta_{a(r+kN)},$$

where

$$c_r = \sum_{\ell \in \mathbb{Z}} f(a(r + \ell N)), \quad r = 0, \dots, N-1.$$

Hence the mild distribution is completely determined by

$$(c_0, \dots, c_{N-1}) \in \mathbb{C}^N \simeq \mathbb{C}^{\mathbb{Z}_N}.$$

More generally, nested lattices lead to functions on a finite Abelian quotient group Λ/Γ .



The DFT and FFT arise on the finite group

On the finite group $\mathbb{Z}_N^d$ the Fourier transform is exactly the discrete Fourier transform

$$\widehat{c}_s = N^{-d/2} \sum_{r \in \mathbb{Z}_N^d} c_r e^{-2\pi i r \cdot s / N}.$$

Thus the route

$$\mathbf{S}_0(\mathbb{R}^d) \longrightarrow \{\text{periodic-discrete mild distributions}\} \simeq \mathbb{C}^{\mathbb{Z}_N^d}$$

turns the ordinary Fourier transform into a finite Fourier transform.

The FFT is simply a fast algorithm for this **exact finite transform**.

The approximation occurs earlier:

continuous signal $\rightsquigarrow$ sampled and periodized finite model.

For $f \in \mathbf{S}_0(\mathbb{R}^d)$, simultaneous decay/localization of f and $\widehat{f}$ makes this passage particularly well behaved.

See Feichtinger–Kozek–Luef (2009) and

Abreu–Balazs–Holighaus–Luef–Speckbacher (2024)



Shannon sampling through the same picture

Sampling with step a gives

$$\widehat{f} \mapsto \sum_{k \in \mathbb{Z}^d} \widehat{f} \left(\cdot - \frac{k}{a} \right).$$

If the shifted spectral copies do not overlap, there is no aliasing. For example, in one dimension a sufficient Nyquist condition is

$$\text{supp } \widehat{f} \subset \left(-\frac{1}{2a}, \frac{1}{2a} \right).$$

Then the samples $f(ak)$ contain the information needed to reconstruct the band-limited signal.

There is an instructive connection with the earlier BOXCAR example: the classical sinc reconstruction kernel corresponds to a **sharp spectral cut-off**. Such a cut-off is not an automorphism of $\mathcal{S}_0(\mathbb{R}^d)$, even though the sampling theorem itself has a very natural formulation in the BGT framework.

For the BGT formulation, Poisson formula, Dirac combs, and Shannon



Operator hierarchy: summary

Class	Examples	What happens
BGT automorphisms	$\mathcal{F}, \mathcal{F}_\alpha$, TF shifts, metaplectic maps, chirps	$\mathbf{S}_0(\mathbb{R}^d)$, L^2 , $\mathbf{S}'_0(\mathbb{R}^d)$ are all preserved
L^2 -bounded, not S_0 -preserving	BOXCAR multiplier	bounded on L^2 , but a jump destroys S_0
general BGT kernel operators	$T : \mathbf{S}_0(\mathbb{R}^d) \rightarrow \mathbf{S}'_0(\mathbb{R}^d)$	kernel $K_T \in S'_0(\mathbb{R}^{2d})$
sampling / periodization	$f \mapsto f \Delta_a$, $f \mapsto f * \Delta_a$	outputs are discrete or periodic and need not lie in L^2

BGT does not merely enlarge Hilbert space theory; it changes which operators are regarded as natural.



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From ℓ^2 to L^2 and the Role of Trace Class Operators I

- In the discrete setting, the natural Banach Gelfand Triple is

$$(\ell^1, \ell^2, \ell^\infty), \quad \text{with} \quad \ell^1(\mathbb{Z}^d) \subset \ell^2(\mathbb{Z}^d) \subset \ell^\infty(\mathbb{Z}^d),$$

with the dual pairing extending the scalar product on ℓ^2 .

- Passing to continuous variables suggests to consider the NON-VALID! spaces: $L^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, with the scalar product for $L^2(\mathbb{R}^d)$ given by $\langle f, g \rangle = \int_{\mathbb{R}^d} f(x) \overline{g(x)} dx$, one better uses the scale, increasing for $p \in [1, \infty]$ and closed under duality, with $1/p + 1/q = 1$ for $p < \infty$:

$$\mathcal{S}'_0(\mathbb{R}^d) = \mathcal{M}^1(\mathbb{R}^d) \hookrightarrow \mathcal{M}^p(\mathbb{R}^d) \hookrightarrow \mathcal{M}^\infty(\mathbb{R}^d) = \mathcal{S}'_0(\mathbb{R}^d).$$



From ℓ^2 to L^2 and the Role of Trace Class Operators II

- In analogy, one considers spaces of operators:
 - Hilbert–Schmidt operators $\mathcal{HS}$ as the analogue of L^2 ,
 - Trace class operators $\mathcal{L}^1$ as the analogue of L^1 .
- The Hilbert–Schmidt class carries a natural inner product:

$$\langle T, S \rangle_{\mathcal{HS}} := \text{trace}(S^* T),$$

which is the operator analogue of the L^2 scalar product.

- Thus:
 - vectors $\leftrightarrow$ functions, or (mild) distributions;
 - matrices $\leftrightarrow$ operators (not only bounded operators on $L^2(\mathbb{R}^d)$)
 - matrix trace $\leftrightarrow$ integral.
- This analogy provides the bridge from finite-dimensional linear algebra to operator theory in harmonic analysis.



What is a signal? — start from measurements

A useful change of viewpoint is to avoid asking first

“Is a signal a function, an L^2 -class, a measure, or a distribution?”

and to ask instead:

What can we measure about a signal?

Following [?], take the functions

$$g \in \mathbf{S}_0(\mathbb{R}^d)$$

as **possible measurements** (test functions).

A signal σ should then assign to every such measurement g a complex number

$$\sigma(g) \in \mathbb{C},$$

the **response of the signal to the measurement**.

Linearity and stability of the measurement process lead naturally to

$$\sigma \in \mathbf{S}'_0(\mathbb{R}^d) = (\mathbf{S}_0(\mathbb{R}^d))'.$$

Thus the proposed model is



The fundamental signal–measurement pairing

The basic object is **not** an L^2 scalar product but the duality

$$\begin{aligned} \mathbf{S}_0(\mathbb{R}^d) \times \mathbf{S}'_0(\mathbb{R}^d) &\longrightarrow \mathbb{C}, \\ (g, \sigma) &\longmapsto \sigma(g). \end{aligned}$$

It is a bilinear pairing, with

$$|\sigma(g)| \leq \|\sigma\|_{S'_0} \|g\|_{S_0}.$$

Interpretation:

- $g \in \mathbf{S}_0(\mathbb{R}^d)$ describes a **measurement procedure**;
- $\sigma \in \mathbf{S}'_0(\mathbb{R}^d)$ describes the **signal**;
- $\sigma(g)$ is the **measurement result**.

Two signals are identical precisely when they give the same result for every admissible measurement:

$$\sigma_1(g) = \sigma_2(g) \quad \forall g \in \mathbf{S}_0(\mathbb{R}^d) \quad \implies \quad \sigma_1 = \sigma_2.$$

Hence a signal is characterized operationally by the collection



Why should the measurements belong to $\mathcal{S}_0(\mathbb{R}^d)$?

The Feichtinger algebra provides a particularly convenient class of “nice probes”:

$$\mathcal{S}_0(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d) \hookrightarrow L^2.$$

Thus a measurement function is

- **continuous**: pointwise values are meaningful;
- **integrable**: ordinary integrals are available;
- **well localized in time and frequency**;
- stable under **translations and modulations**;
- **Fourier invariant**;
- stable under the basic operations of Fourier and Time–Frequency Analysis.

In particular,

$$g \in \mathcal{S}_0(\mathbb{R}^d) \quad \implies \quad \widehat{g} \in \mathcal{S}_0(\mathbb{R}^d).$$

Hence the same class of measurements can be used in the time and frequency domains.



Why not start from L^2 ?

The Hilbert space

$$L^2$$

is indispensable for energy, orthogonality and unitary operators.
But it is not sufficient as a model for *all* signals and measurements.

A basic difficulty is **sampling**:

$$f \in L^2(\mathbb{R}^d)$$

is only an equivalence class modulo sets of measure zero.
Therefore the number

$$f(x_0)$$

is not determined by the L^2 element.

But for $g \in \mathbf{S}_0(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$,

$$\delta_{x_0}(g) = g(x_0)$$

is perfectly meaningful and bounded. Thus

$$\delta_{x_0} \in \mathbf{S}'_0(\mathbb{R}^d), \quad \delta_{x_0} \notin L^2$$



The signal space should contain the objects used in practice

The dual space $\mathbf{S}'_0(\mathbb{R}^d)$ is large enough to contain, in one common framework,

ordinary functions and L^p -signals,
 bounded and periodic functions,
 measures and discrete lattice measures,

$$\delta_x, \quad \sum_{k \in \mathbb{Z}^d} \delta_k.$$

For example,

$$h \in C_b(\mathbb{R}^d) \implies \sigma_h(g) := \int_{\mathbb{R}^d} h(t)g(t) dt \in \mathbf{S}'_0(\mathbb{R}^d).$$

For a point mass,

$$\delta_x(g) = g(x),$$

and for a Dirac comb,



Mild convergence = convergence of measurement results

The duality also provides the appropriate notion of approximation.
For a bounded sequence (σ_n) in $\mathbf{S}'_0(\mathbb{R}^d)$ we write

$$\sigma_n \xrightarrow{w^*} \sigma$$

or say that σ_n converges **mildly** to σ if

$$\boxed{\sigma_n(g) \longrightarrow \sigma(g) \quad \text{for every } g \in \mathbf{S}_0(\mathbb{R}^d).}$$

Thus

two signals are close if all relevant measurements give close results.

This is an operational notion of convergence, rather than merely a distance between abstract functions.

For example,

$$x_n \rightarrow x \quad \implies \quad \delta_{x_n} \xrightarrow{w^*} \delta_x,$$

because

$$g(x_n) \rightarrow g(x) \quad (g \in \mathbf{S}_0(\mathbb{R}^d)).$$



Time–Frequency analysis is a family of measurements

Choose a non-zero window $g \in \mathbf{S}_0(\mathbb{R}^d)$. The STFT of a mild distribution is obtained by probing it with time–frequency shifted copies of g :

$$V_g \sigma(x, \omega) = \sigma(M_\omega T_x g)$$

(up to the chosen conjugation convention).

Thus a spectrogram or a Gabor analysis is literally a **family of measurements of the signal**.

For bounded sequences in $\mathbf{S}'_0(\mathbb{R}^d)$, mild convergence can be recognized through convergence of such Time–Frequency measurements on compact parts of phase space.

A finite digital observation therefore records only finitely many quantities of the form

$$\sigma(M_{\omega_k} T_{x_k} g).$$

Increasing the observed time–frequency region and the resolution gives better finite-dimensional information about the underlying mild distribution.



The Banach Gelfand Triple separates three roles

The resulting framework is

$$\mathcal{S}_0(\mathbb{R}^d) \hookrightarrow L^2 \hookrightarrow \mathcal{S}'_0(\mathbb{R}^d).$$

These three spaces have different natural interpretations:

- $\mathcal{S}_0(\mathbb{R}^d)$: good test signals / measurements,
- L^2 : finite-energy signals and Hilbert-space geometry,
- $\mathcal{S}'_0(\mathbb{R}^d)$: general signals / mild distributions.

The Hilbert space is therefore not discarded; it becomes the *middle layer*.

The outer duality

$$\mathcal{S}_0(\mathbb{R}^d) \times \mathcal{S}'_0(\mathbb{R}^d) \rightarrow \mathbb{C}$$

is more fundamental for sampling, point masses, periodic signals and distributional signals, while the middle layer retains Parseval, orthogonality and unitary Fourier analysis.

The BGT combines test-function duality and Hilbert-space



Why this is a reasonable model of “signal”

The proposal can be summarized in four steps:

- ① A physical signal is known only through **measurements**.
- ② Choose $\mathbf{S}_0(\mathbb{R}^d)$ as a robust class of localized, Fourier-invariant measurement functions.
- ③ Define a signal by its stable response

$$g \longmapsto \sigma(g).$$

The corresponding maximal Banach space is

$$\mathbf{S}'_0(\mathbb{R}^d) = (\mathbf{S}_0(\mathbb{R}^d))'.$$

- ④ Approximation is described by convergence of measurement results:

$$\sigma_n(g) \rightarrow \sigma(g).$$

Consequently,

a signal need not have pointwise values;
it only has to produce consistent measurement values.



Explaining standard results

Many familiar statements from engineering, physics and analysis become particularly transparent in the Banach Gelfand Triple

$$\mathcal{S}_0(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'_0(\mathbb{R}^d).$$

We illustrate this with:

- Plancherel's Theorem;
- translation-invariant systems and convolution;
- Dirac combs, sampling and periodization;
- Shannon's Sampling Theorem;
- periodic-discrete signals, DFT and FFT.
- classical summability theory (Fejer).

One framework for continuous, discrete and periodic signals.



L^p -spaces inside the BGT

For every $1 \leq p \leq \infty$,

$$\mathbf{S}_0(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow \mathbf{S}'_0(\mathbb{R}^d).$$

Moreover,

$$\mathbf{S}_0(\mathbb{R}^d) \text{ is dense in } L^p(\mathbb{R}^d), \quad 1 \leq p < \infty.$$

The second embedding is the natural duality

$$f \longmapsto \sigma_f, \quad \sigma_f(g) = \int_{\mathbb{R}^d} f(x)g(x) \, dx, \quad g \in \mathbf{S}_0(\mathbb{R}^d).$$

Thus ordinary L^p -signals can be seen (naturally) as (*regular*) *mild distributions*.

$\mathbf{S}_0(\mathbb{R}^d)$ is small enough for classical analysis, while $\mathbf{S}'_0(\mathbb{R}^d)$ is large enough for different types of signals usually discussed in application fields.



Plancherel's Theorem from $\mathcal{S}_0(\mathbb{R}^d)$

The Feichtinger algebra is Fourier invariant: $\mathcal{F}(\mathcal{S}_0(\mathbb{R}^d)) = \mathcal{S}_0(\mathbb{R}^d)$. There are several characterizations of $\mathcal{S}_0(\mathbb{R}^d)$ making this easy, e.g. starting from the Fourier invariance of the Gauss function.

Its elements, and their Fourier transforms, are continuous and absolutely Riemann integrable; Fourier inversion is therefore an ordinary pointwise integral formula. We have

$$\langle \widehat{f}, \widehat{g} \rangle_{L^2} = \langle f, g \rangle_{L^2}, \quad f, g \in \mathcal{S}_0(\mathbb{R}^d),$$

hence

$$\|\widehat{f}\|_2 = \|f\|_2.$$

Since $\mathcal{S}_0(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, $\mathcal{F}$ extends uniquely to

$$\boxed{\mathcal{F} : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d) \quad \text{unitarily.}}$$

Plancherel is obtained by completion from $\mathcal{S}_0(\mathbb{R}^d)$.



Translation-Invariant Systems (TILS)

Let $T : \mathcal{S}_0(\mathbb{R}^d) \longrightarrow \mathcal{S}'_0(\mathbb{R}^d)$ be bounded and commute with translations:

$$TT_x = T_x T, \quad x \in \mathbb{R}^d.$$

Then there is a unique $\sigma \in \mathcal{S}'_0(\mathbb{R}^d)$ such that

$$Tf = \sigma * f, \quad f \in \mathcal{S}_0(\mathbb{R}^d).$$

For $f \in \mathcal{S}_0(\mathbb{R}^d)$ this convolution is a bounded uniformly continuous function and can be described pointwise as

$$(Tf)(x) = (\sigma * f)(x) = \sigma(T_x f^\vee).$$

Thus σ is the generalized **impulse response**. It is the mild limit of the output obtained through T when using as input a Dirac sequence.

Linear translation-invariant system = convolution operator.



Recovering the impulse response

If we would now that such a sigma exists it is easy to obtain from T :

$$(Tf)(x) = \sigma(T_x f^\vee),$$

then at $x = 0$, $(Tf)(0) = \sigma(f^\vee)$. Replacing f by $f^\vee$ gives

$$\sigma(f) = T(f^\vee)(0).$$

For an ordinary integrable impulse response h , $Tf = h * f$, but now the same is true in the context $\mathbf{S}'_0(\mathbb{R}^d)$. On the Fourier side, by multiplication with the (generalized) *transfer function* $\hat{\sigma} \in \mathbf{S}'_0(\mathbb{R}^d)$: $\widehat{Tf} = \hat{\sigma} \hat{f}$, or equivalently expressed:

$$Tf = \mathcal{F}^{-1}(\hat{\sigma} \cdot \hat{f}), \quad f \in \mathbf{S}_0(\mathbb{R}^d).$$

For a self-contained proof one show first that T maps automatically $\mathbf{S}_0(\mathbb{R}^d)$ into $\mathbf{C}_b(\mathbb{R}^d)$, using (minimality) properties of $\mathbf{S}_0(\mathbb{R}^d)$.

$$\text{translation invariance} \iff \text{convolution} \xrightarrow{\mathcal{F}} \text{multiplication}.$$



Dirac combs and weighted Dirac combs

For a lattice $\Lambda \subset \mathbb{R}^d$,

$$\Delta_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda \in \mathcal{S}'_0(\mathbb{R}^d).$$

Every bounded coefficient family $(c_\lambda)_{\lambda \in \Lambda}$ defines

$$\sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda \in \mathcal{S}'_0(\mathbb{R}^d).$$

Hence bounded multiplication on the lattice gives weighted Dirac combs. For $h \in \mathcal{S}_0(\mathbb{R}^d)$, we even have

$$h \Delta_\Lambda = \sum_{\lambda \in \Lambda} h(\lambda) \delta_\lambda, \quad \text{with} \quad \sum_{\lambda \in \Lambda} |h(\lambda)| < \infty.$$

Thus $h \Delta_\Lambda$ is a finite discrete measure.

Multiplication by Δ_Λ realizes sampling.



Periodization and the Fourier transform

For $f \in \mathbf{S}_0(\mathbb{R}^d)$, $f * \Delta_\Lambda = \sum_{\lambda \in \Lambda} T_\lambda f$ is a continuous Λ -periodic function. Thus convolution with Δ_Λ realizes **periodization**. For

$$\Lambda^\perp = \{\omega : e^{2\pi i \lambda \cdot \omega} = 1 \text{ for all } \lambda \in \Lambda\},$$

Poisson summation (!valid for any $f \in \mathbf{S}_0(\mathbb{R}^d)$) gives

$$\widehat{\Delta_\Lambda} = \text{dens}(\Lambda) \Delta_{\Lambda^\perp}.$$

Consequently,

$$\mathcal{F}(h \Delta_\Lambda) = \text{dens}(\Lambda) \widehat{h} * \Delta_{\Lambda^\perp},$$

while

$$\mathcal{F}(f * \Delta_\Lambda) = \text{dens}(\Lambda) \widehat{f} \cdot \Delta_{\Lambda^\perp}.$$

Sampling and periodization are interchanged by the Fourier transform.



Shannon's Sampling Theorem

In dimension one assume $\text{supp}(\hat{f}) \subseteq [-1/2, 1/2]$ (band-limitedness)

Sampling with step h gives $f \Delta_{h\mathbb{Z}} = \sum_{k \in \mathbb{Z}} f(kh) \delta_{kh}$.

Fourier transformation yields $\mathcal{F}(f \Delta_{h\mathbb{Z}}) = \frac{1}{h} \sum_{k \in \mathbb{Z}} T_{k/h} \hat{f}$.

Thus sampling periodizes the spectrum. If h is sufficiently small (the sampling frequency $1/h$ sufficiently large), the copies do not overlap and we find $g \in \mathbf{S}_0(\mathbb{R})$ with $\hat{g}(s) = 1$ on $[-0.5, 0.5]$ and small support. Then

$$f(x) = \sum_{k \in \mathbb{Z}} f(kh) g\left(\frac{x - kh}{h}\right).$$

With convergence in $L^2(\mathbb{R})$ for $f \in L^2(\mathbb{R}^d)$ or in $\mathbf{S}_0(\mathbb{R})$ for $f \in L^1(\mathbb{R})$.

No spectral overlap $\implies$ perfect reconstruction.



Sampling followed by periodization

Work on $\mathbb{R}$ and choose

$$p = Nh, \quad N \in \mathbb{N}.$$

Let

$$S_h f = f \Delta_{h\mathbb{Z}}, \quad \mathcal{P}_p \sigma = \sigma * \Delta_{p\mathbb{Z}}.$$

Then

$$\mathcal{A}_{h,N} f := \mathcal{P}_p S_h f$$

is discrete on $h\mathbb{Z}$ and p -periodic.

It has the form

$$\mathcal{A}_{h,N} f = \sum_{r=0}^{N-1} c_r T_{rh} \Delta_{p\mathbb{Z}},$$

with

$$c_r = \sum_{\ell \in \mathbb{Z}} f(h(r + \ell N)).$$

Hence it is determined by the finite vector $(c_0, \dots, c_{N-1}) \in \mathbb{C}^N$.



All periodic-discrete signals are obtained

Let

$$\sigma = \sum_{r=0}^{N-1} c_r T_{rh} \Delta_{p\mathbb{Z}}, \quad p = Nh.$$

Choose $\varphi \in \mathbf{S}_0(\mathbb{R})$ with $\varphi(0) = 1$, $\text{supp}(\varphi) \subset (-h/2, h/2)$, and put

$$f(x) = \sum_{r=0}^{N-1} c_r \varphi(x - rh).$$

Then $f \in \mathbf{S}_0(\mathbb{R})$ and

$$\mathcal{P}_p \mathcal{S}_h f = \sigma.$$

Therefore

$$\mathcal{A}_{h,N}(\mathbf{S}_0(\mathbb{R})) = \{p\text{-periodic discrete signals on } h\mathbb{Z}\}.$$

This space is naturally identified with function on the subquotient:

$$\mathbb{C}^N \simeq \ell^2(\mathbb{Z}_N)$$



The Fourier transform becomes the DFT

For

$$\sigma = \sum_{r=0}^{N-1} c_r T_{rh} \Delta_{p\mathbb{Z}}, \quad p = Nh,$$

use

$$\widehat{\Delta_{p\mathbb{Z}}} = \frac{1}{p} \Delta_{\frac{1}{p}\mathbb{Z}}.$$

At the frequency points m/p the coefficients are $\sum_{r=0}^{N-1} c_r e^{-2\pi i r m / N}$.
From

$$C_s = \sum_{r=0}^{N-1} c_r e^{-2\pi i r s / N}, \quad s = 0, \dots, N-1,$$

we obtain

$$\widehat{\sigma} = \frac{1}{p} \sum_{s=0}^{N-1} C_s T_{s/p} \Delta_{\frac{1}{h}\mathbb{Z}}.$$

The distributional FT is exactly the DFT on $\mathbb{Z}_N$.

The FFT is an efficient algorithm for computing this DFT.



From the DFT to the continuous Fourier transform

For $f \in \mathbf{S}_0(\mathbb{R})$ the DFT coefficients satisfy

$$C_s = \sum_{k \in \mathbb{Z}} f(kh) e^{-2\pi i ks/N}.$$

Since $p = Nh$, Poisson summation gives the exact identity

$$hC_s = \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{s}{p} + \frac{m}{h}\right).$$

Thus the FFT computes samples of a **periodized Fourier transform**.

If h is small, the unwanted terms with $m \neq 0$ are small in the frequency region of interest. Therefore

$$hC_s \approx \hat{f}\left(\frac{s}{p}\right).$$

At the same time, taking $p = Nh$ large reduces the effect of periodization in the time variable.



What the FFT approximates

An N -point FFT gives values on the frequency grid

$$\frac{1}{p}\mathbb{Z},$$

over a frequency interval of length approximately $1/h$.

The two parameters have different roles:

p large $\implies$ small time-periodization error,

h small $\implies$ small frequency-aliasing error.

The FFT itself introduces no approximation.

The approximation occurs when a function on $\mathbb{R}$ is replaced by a finite periodic-discrete signal on $\mathbb{Z}_N$.

The simultaneous localization of f and $\hat{f}$ for $f \in \mathbf{S}_0(\mathbb{R})$ makes this passage particularly natural.



Classical Summability I

Finally let us mention the problem of summability. Already in the classical (periodic) situation the naive inversion of the Fourier transform, by taking partial sums of the Fourier series of a function $f \in L^1$ may cause problems (divergence, even pointwise everywhere). Starting with Fejer a large number of summability method has been invented which allows to overcome the deficiency arising from the fact that $f \in L^1(\mathbb{R}^d)$ does NOT imply $\hat{f} \in L^1(\mathbb{R}^d)$ (unless $f \in \mathcal{S}_0(\mathbb{R}^d)$!)

The method consists in multiplying $\hat{f}$ first with some (widely) dilated version of some $h = \hat{g}$, with $g \in \mathcal{S}_0(\mathbb{R}^d)$ and $h(0) = \int_{\mathbb{R}^d} g(x) dx = 1$. Then the product is integrable and the inversion formula applies. On the time-side this corresponds to choose (using compression) a Dirac sequence generated by g , and thus one obtains $f \in L^1(\mathbb{R}^d)$ in the limit (taken in the L^1 -norm).



Classical Summability II

Ferenc Weisz has investigated the question, which on of the classical summability kernels belong to $\mathbf{S}_0(\mathbb{R}^d)$ and his finding was: All of them belong to $\mathbf{S}_0(\mathbb{R}^d)$, and thus the reasoning: Given a candidate, check whether it belongs to $\mathbf{S}_0(\mathbb{R}^d)$ and then use the properties of $\mathbf{S}_0(\mathbb{R}^d)$ allows to verify the validity of all the classical summability kernels!!

Another, related question is the validity of Poisson's formula. Clearly it is valid for Schwartz functions, and thus the Fourier invariance of the standard Dirac comb $\sqcup = \sum_{k \in \mathbb{Z}^d} \delta_k$ in the sense of tempered distributions appears, but Fourier invariance in the sense of $\mathbf{S}'_0(\mathbb{R}^d)$ is just the stronger statement: Poisson's formula is valid for all $f \in \mathbf{S}_0(\mathbb{R}^d)$.

There are “counter examples” of spaces with mild decay in the time and the frequency domain, but all of them are not included in $\mathbf{S}_0(\mathbb{R}^d)$.



Why $S_0(\mathbb{R}^d)$ and mild distributions?

The examples use all three levels of

$$S_0(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow S'_0(\mathbb{R}^d).$$

- $S_0(\mathbb{R}^d)$: continuous, integrable, Fourier invariant, restrictions to lattices are absolutely summable. and well localized in the TF-sense.
- $L^2(\mathbb{R}^d)$: Hilbert-space geometry and Plancherel.
- $S'_0(\mathbb{R}^d)$: Dirac masses, Dirac combs, periodic signals, sampling and periodization.

The Fourier transform acts compatibly on all three spaces. It defines a **unitary Banach Gelfand Triple automorphism**.

Thus one obtains in a common setting:

Plancherel, convolution systems,
sampling/periodization, Shannon reconstruction,
finite Fourier analysis and FFT approximation.

The BGT explains why these standard formulas fit together.



BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\mathbf{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\mathbf{B}'$ (hence w^* -dense there) is called a **Banach Gelfand triple**.

Definition

If $(\mathbf{B}_1, \mathcal{H}_1, \mathbf{B}'_1)$ and $(\mathbf{B}_2, \mathcal{H}_2, \mathbf{B}'_2)$ are Gelfand triples then a linear operator T is called a **[unitary] Gelfand triple isomorphism** if

- ① A is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$.
- ② A is a unitary isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
- ③ A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\mathbf{B}'_1$ and $\mathbf{B}'_2$.



Banach Gelfand Triples

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as **B** the space of elements within $\mathcal{H}$ which have an **absolutely convergent expansion**, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.

For the case of the Fourier system as CONB for $\mathcal{H} = L^2([0, 1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\mathbf{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\mathbf{PM}(\mathbb{T}) = \mathbf{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\mathbf{A}, L^2, \mathbf{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.



Sufficient conditions for BGT-automorphisms

A couple of basic facts resulting direct from the definition, combined with the general fact that the space of w^* -continuous linear functionals on a dual space $\mathbf{B}'$ coincides naturally with the original Banach space, gives the following facts.

We apply them to the concrete Banach Gelfand Triple $(\mathbf{S}_0, L^2, \mathbf{S}'_0)$.

- For every BGTr-mapping there is an adjoint BGTr-mapping T^* , and $T^{**} = T$ for each of them;
- If a bounded linear mapping on $(L^2(\mathbb{R}^d), \|\cdot\|_2)$ leaves $\mathbf{S}_0(\mathbb{R}^d)$ invariant and so does T^* (the Hilbert adjoint), then T (and also T^*) define BGTr-homomorphism.

Among others this principle can be used to show that the even *fractional Fourier transforms* define (unitary) BGTr-automorphisms on $(\mathbf{S}_0, L^2, \mathbf{S}'_0)(\mathbb{R}^d)$.



Some general scenarios arising in Gabor Analysis

There are situations, where one is forced to view even more general situations than BGTr-operators (BGOs), even if one is a priori interested only in the Hilbert space behaviour:

Here are some examples:

- 1 Let $g \in L^2(\mathbb{R}^d)$ be given, and any lattice $\Lambda \triangleleft \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then the coefficient operator

$$C : f \mapsto (V_g f(\lambda))_{\lambda \in \Lambda}$$

only maps $L^2(\mathbb{R}^d)$ into $\text{cosp}(\Lambda)$, but $\mathcal{S}_0(\mathbb{R}^d)$ into $\ell^2(\Lambda)$.

- 2 similar situation for the synthesis operator, which is its adjoint.
- 3 Finally the frame-operator is a priori an operator which maps $(\mathcal{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathcal{S}_0})$ into $(\mathcal{S}'_0(\mathbb{R}^d), \|\cdot\|_{\mathcal{S}'_0})$.

But this is good enough to show that it has a Janssen representation.



Regularizing Operators

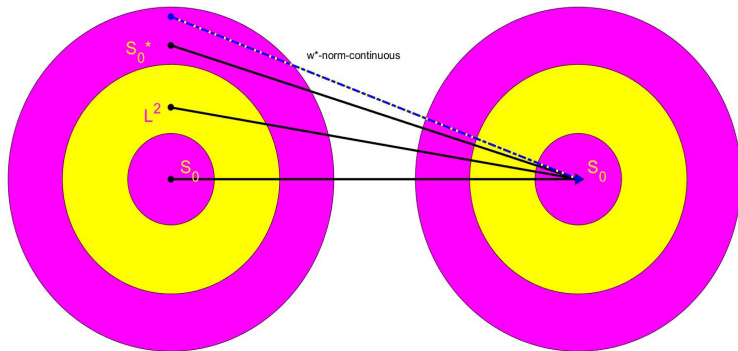


Figure: BGTREG1.jpg



Regularizers help to approximate distributions

An important family of regularizing operators are those bounded families of BGOs, where each of them maps $(\mathbf{S}'_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}'_0})$ into $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$, and which form an approximating the identity. Abstractly speaking they form bounded nets of BGOs which converge strongly (e.g. in their action on $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$) to the identity. Since we have

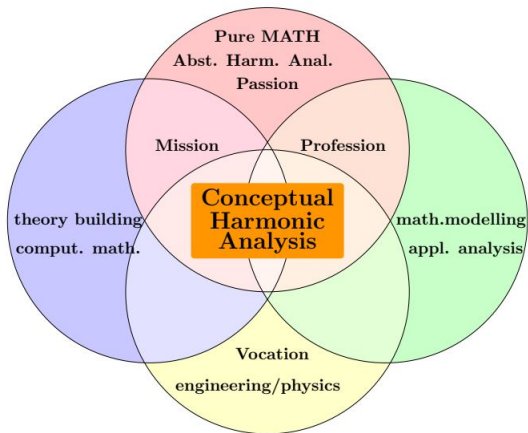
$$(\mathbf{S}'_0 * \mathbf{S}_0) \cdot \mathbf{S}_0 \subset \mathbf{S}_0 \quad \text{and} \quad (\mathbf{S}'_0 \cdot \mathbf{S}_0) * \mathbf{S}_0 \subset \mathbf{S}_0$$

this can be product-convolution (or convolution-product) operators.

A similar property is shared by Gabor multipliers with finitely many symbols (using lattice points in a bounded domain).



IKIGAI on Fourier Analysis



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- **K. Gröchenig:** *Foundations of Time–Frequency Analysis*, Birkhäuser, 2001.
A standard reference for Gabor frames, modulation spaces, the Zak transform, density questions, and time–frequency methods.
- **H. G. Feichtinger and T. Strohmer (eds.):** *Gabor Analysis and Algorithms: Theory and Applications*, Birkhäuser, 1998.
A broad collection covering Gabor analysis, algorithms, LCA groups, sampling, and operator-theoretic aspects.
- **H. G. Feichtinger and T. Strohmer (eds.):** *Advances in Gabor Analysis*, Birkhäuser, 2003.
Further developments on Gabor frames, pseudodifferential operators, irregular systems, and applications.
- **O. Christensen:** *An Introduction to Frames and Riesz Bases*, Birkhäuser.
General frame theory, with substantial material on Gabor systems and duality.
- **G. B. Folland:** *Harmonic Analysis in Phase Space*, Princeton University Press, 1989.
A classical source for the Heisenberg group, Weyl calculus, phase space, and the operator-theoretic background of TF analysis.



Links and references I

Thanks for your attention!

Let us provide a few links.

ETH Course Notes are found at:

www.nuhag.eu/ETH20

At the page

www.nuhag.eu/feitalks

you find my talks and also recorded talks from some courses, for example

From linear algebra to Gelfand triples.



Links and references II

Various [Lecture Notes](#) can be downloaded from

<https://www.univie.ac.at/nuhag-php/home/skripten.php>

The public version of my publication list can be provided upon request via

hans.feichtinger@univie.ac.at.

This talk provides some thoughts related to the idea of

Conceptual harmonic analysis.

