**Hans G. Feichtinger**
**NuHAG - University of Vienna**

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# Banach Gelfand Triples for Harmonic Analysis

Harmonic Analysis and its Applications at Tokyo
Tokyo Woman's Christian University

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March 24 (Sat)–26 (Mon), 2007

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http://www.nuhag.eu

March 2007

# Second Talk:
# March 25th, 2007

# Beyond Banach Gelfand Triples:
# Modulation spaces and Gabor Analysis

The second talk will show that in addition to the triple of Banach space $\big(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)\big)$ there is a whole family of Banach spaces "around" these spaces, in particular the *by now classical* spaces $\boldsymbol{M}^s_{p,q}(\mathbb{R})$ or the space $\boldsymbol{M}^p_{v_s}(\mathbb{R}^d)$ or $\boldsymbol{M}^p_s(\mathbb{R}^d)$ which are obtained using radial symmetric weights of polynomial growths of order $s$, $s \in \mathbb{R}$ on phase space.

## Outline of the first TALK

- Typical questions of (classical and modern) Fourier analysis
- Fourier transforms, convolution, impulse response, transfer function
- The Gelfand triple $(\boldsymbol{\mathcal{S}}(\mathbb{R}), \boldsymbol{L}^2(\mathbb{R}), \boldsymbol{\mathcal{S}}'(\mathbb{R}))$ resp. $(\boldsymbol{\mathcal{S}}, \boldsymbol{L}^2, \boldsymbol{\mathcal{S}}')(\mathbb{R}^d)$
- The Banach Gelfand Triples $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ and their use;
- various (unitary) Gelfand triple isomorphism involving $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$

**Definition 1.** *A triple* $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$*, consisting of a Banach space* $\boldsymbol{B}$*, which is dense in some Hilbert space* $\mathcal{H}$*, which in turn is contained in* $\boldsymbol{B}'$ *is called a Banach Gelfand triple.*

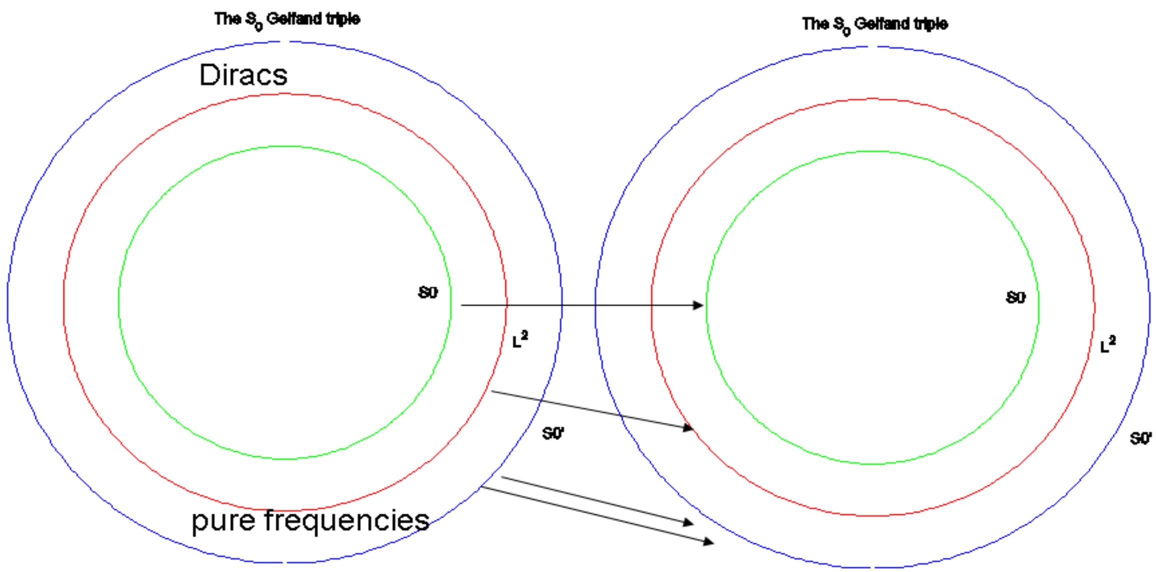
**Definition 2.** *If* $(\boldsymbol{B}^1, \mathcal{H}_1, \boldsymbol{B}_1')$ *and* $(\boldsymbol{B}^2, \mathcal{H}_1, \boldsymbol{B}_2')$ *are Gelfand triples then a linear operator* $T$ *is called a [unitary] Gelfand triple isomorphism if*

1. *A is an isomorphism between* $\boldsymbol{B}^1$ *and* $\boldsymbol{B}^2$*.*
2. *A is a [unitary operator resp.] isomorphism between* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*.*
3. *A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between* $\boldsymbol{B}_1'$ *and* $\boldsymbol{B}_2'$*.*

The prototype is $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$. $w^*$-convergence corresponds to coordinate convergence in $\ell^\infty$. It can be transferred to "abstract Hilbert spaces" $\mathcal{H}$. Given any orthonormal basis $(h_n)$ one can relate $\boldsymbol{\ell}^1$ to the set of all elements $f \in \mathcal{H}$ which have an *absolutely convergent* series expansions with respect to this basis. In fact, in the classical case of $\mathcal{H} = \boldsymbol{L}^2(\mathbb{T})$, with the usual

Fourier basis the corresponding spaces are known as Wiener's $\boldsymbol{A}(\mathbb{T})$. The dual space is then $\mathcal{P}_M$, the space of pseudo-measures $= \mathcal{F}^{-1}[\boldsymbol{\ell}^\infty(\mathbb{Z})]$.

## Gelfand triple mapping



## Realization of a GT-homomorphism

Very often a Gelfand-Triple homomorphism $T$ can be *realized with the help of some kind of "summability methods"*. In the abstract setting this is a sequence [1] $A_n$, having the following property:

- each of the operators maps $\boldsymbol{B}'_1$ into $\boldsymbol{B}^1$;
- they are a uniformly bounded family of Gelfand-triple homomorphism on $(\boldsymbol{B}^1, \mathcal{H}_1, \boldsymbol{B}'_1)$;
- $A_n f \to f$ in $\boldsymbol{B}^1$ for any $f \in \boldsymbol{B}^1$;

It then *follows* that the limit $T(A_n f)$ exists in $\mathcal{H}_2$ respectively in $\boldsymbol{B}'_2$ (in the $w^*$-sense) for $f \in \mathcal{H}_1$ resp. $f \in \boldsymbol{B}'_1$ and thus describes concretely the prolongation to the full Gelfand triple. This continuation is unique due to the $w^*$-properties assumed for $T$ (and the $w^*$-density of $\boldsymbol{B}^1$ in $\boldsymbol{B}'_1$).

[1] or more generally a net

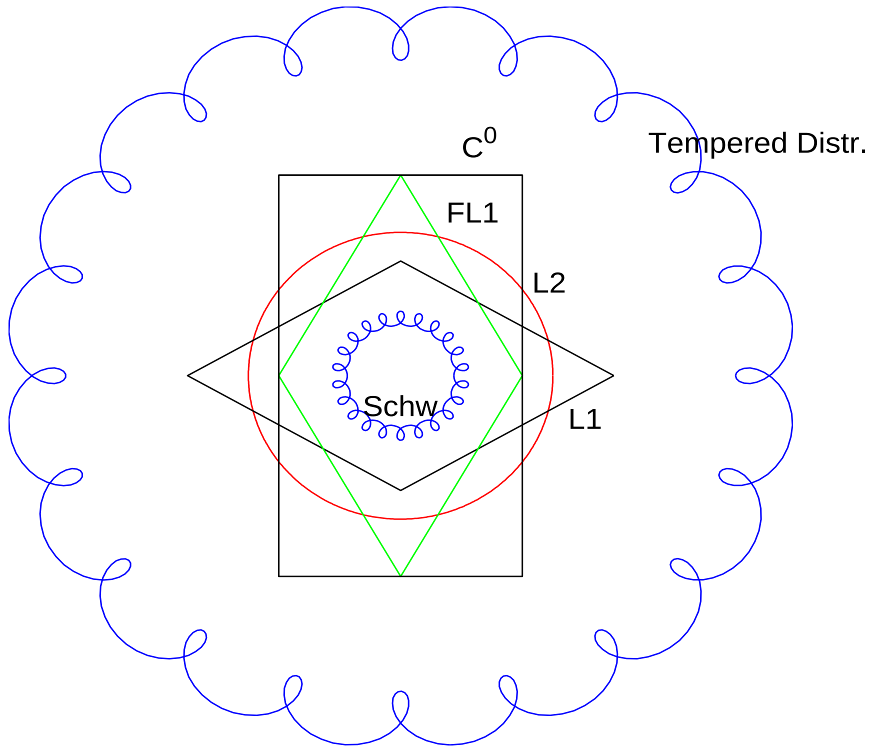
# Classical Approach to Fourier Analysis

- Fourier Series (periodic functions), summability methods;
- Fourier Transform on $\mathbb{R}^d$, using Lebesgue integration;
- Theory of Almost Periodic Functions;
- Generalized functions, tempered distributions;
- Discrete Fourier transform, FFT;
- Abstract ($>>$ Conceptional) Harmonic Analysis over LCA groups;
- . . . but what are the connections??

## Tempered distributions as unifying tool:

- Fourier Series (periodic functions), summability methods;
- Fourier Transform on $\mathbb{R}^d$, using Lebesgue integration;
- Theory of Almost Periodic Functions;
- Generalized functions, tempered distributions ;
- Discrete Fourier transform, FFT;
- Conceptional Harmonic Analysis over LCA groups;

# The classical view on the Fourier Transform



## What are our goals when doing Fourier analysis

- find relevant "harmonic components" in [almost] periodic functions;
- define the Fourier transform (first $\boldsymbol{L}^1(\mathbb{R}^d)$, then $\boldsymbol{L}^2(\mathbb{R}^d)$, etc.);
- describe time-invariant linear systems as convolution operators;
- describe such system as Fourier multipliers (transfer function);
- deal with (slowly) time-variant channels (communications);
- describe changing frequency content ("musical transcription");
- define FT on $\boldsymbol{L}^p$-spaces, or more general functions/distributions;

## The GOAL of this presentation:

- Provide a relative simple minded approach to Fourier analysis;
- based on standard arguments from functional analysis only;
- provide clear rules, based on basic Banach space theory;
- comparison with extensions $\mathbb{Q} >> \mathbb{R}$ resp. $\mathbb{R} >> \mathbb{C}$;
- provide confidence that "generalized functions" really exist;
- provide simple descriptions to the above list of questions!

# Key Players for Time-Frequency Analysis

## Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \, .$$

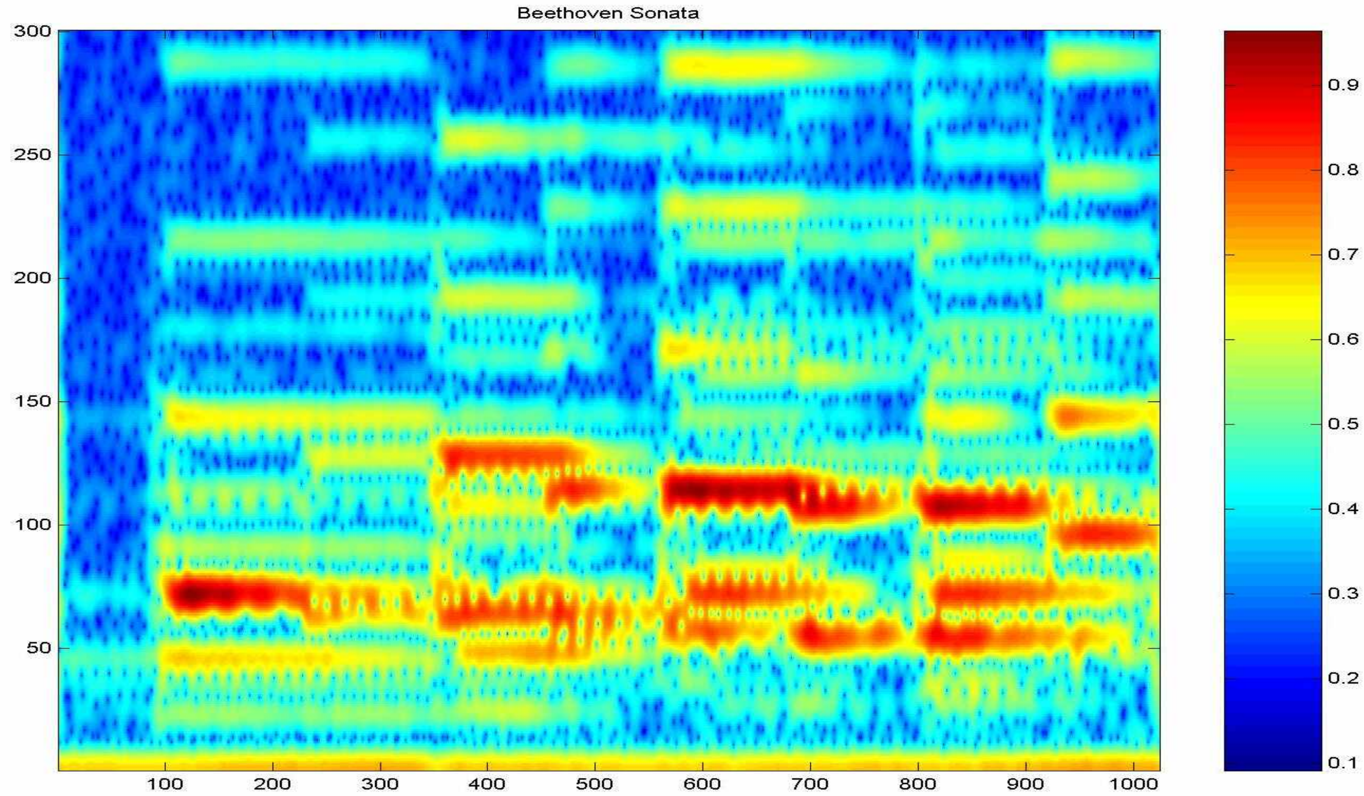
Behavior under Fourier transform

$$(T_x f)\widehat{} = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{} = T_\omega \hat{f}$$

## The Short-Time Fourier Transform

$$V_g f(\lambda) = V_g f(t,\omega) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle, \quad \lambda = (t,\omega);$$

# A Typical Musical STFT



## $\boldsymbol{S}_0(\mathbb{R}^d) = M^1(\mathbb{R}^d) := M^0_{1,1}(\mathbb{R}^d)$

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is (by definition) in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero $g$ (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is a Banach space, for any fixed, non-zero $g \in S_0(\mathbb{R}^d)$, and different windows $g$ define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable windows. Often the Gaussian is used as a window. Note that

$$V_g f(x,\omega) = (\widehat{f \cdot T_x g})(\omega), \quad \text{i.e., } g \text{ localizes } f \text{ near } x.$$

**Lemma 1.** *Let* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, *then the following holds:*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

**Remark 2.** *Moreover one can show that* $\boldsymbol{S}_0(\mathbb{R}^d)$ *is the smallest non-trivial Banach spaces with this property, i.e., it is continuously embedded into any such Banach space. As a formal argument one can use the continuous inversion formula for the STFT:*

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(\lambda)\pi(\lambda) g d\lambda$$

*which implies*

$$\|f\|_{\boldsymbol{B}} \leq \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(\lambda)| \|\pi(\lambda)g\|_{\boldsymbol{B}}\, d\lambda = \|g\|_{\boldsymbol{B}} \|f\|_{\boldsymbol{S}_0}.$$

## Basic properties of $S_0(\mathbb{R}^d)$ resp. $S_0(G)$

**THEOREM:**

- For any automorphism $\alpha$ of $G$ the mapping $f \mapsto \alpha^*(f)$ is an isomorphism on $\boldsymbol{S}_0(G)$; $[with(\alpha^* f)(x) = f(\alpha(x))], x \in G$.
- $\mathcal{F}\boldsymbol{S}_0(G) = \boldsymbol{S}_0(\hat{G})$; (Invariance under the Fourier Transform)
- $T_H \boldsymbol{S}_0(G) = \boldsymbol{S}_0(G/H)$; (Integration along subgroups)
- $R_H \boldsymbol{S}_0(G) = \boldsymbol{S}_0(H)$; (Restriction to subgroups)
- $\boldsymbol{S}_0(G_1)\hat{\otimes}\boldsymbol{S}_0(G_2) = \boldsymbol{S}_0(G_1 \times G_2)$. (tensor product stability);

## Basic properties of $S_0'(\mathbb{R}^d)$ resp. $S_0'(G)$

**THEOREM:** (Consequences for the dual space)

- $\sigma \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is in $\boldsymbol{S_0'}(\mathbb{R}^d)$ if and only if $V_g\sigma$ is bounded;

- $w^*$-convergence in $\boldsymbol{S_0'}(\mathbb{R}^d)$ is equivalent to pointwise convergence of $V_g\sigma$;

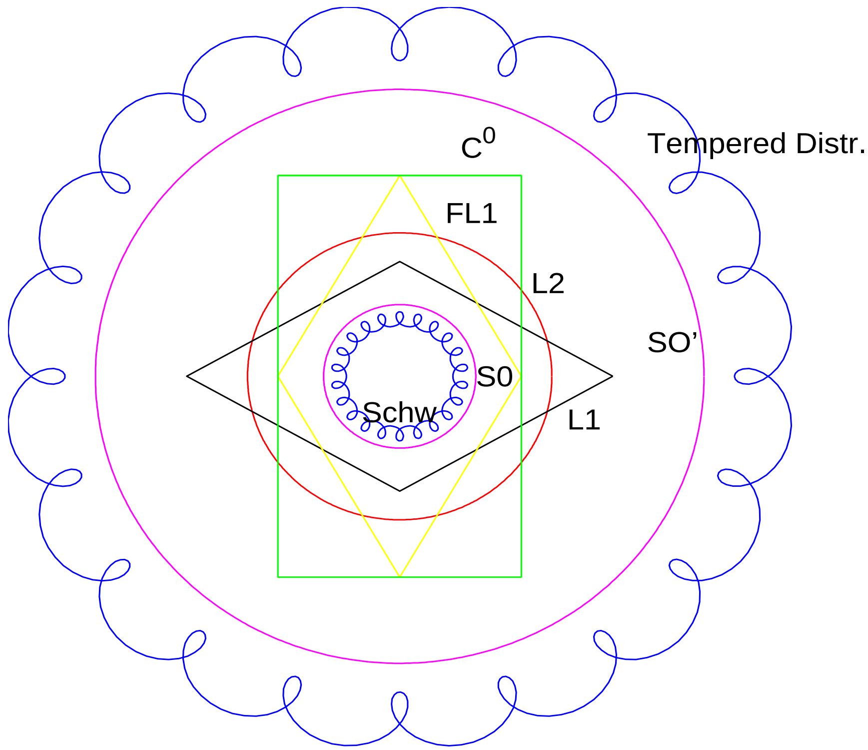
- $\left(\boldsymbol{S_0'}(G), \|\cdot\|_{\boldsymbol{S_0'}}\right)$ is a Banach space with a translation invariant norm;

- $\boldsymbol{S_0'}(G) \subseteq \mathcal{S}'(G)$, i.e. $\boldsymbol{S_0'}(G)$ consists of tempered distributions;

- $\boldsymbol{P}(G) \subseteq \boldsymbol{S_0'}(G) \subseteq \boldsymbol{Q}(G)$; (sits between pseudo- and quasimeasures)

- $T(G) = W(G)' \subseteq \boldsymbol{S_0'}(G)$; (contains translation bounded measures);

# Schwartz space, $S_0$, $L^2$, $S_0'$, tempered distributions



# Basic properties of $S_0(\mathbb{R}^d)$ continued

**THEOREM:**

- the Generalized Fourier Transforms, defined by transposition
$$\langle \hat{\sigma}, f \rangle = \langle \sigma, \hat{f} \rangle,$$
for $f \in \boldsymbol{S}_0(\hat{G}), \sigma \in \boldsymbol{S}_0{}'(G)$, satisfies $\mathcal{F}(\boldsymbol{S}_0'(G)) = \boldsymbol{S}_0'(\hat{G})$.

- $\sigma \in \boldsymbol{S}_0'(G)$ is H-periodic, i.e. $\sigma(f) = \sigma(T_h f)$ for all $h \in H$, iff there exists $\dot{\sigma} \in \boldsymbol{S}_0'(G/H)$ such that $\langle \sigma, f \rangle = \langle \sigma, T_H f \rangle$.

- $\boldsymbol{S}_0'(H)$ can be identified with a subspace of $\boldsymbol{S}_0'(G)$, the injection $i_H$ being given by
$$\langle i_H \sigma, f \rangle := \langle \sigma, R_H f \rangle.$$
For $\sigma \in \boldsymbol{S}_0'(G)$ one has $\sigma \in i_H(\boldsymbol{S}_0'(H))$ iff $\mathrm{supp}(\sigma) \subseteq H$.

## The Usefulness of $S_0(\mathbb{R}^d)$

**Theorem 1.** (**Poisson's formula**) *For $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ and any discrete subgroup $H$ of $\mathbb{R}^d$ with compact quotient the following holds true: There is a constant $C_H > 0$ such that*

$$\sum_{h \in H} f(h) = C_H \sum_{l \in H^\perp} \hat{f}(l) \tag{1}$$

*with absolute convergence of the series on both sides.*

By duality one can express this situation as the fact that the Comb-distribution $\mu_{\mathbb{Z}^d} = \sum_{k \in Z^d} \delta_k$, as an element of $\boldsymbol{S}_0'(\mathbb{R}^d)$ is invariant under the (generalized) Fourier transform. Sampling corresponds to the mapping $f \mapsto f \cdot \mu_{\mathbb{Z}^d} = \sum_{k \in \mathbb{Z}^d} f(k)\delta_k$, while it corresponds to convolution with $\mu_{\mathbb{Z}^d}$ on the Fourier transform side = periodization along $(\mathbb{Z}^d)^\perp = \mathbb{Z}^d$ of the Fourier transform $\hat{f}$. For $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ all this makes perfect sense.

## Regularizing sequences for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$

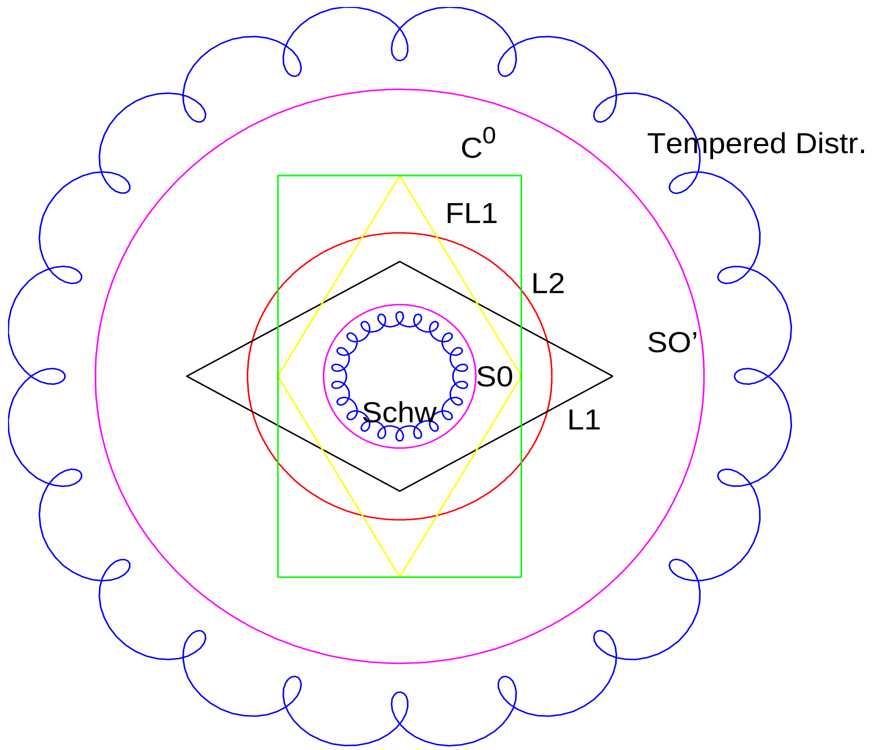
Wiener amalgam convolution and pointwise multiplier results imply that

$$\boldsymbol{S}_0(\mathbb{R}^d) \cdot (\boldsymbol{S}_0'(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d) \quad \boldsymbol{S}_0(\mathbb{R}^d) * (\boldsymbol{S}_0'(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d)$$

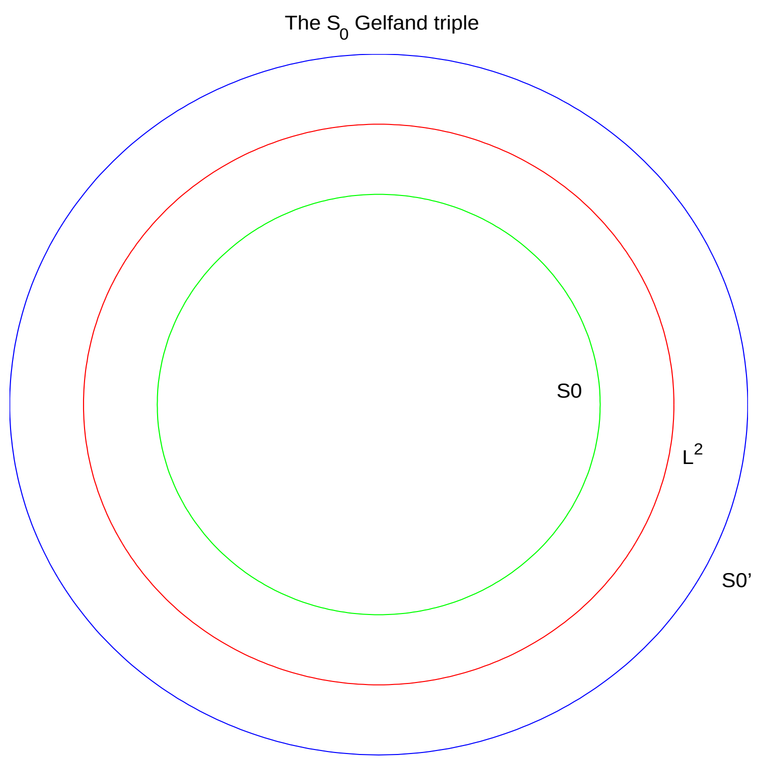
e.g. $\boldsymbol{S}_0(\mathbb{R}^d) * \boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^\infty) \subseteq \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^\infty)$.

Let now $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ be given with $h(0) = 1$. Then the dilated version $h_n(t) = h(t/n)$ are a uniformly bounded family of multipliers on $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$, tending to the identity operator in a suitable way. Similarly, the usual Dirac sequences, obtained by compressing a function $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(x)dx = 1$ are showing a similar behavior: $g_n(t) = n \cdot g(nt)$
Following the above rules the combination of the two procedures, i.e. product-convolution or convolution-product operators of the form provide suitable regularizers: $A_n f = g_n * (h_n \cdot f)$ or $B_n f = h_n \cdot (g_n * f)$.

# Schwartz space, $S_0$, $L^2$, $S_0'$, tempered distributions



# The Gelfand Triple $(S_0, L^2, S_0')$




The Fourier transform is a prototype of a Gelfand triple isomorphism.

# The Fourier transform as Gelfand Triple Automorphism

**Theorem 2.** *Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:*

(1) $\mathcal{F}$ *is an isomorphism from* $\boldsymbol{S}_0(\mathbb{R}^d)$ *to* $\boldsymbol{S}_0(\widehat{\mathbb{R}^d})$,

(2) $\mathcal{F}$ *is a unitary map between* $\boldsymbol{L}^2(\mathbb{R}^d)$ *and* $\boldsymbol{L}^2(\widehat{\mathbb{R}^d})$,

(3) $\mathcal{F}$ *is a weak* (and norm-to-norm) continuous bijection from* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *onto* $\boldsymbol{S}_0'(\widehat{\mathbb{R}^d})$.

*Furthermore we have that Parseval's formula*

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \tag{2}$$

*is valid for* $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, *and therefore on each level of the Gelfand triple* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

The properties of Fourier transform can be expressed by a Gelfand bracket

$$\langle f, g \rangle_{(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')} = \langle \hat{f}, \hat{g} \rangle_{(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')} \tag{3}$$

which combines the functional brackets of dual pairs of Banach spaces and of the inner-product for the Hilbert space.

One can characterize the Fourier transform as the *uniquely determined* unitary Gelfand triple automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ which maps pure frequencies into the corresponding Dirac measures [2]

[2] as one would expect in the case of a finite Abelian group.

## The Kernel Theorem for general operators in $\mathcal{L}(S_0, S_0')$

**Theorem 3.** *If $K$ is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x, y) = g(x)f(y)$.*

# The Kernel Theorem for general operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$

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Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x, y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)f(y)dyg(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)g(x)f(y)dxdy.$$

This result is the "outer shell of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.

Again the complete picture can again be best expressed by a unitary Gelfand triple isomorphism. We first describe the innermost shell:

**Theorem 4.** *The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ on the kernels.*
*Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if the corresponding operator $K$ maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, but not only in a bounded way, but also continuously from $w^*-$topology into the norm topology of $\boldsymbol{S}_0(\mathbb{R}^d)$.*

Remark: Note that for "regularizing" kernels in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ the usual identification (recall that the entry of a matrix $a_{n,k}$ is the coordinate number $n$ of the image of the $n-$th unit vector under that action of the matrix $A = (a_{n,k})$:

$$k(x, y) = K(\delta_y)(x) = \delta_x(K(\delta_y).$$

Since $\delta_y \in \boldsymbol{S}_0'(\mathbb{R}^d)$ and consequently $K(\delta_y) \in \boldsymbol{S}_0(\mathbb{R}^d)$ the pointwise evaluation makes sense.

With this understanding our claim is that the kernel theorem provides a (unitary) isomorphism between the Gelfand triple (of kernels) $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ into the Gelfand triple of operator spaces

$$\left(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)), \, \mathcal{HS}, \, \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\right).$$

# The Kohn Nirenberg Symbol and Spreading Function

The Kohn-Nirenberg symbol $\sigma(T)$ of an operator $T$ (respectively its *symplectic* Fourier transform, the *spreading distribution* $\eta(T)$ of $T$) can be obtained from the kernel using some automorphism and a partial Fourier transform, which again provide unitary Gelfand isomorphisms. In fact, the symplectic Fourier transform is another unitary Gelfand Triple (involutive) automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

**Theorem 5.** *The correspondence between an operator $T$ with kernel $K$ from the Banach Gelfand triple $\big(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\big)$ and the corresponding spreading distribution $\eta(T) = \eta(K)$ in $\boldsymbol{S}_0'(\mathbb{R}^{2d})$ is the uniquely defined Gelfand triple isomorphism between $\big(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\big)$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ which maps the time-frequency shift operators $M_y \circ T_x$ onto the Dirac measure $\delta_{(x,y)}$.*

## Kohn-Nirenberg and Spreading Symbols of Operators

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x + \frac{y}{2}, x - \frac{y}{2})$
- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y-x)$
- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$
- *partial Fourier transform in the first variable:* $\mathcal{F}_1$
- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

## Kohn-Nirenberg correspondence

1. Let $\sigma$ be a tempered distribution on $\mathbb{R}^d$ then the operator with *symbol* $\sigma$

$$K_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}d\omega$$

is the *pseudodifferential operator* with Kohn-Nirenberg symbol $\sigma$.

$$\begin{aligned} K_\sigma f(x) &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\sigma(x,\omega)e^{-2\pi i(y-x)\cdot\omega}d\omega\Big)f(y)dy \\ &= \int_{\mathbb{R}^d} k(x,y)f(y)dy. \end{aligned}$$

2. Formulas for the (integral) kernel $k$: $k = \mathcal{T}_a\mathcal{F}_2\sigma$

$$\begin{aligned} k(x,y) &= \mathcal{F}_2\sigma(\eta, y-x) = \mathcal{F}_1^{-1}\widehat{\sigma}(x, y-x) \\ &= \widehat{\sigma}(\eta, y-x)e^{2\pi i\eta\cdot x}d\eta. \end{aligned}$$

3. The *spreading representation* of the same operator arises from the identity

$$K_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta, u) M_\eta T_{-u} f(x) du d\eta.$$

$\widehat{\sigma}$ is called the spreading function of the operator $K_\sigma$.

If $f, g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, then the so-called *Rihaczek distribution* is defined by

$$R(f,g)(x,\omega) = e^{-2\pi i x \cdot \omega} \overline{\widehat{f}(\omega)} g(x).$$

and belongs to $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, for any $\sigma \in \mathcal{S}'(\mathbb{R}^d)$

$$\langle \sigma, R(f,g) \rangle = \langle K_\sigma f, g \rangle$$

is well-defined and describes a uniquely defined operator from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

## Weyl correspondence

1. Let $\sigma$ be a tempered distribution on $\mathbb{R}^d$ then the operator

$$L_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\xi, u) e^{-\pi i \xi \cdot u} f(x) du d\xi$$

is called the *pseudodifferential operator* with *symbol* $\sigma$. The map $\sigma \mapsto L_\sigma$ is called the *Weyl transform* and $\sigma$ the Weyl symbol of the operator $L_\sigma$.

$$\begin{aligned} L_\sigma f(x) &= \iint_{\mathbb{R}^{2d}} \widehat{\sigma} e^{-\pi i u \cdot \xi} T_{-u} M_\xi f(x) du d\xi \\ &= \int_{\mathbb{R}^d} \Big( \int_{\mathbb{R}^d} \widehat{\sigma}(\xi, y - x) e^{-2\pi i \xi \frac{x+y}{2}} \Big) f(y) dy. \end{aligned}$$

2. Formulas for the kernel $k$ from the KN-symbol: $k = \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma$

$$\begin{aligned} k(x,y) &= \mathcal{F}_1^{-1}\widehat{\sigma}\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{F}_2\sigma\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{F}_2^{-1}\sigma\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma. \end{aligned}$$

3. $\langle L_\sigma f, g\rangle = \langle k, g \otimes \overline{f}\rangle$. (Weyl operator vs. kernel)

If $f, g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, then the *cross Wigner distribution* of $f, g$ is defined by

$$W(f,g)(x,y) = \int_{\mathbb{R}^d} f(x+t/2)\overline{g}(x-t/2)e^{-2\pi i\omega\cdot t}dt = \mathcal{F}_2\mathcal{T}_s(f \otimes \overline{g})(x,\omega).$$

and belongs to $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, for any $\sigma \in \mathcal{S}'(\mathbb{R}^d)$

$$\langle \sigma, W(f,g) \rangle = \langle L_\sigma f, g \rangle$$

is well-defined and describes a uniquely defined operator $L_\sigma$ from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

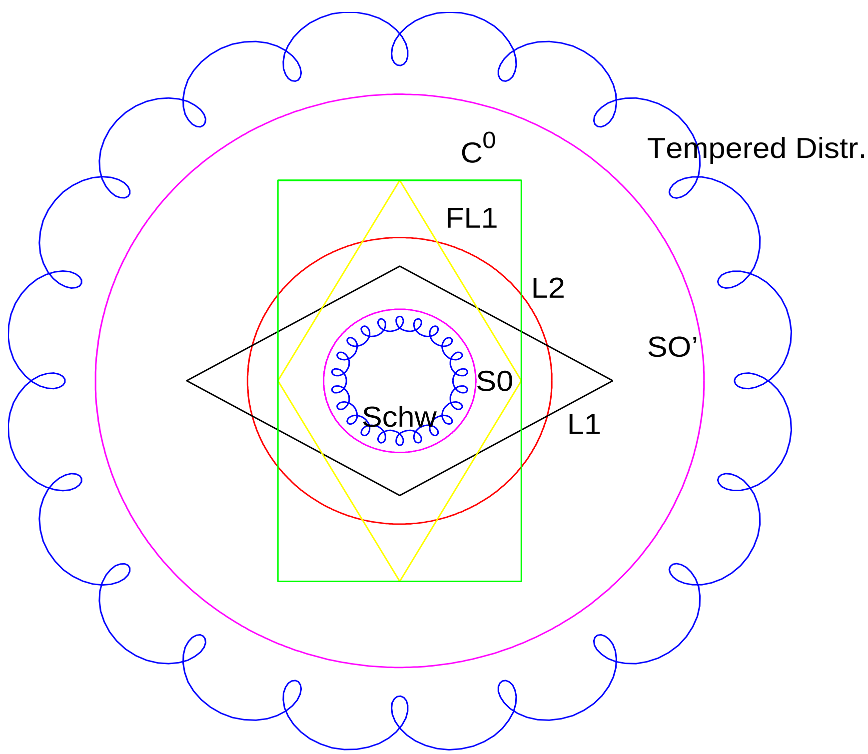
$$(\mathcal{U}\sigma)(\xi, u) = \mathcal{F}^{-1}(e^{\pi i u \cdot \xi} \widehat{\sigma}(\xi, u)).$$

$$K_{\mathcal{U}\sigma} = L_\sigma$$

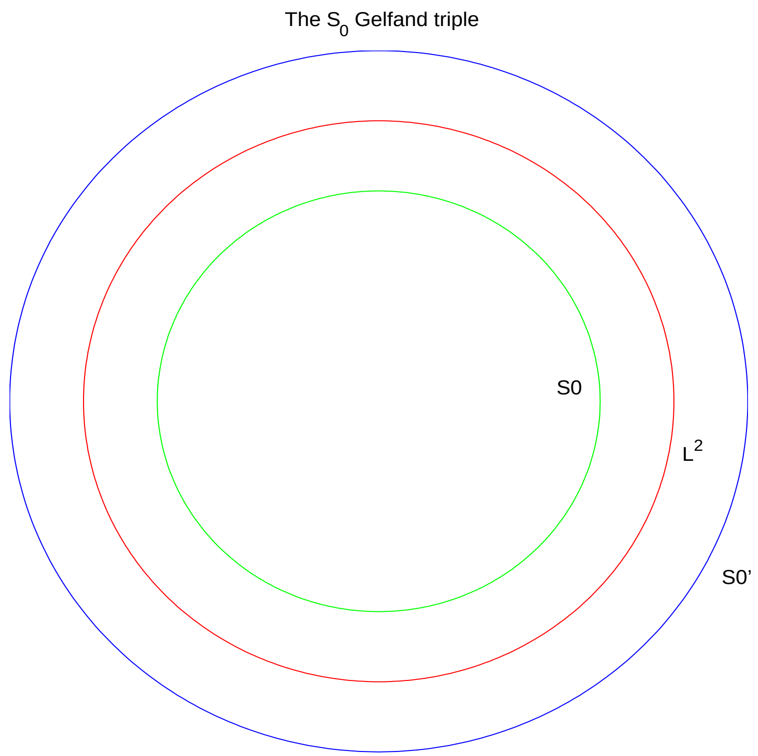
describes the connection between the Weyl symbol and the operator kernel.

In all these considerations the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ can be correctly replaced by $\boldsymbol{S}_0(\mathbb{R}^d)$ and the tempered distributions by $\boldsymbol{S}_0'(\mathbb{R}^d)$.

# Schwartz space, $S_0$, $L^2$, $S_0'$, tempered distributions



# The Gelfand Triple $(S_0, L^2, S_0')$




Fourier transform is a prototype of a unitary Gelfand triple isomorphism.

## Back to the Classical Problems

We want to give an interpretation of the usual summability methods showing the relevance of $\boldsymbol{S}_0(\mathbb{R}^d)$ in this business for a number of sufficient conditions for $f$ to belong to $\boldsymbol{S}_0(\mathbb{R}^d)$: in the case of $d = 1$ a sufficient condition is that $f$ is an integrable and piecewise linear function with not too irregular nodes, or $f, f', f'' \in \boldsymbol{L}^1(\mathbb{R})$. Recall $\boldsymbol{S}_0 \cdot \mathcal{F}\boldsymbol{L}^1 \subseteq \boldsymbol{S}_0$.
The typical reasoning where summability methods are applied is in order to give the usual inversion formula $f(t) = \int_{\mathbb{R}^d} \hat{f}(s) e^{2\pi i s t} ds$ a meaning, even if $\hat{f} \notin \boldsymbol{L}^1(\mathbb{R}^d)$. This is done by multiplying it with some integrable and continuous function $h$, with $h(0) = 1$, which is then dilated. In other words, one replaces the integrand $f(s)$ by $f(s)h(\varrho s)$, for some small value of $\varrho$.
It can be shown for all the "good classical kernels" that they are of this form, for some $h \in \boldsymbol{S}_0(\mathbb{R}^d)$. This means of course that $s \mapsto h(\varrho s)$ is the Fourier transform of some compressed $\boldsymbol{S}_0(\mathbb{R}^d)$ version $St_\varrho g$ of some function $g$ (with $\hat{g} = h$) and hence $St_\varrho g * f$ converges to $f$ in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$.

## Characterize translation invariant operators as convolution operators

Let us start by citing the introduction of Larsen's book [5]: Given any pair of Banach space of [equivalence classes] of functions on a locally compact Abelian group one may ask: "what are the bounded linear operators between them which are also commuting with translations". Let us call those spaces $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ and $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$, and ask for $H_{\mathcal{G}}(\boldsymbol{B}^1, \boldsymbol{B}^2)$:

$$H_{\mathcal{G}}(\boldsymbol{B}^1, \boldsymbol{B}^2) = \{T : \boldsymbol{B}^1 \mapsto \boldsymbol{B}^2, \text{bd. and linear}, \, T_x \circ T = T \circ T_x, \, \forall x \in \mathcal{G}\,\}. \tag{4}$$

In most cases one shows that it equals $H_{\boldsymbol{L}^1}(\boldsymbol{B}^1, \boldsymbol{B}^2)$, defined as follows:

$$H_{\boldsymbol{L}^1}(\boldsymbol{B}^1, \boldsymbol{B}^2) = \{T : \boldsymbol{B}^1 \mapsto \boldsymbol{B}^2, \text{bd. and linear}, \, T(g*f) = g*Tf\,, \, \forall g \in \boldsymbol{L}^1\,\}, \tag{5}$$

which will be called the space of all $\boldsymbol{L}^1$-module homomorphism between $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ and $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ (cf. Rieffel! [6]). There are not too many cases where this space can be identified in an easy and complete way:

1. Wendel's Theorem, $p = 1$, ([5])

$$\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1, \boldsymbol{L}^1)(G) \approx \boldsymbol{M}_b(G)$$

   or in words: The bounded operator on $\boldsymbol{L}^1$ commuting with $\boldsymbol{L}^1$-convolution are exactly the convolution operators with bounded measures $\mu \in \boldsymbol{M}_b(G)$.

2. $p = 2$ :

$$\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^2, \boldsymbol{L}^2)(G) \approx \mathcal{F}\boldsymbol{L}^\infty$$

   i.e. the bounded $\boldsymbol{L}^1$-homomorphism on $\boldsymbol{L}^2(G)$ are exactly the operators of the form $f \mapsto \mathcal{F}^{-1}(h\hat{f})$, for some $h \in \boldsymbol{L}^\infty$. By a suitable interpretation of $\mathcal{F}\boldsymbol{L}^\infty$ it can is called the space $\boldsymbol{P}(G)$ of pseudo-measures, and $T$ is represented as convolution with a pseudo-measure.

3. For general $p \in (1, \infty)$ one can show that $H_{\boldsymbol{L}^1}(\boldsymbol{L}^p, \boldsymbol{L}^p)$ equals $H_{\boldsymbol{L}^1}(\boldsymbol{L}^{p'}, \boldsymbol{L}^{p'})$ for $1/p + 1/p' = 1$. It follows therefrom via complex interpolation (with the choice $\theta = 0.5$) that

$$H_{\boldsymbol{L}^1}(\boldsymbol{L}^p, \boldsymbol{L}^p) \subseteq \mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^2, \boldsymbol{L}^2) = \mathcal{F}\boldsymbol{L}^\infty.$$

   This implies that in the context of $\boldsymbol{L}^p$-spaces (except for $p = \infty$) one can describe $\boldsymbol{L}^1$-homomorphism as convolution operators with a pseudo-measure.

4. As soon as one wants to generalize this characterization of $\boldsymbol{L}^1$-homomorphism to the case where the two spaces are not equal anymore, i.e. when one is interested in the characterization of $H_{\boldsymbol{L}^1}(\boldsymbol{L}^p, \boldsymbol{L}^q)$, for some pair of values $p$ and $q$ one finds that pseudo-measures are not sufficient anymore! Just note that obviously any $\boldsymbol{L}^2$-function $h$ defines a

bounded linear operator from $\boldsymbol{L}^1$ into $\boldsymbol{L}^2$ via convolution, since obviously $\widehat{h * f} = \widehat{h} \cdot \widehat{f} \in \mathcal{F}\boldsymbol{L}^2 \cdot \mathcal{F}\boldsymbol{L}^1 \subseteq \mathcal{F}\boldsymbol{L}^2$. [3]

The theory of quasi-measures was a vehicle providing a way out of this dilemma. From todays view-point quasi-measures are exactly the (tempered) distributions which equal locally pseudo-measures, but the original definition was much more involved (going back to Gaudry, see [4] the equivalence was established by Cowling in [2]).

In contrast, from the point of view of the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ this question has a fairly simple answer, however. Since $\boldsymbol{S}_0(G) \subseteq \boldsymbol{L}^p(G) \subseteq \boldsymbol{S}_0'(G)$ for any value of $p \in [1, \infty]$ (due to the minimality of $\boldsymbol{S}_0(G)$, hence the maximility of $\boldsymbol{S}_0'(G)$) it is easy to observe the following natural embeddings:

$$H_{\boldsymbol{L}^1}(\boldsymbol{L}^p, \boldsymbol{L}^q) \hookrightarrow H_{\boldsymbol{L}^1}(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(G). \tag{6}$$

[3] We only need $\boldsymbol{L}^1 * \boldsymbol{L}^2 \subseteq \boldsymbol{L}^2$ !?

We only have to recall the definition of the convolution of $\sigma \in \boldsymbol{S}_0'(G)$ with $f \in \boldsymbol{S}_0(G)$, indeed the standard interpretation of the convolution of a bounded linear functional with a test function applies:

$$\sigma * f(x) = \sigma(T_x \check{f}).$$

This also implies that $\boldsymbol{S}_0'(G) * \boldsymbol{S}_0(G) \subseteq \boldsymbol{C}_b(G)$. Hence it in fact possible to recover $\sigma$, given the operator $T : f \mapsto \sigma * f$, by means of the identity $\sigma(f) = T(\check{\sigma})(0)$.

It is of course not difficult to show that the generalized FT on $\boldsymbol{S}_0'(G)$ allows to describe $T$ as a "multiplication operator on the FT side", by giving a meaning to the formula: $T(f) = \mathcal{F}^{-1}(\hat{\sigma} \cdot \hat{f})$. The transfer function $\hat{\sigma}$ is therefore an element of $\boldsymbol{S}_0'$, hence a quasi-measure. This fact has to be proven separately in the book of Larsen ([**?**]), because the space $\boldsymbol{Q}(\mathbb{R}^d)$ of quasi-measures is too large in order to be invariant with respect to the Fourier transform (leave alone the fact that the original definition of the space of quasi-measures was a quite complicated one).

5. Sometimes unbounded measures still have measures as Fourier transforms. The so-called chirp function $x \mapsto e^{\pi i |x|^2}$ is an excellent example, because it is even invariant under the Fourier transform. Dilated version therefore are mapped onto correspondlingly inversly dilated chirp functions. The most general theory in this direction has been developed by Argabright and Gil de Lamadrid ([1]) in the 1970-th. It can also be subsumed in the $\boldsymbol{S}_0'$-context.

# END OF THE FIRST LECTURE

THANK you for your attention! HGFei

# Modulation space for Gabor Analysis

March 25th, 2007

Beyond Banach Gelfand Triples:
Modulation spaces and Gabor Analysis

The second talk will show that in addition to the triple of Banach space $\big(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)\big)$ there is a whole family of Banach spaces "around" these spaces, in particular the *by now classical* spaces $\boldsymbol{M}^s_{p,q}(\mathbb{R})$ or the space $\boldsymbol{M}^p_{v_s}(\mathbb{R}^d)$ or $\boldsymbol{M}^p_s(\mathbb{R}^d)$ which are obtained using radial symmetric weights of polynomial growths of order $s$, $s \in \mathbb{R}$ on phase space.

# Preview

- Wiener amalgams and modulation spaces;
- (Banach) frames and Riesz basis viewed as retracts;
- Retracts in Gabor Analysis;
- spline type spaces;
- Gabor multipliers and their properties;

# What are function spaces good for?

- Describe the smoothness or variation/oscillation of functions;
- Describe (rate of) decay of functions, summability properties;
- Describe the mapping properties of linear operators, domains of unbounded operators;

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- Describe the mapping properties of linear operators, domains of unbounded operators;

There is a huge zoo of Banach spaces of functions or distributions used in the literature:

- the classical $\boldsymbol{L}^p$-spaces, but also Lorentz or Orlicz spaces (typically defined by the distribution of their values, hence rearrangement invariant);
- Lipschitz spaces, Besov-Triebel-Lizorkin spaces, Bessel potential spaces;
- weighted spaces, mixed norm spaces;
- spaces describing bounded variation, Morrey-Campanato spaces;
- Hardy spaces, characterized by atomic decompositions;
- Herz spaces, defined by decompositions;

# Wiener Amalgams: Wiener's Role

First appearance in Norbert Wiener's theory of generalized harmonic analysis ("The Fourier Transform and Certain of its Applications") and Tauberian Theorems around 1929-1932: [$W(L^1, L^2)$ and $W(L^2, L^1)$, $W(L^1, L^\infty)$ and a bit later $W(L^\infty, L^1)$], using the discrete norm for these spaces:

$$\|f\|_{W(L^p,\ell^q)} = \left( \sum_{n \in \mathbb{Z}} \left( \int_n^{n+1} |f(t)|^p \, dt \right)^{q/p} \right)^{1/q}, \tag{7}$$

with the usual adjustments if $p$ or $q$ is infinity. Advantage over ordinary $L^p$-spaces: natural inclusions, in the local component as over the torus, while globally on has the natural inclusions between sequence spaces, with opposite orientation.

Hence $W(L^\infty, \ell^1)$ is the smallest within *this* family and $W(L^1, \ell^\infty)$ is the largest.
The closure of test functions in this space (resp. the continuous functions in this space) forms Wiener's algebra, which we denote by $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$.

# CLASSICAL Wiener Amalgams: Basic Properties

The use of amalgam spaces (cf. e.g. the survey article by Fournier and Stewart, Bull. Amer. Math. Soc., 1980) shows their usability in a wide range of problems of analysis. In most cases one can just argue, that one has to think *coordinatewise*.

# CLASSICAL Wiener Amalgams: Basic Properties

The use of amalgam spaces (cf. e.g. the survey article by Fournier and Stewart, Bull. Amer. Math. Soc., 1980) shows their usability in a wide range of problems of analysis. In most cases one can just argue, that one has to think *coordinatewise*.
For example, with respect to duality, pointwise multiplication, or a Hausdorff-Young type statement for the Fourier transform:

$$W(L^p, \ell^q)' = W(L^{p'}, \ell^{q'}), \quad 1 \leq p, q, < \infty$$

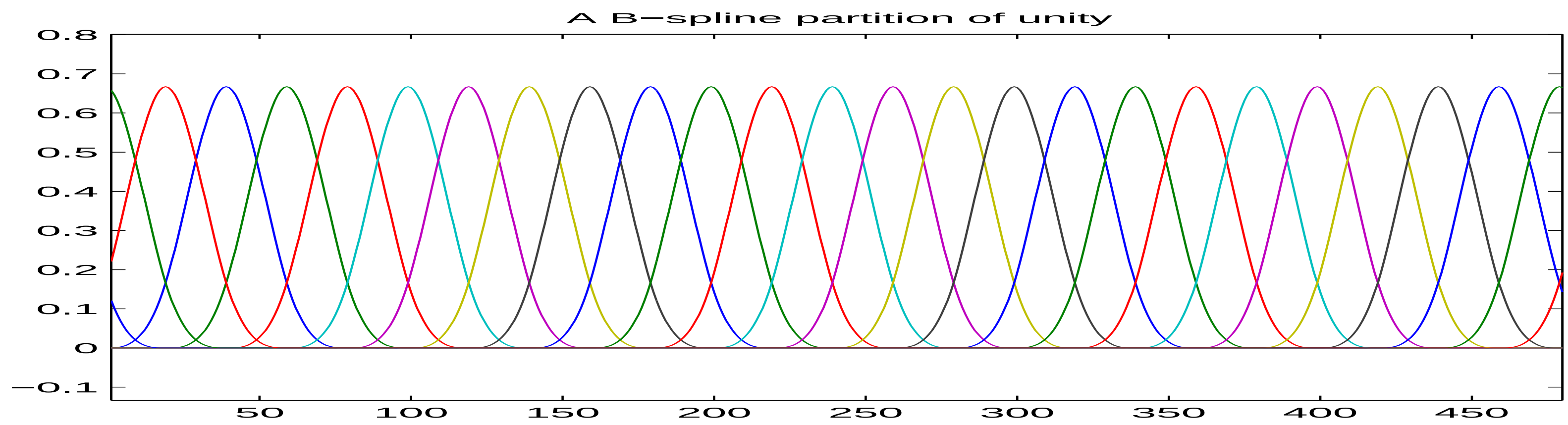
$$\mathcal{F}W(L^p, \ell^q) \subseteq W(L^{q'}, \ell^{p'}), \quad 1 \leq p, q, \leq 2$$

However, if one uses smooth partitions of unity, one can also characterize $\boldsymbol{S}_0(\mathbb{R}^d)$ as a (generalized) Wiener Amalgam space of the form $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^1)(\mathbb{R}^d)$. In other words, given a continuous function $\varphi$ with compact support and $\hat{\varphi} \in \boldsymbol{L}^1(\mathbb{R}^d)$, satisfying the BUPU-condition $\sum_{k \in \mathbb{Z}^d} T_k \varphi \equiv 1$, we have $f \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ if and only if $\sum_k \|f.T_k\varphi\|_{\mathcal{F}\boldsymbol{L}^1} = \sum_k \|M_k\hat{\varphi} * \hat{f}\|_{\boldsymbol{L}^1} < \infty$.

# Bounded Uniform Partitions of Unity

**Definition 3.** *A bounded family* $\Psi = (\psi_n)_{n \in \mathbb{Z}^d}$ *in a Banach algebra* $(\mathbf{A}, \|\cdot\|_A)$ *is a regular A-Bounded Uniform Partition of Unity if*

$$\sum_{n \in \mathbb{Z}^d} \psi(x - n) = 1 \quad \textit{for all} \quad x \in \mathbb{R}^d$$



# Wiener Amalgam Convolution Theorem

**Theorem 1.** *Assume the indices* $p_i$, $q_i$ *and the moderate weights* $w_i$ *are such that there exist constants* $C_1$, $C_2 > 0$ *so that*

$$\forall\, h \in L^{p1}, \quad \forall\, k \in L^{p2}, \quad \|h * k\|_{L^{p3}} \;\leq\; C_1 \,\|h\|_{L^{p1}} \,\|k\|_{L^{p2}},$$

$$\forall\, h \in L^{q1}_{w_1}, \quad \forall\, k \in L^{q2}_{w_2}, \quad \|h * k\|_{L^{q3}_{w_3}} \;\leq\; C_2 \,\|h\|_{L^{q1}_{w_1}} \,\|k\|_{L^{q2}_{w_2}}.$$

*Then there is a constant* $C > 0$ *such that for all* $f \in W(L^{p1}, L^{q1}_{w_1})$ *and* $g \in W(L^{p2}, L^{q2}_{w_2})$ *we have*

$$\|f * g\|_{L^{q3}_{w_3}} \;\leq\; C \,\|f\|_{W(L^{p1}, L^{q1}_{w_1})} \,\|g\|_{W(L^{p2}, L^{q2}_{w_2})}.$$

*In other words, if* $L^{p1} * L^{p2} \subseteq L^{p3}$ *and* $L^{q1}_{w_1} * L^{q2}_{w_2} \subseteq L^{q3}_{w_3}$, *then*

$$W(L^{p1}, L^{q1}_{w_1}) * W(L^{p2}, L^{q2}_{w_2}) \subseteq W(L^{p3}, L^{q3}_{w_3}).$$

# Wiener Amalgam Space: General local/global components

**Definition 4.** *A Banach space* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *of tempered distributions is called a* **standard space** *if it satisfies the following conditions:*

1. $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow \boldsymbol{B} \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$,
2. $\boldsymbol{B}$ *is translation and modulation invariant* $T_x\boldsymbol{B} = \boldsymbol{B}$ *and* $M_y\boldsymbol{B} = \boldsymbol{B}$ *for all* $x, y \in \mathbb{R}^d$.

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3. *The Banach algebra* $\boldsymbol{A}$ *of pointwise multipliers of* $\boldsymbol{B}$ *contains* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.
4. *There is some Beurling algebra* $L^1_w(\mathbb{R}^d)$ *which acts boundedly on* $\boldsymbol{B}$ *through convolution, i.e.*

$$\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{1,w}\|f\|_{\boldsymbol{B}} \quad \forall f \in \boldsymbol{B}, g \in L^1_w.$$

A class of spaces as above are the space $\mathcal{F}\boldsymbol{L}^q$, the image of $\boldsymbol{L}^q(\mathbb{R}^d)$ under the Fourier transform (in the spirit of $\boldsymbol{S}_0'(\mathbb{R}^d)$ or $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$). Obviously it suffices to take $\boldsymbol{A} = \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ in this case, since $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{L}^q(\mathbb{R}^d)$ implies $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \mathcal{F}\boldsymbol{L}^q(\mathbb{R}^d) \subseteq \mathcal{F}\boldsymbol{L}^q(\mathbb{R}^d)$.

# Selective, Continuous Description of Wiener Amalgam Spaces

**Definition 5.** *(Wiener Amalgam spaces) Let* $(\mathbf{B}, \|.\|_{\mathbf{B}})$ *be a standard space and* $(\mathbf{C}, \|.\|_{\mathbf{C}})$ *a solid and translation invariant Banach space of functions, i.e., a complete space of measurable functions, such that* $f \in \mathbf{C}$*,* $g$ *measurable and* $|g(x)| \leq |f(x)|$ *for all* $X$*, implies* $g \in \mathbf{C}$ *and* $\|g\|_{\mathbf{C}} \leq \|f\|_{\mathbf{C}}$ *as well as* $T_x\mathbf{C} = \mathbf{C}$*. Then we define for* $f \in \mathbf{B}_{loc}$ *and some compactly supported "window"* $k \in A$ *the so-called control function with respect to the* $\mathbf{B}$*-norm:*

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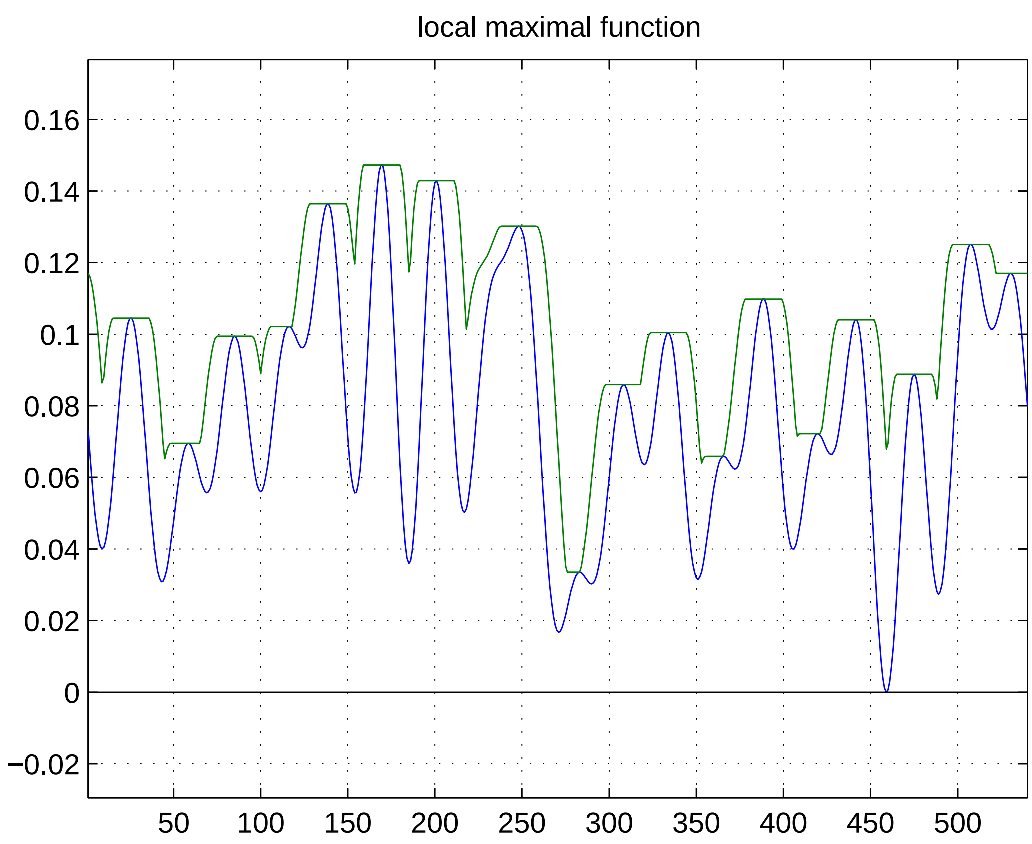
$$K(f, k) : x \mapsto \|(T_x k) \cdot f\|_{\mathbf{B}}.$$

*On the basis of this control function a linear space, the Wiener amalgam space with local component* $\mathbf{B}$ *and global component* $\mathbf{C}$*, denoted by* $\boldsymbol{W}(\mathbf{B}, \mathbf{C})$ *is defined as follows:*

$$\boldsymbol{W}(\mathbf{B}, \mathbf{C}) := \{f \in \mathbf{B}_{loc} | K(f, k) \in \mathbf{C}\}.$$

*Different windows* $k$ *define the* same *space and equivalent norms.*
If $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is dense in $\boldsymbol{B}$ and $\boldsymbol{C}$, than $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{C})' = \boldsymbol{W}(\boldsymbol{B}', \boldsymbol{C}')$.

# A Typical Control Function



# Selective, Discrete Description of Wiener Amalgam Spaces

**Theorem 6.** *Assume that* $\mathbf{A} \cdot \mathbf{B} \subseteq \mathbf{B}$, *with* $\|h \cdot f\|_B \leq \|h\|_A \|f\|_B$ *for all* $h \in \mathbf{A}$, $f \in \mathbf{B}$. *Then* $f \in W(\mathbf{B}, L^q_w)$, $1 \leq q < \infty$, *if and only if for each (or just for one individual)* $\mathbf{A}$*-BUPU* $\Psi$ *one has*

$$\|f\|'_W = \left( \sum_{i \in I} \|f \psi_i\|_B^q w^q(x_i) \right)^{1/q} < \infty$$

## Modulation Spaces (HF: around 1983)

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$$\boldsymbol{M}^s_{p,q}(\mathbb{R}^d) = \mathcal{F}^{-1}\left( \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q_s) \right)$$

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$$V_gf(t,\omega) = \langle f, M_\omega T_t g\rangle$$

# Modulation Spaces

The modulation spaces occur in the study of the concentration of a function in the time-frequency plane. They are defined in the following way:
Let $g \in \mathcal{S}$ be a Schwartz function, $1 \leq p, q < \infty$, $s \in R$, then

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i.e. for which $V_g f$ belongs to some weighted mixed norm space over phase space. In the "classical" case the weight depends only on frequency, hence the spaces are isometrically translation invariant. The only important facts about the constraint imposed on $V_g f$ is the membership in some *solid and translation invariant* Banach space of functions.

The modulation space $M^s_{pq}(\mathbb{R})$ is a Banach space of tempered distributions, the definition is independent of the analyzing function $g$, and different $g$'s yield equivalent norms on these spaces.

Among the modulation spaces are the following important function spaces:

(a) the Segal algebra $S_0(R)$ as $S_0 = M^0_{1,1}$.

(b) $L^2(R) = M^0_{2,2}$;

(c) the Bessel potential spaces as $M^s_{2,2}$;

(d) the Shubin classes $\boldsymbol{Q}_s(\mathbb{R}^d)$ for the weighted $\boldsymbol{L}^2(\mathbb{R}^{2d})$ spaces, with radial symmetric weights $v_s(\lambda) = (1 + |\lambda|^2)^{s/2}$.
A lot of details on these spaces can be found in the book of Gröchenig, and in the survey note (written in 1983, and published in 2003) [3].

# A short reminder about frames and Riesz bases in $\mathbb{C}^m$

A family of vectors $(a_k)_{1\leq k\leq n}$ is a generating set in $\mathbb{C}^m$ if and only if $x \mapsto \mathbf{A} * x$ is *surjective* (the columns of $\mathbf{A}$ span $\mathbb{C}^m$), or equivalently if $y \mapsto (\langle y, a_k\rangle)_{1\leq k\leq n}$ is injective. By a compactness argument this is equivalent to the existence of positive numbers $A, B > 0$ such that

$$A\|y\|^2 \leq \sum_k |\langle y, a_k\rangle|^2 \leq B\|y\|^2$$

This is of course in a Hilbert space an appropriate definition for a **frame**.
A family of vectors in $\mathbb{C}^m$ is linear independent if the mapping $x \mapsto \mathbf{A} * x$ is *injective* which again is equivalent to the validity of an estimate, for some constants $C, D > 0$ of the following form:

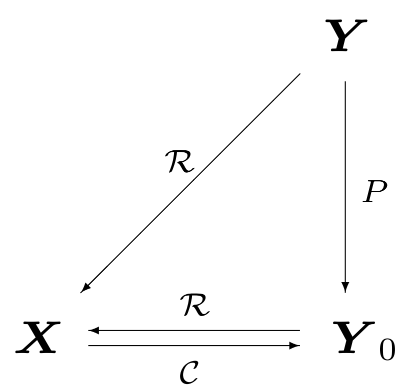
$$C\|x\|^2 \leq \|\sum_k x_k a_k\|^2 \leq D\|x\|^2$$

which is of course a suitable definition of **Riesz bases** resp. Riesz basic sequences.

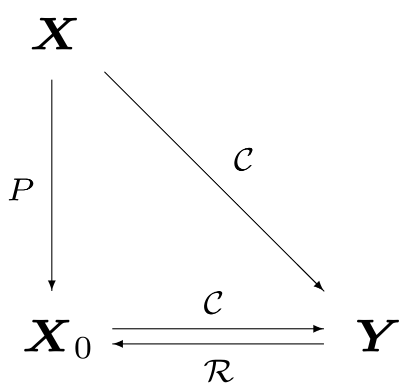
# Frames and Riesz Bases: Commutative Diagrams

Think of $\boldsymbol{X}$ as something like $\boldsymbol{L}^p(\mathbb{R}^d)$, and $\boldsymbol{Y} = \ell^p$:

Frame case: $\mathcal{C}$ is injective, but not surjective, and $\mathcal{R}$ is a left inverse of $\mathcal{C}$. This implies: $P = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of $\mathcal{C}$ in $\boldsymbol{Y}$:



<u>Riesz Basis case:</u> E.g. $\boldsymbol{X}_0 \subset \boldsymbol{X} = L^p$, and $\boldsymbol{Y} = \ell^p$: $\mathcal{R}$ is injective, but not surjective, and $\mathcal{C}$ is a left inverse of $\mathcal{R}$. This implies: $P = \mathcal{R} \circ \mathcal{C}$ is a projection in $\boldsymbol{X}$ onto the range $\boldsymbol{X}_0$ of $\mathcal{R}$ in $\boldsymbol{X}$:




An instructive example for this case are (say - cubic) spline type space. Here $\mathcal{R}$ maps as sequence $\mathbf{c} = (c_n)_{n \in \mathbb{Z}^d}$ into a sum of the form $f = \sum_n c_n T_n \varphi$ (for suitable B-splines $\varphi$). The range is closed within $\boldsymbol{L}^p(\mathbb{R}^d)$. In this case the coefficients (even for a general $f \in \boldsymbol{L}^p(\mathbb{R}^d)$) can be determined via $c_n = f * \widetilde{\varphi}(n), n \in \mathbb{Z}^d$, where $\widetilde{\varphi}$ is the so-called dual generator or the spline-type space.

# Unconditional Banach Frames

A suggestion for bringing the well established notion of Banach frames closer to the setting we are used from the Hilbert space and $\ell^2$-setting:

**Definition 7.** *A mapping $\mathcal{C} : \mathbf{B} \to \mathbf{Y}$ defines an unconditional (or solid) Banach frame for $\mathbf{B}$ w.r.t. the sequence space $\mathbf{Y}$ if*

1. $\exists\, \mathcal{R} : \mathbf{Y} \to \mathbf{B}$, *with* $\mathcal{R} \circ \mathcal{C} = Id_{\mathbf{B}}$,
2. $(\mathbf{Y}, \|\|_{\mathbf{Y}})$ *is a solid Banach space of sequences over* $I$, *with* $\mathbf{c} \mapsto c_i$ *being continuous from* $\mathbf{Y}$ *to* $\mathbb{C}$ *and solid, i.e. satisfying* $\mathbf{z} \in \mathbf{Y}, \mathbf{x} : |x_i| \leq |z_i| \forall i \in I \Rightarrow \mathbf{x} \in \mathbf{Y}, \|\mathbf{x}\|_{\mathbf{Y}} \leq \|\mathbf{z}\|_{\mathbf{Y}}$ *(hence, w.l.o.g.,* $\mathbf{e}_i \in \mathbf{Y}$*),*
3. *finite sequences are dense in* $\mathbf{Y}$ *(at least in the* $w^*$*-sense).*

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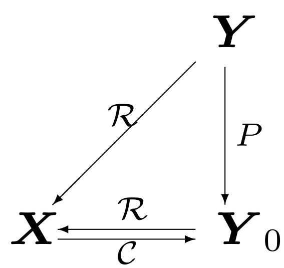
3. *finite sequences are dense in* $\mathbf{Y}$ *(at least in the* $w^*$*-sense).*

**Corollary 3.** *By setting* $h_i := \mathcal{R}e_i$ *we have* $\mathcal{R}\mathbf{c} = \mathcal{R}(\sum e_i \mathbf{e}_i) = \sum_{i \in I} c_i h_i$ *unconditional in* $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$*, hence* $f = \sum_{i \in I} T(f)_i h_i$ *as unconditional series.*

We may talk about Gelfand frames (or Banach frames for Gelfand triples) resp. Gelfand Riesz bases (as opposed to a Riesz projection basis for a given pair of Banach spaces).

# Gelfand Triples, Modulations Spaces, . . . continued

We have seen so far, that the notions of "generating sets of vectors in $\mathbb{R}^d$" ($>$ frames, Banach frames) resp. that of a linear independent set ($>$ Riesz bases, Riesz projection bases) are characterized by a "triangular diagram" [4] The combination of both properties (i.e. the case $\boldsymbol{Y}_0 = \boldsymbol{Y}$ exactly describes *exact frames* resp. unconditional bases. Such a diagram - characterizing a retract - also makes sense for Banach Gelfand triples (even families), leading to Banach frames and Riesz projection bases.




The interpretation of e.g. Besov (or also Wiener amalgam) spaces as retracts of vector-valued sequence spaces is the usual way of proving complex interpolation results!!

[4] representing the fact that the range of a $5 \times 3$-matrix $A$ in $\mathbb{R}^5$, i.e. the column space of $A$, can be identified with $\mathbb{R}^3$ if $A$ has maximal rank, and sits within $\mathbb{R}5$ as a complemented subspace. Moreover the so-called pseudo-inverse (denoted by PINV in MATAB) describing the minimal norm least square solution of $A * x = b$ defines a left inverse $\mathcal{R}$ to $\mathcal{C} : x \mapsto A * x$, completing the diagram.

# Examples of Gelfand Triple Isomorphisms

1. The standard Gelfand triple $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$.
2. The family of orthonormal Wilson bases (obtained from Gabor families by suitable pairwise linear-combinations of terms with the same absolute frequency) extends the natural unitary identification of $\boldsymbol{L}^2(\mathbb{R}^d)$ with $\ell^(I)$ to a unitary Banach Gelfand Triple isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ and $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(I)$.
3. The Fourier transform is a prototype of a unitary GT-automorphism for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$.
4. There is an important Gelfand triple of Operator spaces, namely $\big(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)),\, \mathcal{HS},\, \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\big)$, which is characterized by its mapping property, but due to suitable unitary Gelfand triple isomorphisms to $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ (kernel theorem), or $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ (using the spreading $\eta(T)$ or the Kohn-Nirenberg $\sigma(T)$ relation, which are connected between the symplectic Fourier transform over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$).

The following lemma is more or less a reformulation of the definition of $\boldsymbol{S}_0(\mathbb{R}^d)$ (or rather one of the many equivalent characterizations of this space).

**Lemma 4.** *For any* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *the short-time Fourier transform* $f \mapsto V_g f$ *establishes a retract from* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ *into* $(\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, *with left inverse*

$$V_g^* : F \mapsto \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \; d\lambda$$

One can however show that $V_g f \in \boldsymbol{S}_0'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ if and only if $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and therefore one can also formulate the following claim:

**Lemma 5.** *For any* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *the short-time Fourier transform* $f \mapsto V_g f$ *establishes a retract from* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ *into* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, *with left inverse:*

$$V_g^* : F \mapsto \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \; d\lambda.$$

It is even more interesting that for any TF-lattice $\Lambda$ the restriction of $V_g f$ to $\Lambda$, i.e.

**Theorem 7.** *For $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ the mapping $C : f \mapsto (V_g f(\lambda))_{\lambda\in\Lambda}$ is a bounded Gelfand triple morphism into $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(\Lambda)$.*
*The mapping $C$ defined above is a retract if and only if the family $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a (Gabor) frame. Indeed, the dual Gabor atom $\widetilde{g}$ is automatically!! in $\boldsymbol{S}_0(\mathbb{R}^d)$ if the frame operator is invertible. Consequently the mapping $\mathcal{R} : \mathbf{c} \mapsto \sum_\Lambda c_\lambda \pi(\lambda)\widetilde{g}$ is also a Gelfand triple morphism from*

$$(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(\Lambda) \quad \textit{into} \quad (\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d).$$

**Remark 6.** *The above statement is really a Banach Gelfand triple version of the usual frame characterization. While the Hilbert space case only emphasizes that the coefficient mapping $f \mapsto V_g f|_\Lambda$ is a retract from $\boldsymbol{L}^2(\mathbb{R}^d)$ into $\boldsymbol{\ell}^2(\Lambda)$, we see that it extends to all three levels, and allows, among others, to characterize $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ by the property that $\sum_{\lambda\in\Lambda} |V_g f(\lambda)| < \infty$*

In the background the following important result by Gröchenig and Leinert has to be formulated:

**Theorem 8.** *Assume that for* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *the Gabor frame operator*

$$S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g$$

*is invertible as an operator on* $\boldsymbol{L}^2(\mathbb{R}^d)$*, then it is also invertible on* $\boldsymbol{S}_0(\mathbb{R}^d)$ *and in fact on* $\boldsymbol{S}_0'(\mathbb{R}^d)$*.*
*In other words: Invertibility at the level of the Hilbert space automatically !! implies that* $S$ *is (resp. extends to ) an isomorphism of the Gelfand triple automorphism for* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$*.*

To find other situations where this happens is an interesting task.

**Remark 7.** *Recall also that the Balian-Low principle prohibits the existence of a "Gaborian Riesz basis" for* $\boldsymbol{L}^2(\mathbb{R}^d)$ *with an atom* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$*!*

# Gabor Riesz Projection Bases

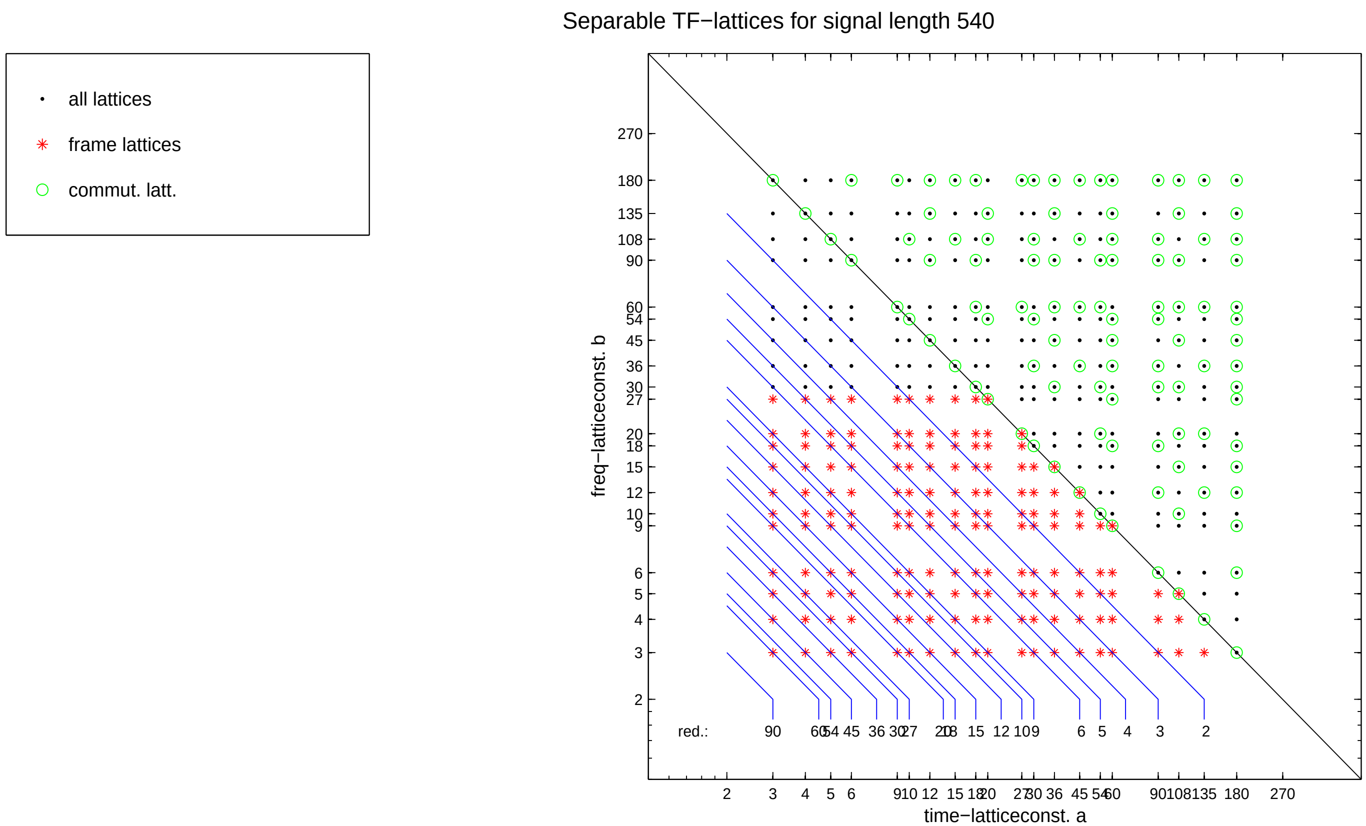
In some other cases, e.g. for applications in applications such as mobile communication, one would like to recover coefficents from linear combinations of the form $\sum_{\Lambda}^{\circ} c_{\lambda^\circ} g_{\lambda^\circ}$, in other words, one needs Gabor Riesz bases.

**Theorem 9.** *For $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ the mapping $\mathcal{C} : \mathbf{c} \mapsto \sum_{\Lambda^\circ} c_{\lambda^\circ} g_{\lambda^\circ}$ is a bounded Gelfand triple morphism $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(\Lambda^\circ)$ into $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$.*
*The mapping $\mathcal{C}$ defined above is a retract if and only if the family $(\pi(\lambda^\circ)g)_{\lambda^\circ \in \Lambda^\circ}$ is a (Gabor) Riesz basis. Indeed, the generator of the biorthogonal Gabor atom $\widetilde{g}$ is automatically in $\boldsymbol{S}_0(\mathbb{R}^d)$ and the Gram operator is invertible on $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(\Lambda^\circ)$. Consequently the mapping $\mathcal{R} : f \mapsto V_{\widetilde{g}} f(\lambda^\circ)$ is a Gelfand triple morphism from*

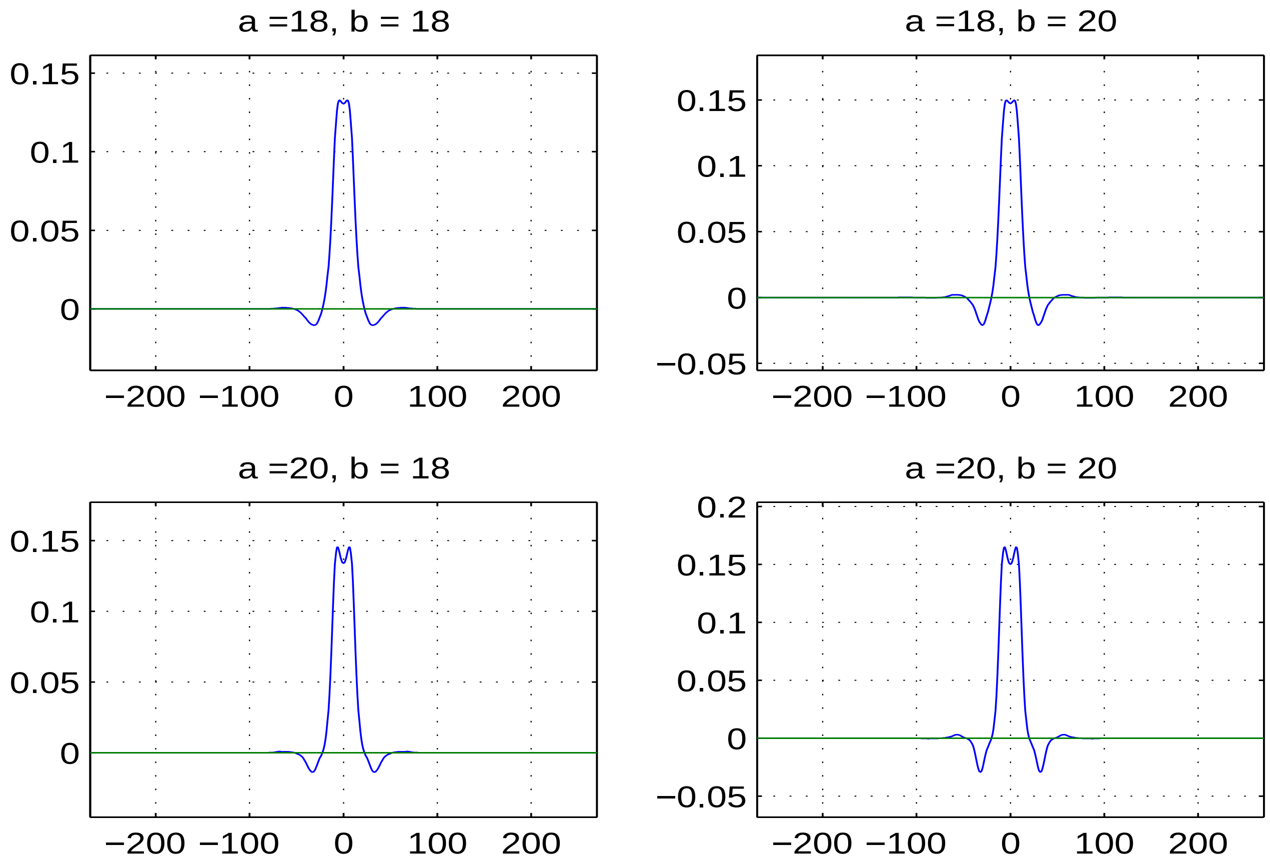
$$(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d) \quad \textit{into} \quad (\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty).$$

The Ron-Shen principle gives more details about the relation between $\Lambda$ and $\Lambda^\circ$.

# A Discrete Version: Each Point "is" a Lattice, $n = 540$



# On the continuous dependence of dual atoms on the TF-lattice



# Stability of Gabor Frames with respect to Dilation

Recent results (Trans. Amer. Math. Math. Soc.), obtained together with N. Kaiblinger.

For a subspace $X \subseteq \boldsymbol{L}^2(\mathbb{R}^d)$ define the set

$$F_g = \big\{(g, L) \in X \times \mathrm{GL}(\mathbb{R}^{2d}) \text{ which generate a Gabor frame } \{\pi(Lk)g\}_{k\in\mathbb{Z}^{2d}}\big\}. \tag{8}$$

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**Theorem 10.** (i) *The set* $F_{\boldsymbol{S}_0(\mathbb{R}^d)}$ *is open in* $\boldsymbol{S}_0(\mathbb{R}^d) \times \mathrm{GL}(\mathbb{R}^{2d})$.
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There is an analogous result for the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

**Corollary 8.** (i) *The set* $F_{\mathcal{S}}$ *is open in* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \times \mathrm{GL}(\mathbb{R}^{2d})$.
(ii) *The mapping* $(g, L) \mapsto \widetilde{g}$ *is continuous from* $F_{\mathcal{S}}$ *into* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

# Why is it so relevant to know it for the $S_0(\mathbb{R}^d)$ norm?

*Isn't the description above self-referential? Wouldn't it be reasonable to look out for the same results for "more standard" function spaces? (assuming that we are only interested in the* $\boldsymbol{L}^2$*-setting*

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NO!

- Simply because the continuous dependence is not valid in the ordinary $\boldsymbol{L}^2$-setting!
- Even if it was true for some other norm it would not imply, that the overall system, i.e. the mapping

$$f \mapsto \sum_{\lambda \in \Lambda} V_h(f) g_\lambda$$

would *not* be close to the Identity operator in the *operator norm on* $\boldsymbol{L}^2(\mathbb{R}^d)$, for all functions $h$ which arise as dual windows for a pair $(g', \Lambda')$, with $g'$ close to $g$ and $\Lambda'$ close to $\Lambda$ (in the sense of having very similar generator!).

A result of this type is of course the basis for many results about Gabor multipliers, arising by multiplying the Gabor coefficients with some sequence, to be called upper symbol. So every bounded sequence $m \in \ell^\infty(\mathbb{Z}^{2d})$ defines a bounded linear operator. Even more is true:

**Theorem 11.** *For a fixed pair of $\boldsymbol{S}_0(\mathbb{R}^d)$-functions $g, \gamma$, the mapping from the upper symbol $m \in (\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$ to the Gabor multiplier*

$$f \mapsto GM_m(f) := \sum_{\lambda \in \Lambda} V_\gamma f(\lambda) m_\lambda g_\lambda$$

*is a GT-morphism into* $\big(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)),\ \mathcal{HS},\ \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\big)$.

There is an alternative description of Gabor multipliers $GM_m$ is to express it as a sum of rank-one operators $P_\lambda : f \mapsto \langle f, g_\lambda \rangle g_\lambda$:

$$GM_m = \sum_{\lambda \in \Lambda} m_\lambda P_\lambda.$$

# Riesz Projection bases for Spline-type spaces

Think of a translation invariant (say wavelet) closed subspace $V$ with a Riesz (or even orthonormal) basis of the form $(T_\lambda \varphi)_{\lambda \in \Lambda}$. If $\varphi$ is of some mild quality, namely $\varphi \in \boldsymbol{W}(\boldsymbol{L}^2, \ell^1)$ then we have $\varphi * \varphi^* \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) = \boldsymbol{S}_0(\mathbb{R}^d)$, hence the sampled autocorrelation function is in $\ell^1$.

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The orthonormal projection from the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$: $f \mapsto P_V$ onto the spline-type space is obtained by the mapping

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But this mapping is not only well defined on $\boldsymbol{L}^2$, but also on a $\boldsymbol{L}^p$, for the full range of $1 \leq p \leq \infty$ and - again due to the properties of Wiener amalgams brings us for $f \in (\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)$ coefficients which are in $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$, which in turn implies that the function $\sum_{\lambda \in \Lambda} (\widetilde{\varphi} * f)(\lambda) T_\lambda \varphi$ is a well defined element of $\boldsymbol{W}(\boldsymbol{C}^0, (\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty))$, hence in particular $(\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)(\mathbb{R}^d)$.

# The Gabor Multipliers and Spline Type Spaces

The question, whether a Gabor multiplier $GM_m$ is uniquely determined can be recast into a question about the property of the family of rank 1 operators. Let us assume for simplicity that $\gamma = g$ (and perhaps that we have tight Gabor atoms, so that the constant multiplier $m(\lambda) \equiv 1$ gives the $Id$-operator.

**Theorem 12.** *The family of rank-1 operators* $P_\lambda$, $f \mapsto \langle f, g_\lambda \rangle g_\lambda$ *is a Riesz basis for its closed linear span in* $\mathcal{HS}$ *if and only if the circulant matrix generated from the "vector"* $|V_g g(\lambda)|^2$ *is invertible.*

*If this is the case, the family* $(P_\lambda)_{\lambda \in \Lambda}$ *is in fact a Riesz projection basis for* $\big(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)),\ \mathcal{HS},\ \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\big)$, *i.e. the mapping* $Q : T \mapsto$ *best-approximation-to* $T$ *in the* $\mathcal{HS}$ *sense extends to a retract from* $\big(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)),\ \mathcal{HS},\ \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\big)$ *into the spaces of Gabor multipliers with multiplier symbols in* $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(\Lambda)$ *(which is surjective).*

*There exists a "bi-orthogonal" (in the* $\mathcal{HS}$*-sense) family* $(Q_\lambda)_{\lambda \in \Lambda}$ *in* $\mathcal{L}(S_0', S_0)$, *in the sense that the best-approximation operator is of the form* $T \mapsto \sum_{\lambda \in \Lambda} \langle T, Q_\lambda \rangle_{\mathcal{HS}} P_\lambda$.

# Local properties of STFTs with $S_0(\mathbb{R}^d)$- windows

**Corollary 9.** *Let* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$. *Then* $|V_g f|^2 \in \boldsymbol{S}_0(\mathbb{R}^{2d}) \subset \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)$ *for* $f \in L^2(\mathbb{R}^d)$.

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This will be an important fact in the background of the following study: Consider the rank-one operators $P_\lambda : f \mapsto \langle f, g_\lambda \rangle g_\lambda$, for $\lambda \in \Lambda$. For $g$ normalized in $\boldsymbol{L}^2$ these are the projections on the 1D-space generated by $g_\lambda$, and for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ they are "good quality operators in $\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d))$.

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We them as elements of $\mathcal{HS}$ and want to find out, whether they are a Riesz basis in $\mathcal{HS}$ by checking for the invertibility of their Gram matrix.

$$\langle P_\lambda, P_{\lambda'} \rangle_{\mathcal{HS}} = |\langle g_\lambda, g_{\lambda'} \rangle|^2_{\boldsymbol{L}^2} = |V_g g(\lambda - \lambda')|^2$$

this in turn is a circulant matrix, and its invertibility is equivalent to the fact that the $\Lambda^\perp$ periodic version of $\mathcal{F}_\Lambda(|V_g g|^2)$ is free of zeros (note that we can apply Wiener's inversion theorem because $\mathcal{F}_\Lambda(|V_g g|^2) \in \boldsymbol{S}_0(\mathbb{R}^d)$, hence its periodization has an absolutely convergent Fourier series (as well as its inverse with respect to convolution).

# Gabor Multipliers: Overview of Questions

$$G_m(f) = \sum_{\lambda\in\Lambda} m_\lambda V_g f(\lambda) g_\lambda = \sum_{\lambda\in\Lambda} m_\lambda P_\lambda(f)$$

where we assume for simplicity that $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ generates a tight Gabor frame, or equivalently, we assume that $m \equiv 1$ gives us the identity operator.

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- Gabor multiplier results can be obtained from the mapping properties of the $\mathcal{C} : f \mapsto V_g f|_\Lambda$ and the synthesis mapping $\mathcal{R} : c \mapsto \sum c_\lambda g_\lambda$.
- in addition one may ask in which sense the quality of the Gabor multipliers depends on the ingredients;
- what can be said about the linear mapping from sequences $(m_\lambda)$ to operators $G_m$ (injectivity, etc.);
- best approximation by Gabor multipliers;
- questions of stability (condition numbers);

# Gabor Multipliers: A summary of facts

**Theorem 13.** *The mapping $GM$ from the "upper symbol" $(m_\lambda)$ to the Gabor multiplier $G_m$ (for arbitrary $g \in \boldsymbol{L}^2(\mathbb{R}^d)$) is a Gelfand triple isomorphism from the Gelfand triple $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$ to the Gelfand triple of operator spaces on $\boldsymbol{L}^2(\mathbb{R}^d)$ consisting of $(\mathcal{S}_1, \mathcal{HS}, \mathcal{B}(\boldsymbol{L}^2))$.*

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For $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ we have something stronger:

**Theorem 14.** *Assume $g \in \boldsymbol{S}_0(\mathbb{R}^d)$. Then the mapping $GM$ from the "upper symbol" $(m_\lambda)$ to the Gabor multiplier $G_m$ is a bounded linear Gelfand triple mapping from the Gelfand triple $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$ to the Gelfand triple of operator spaces $\big(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)),\ \mathcal{HS},\ \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\big)$.*

**Theorem 15.** *Assume that $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and that $(P_\lambda)$ is a Riesz basis in $\mathcal{HS}$, then the mapping $GM$ defines a Gelfand Riesz basis for $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$ into $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$.*
*In particuar, the orthogonal projection $T \mapsto P(T)$, mapping a given Hilbert Schmidt operator to its coefficients of the best approximation by Gabor multipliers with respect to the given Gabor frame generated from $(g, \Lambda)$ is extending to a Gelfand triple mapping from $\big(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)),\ \mathcal{HS},\ \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\big)$ to $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$.*

# Robustness and Approximation

The $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$-framework also guarantees various types of robustness, such as

- stability with respect to jitter error (positions are changing);
- even irregular Gabor multipliers can be treated (P. Balazs);
- changing the lattice;
- let lattice converge towards each other;
- let the lattice "tend to the full TF-plane" (Anti-Wick calculus);
- function spaces (again Wiener amalgams) in order to describe the "rough" symbols defining good STFT-multpliers;
- etc. . . . etc. . . .

# Modulation spaces and pseudo-differential operators

As we have seen modulation spaces are also quite interesting objects for the discussion of pseudo-differential operators. This movement has been started in 1994 with the work of Tachizawa, considering classical pseudo-differential operators on modulation spaces.

On the other hand Gröchenig/Heil have been able to improve the classical Calderon-Vaillancourt theorem using modulation spaces arguments. More recently it has been recognized that e.g. Sjöstrands work makes implicitly use of modulation space descriptions of kernels or symbols of pseudo-differential operators. The relevant papers (in particular by Gröchenig) are published under the name of "localization" of frames.

# Coorbit Theory and Atomic Decompositions

In the late 80's together with K. Gröchenig a joint viewpoint on Besov-Triebel-Lizorkin spaces (characterized via the continuous wavelet transform) and the modulation spaces (characterized via the STFT) has been given, using the theory of (integrable) group representations. Banach spaces of analytic functions on the unit disk invariant under the Moebius transformation are another instance of this abstract setting.

In that context on can proof that one can recover a generalized wavelet transform $V_g f$ (for a sufficiently nice "window" $g$) from a sufficiently dense subset of sampling points. Equivalently, once can decompose every distribution $f$ with $V_g f$ in some weighted (mix-norm) $\boldsymbol{L}^p$-space as a sum of atoms of the form $\pi(\lambda_i)g$, with $\boldsymbol{\ell}^p$-coefficients.

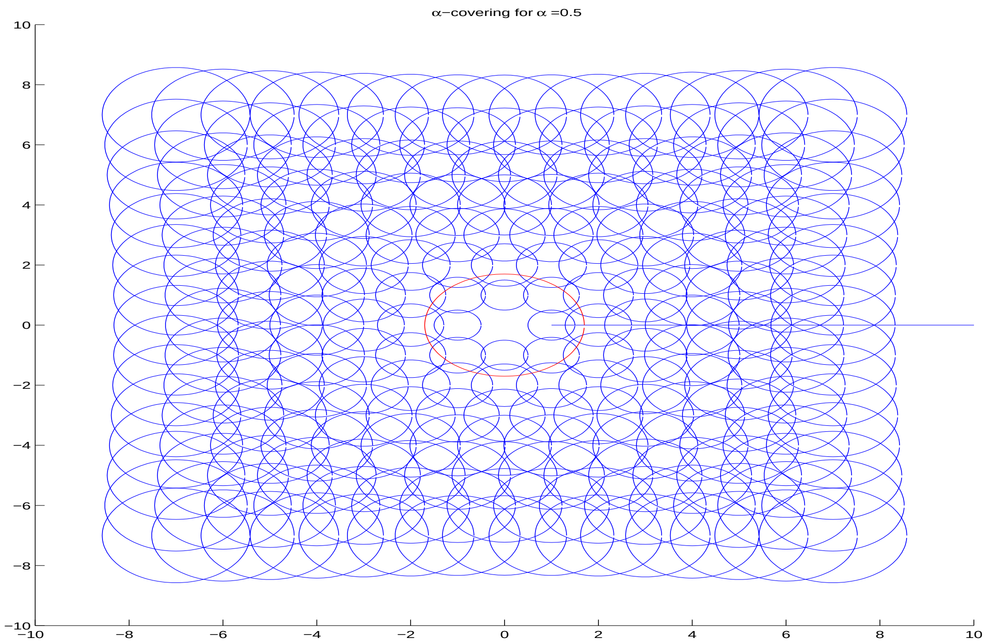
# Alpha-modulation Spaces

After discussions with H. Triebel starting with a PhD thesis of Peter Gröbner (1992, University Vienna) a new family of function spaces has been designed, which intermediates between modulation spaces and Besov spaces.

They can be characterized by (admissible) partitions of unity on the Fourier transform side which are in between uniform (= modulation spaces) and dyadic (Besov spaces). Roughly speaking the length of the support of $\phi_j$ is related to $(1 + |s_j|)^\alpha$, for some $\alpha \in [0, 1)$, if $s_j$ described the center of the support of $\phi_j$.

All the basic results about these spaces (norm-equivalence, optimal embedding, etc.) have been discussed in the PhD thesis of Peter Gröbner, while Banach frame decompositions (atomic decompositions) have been obtained in recent work of Massimo Fornasier and Lasse Borup and Morten Nielsen, respectively, very recently.

# Admissible covering in $\mathbb{R}^2$ for alpha-modulation spaces



# Time-Frequency Concentration via Modulation Spaces

Recall that the Gauss function is given by $g_0(t) = e^{-\pi t^2}$, $z = (x, \xi) \in \mathbb{R}^d$. The short time Fourier transform (STFT) with Gaussian window is therefore

$$V_{g_0} f(z) = \int_{\mathbb{R}^d} f(t)\, g_0(t-x)\, e^{2\pi i t \cdot \xi}\, dt = \langle f, M_\xi T_x g_0 \rangle$$

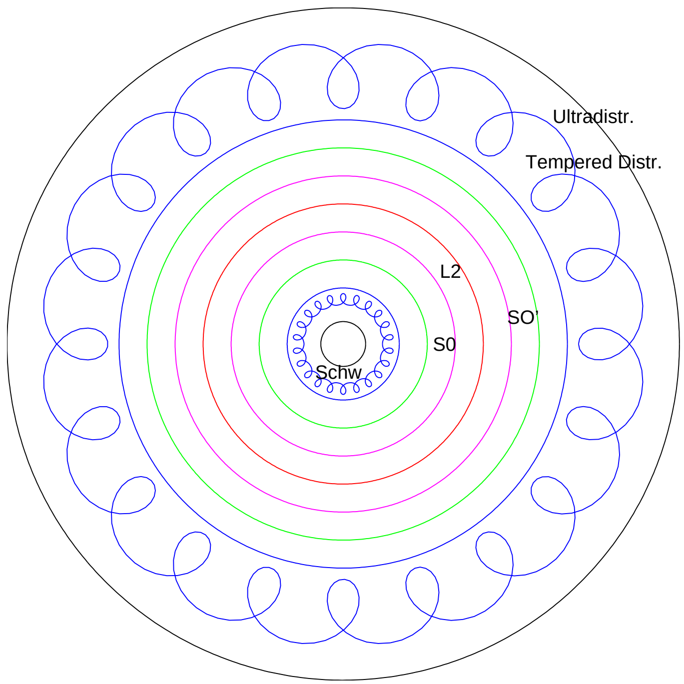
Modulation space $M^p_v$ ("good pulses") consists of all functions such that

$$\|f\|_{M^1_v} := (\int_{\mathbb{R}^d} |V_{g_0} f(z)|^p\, v^p(z)\, dz)^{1/p} < \infty$$

Typical examples for such weight functions $v$ on phase space are either weights depending on frequency only, such as $v(x, \xi) = (1 + |x|)^s$ (leading to the "classical" modulation spaces), or more interesting (because they lead to Fourier invariant spaces) weights which are radial: $v_s := (1 + |x|^2 + |\xi|^2)^{s/2}$. The intersection of all spaces $M^p_{v_s}$ is just the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. $M^2_s = Q_s =$ (Shubin Class, having a characterization in terms of Hermite coefficients).

# A Collection of Fourier Invariant Spaces

It is also possible to make use of radial weights of sub-exponential growth, and obtain in this way a family of Fourier invariant Banach spaces of test functions and corresponding spaces of ultra-distributions.



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