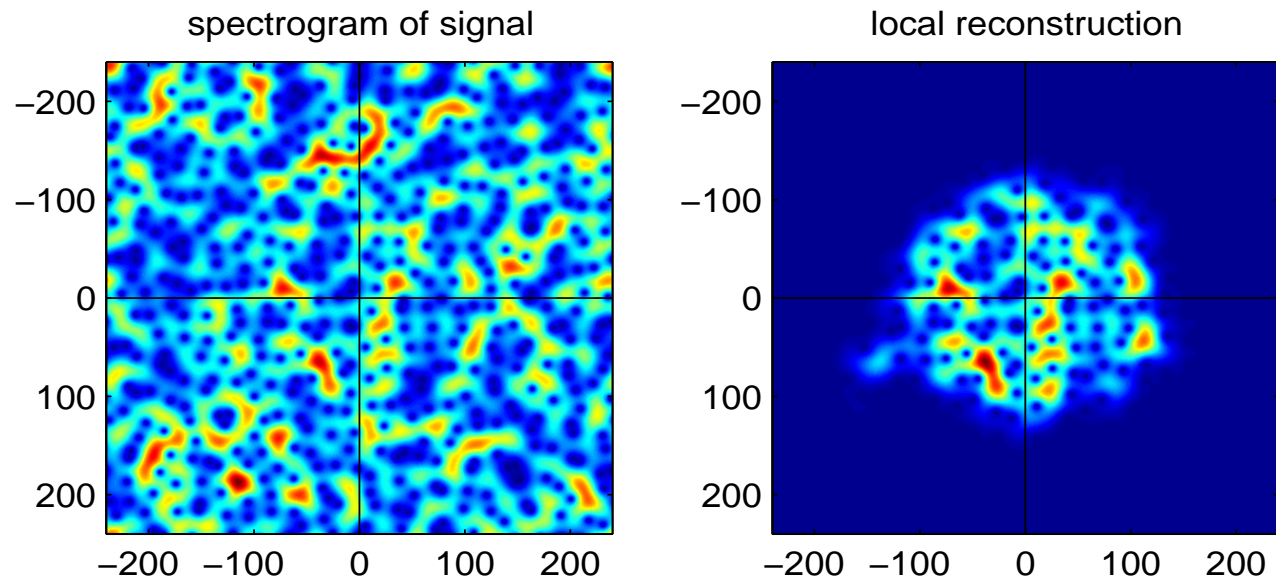


## Localized Frames via Coorbit Theory

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Marie Curie Excellence Grant  
NuHAG, Vienna  
SampTA07, Thessaloniki, June 2007

It is the purpose of this talk to point out certain connections between the early **coorbit theory** ([3, 4]) and the more recent theory of **localized frames** (see [11]). Essentially one can say, that in the situation which allows to obtain an atomic decomposition for a whole family of Banach space (in fact coorbit spaces), then one has implicitly the guarantee that the corresponding reconstruction operators (they are not necessarily the same as the one obtained by the standard frame method) are a localized. This implies that one has also automatically the so-called HAP (= **homogeneous approximation property**), even in the context of general coorbit spaces, not only for  $\mathbf{L}^2(\mathbf{R})$ . One possible way to obtain these results is to follow the program outlined in [5] in the coorbit context (instead of the original irregular sampling setting).



## Frames, Banach Frames, Coorbit Spaces

By now the concept of **frames in Hilbert spaces**  $\mathcal{H}$  is well known and widely used. Given a frame  $(g_i)_{i \in I}$  there is a well-defined (canonical) dual frame  $(\tilde{g}_i)_{i \in I}$ , such that every  $f \in \mathcal{H}$  has a representation as

$$f = \sum_i \langle f, g_i \rangle \tilde{g}_i = \sum_i \langle f, \tilde{g}_i \rangle g_i$$

Because technically frames can be characterized by the boundedness and invertibility of the frame operator  $S : f \mapsto Sf := \sum_i \langle f, g_i \rangle g_i$  the focus of interest is often on the establishment of a simple norm equivalence, namely that of the Hilbert space norm, and the coefficient energy, given by  $\sum_{i \in I} |\langle f, \tilde{g}_i \rangle|^2$ .

From this point of view it appears as natural to generalize the concept to a Banach space setting by claiming that for certain Banach spaces (such as Besov or modulation spaces) one has a norm equivalence between the Banach space norm and a corresponding sequence space norm (typically a weighted mixed-norm Banach sequence space): **Banach Frames**.

## Gabor frames and modulation spaces

In this talk we will concentrate on (regular or irregular) **Gabor frames**, hence the corresponding natural family of Banach spaces of (tempered) distributions is the family of modulation spaces. Given a non-zero window  $g \in \mathcal{S}(\mathbb{R}^d)$ , a  $v$ -moderate weight function  $m$  on  $\mathbb{R}^{2d}$  of polynomial growth, and  $1 \leq p, q \leq \infty$ , the **modulation space**  $\mathbf{M}_m^{p,q}(\mathbb{R}^d)$  consists of all tempered distributions  $f \in \mathcal{S}'(\mathbb{R}^d)$  such that  $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$  (weighted mixed-norm spaces). The norm on  $M_m^{p,q}$  is

$$\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p m(x, \omega)^p dx \right)^{q/p} d\omega \right)^{1/p}.$$

If  $p = q$ , we write  $M_m^p$  instead of  $M_m^{p,p}$ , and if  $m(z) \equiv 1$  on  $\mathbb{R}^{2d}$ , then we write  $\mathbf{M}^{p,q}$  and  $\mathbf{M}^p$  for  $M_m^{p,q}$  and  $M_m^{p,p}$ .

Then  $\mathbf{M}_m^{p,q}(\mathbb{R}^d)$  is a Banach space whose definition is independent of the choice of the window  $g$ .

Moreover, if  $m$  is  $v$ -moderate and  $g \in M_v^1 \setminus \{0\}$ , then  $\|V_g f\|_{L_m^{p,q}}$  is an equivalent norm for  $M_m^{p,q}(\mathbb{R}^d)$ . We always measure the  $M_m^{p,q}$ -norm with a fixed non-zero window  $g \in \mathcal{S}(\mathbb{R}^d)$  and repeatedly use the fact that for any  $g_1 \in M_v^1(\mathbb{R}^d)$  the norm equivalence

$$\|f\|_{M_m^{p,q}} \asymp \|V_{g_1} f\|_{L_m^{p,q}}$$

holds.

We have Hilbert spaces for the case  $p = 2 = q$ . For the choice  $m(x, \omega) = (1 + |\omega|)^s$ , with  $s \in \mathbb{R}$  these are just the classical  $L^2$ -Sobolev space, while we obtain so the called Shubin classes  $\mathcal{Q}_s(\mathbb{R}^d)$  by choosing a radial weight  $v_s(x, \omega) := (1 + x^2 + \omega^2)^{s/2}$ . On the other hand the minimal space among all the (non-trivial) Banach space which are isometrically invariant under TF-shifts is the space  $M^1(\mathbb{R}^d)$  which appeared first as a minimal Segal algebra (in 1979), when it was denoted by the symbol  $\mathcal{S}_0(\mathbb{R}^d)$ . The triple  $(\mathcal{S}_0, L^2, \mathcal{S}'_0)$ , is a very convenient tool for the description of operators arising in Fourier analysis (e.g., the Fourier transform itself). The space  $M_{v_s}^1(\mathbb{R}^d)$ , with  $v_s$  as above, are important as families of “atoms”. Most recently “costumized” modulation spaces are considered (in the PhD thesis of Roza Acesca), which allow to model functions of variable band-width. [2, 1, 6, 12, 13]

## General modulation spaces

$$\mathbf{B} = \mathcal{M}\mathbf{o}(\mathbf{Y}) := \{\sigma \in \mathcal{S}(\mathbb{R}^d) \mid V_g \sigma \in \mathbf{Y}\}$$

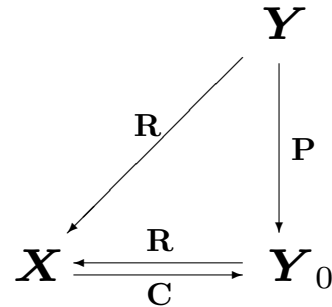
where  $\mathbf{Y}$  is some solid, translation invariant Banach space of (continuous) functions on  $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ , such as a weighted (mixed norm)  $L^p$ -space over  $\mathbb{R}^{2d}$ , or a customized function space (in the case of functions of variable band-width), i.e., we require

- $H \in \mathbf{Y}, |G(x)| \leq |H(\lambda)| \quad \forall \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d \Rightarrow \|G\|_{\mathbf{Y}} \leq \|H\|_{\mathbf{Y}}$
- $\|T_\lambda H\|_{\mathbf{Y}} \leq w(\lambda) \|H\|_{\mathbf{Y}} \quad \forall H \in \mathbf{Y};$

The growth of the weight function  $w(\lambda)$  will dictate the necessary quality of the Gabor atom  $g$  allowed in the Gabor reconstruction theorem.

## Better a diagram

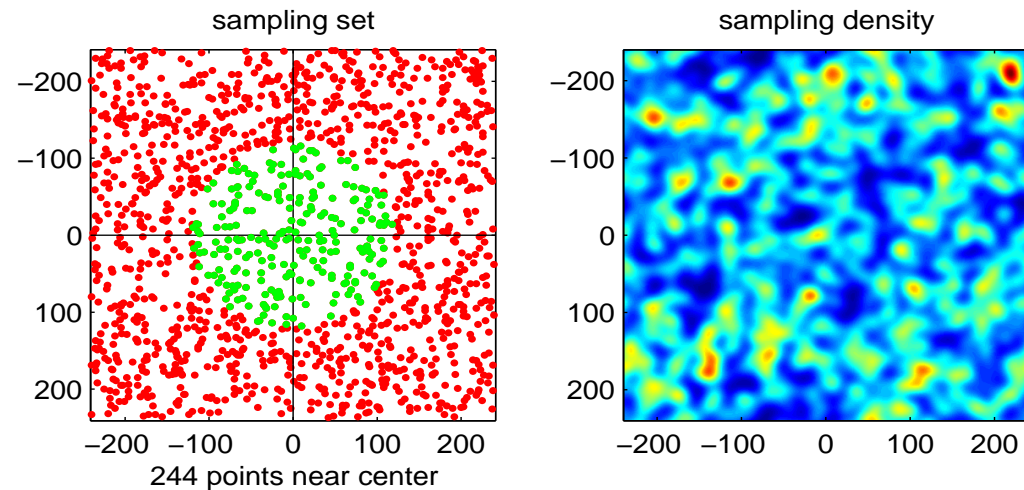
$\mathbf{R}$  being a left inverse of  $\mathbf{C}$  implies that  $\mathbf{P} = \mathbf{C} \circ \mathbf{R}$  is a projection in  $\mathbf{Y}$  onto the range  $\mathbf{Y}_0$  of  $\mathbf{C}$ , thus we have the following commutative diagram.



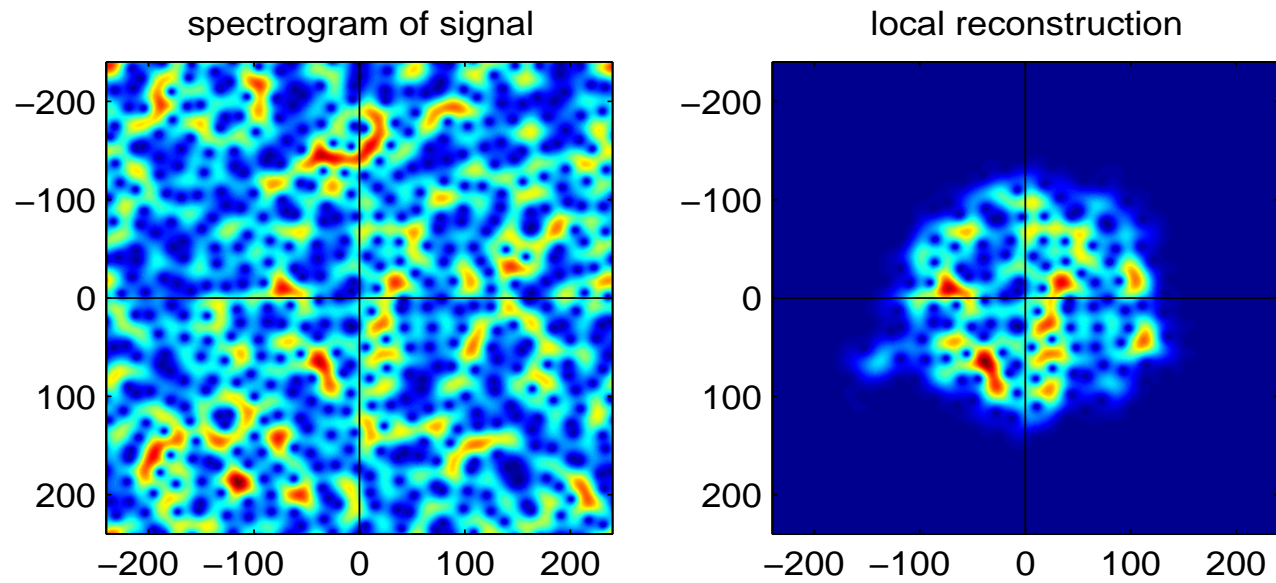
Here  $\mathbf{X}$  is the Banach space under consideration, on which the frame (coefficient or analysis) mapping  $\mathbf{C} : f \mapsto (\langle f, g_i \rangle)_{i \in I}$ , which is injective and has closed range  $\mathbf{Y}_0$  within the target sequence space  $\mathbf{Y}$ . The reconstruction mapping is then typically given via  $\mathbf{R}(\mathbf{e}_i) := \tilde{g}_i$ . It's existence is equivalent to the fact that  $\mathbf{P}$  is a projection with range  $\mathbf{Y}_0$ . This larger picture determines so-called Banach frames ([10]).

## A typical example

$n = 480$ , and an irregular Gabor family (using a Gaussian atom) with 1392 Gabor atoms, out of which 244 ( $M \subset I$ ) are located near the center (17%).

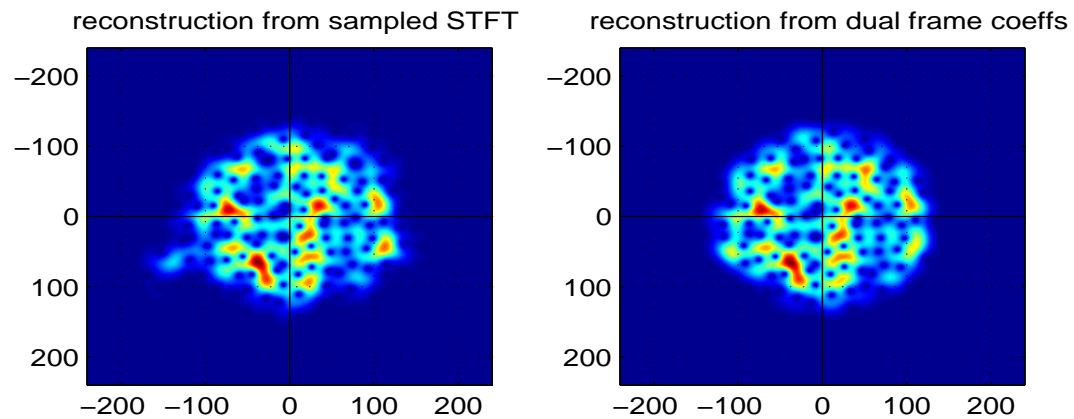




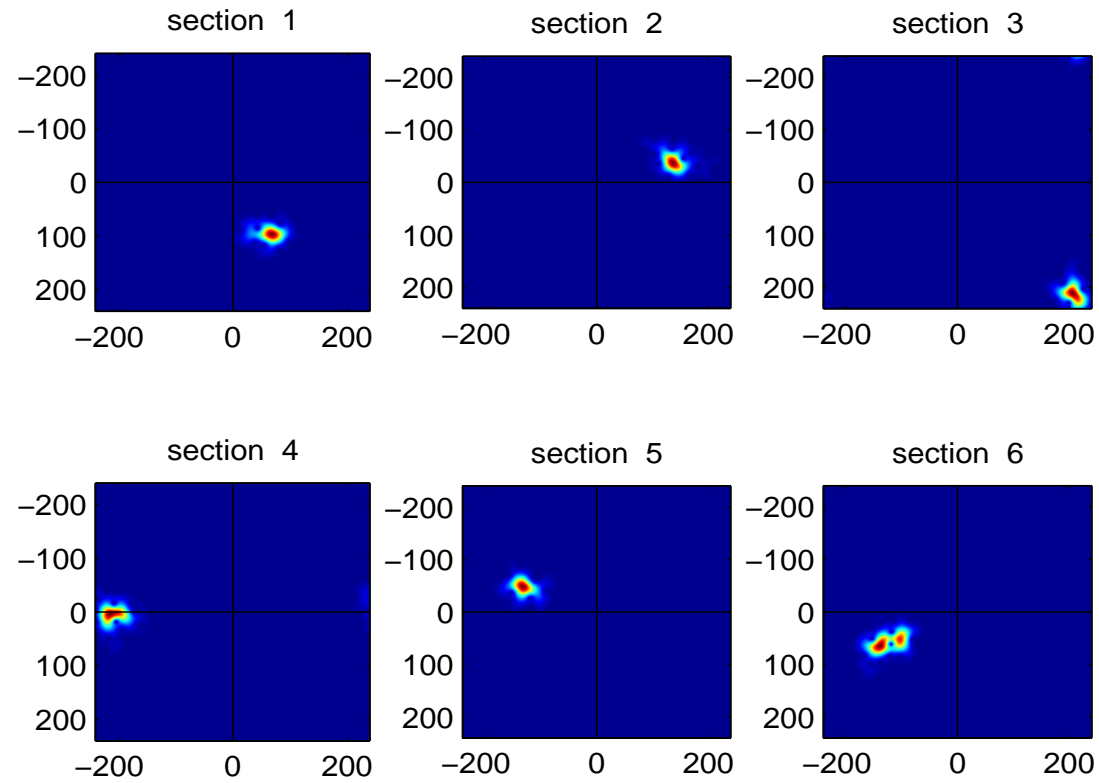


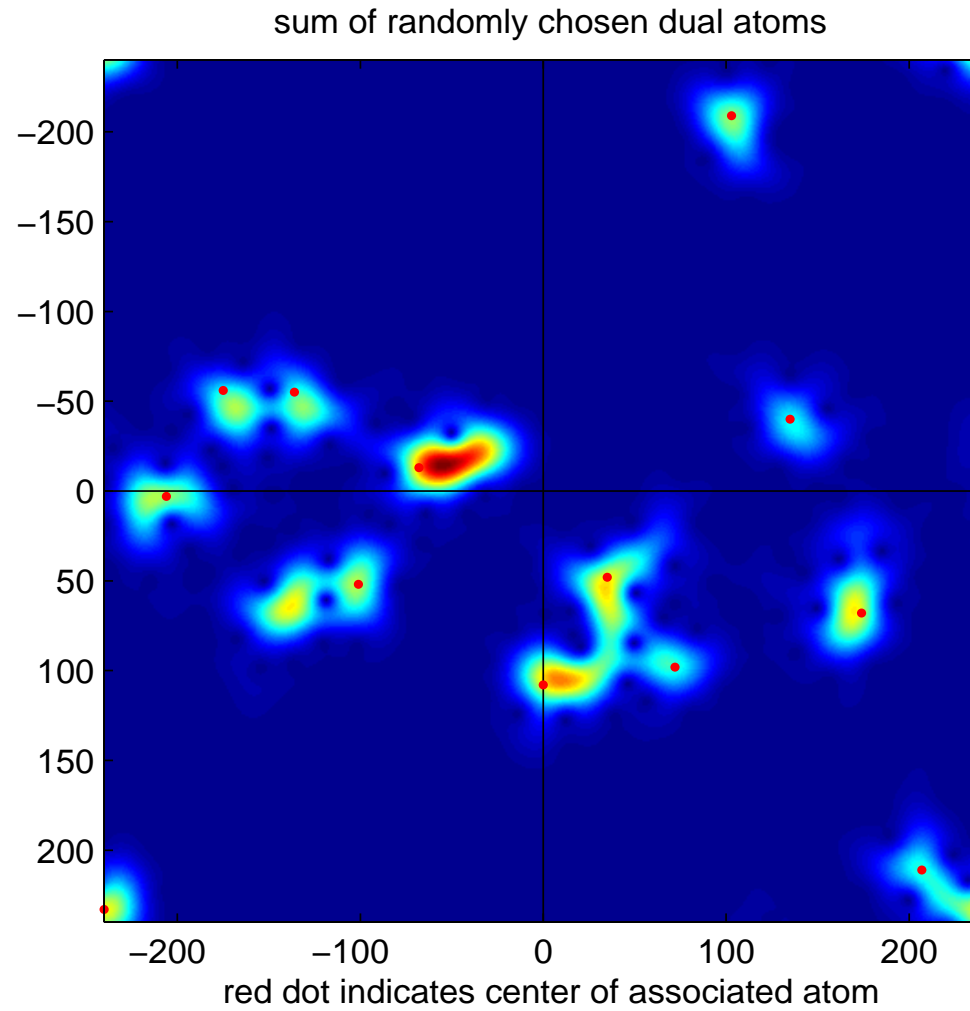
## Local reconstructions, using dual frame or frame

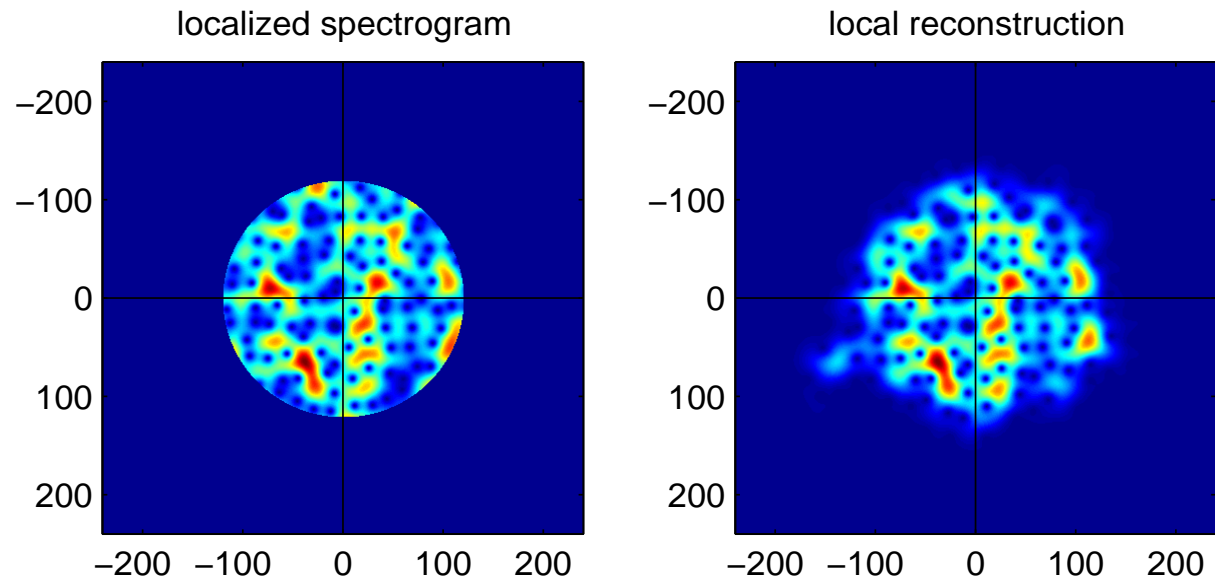
$$f_a = \sum_{i \in M} \langle f, g_i \rangle \tilde{g}_i \quad \text{versus} \quad f_b = \sum_{i \in M} \langle f, \tilde{g}_i \rangle g_i$$

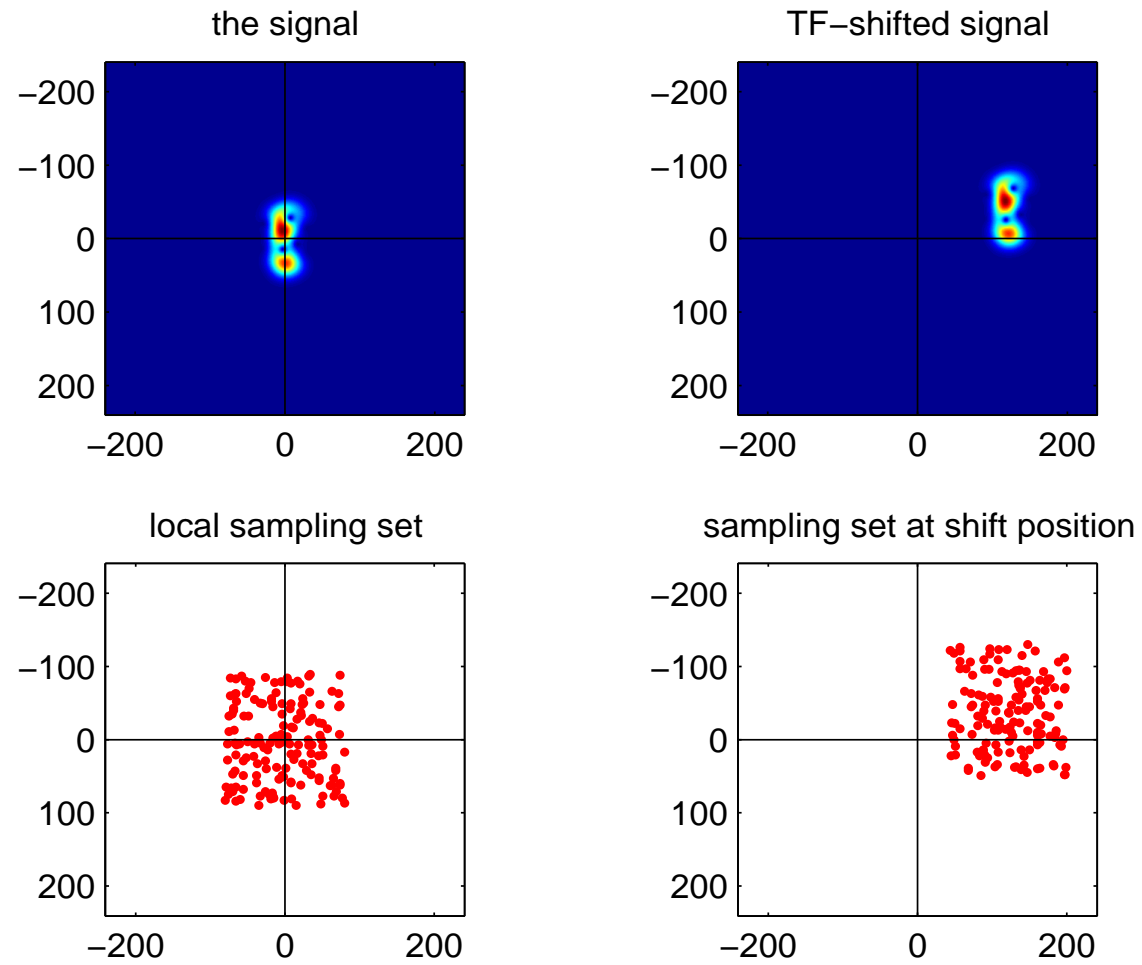


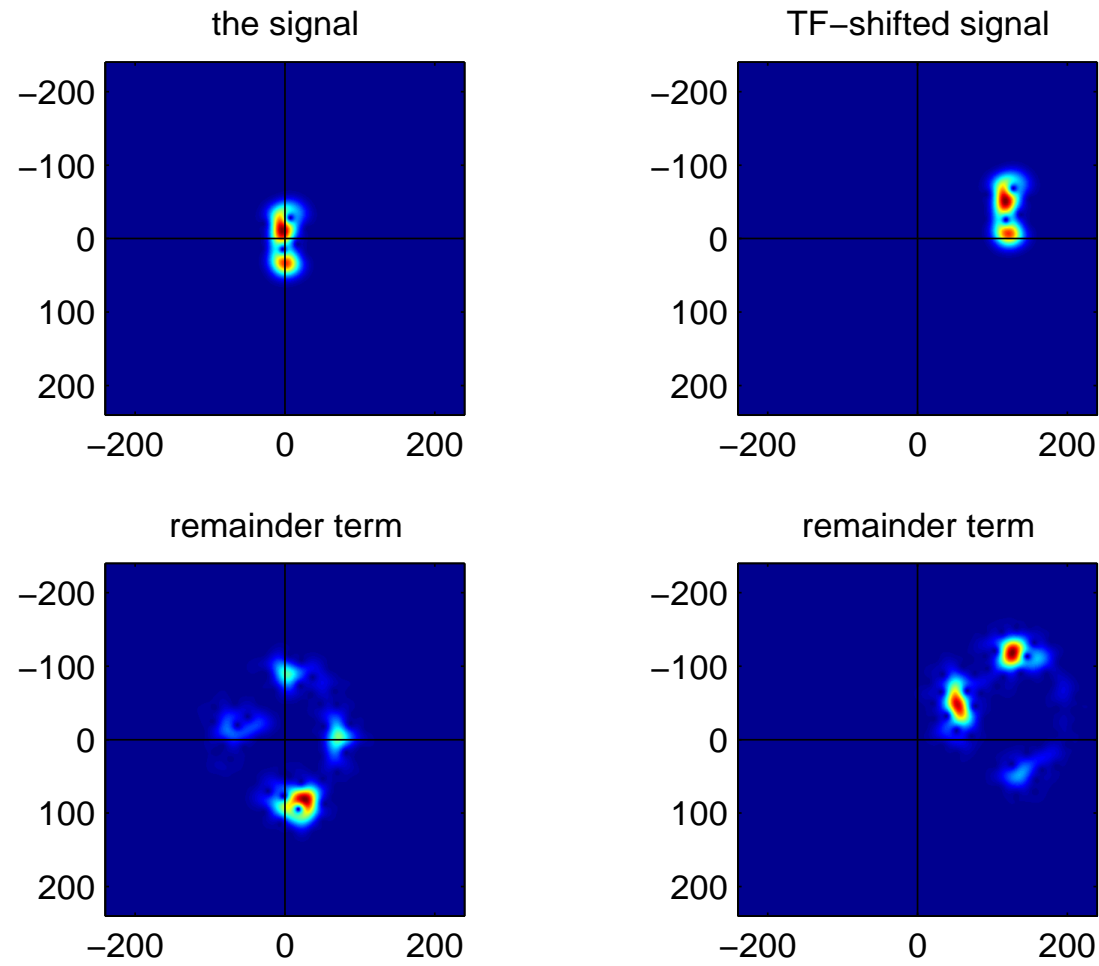
## Inspection of dual frame TF-concentrations











## The Uniform Homogeneous Approximation Property

**Definition 1.** *Given an atomic decomposition of the form*

$$f = \sum_{i \in I} \langle f, g_i \rangle \tilde{g}_i,$$

*with atoms  $g_i$  centered at  $\lambda_i$  (e.g.  $g_i = \pi(\lambda_i)g$  for some nice  $g$ , and for a family of coorbit space  $\mathbf{B}$ , valid for a family of coorbit spaces, one says that the pair of families  $(g_i, \tilde{g}_i)$  ( $\tilde{g}_i$  which need not be the “canonical dual frame”) is said to have the **UHAP** (**:= the uniform homogeneous approximation property**) if for every  $B = Co(Y)$  in this family, and for every  $f \in B$  and  $\varepsilon > 0$  there exists some compact set  $Q \subset \mathcal{G}$  such that*

$$\|f - \pi(\lambda_0)^{-1} \sum_{i \in I_{\lambda_0+Q}} \langle \pi(\lambda_0)f, g_i \rangle \tilde{g}_i\|_B < \varepsilon$$

*for all  $\lambda_0 \in \mathcal{G}$ , where the finite set  $I_{\lambda_0+Q}$  is defined as the subset of  $I$  of indices such that  $\lambda_i \in \lambda_0 + Q$ .*

*If  $\pi(\lambda)$  acts isometrically on  $B$  we are back to the original definition of (HAP).*



In other words, the size of the set of relevant sampling points for a reconstruction of  $\pi(\lambda)f$  (around its center, which moves with  $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ ) can be assumed to be essentially of fixed size, for the given level of precision. Of course, if  $(g_i)_{i \in I}$  is just a coherent frame the scalar products  $(\langle f, g_i \rangle)_{i \in I}$  are just the samples of the continuous transform (e.g. the continuous STFT, or the continuous wavelet transform).

Distinguish between the following views on [U]HAP:

- The HAP, which considers the case  $\mathcal{C}o\mathbf{L}^2 = \mathbf{L}^2(\mathbb{R}^d)$  only;
- The HAP for coorbit spaces which are isometrically  $\pi(\lambda)$ -invariant, such as the modulation spaces  $\mathbf{M}_{p,q}^0$ .
- the claim that the *canonical* dual frame, together with its original satisfies the HAP (cf. [15, 20])
- the claim that - for a given quality of the frame (expressed in term of “localization” of the frame, resp. low correlation between the elements of the frame elements which are at some distance in phase space (uniformly over the family).
- claims which use (strict) coherence on the side of the analyzing frame (i.e.  $g_i = \pi(\lambda_i)g$ ), or those allowing to switch the roles of  $\tilde{g}_i$  and  $g_i$ .
- claims valid for  $\tilde{g}_i$  obtained by a particular (family) of reconstruction method(s).

## Equivalent Characterizations of HAP

The following theorem describes a number of equivalent properties:

**Theorem 1.** *The following conditions for a pair of dual frames  $(g_i, \tilde{g}_i)_{i \in I}$ :*

- *the HAP conditions is satisfied for every element of  $f \in \mathbf{B}$ ;*
- *the HAP conditions is satisfied simultaneously for every **finite** set  $\{f_1, \dots, f_k\}$  of elements from  $\mathbf{B}$  simultaneously;*
- *the HAP is valid for a **compact** subset  $M$  of elements in  $\mathbf{B}$ .*
- *For every compact operator  $T$  one has: There exists some compact set  $Q \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ , such that for any  $T_\lambda := \pi(\lambda) \circ T \circ \pi(\lambda)^{-1}$  and any  $f \in \mathbf{B}$*

$$\|T_\lambda f - \sum_{i \in I_Q} \langle T_\lambda f, g_i \rangle \tilde{g}_i\|_{\mathbf{B}} < \varepsilon \|f\|_{\mathbf{B}}$$

Note: It is easy to show that (i) above is (technically) equivalent to the choice of a rank 1 projection operator:  $Th = \langle h, f \rangle f$ , for some function  $f \in \mathbf{L}^2(\mathbb{R}^d)$  with  $\|f\|_2 = 1$ .

## UHAP is valid in the context of coorbit theory

Coorbit theory deals with abstract continuous transform, related to integrable group representations on some Hilbert space. It makes use of function spaces of those acting groups and allows a uniformly discrete sampling reconstruction theorem for large families of Banach spaces of functions and distributions.

For the case of the  $ax + b$  group we have the continuous wavelet transform, and a detailed study of HAP (for the Hilbert space  $L^2(\mathbb{R}^d)$ ) is given in [15] and [20].

We will stay with the (usual) Schrödinger representation of the reduced Heisenberg group, or in the engineering terminology, work with the STFT, understood as a function over the TF (= time-frequency) plane or phase space (in the context of coherent states). We allow quite general windows, and consider the appropriate class of function spaces, which are the (generalized, ultra-, classical, . . . ) **modulation spaces**.

There is the “frame” (or sampling) view-point, which describes the problem as a reconstruction problem from a sampled STFT, and the alternative “atomic decomposition view points”, where the Gabor atoms are used as building blocks.

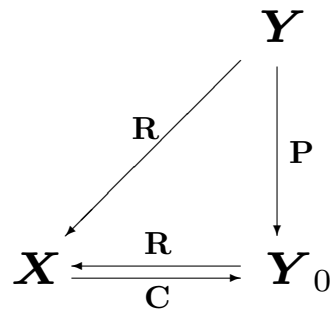
Then the coorbit theory tells us, that for a given “quality” of the Gabor atom  $g$  (i.e. knowing the norm in some function space describing smoothness and decay) we can guarantee that every sufficiently dense (and relatively separated) family of sampling points  $(\lambda_i)_{i \in I} = (x_i, \omega_i)$  allows to reconstruct functions  $f$  from any of those modulation spaces via

$$f = \sum_{i \in I} V_g(\lambda_i) \tilde{g}_i$$

with unconditional convergence of the infinite sum (in the norm topology of  $\mathcal{S}(\mathbb{R}^d)$  is dense in  $\mathbf{B}$ , and in the sense of the  $w^*$ -topology otherwise, i.e., in the sense of uniform convergence of the STFT over compact sets. In other words, the spectrogram of the partial sums will look more and more similar to the spectrogram of the overall function or distribution, as we take more and more terms (even if the norm of the remainder term does not go to zero).

## Once more the diagram

$\mathbf{R}$  being a left inverse of  $\mathbf{C}$  implies that  $\mathbf{P} = \mathbf{C} \circ \mathbf{R}$  is a projection in  $\mathbf{Y}$  onto the range  $\mathbf{Y}_0$  of  $\mathbf{C}$ , thus we have the following commutative diagram. Think of  $\mathbf{X}$  as one of the modulation spaces, of  $\mathbf{C}$  the operator sending  $f$  to the set of sampling values  $(V_g f(\lambda_i))_{i \in I}$  which is part of the appropriate (mixed norm, weighted) sequence space  $\mathbf{Y}$ , and  $\mathbf{R}$  the synthesis mapping  $(c_i)_{i \in I} \mapsto \sum_{i \in I} c_i \tilde{g}_i$ .



Note that it is important for applications that one has not only reconstruction from sequences  $(c_i)_{i \in I}$  exactly in the range of  $\mathbf{C}$  (which is denoted by  $\mathbf{Y}_0$  here), but also from the whole (solid) sequence space  $\mathbf{Y}$ , think of noise samples or samples followed by thresholding.

## What can Coorbit theory give to you?

The first (*and naive*) view on coorbit theory is to view it as a generalization of the ordinary frame theory to the setting of reconstruction in Hilbert spaces (as a consequence of the norm-equivalence between the Hilbert space norm and the coefficient energy norm) to a Banach space setting. This is also, how **Banach frames** are defined.

**Proposition 1.** (A) Given *any modulation spaces*  $\mathbf{B}$  there is a certain minimal quality required for the Gabor atom  $g$ , such that for such an atom  $g$  we can find a density measure  $\delta > 0$  such that any relatively separated allows stable reconstruction of any  $f \in \mathbf{B}$  in the form

$$f = \sum_i V_g f(\lambda_i) \tilde{g}_i \dots$$

**Proposition 2.** (B) Given **a family of modulation spaces**  $\mathbf{B}$  there is a certain minimal quality required for the Gabor atom  $g$ , such that for such an atom  $g$  we can find a density measure  $\delta > 0$  such that any relatively separated ...

$$f = \sum_i V_g f(\lambda_i) \tilde{g}_i \quad \forall f \in \mathbf{B}$$

## Uniformity with respect to location

**Theorem 2.** (*exploiting the details of coorbit theory*)

(C) Given **a family of modulation spaces  $\mathbf{B}$**  there is a certain minimal quality required for the Gabor atom  $g$ , and a family of iterative reconstruction methods which allow the reconstruction of any  $f$  from any of the modulation spaces  $\mathbf{B}$  such that we have UNIFORM bounds on the analysis and synthesis operators for all the  $\delta$ -dense and (uniformly, relatively) separated sampling sets  $(\lambda_i)_{i \in I}$ , (independent of the space  $\mathbf{B}$  chosen, independent of the atom chosen within the family), such that

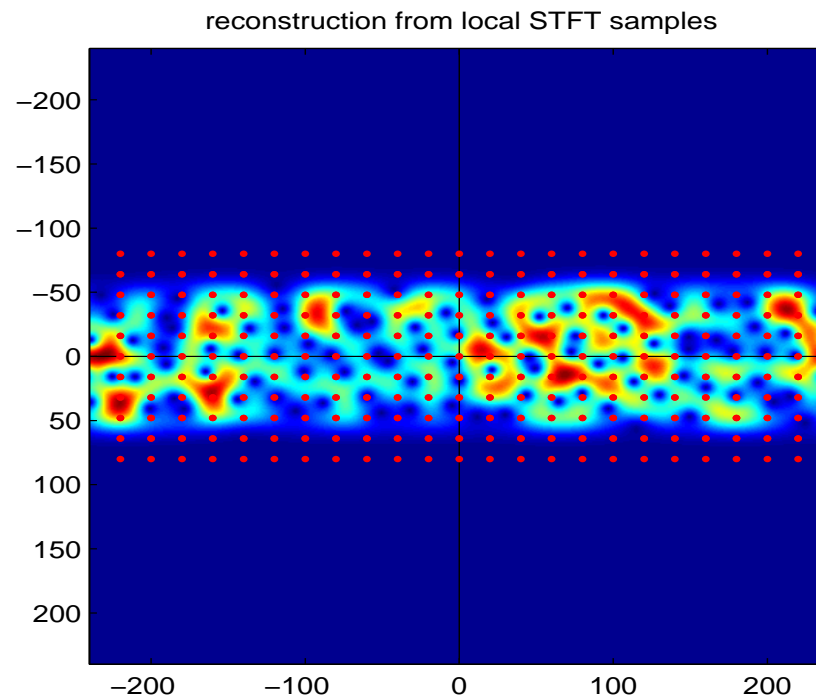
$$f = \pi(\lambda_0)^{-1} \sum_{i \in I} V_g(\pi(\lambda_0)f(\lambda_i)\tilde{g}_i)$$

UHAP then corresponds to the fact that we can find (again uniformly over all these possible choices) for every  $\varepsilon > 0$  some compact set  $Q \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ , with

$$\|f - \pi(\lambda_0)^{-1} \left( \sum_{i \in I_{\lambda_0+Q}} V_g[\pi(\lambda_0)f(\lambda_i)]\tilde{g}_i \right)\|_{\mathbf{B}} < \varepsilon$$

## Localized reconstruction of band-limited functions

Illustration of a result from [9]



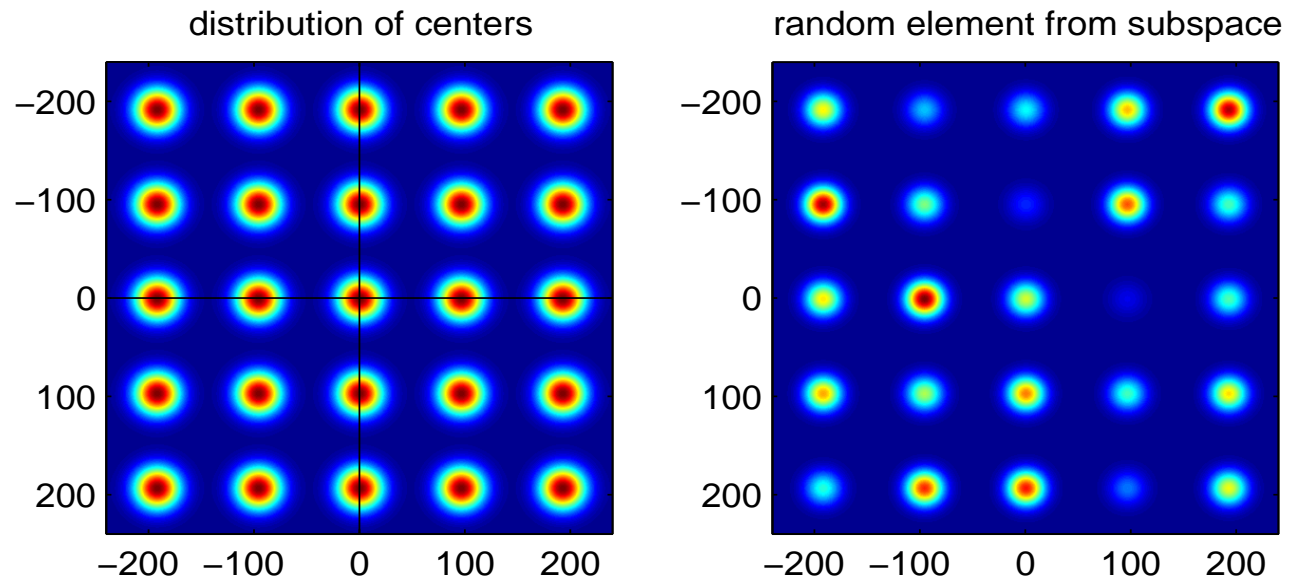


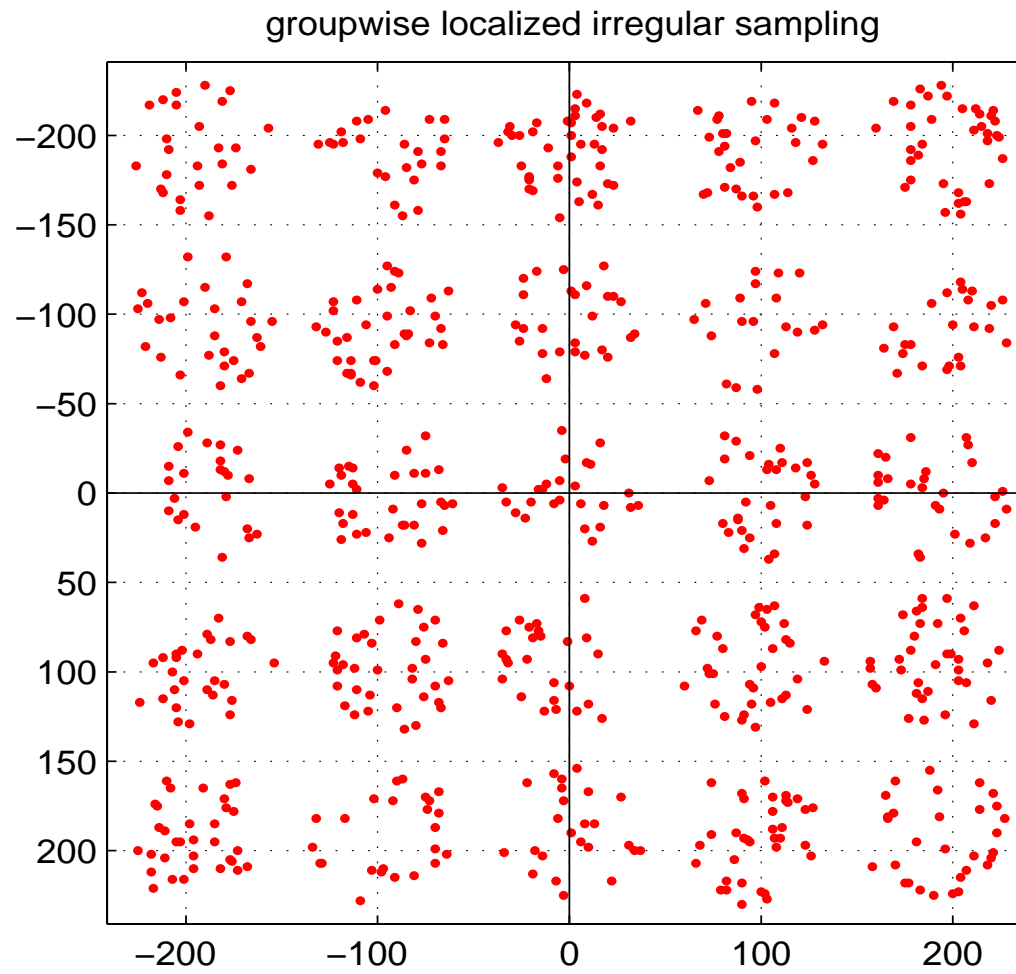
## The piano reconstruction theorem revisited

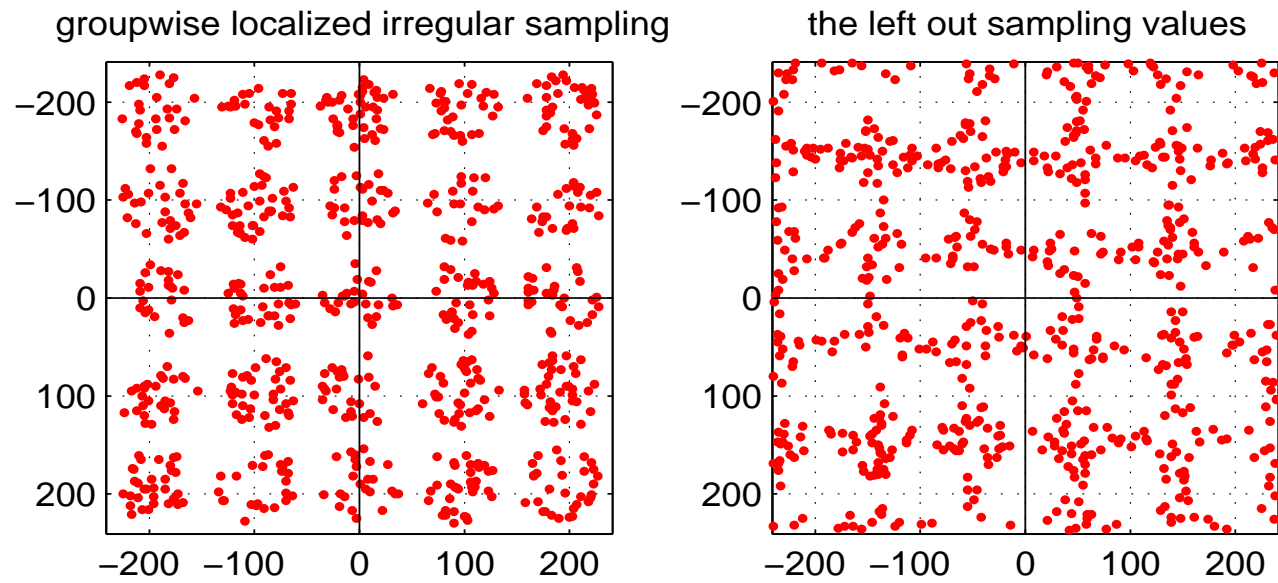
**Theorem 3.** *Given a class of modulation spaces and some  $g \in \mathcal{S}(\mathbb{R}^d)$ , there exists some  $R_0 > 0$  such that any family of the form  $(\pi(\lambda_j)g)_{j \in J}$ , with  $\text{dist}(\lambda_j, \lambda'_j) \geq R_0$  is a Riesz (projection) basis, with Riesz bounds depending on  $g$  and  $R_0$  only (but not on the individual choice of the family  $(\lambda_j)_{j \in J}$ ).*

*In particular, we have - uniformly over the class of modulation spaces, equivalence constants between the  $\mathbf{Y}_d$ -norms of the sequence  $(c_j)_{j \in J}$  and the  $\mathbf{B}$ - modulation space norm  $f = \mathbf{R}(c) = \sum_{j \in J} c_j \pi(\lambda_j)g$ .*

It is therefore natural to ask whether it is enough to know the (irregular) samples of a STFT “around the centers” of such a family (and not in the whole TF-plane) in order to be able to recover the signal  $f$  completely. The positive answer to this question is given in the next theorem.







**Theorem 4.** *Given a class of modulation spaces and some  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ , there exists some  $R_1 > 0$  and  $S \geq R_1$  such that any family of the form  $(\pi(\lambda_j)\varphi)_{j \in J}$ , with  $\text{dist}(\lambda_j, \lambda'_j) \geq R_1$  is a Riesz (projection) basis for its closed linear span in  $\mathbf{B}$ , let us call it  $V$ . The Riesz bounds depending on  $g$  and  $R_1$  only (but not on the individual choice of the family  $(\lambda_j)_{j \in J}$ ). Furthermore it is enough to know the sampling values of  $V_g(f)$  only at the (irregular) sampling values located with the set  $\bigcup_{j \in J} B_S(\lambda_j) \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$  in order to recover  $f$  completely.*

**Proof:** The idea is similar to the reconstruction of band-limited functions from (regular) samples taken from a horizontal strip in phase space, as given in [9].

The argument makes of course use of the fact that there is a biorthogonal family for the given family  $(\pi(\lambda_j)\varphi)_{j \in J} := (\varphi_{\lambda_j})_{j \in J}$ , let us call it  $(\tilde{\varphi}_j)_{j \in J}$ . The projection onto the closed linear span  $V$  of the atoms  $(\varphi_{\lambda_j})_{j \in J}$  within  $\mathbf{B}$  is then obtained by the projection mapping (which is continuous also on  $\mathbf{B}$ ):

$$f \mapsto P(f) = \sum_{j \in J} \langle f, \tilde{\varphi}_j \rangle \varphi_{\lambda_j}$$

and we have (uniform) control over the norm of  $P$  (over the whole family, say  $\|P(f)\|_B \leq C_P \|f\|_B$ ).

It therefore remains to show that we can approximate the identity operator sufficiently well, using only the described sampling values only. In fact, it is enough to have

$$\|f - \sum_{i \in I_S} \langle f, \tilde{g}_i \rangle g_{\lambda_i}\| \leq \gamma \|f\|_B,$$

for all  $f \in V$ , and some  $\gamma < 1/C_P$ , where

$$I_S = \{i \in I \mid x_i \in \bigcup_{j \in J} B_S(\lambda_j) \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}.$$

Clear enough this allows us to show that the linear mapping

$$f \mapsto P\left(\sum_{i \in I_S} \langle f, \tilde{g}_i \rangle g_{\lambda_i}\right)$$

which maps  $V$  into itself is close enough to the identity operator, and that it is invertible as a consequence.  $\square$

**Remark 3.** *There is an alternative viewpoint: Given the space  $V$  and the projection operator  $P$  one can modify the original biorthogonal family in such a way that it is not necessary to know all the sampling values of  $(V_g(f)(x_i))_{i \in I}$ , but only those belonging to the relevant subset  $I_S$  used in the proof. Moreover one can argue (and verify also numerically) that those “perfect reconstructions” for  $V$  are quite similar to the original biorthogonal functions in the “interior” of the relevant index sets (i.e., near the centers  $(\lambda_j)_{j \in J}$ , while they may have to be deformed more seriously near the boundaries.*

Further references: [10], [7, 8],  
 Maybe [17, 16, 18], [15, 14, 20, 19]

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