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Time-frequency analysis and Gabor multipliers: From numerical linear algebra to conceptual harmonic analysis

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EE Dept., Aston University
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ABSTRACT

(see www.nuhag.eu) $>>$ DB+tools $>>$ Talks

My Personal Background (from AHA to CHA)

- Trained as an **abstract harmonic analyst** (Advisor Hans Reiter)
- working on function spaces on locally compact groups, distribution theory
- turning to applications (signal processing, image processing), wavelets
- doing **numerical work** on scattered data approximation, **Gabor analysis**
- heading nowadays a group of some 20 researches at the University of Vienna: www.nuhag.eu

OUTLINE of the TALK:

1. short linear algebra background
2. Time-Frequency Analysis is often explained via Spectrogram
3. Gabor analysis has two aspects:
4. the synthetic viewpoint of Gabor: microtonal piano
5. the analysis viewpoint: recovery from sampled spectrogram
6. action on signals via multiplication of Gabor coefficient: Gabor multipliers

What is GABOR ANALYSIS about?

The original idea (D. Gabor, 1946) was to expand every signal into a sum with uniquely determined coefficients, by choosing integer time-frequency shifted copies of the normalized Gauss-function. Unfortunately there are many problems, and one has to resort to TF-lattices of the form $a\mathbb{Z} \times b\mathbb{Z}$, with $ab < 1$, which gives us some "redundancy".
PROBLEMS:

1. we have to work in an infinite dimensional Hilbert space ($\boldsymbol{L}^2(\mathbb{R}^d)$);
2. the system is non-orthogonal (hence it is not clear how to get coefficients a priori);
3. when we do computations we have to resort to "finite models"
4. how should one approximate the continuous situation by the finite one?

CLAIM: What is really needed!

In contrast to all this the CLAIM is that just a bare-bone version of functional analytic terminology is needed (including basic concepts from Banach space theory, up to w^*-convergence of sequences and basic operator theory), and that the concept of Banach Gelfand triples is maybe quite useful for this purpose. So STUDENTS SHOULD LEARN ABOUT:

- refresh their linear algebra knowledge (ONB, **SVD**!!!, linear independence, generating set of vectors), and matrix representations of linear mappings between finite dimensional vector spaces;
- Banach spaces, bd. operators, dual spaces norm and w^*-convergence;
- about Hilbert spaces, orthonormal bases and unitary operators;
- about frames and Riesz basis (resp. matrices of maximal rank);

Key Players for Time-Frequency Analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

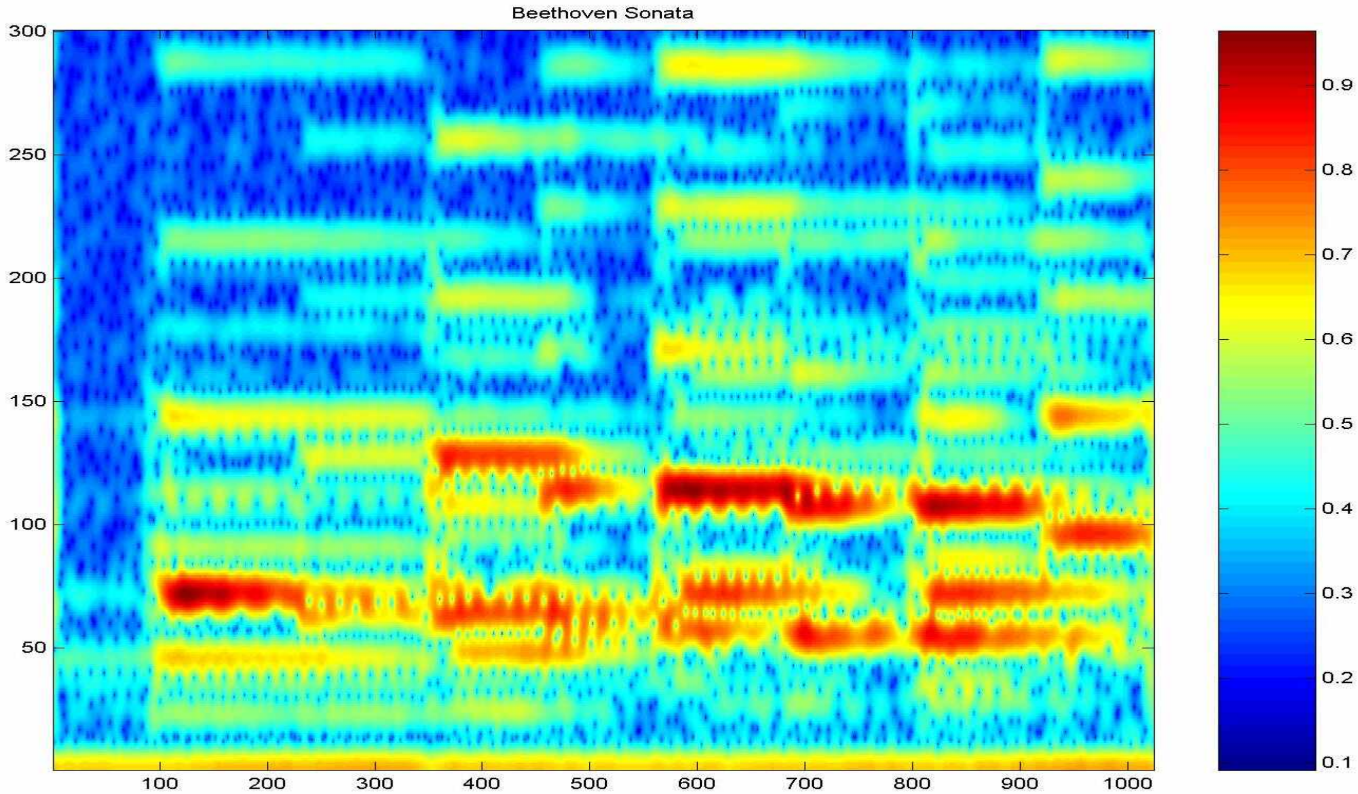
Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = V_g f(t, \omega) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \quad \lambda = (t, \omega);$$

A Typical Musical STFT



Some algebra in the background: The Heisenberg group

Weyl commutation relation

$$T_x M_\omega = e^{-2\pi i x\cdot\omega} M_\omega T_x, \quad (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

$\{M_\omega T_x : (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$ is a **projective representation** of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ on $L^2(\mathbb{R}^d)$. **Heisenberg group** $\mathbb{H} := \{\tau M_\omega T_x : \tau \in \mathbb{T}, (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$

Schrödinger representation $\{\tau M_\omega T_x : (x,\omega,\tau) \in \mathbb{H}\}$ is a square-integrable (irreducible) group representation of $\mathbb{H}$ on the Hilbert space $L^2(\mathbb{R}^d)$. Then the STFT $V_g f$ is a representation coefficient.

Moyal's formula or orthogonality relations for STFTs:
Let f_1, f_2, g_1, g_2 be in $L^2(\mathbb{R}^d)$. Then

$$\left\langle V_{g_1} f_1, V_{g_2} f_2 \right\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \langle g_2, g_1 \rangle_{L^2(\mathbb{R}^d)}.$$

Reconstruction formula
Let $g, \gamma \in L^2(\mathbb{R}^d)$ with $\langle g, \gamma \rangle \neq 0$. Then for $f \in L^2(\mathbb{R}^d)$ we have

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(x, \omega) \pi(x, \omega) \gamma \, dx d\omega.$$

So typically one chooses $\gamma = g$ with $\|g\|_2 = 1$.

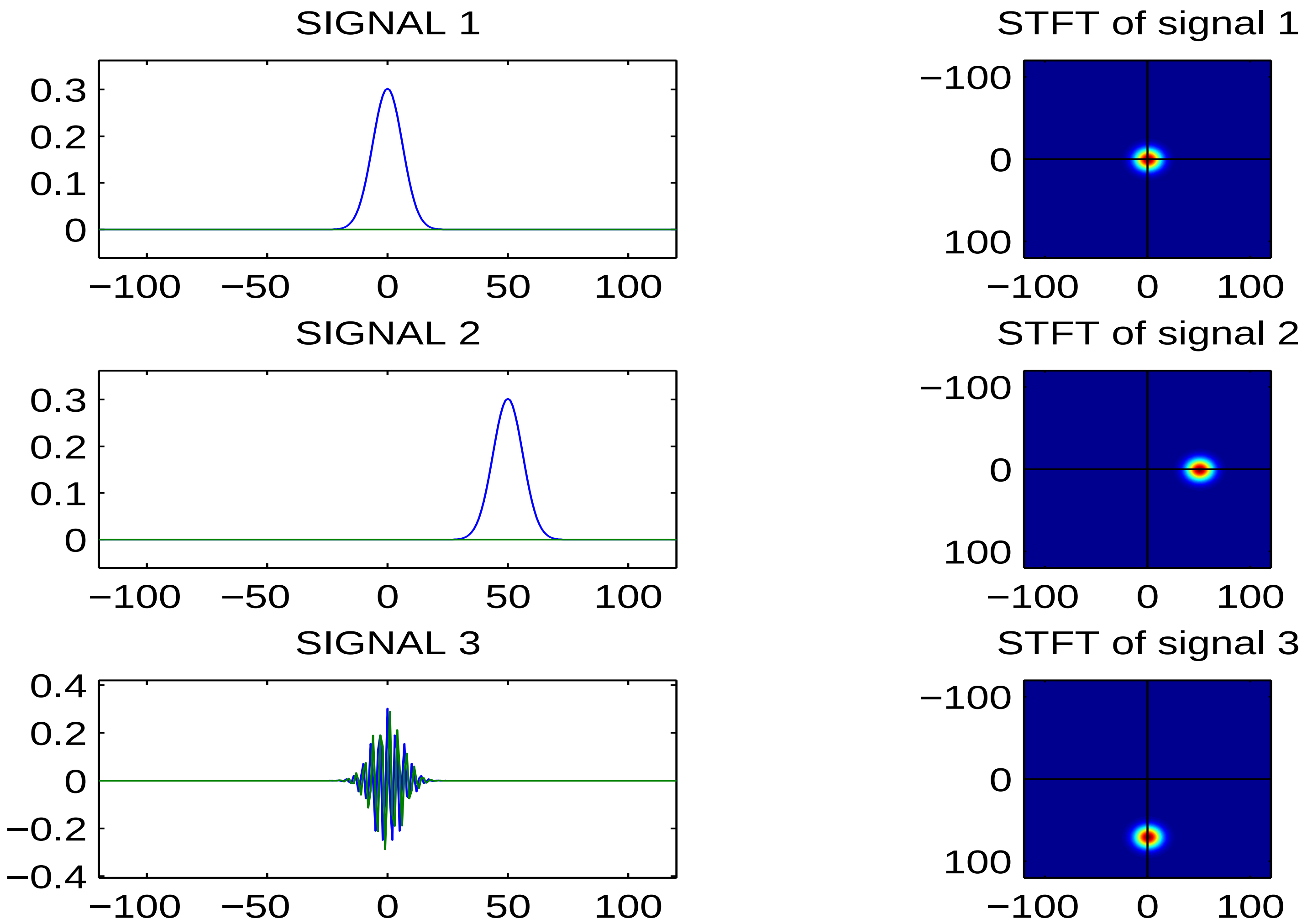
Primer on Gabor analysis: Atomic Viewpoint

D.GABOR's suggested to replace the continuous integral representation by a discrete series and still claim that one should have a representation of arbitrary elements of $\boldsymbol{L}^2(\mathbb{R})$!
Let $g \in L^2(\mathbb{R}^d)$ and Λ a lattice in time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

$$\mathbf{f} = \sum_{\lambda \in \boldsymbol{\Lambda}} \mathbf{a}(\lambda)\pi(\lambda)\mathbf{g}, \quad \text{for some} \quad \mathbf{a} = (\mathbf{a}(\lambda))_{\lambda \in \boldsymbol{\Lambda}}$$

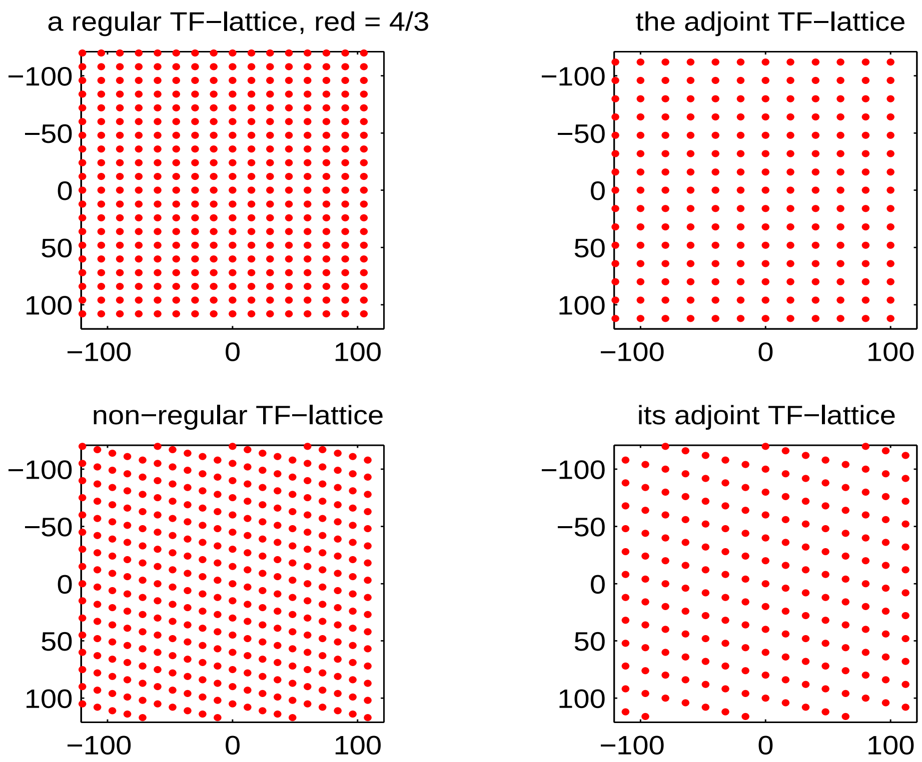
is a so-called **Gabor expansion** of $f \in L^2(\mathbb{R}^d)$ for the **Gabor atom** g.

1946 - D. Gabor: $\Lambda = \mathbb{Z}^2$ and Gabor atom $g(t) = e^{-\pi t^2}$.

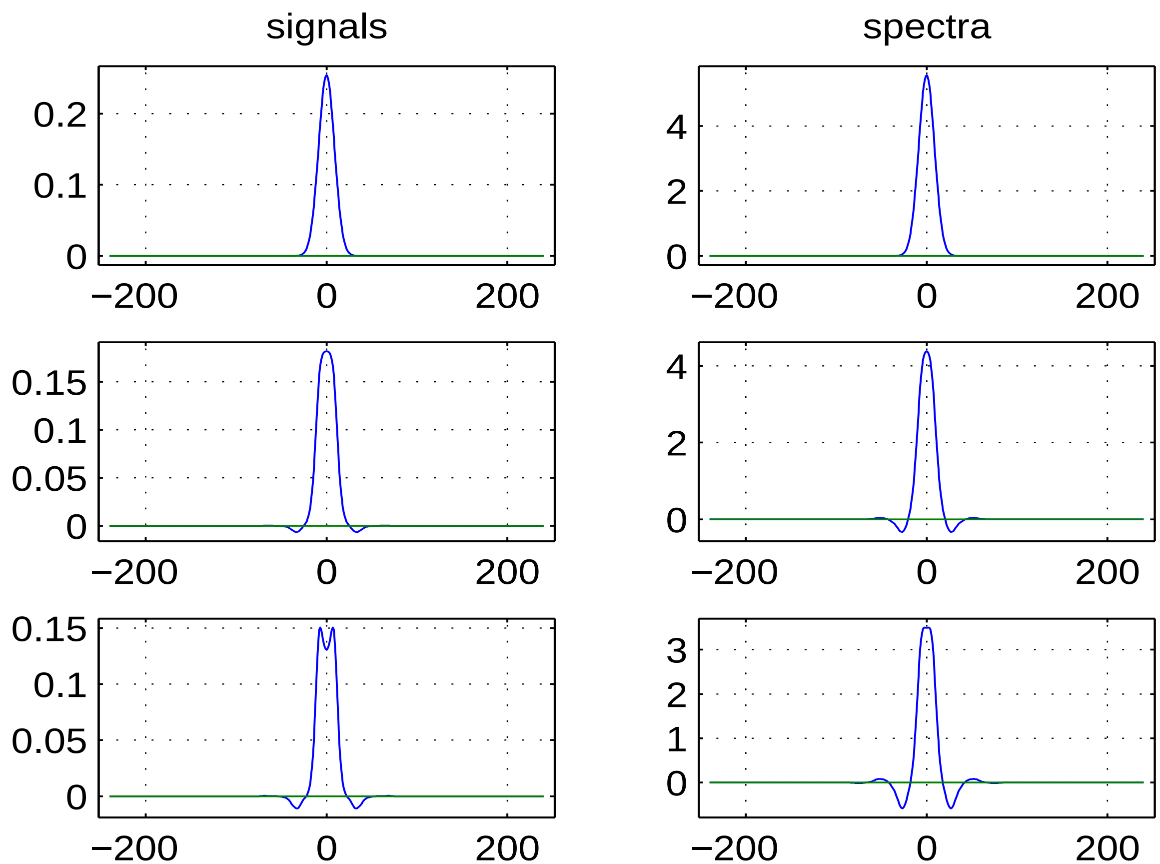

SIGNAL 1
0.3
0.2
0.1
0
−100
−50
0
50
100
STFT of signal 1
−100
0
100
−100
0
100
SIGNAL 2
0.3
0.2
0.1
0
−100
−50
0
50
100
STFT of signal 2
−100
0
100
−100
0
100
SIGNAL 3
0.4
0.2
0
−0.2
−0.4
−100
−50
0
50
100
STFT of signal 3
−100
0
100
−100
0
100

Examples of finite Gabor families

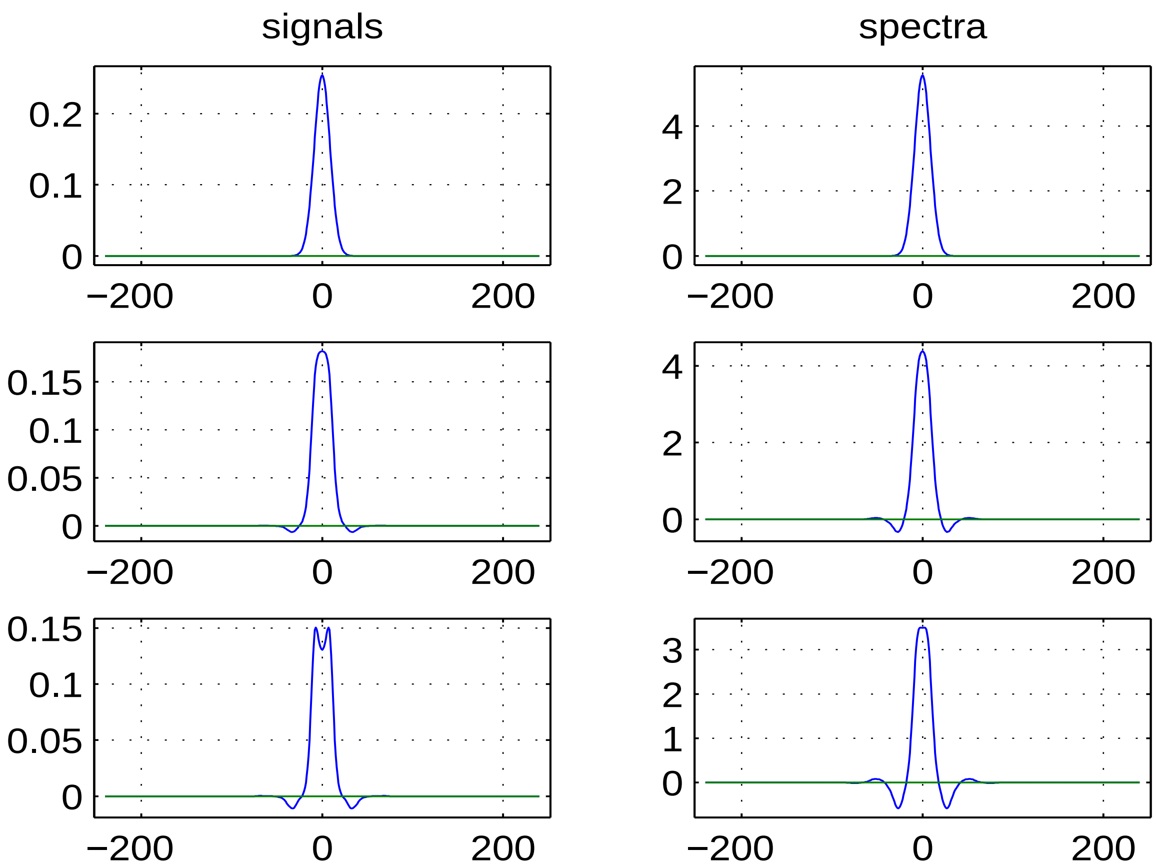
Signal length $n = 240$, lattice Λ with $320 = 4/3 * n$ [$180 = 3/4 * n$] points.



Gabor atom, with canonical tight and dual Gabor atoms



Gabor atom, with canonical tight and dual Gabor atoms



The benefit of having a dual Gabor atom (and duality is a symmetric relationship because the frame operator induced by t $\tilde{g}$ is just the inverse of the frame operator!) is that one can use one for analysis and the other for synthesis as follows:

Seen as a sampling problem, one reconstructs the signal f from the samples of $V_g(f)$ over Λ by the formula $f = S^{-1}S(f) = \sum_\lambda V_g f(\lambda)\pi(\lambda)\tilde{g}$.

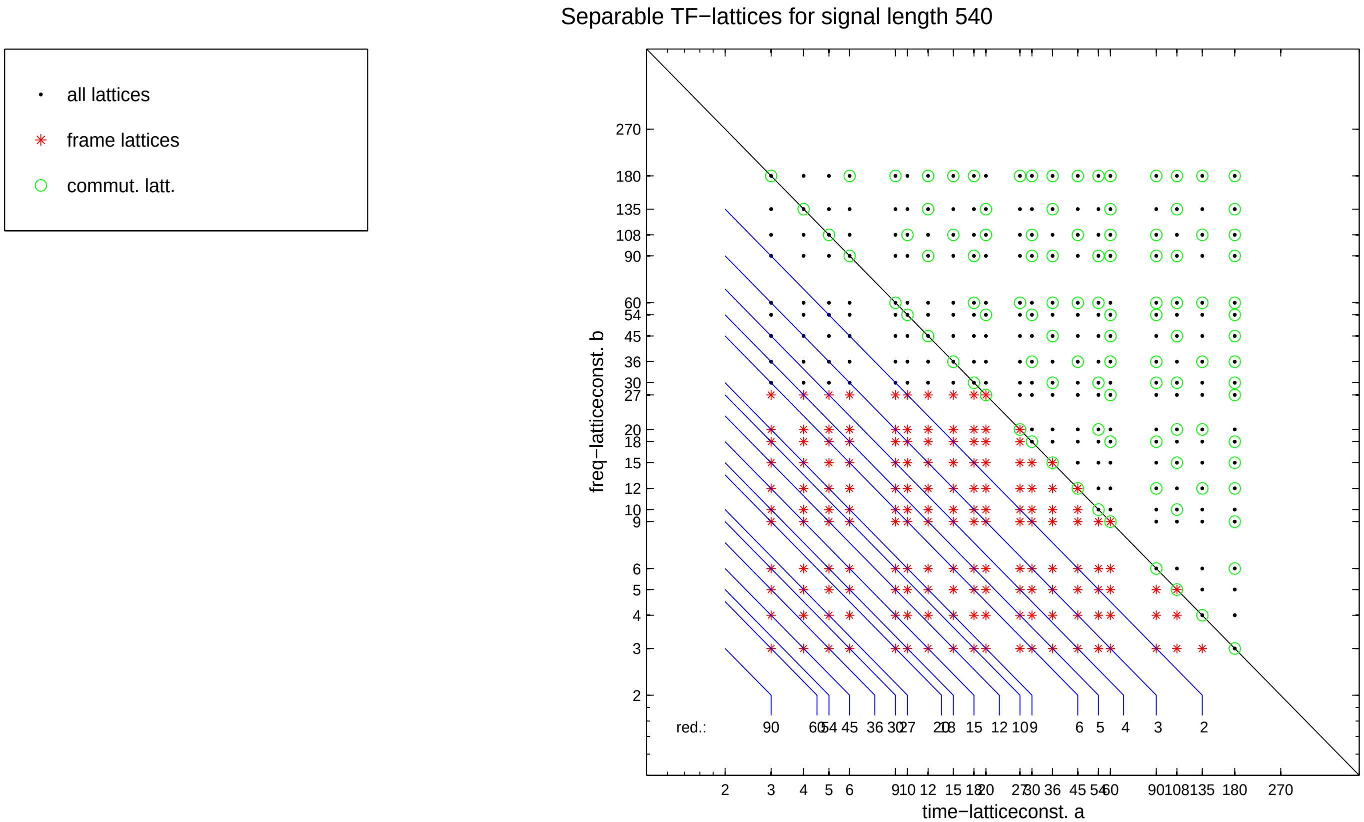
On the other hand, if one takes the atomic point of view, i.e. if one want to fulfill Gabor's wishes by providing in a most efficient ways coefficients for a given function f in order to write it as an (unconditionally convergent) Gabor sum, then one will prefer the formula $f = S^{-1}S(f) = \sum_\lambda V_{\tilde{g}} f(\lambda)\pi(\lambda)g$.

There is also a symmetric way, of modifying both the analysis and synthesis operator in order to (by choosing $h = S^{-1/2}g$)

$$f = \sum_\lambda V_h f(\lambda)\pi(\lambda)h = \sum_\lambda \langle f, h_\lambda \rangle h_\lambda.$$

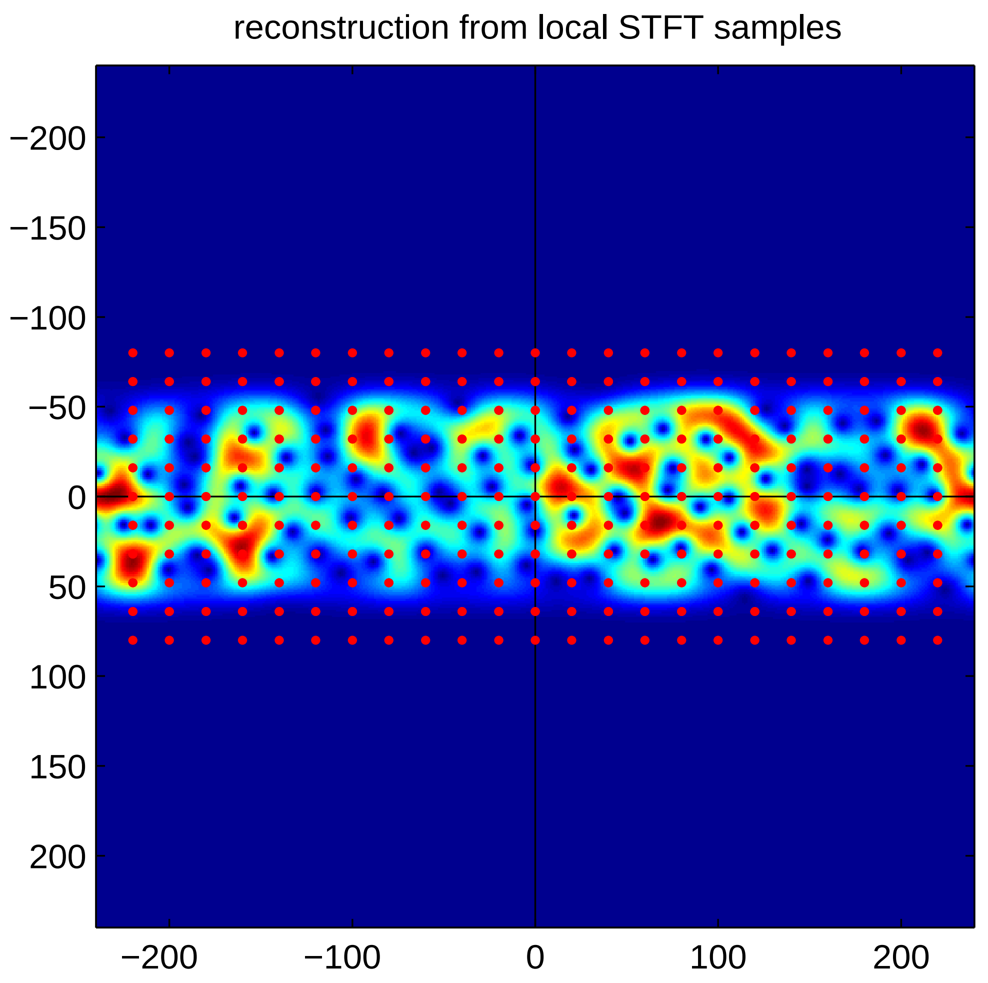
This looks very much like an orthonormal expansion (although it is not), and h is called a tight Gabor atom.

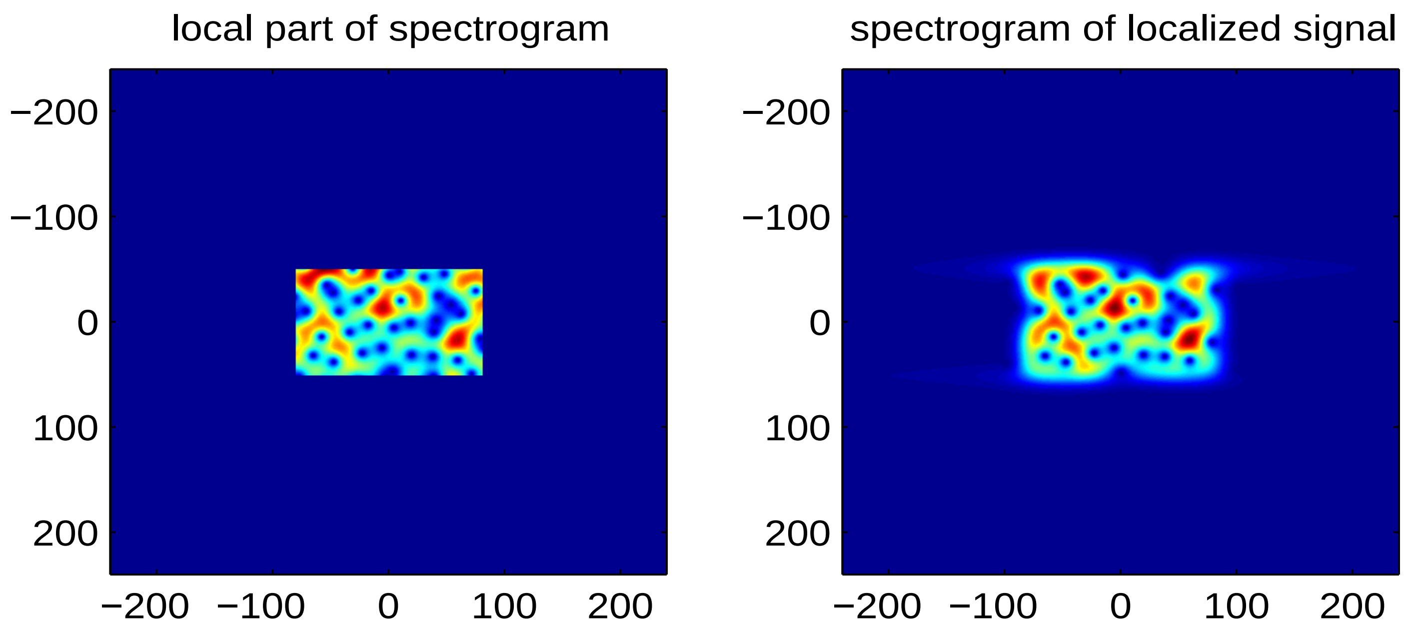
A Discrete Version: Each Point "is" a Lattice, $n = 540$

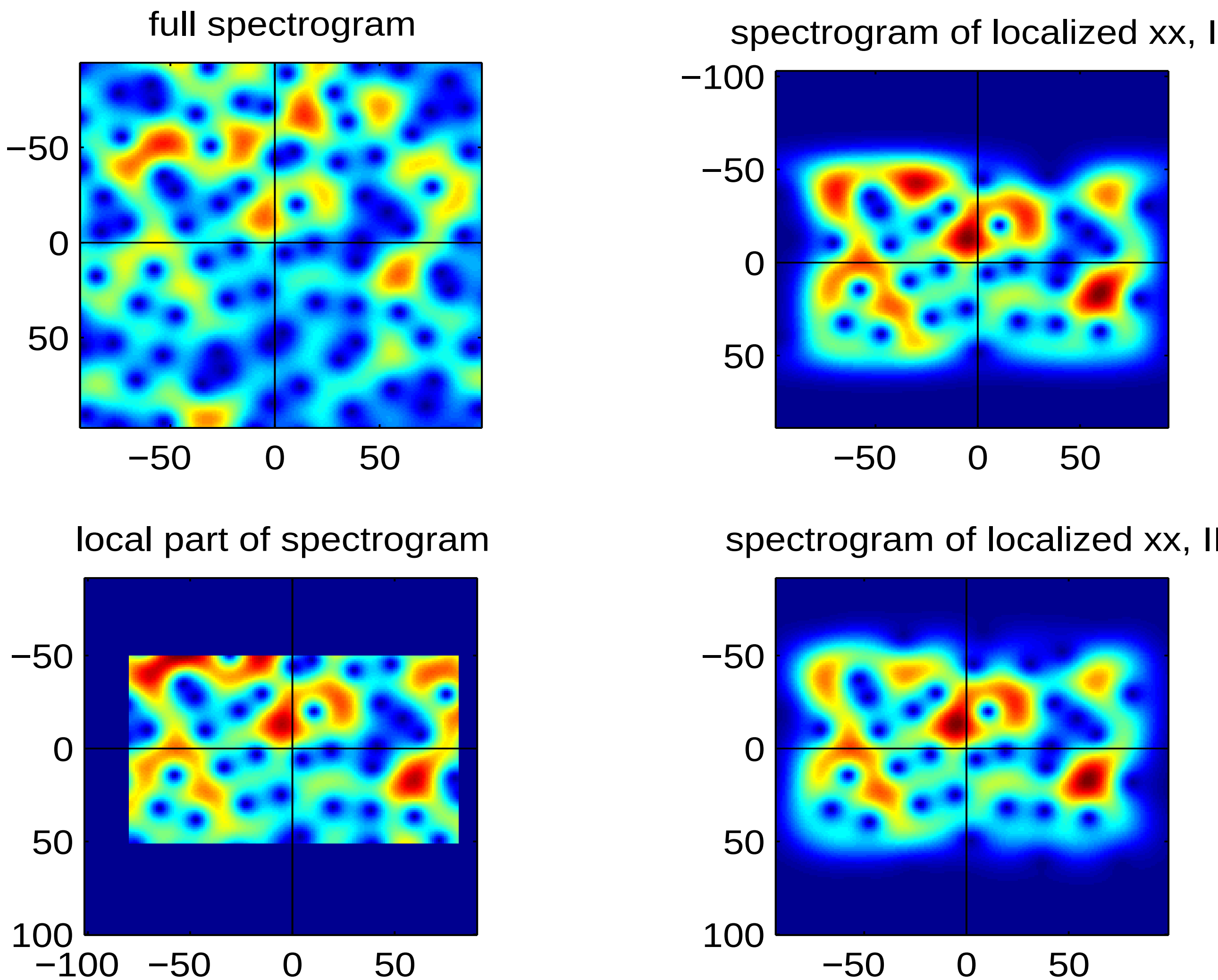


Exact recovery for elements from a subspace

When looking at the above images it is natural to assume that one can have perfect reconstruction of all the signals which are concentrated with the region of interest (looked at from a time-frequency view-point). Unfortunately no single function has its STFT concentrated (for whatever window) in a bounded domain of the time-frequency plane, because that would imply that such a function is both time- and frequency-limited
We are presently investigating (PhD thesis of Roza Acesca) a mathematical clean description for the *idea* of *functions of variable band-width*. The problem with such a concept is that it has to respect the uncertainty principle (which for me implies: one cannot talk about the exact frequency content of a function at a given time, at a precise frequency level!). Also *THERE IS NO SPACE of functions having a their spectrogram in a strip* (of variable width)!


reconstruction from local STFT samples
−200
−150
−100
−50
0
50
100
150
200
−200
−100
0
100
200

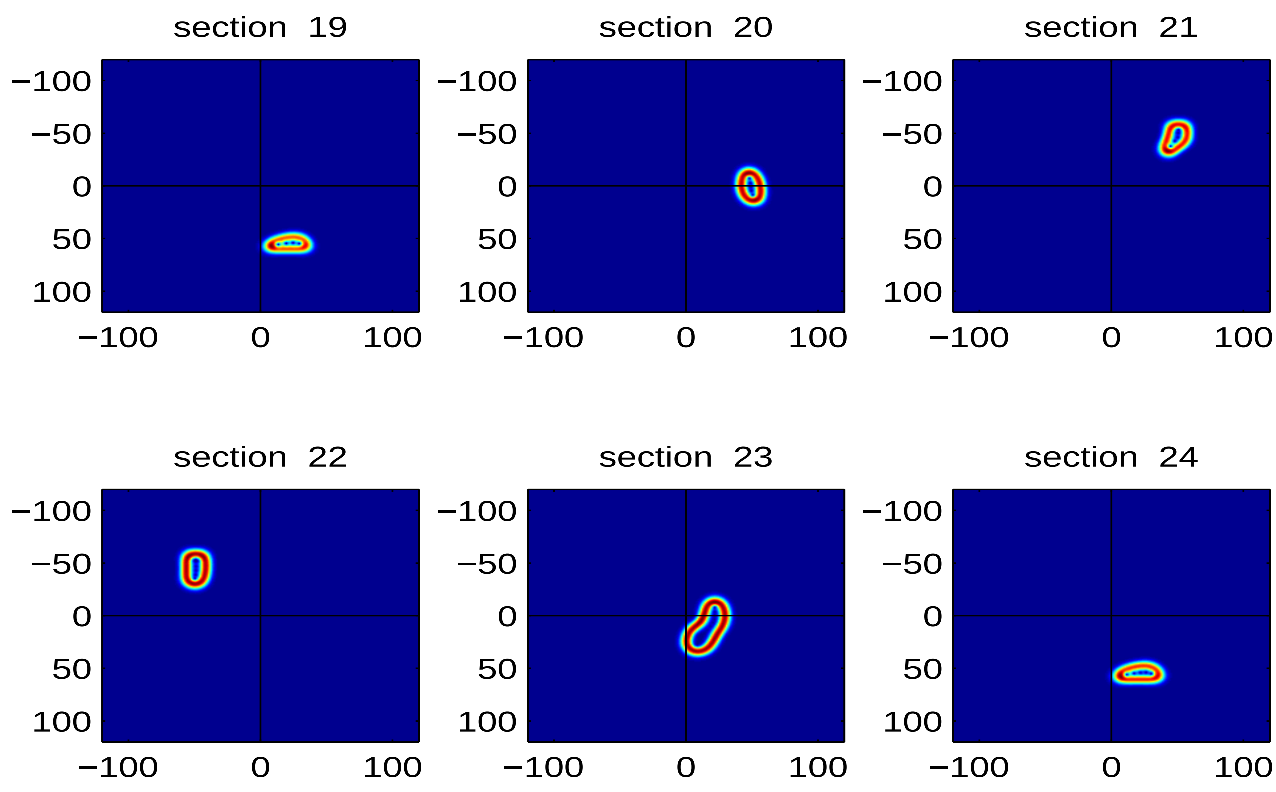

local part of spectrogram
spectrogram of localized signal
−200
−100
0
100
200
−200 −100 0 100 200

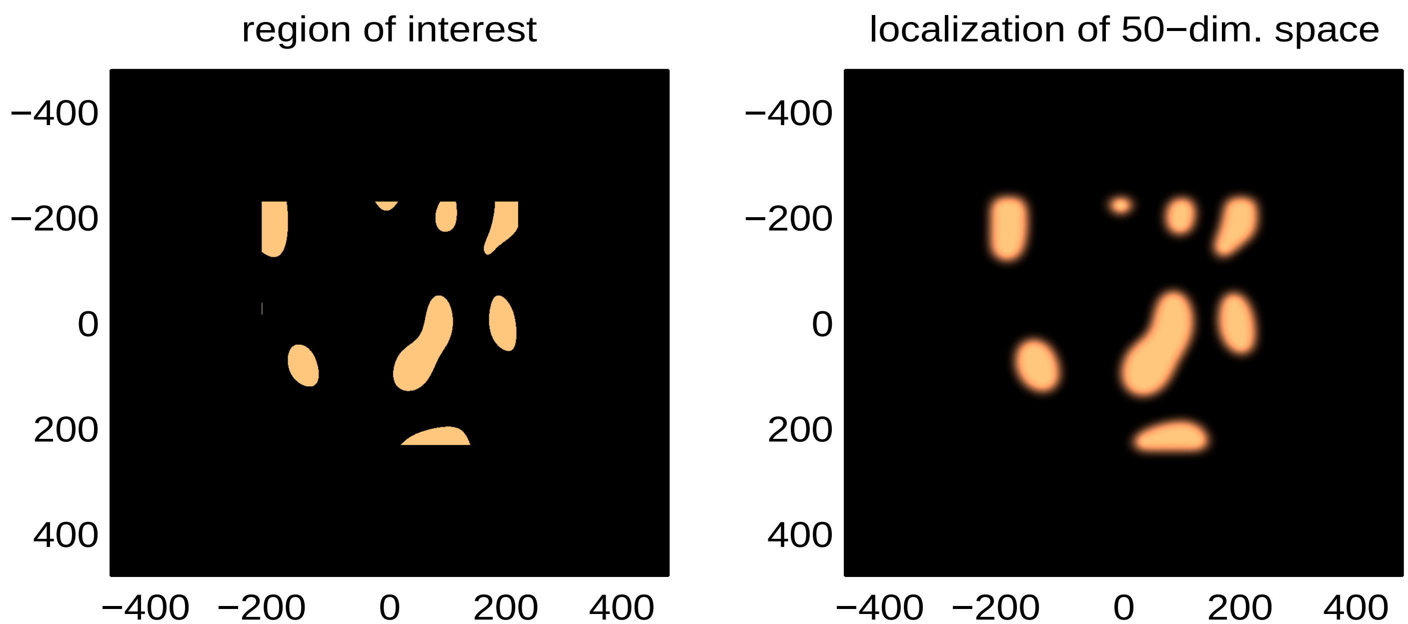

full spectrogram
−50
0
50
−50
0
50
spectrogram of localized xx, I
−100
−50
0
50
−50
0
50
local part of spectrogram
−50
0
50
100
−100
−50
0
50
spectrogram of localized xx, II
−50
0
50
100
−50
0
50

How can one localize a signal to a region of interest?

It is natural to restrict the (regular or irregular) Gabor expansion of a given signal to the region of interest and take this as a kind of projection operator. Alternatively one can take the full STFT and set it to zero outside the region of interest. The disadvantage of such a procedure (which is simple to implement!) is that fact that it does not really give us an (orthogonal) projection operator. In other words, if we apply the same operation twice are a few times there will still be further changes. In fact, the STFT-multipliers (with some $0/1$-mask) all are (mathematically) strict contractions, with a maximal eigenvalue of maybe 0.99. Although iterated application of this denoising procedure (by masking the spectrogram) appears to be useful in many cases it is of interest to find a correct **projection operator**.

Study of the localization operators: eigenvalues and eigenvectors




region of interest
localization of 50−dim. space
−400
−200
0
200
400
−400 −200 0 200 400

Best approximation of a given matrix by Gabor multiplier

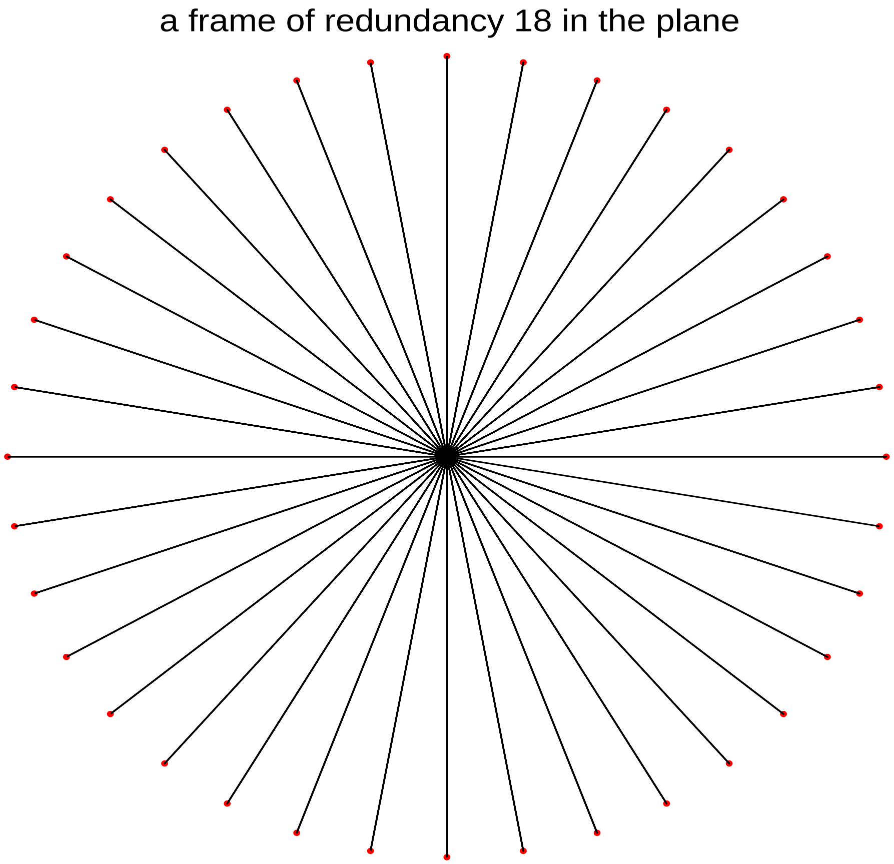
In many cases, even if the building blocks (g_λ) of a Gabor frame are (of course) a linear dependent set of atoms in our signal space, the corresponding set of projection operators (P_λ), given by $h \mapsto \langle h, g_\lambda \rangle g_\lambda$ has good chances to be a linear independent set (in the continuous case: a Riesz basis within the class of Hilbert Schmidt operators, with the scalar product $\langle A, B \rangle_{\mathcal{HS}} = trace(AB^*)$).
This means that the mapping from the sequence (m_λ) to the operator

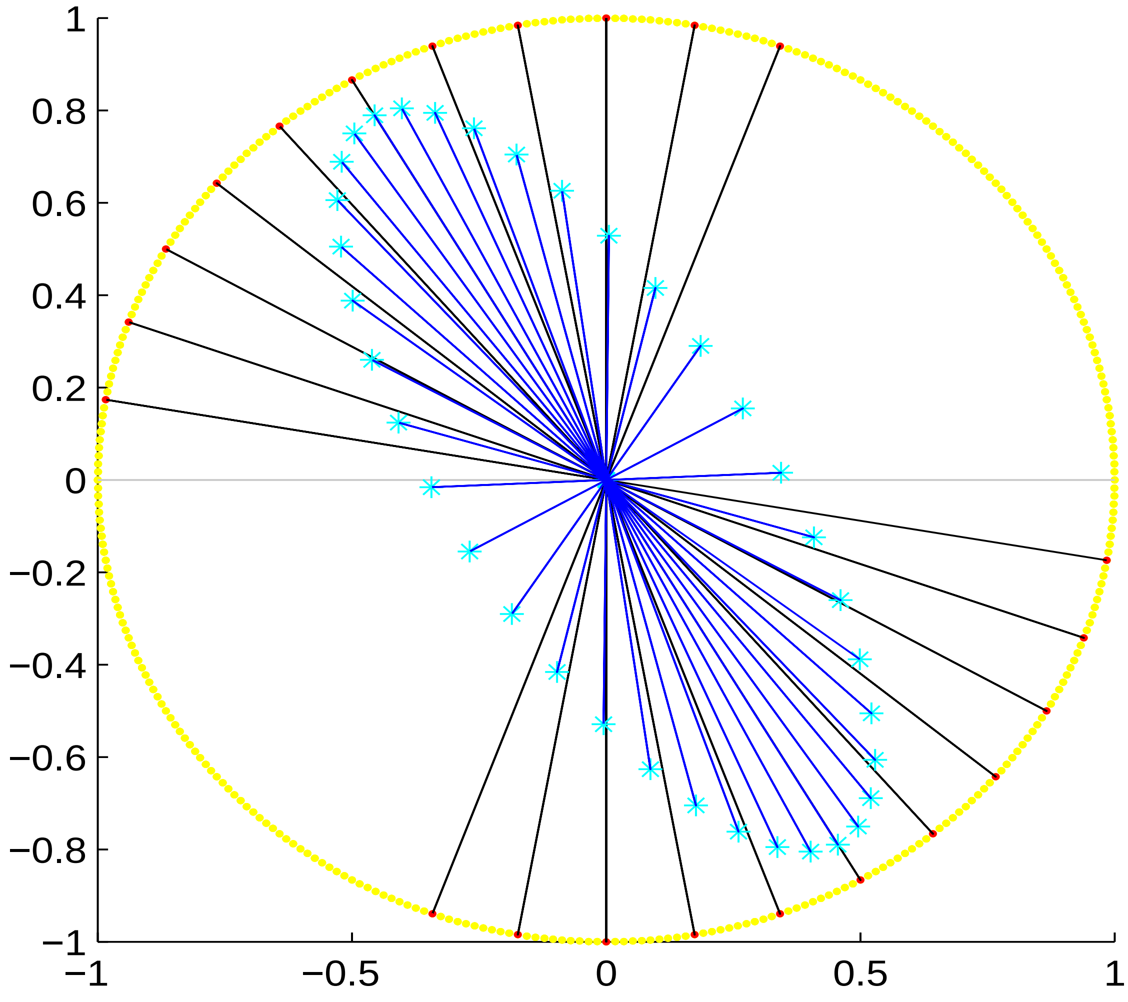
$$Th = \sum_\lambda m_\lambda \langle h, g_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda} m_\lambda P_\lambda(h)$$

is one to one, in other words, the "upper symbol" of a Gabor multiplier is uniquely determined, and the set of Gabor multipliers is closed within the space of all Hilbert Schmidt operators.

Consequently every Hilbert Schmidt operator has a best approximation (in the HS-norm) by a Gabor multiplier (with upper symbol $(m_\lambda \in \ell^2(\Lambda)$.
Although Gabor multipliers with respect to "nice atoms" g will have matrices closely concentrated near the main diagonal (both in the time and in the frequency representation!) it is not at all obvious, but still true that the operator T can be identified (and this best approximation can be determined) from the scalar products $(\langle T(g_\lambda), g_\lambda\rangle)$. This sequence is called the "lower symbol" of the operator T.
In fact, this best approximation procedure extends to a much larger class of symbols, including Gabor multipliers with just bounded symbols (m_λ) (which are not Hilbert Schmidt, but may be invertible, for example). In an ongoing project with the EE Dept. (TU Vienna, Franz Hlawatsch) we are studying the approximation of the inverse of a Gabor multiplier (which by itself is NOT! a Gabor multiplier) or more general the inverse of a slowly varying channel by a (generalized) Gabor multiplier. The idea being that the implementation of such operators should be computationally cheap.

Some idea about frames and frame multipliers




1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.5
0
0.5
1

- Typical questions of (classical and modern) Fourier analysis
- Fourier transforms, convolution, impulse response, transfer function
- The Gelfand triple $(\boldsymbol{\mathcal{S}}, \boldsymbol{L}^2, \boldsymbol{\mathcal{S}}')(\mathbb{R}^d)$, of Schwartz functions and tempered distributions; maybe *rigged Hilbert spaces*;

WHAT WE WANT TO DO TODAY:

- The Banach Gelfand Triples $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ and its use;
- various (unitary) Gelfand triple isomorphisms involving $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$

LET US START WITH SOME FORMAL DEFINITIONS:

Definition 1. *A triple* $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$, *consisting of a Banach space* $\boldsymbol{B}$, *which is dense in some Hilbert space* $\mathcal{H}$, *which in turn is contained in* $\boldsymbol{B}'$ *is called a Banach Gelfand triple.*

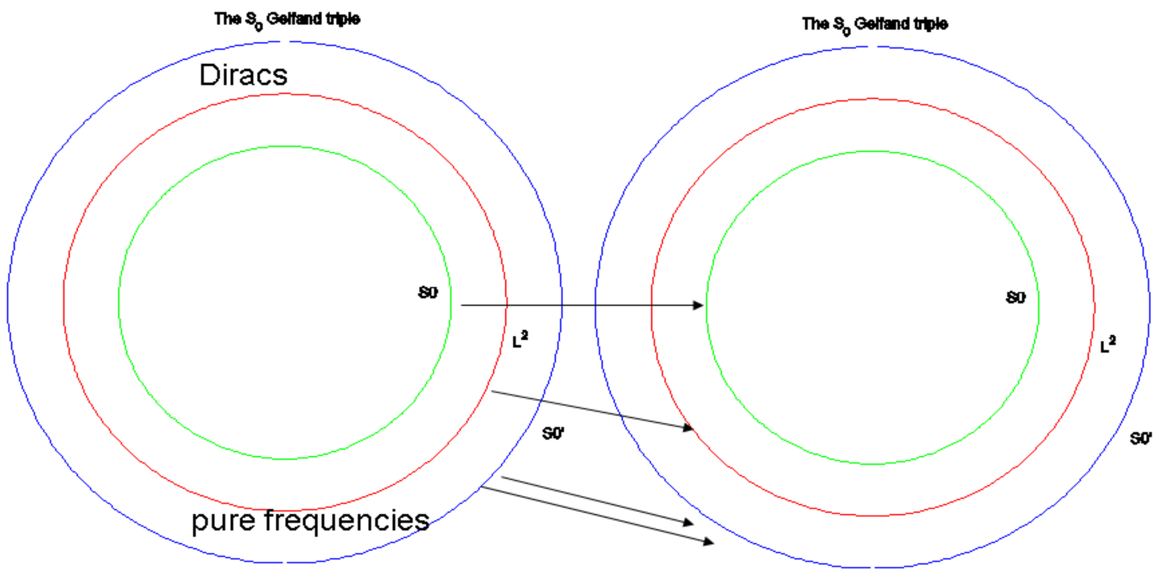
Definition 2. *If* $(\boldsymbol{B}^1, \mathcal{H}_1, \boldsymbol{B}_1')$ *and* $(\boldsymbol{B}^2, \mathcal{H}_2, \boldsymbol{B}_2')$ *are Gelfand triples then a linear operator* T *is called a [unitary] Gelfand triple isomorphism if*

1. *A is an isomorphism between* $\boldsymbol{B}^1$ *and* $\boldsymbol{B}^2$.
2. *A is a [unitary operator resp.] isomorphism between* $\mathcal{H}_1$ *and* $\mathcal{H}_2$.
3. *A extends to a weak** *isomorphism as well as a norm-to-norm continuous isomorphism between* $\boldsymbol{B}_1'$ *and* $\boldsymbol{B}_2'$.

The prototype is $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$. w^*-convergence corresponds to coordinate convergence in ℓ^∞. It can be transferred to "abstract Hilbert spaces" $\mathcal{H}$. Given any orthonormal basis (h_n) one can relate $\boldsymbol{\ell}^1$ to the set of all elements $f \in \mathcal{H}$ which have an *absolutely convergent* series expansions with respect to this basis. In fact, in the classical case of $\mathcal{H} = \boldsymbol{L}^2(\mathbb{T})$, with the usual

Fourier basis the corresponding spaces are known as Wiener's $\boldsymbol{A}(\mathbb{T})$. The dual space is then $\mathcal{P}_M$, the space of pseudo-measures $= \mathcal{F}^{-1}[\boldsymbol{\ell}^\infty(\mathbb{Z})]$.

Gelfand triple mapping



Realization of a GT-homomorphism

Very often a Gelfand-Triple homomorphism T can be *realized with the help of some kind of "summability methods"*. In the abstract setting this is a sequence (or more generally a net) A_n, having the following property:

- each of the operators maps $\boldsymbol{B}_1'$ into $\boldsymbol{B}^1$;
- they are a uniformly bounded family of Gelfand-triple homomorphism on $(\boldsymbol{B}^1, \mathcal{H}_1, \boldsymbol{B}_1')$;
- $A_n f \to f$ in $\boldsymbol{B}^1$ for any $f \in \boldsymbol{B}^1$;

It then *follows* that the limit $T(A_n f)$ exists in $\mathcal{H}_2$ respectively in $\boldsymbol{B}_2'$ (in the w^*-sense) for $f \in \mathcal{H}_1$ resp. $f \in \boldsymbol{B}_1'$ and thus describes concretely the prolongation to the full Gelfand triple. This continuation is unique due to the w^*-properties assumed for T (and the w^*-density of $\boldsymbol{B}^1$ in $\boldsymbol{B}_1'$).

Typical Philosophy

One may think of $\boldsymbol{B}^1$ as a (Banach) space of test functions, consisting of "decent functions" (continuous and integrable), hence $\boldsymbol{B}^{1'}$ is a space of "generalized functions, containing at least all the $\boldsymbol{L}^p$-spaces as well as all the bounded measures, hence in particular finite discrete measures (linear combinations of Dirac measures).

At the INNER = test function level every "transformation" can be carried out very much as if one was in the situation of a finite Abelian group, where sums are convergent, integration order can be interchanged, etc.. At the INTERMEDIATE level of the Hilbert space one has very often a unitary mapping, while only the OUTER LAYER allows to really describe what is going on in the *ideal limit case*, because instead of unit vectors for the finite case one has to deal with Dirac measures, which are only found in the big dual spaces (but not in the Hilbert space!).

Introducing $S_0(\mathbb{R}^d) = M^1(\mathbb{R}^d) := M^0_{1,1}(\mathbb{R}^d)$ $(Fei, 1979)$

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is (by definition) in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $(S_0(\mathbb{R}^d), \| \cdot \|_{S_0})$ is a Banach space, for any fixed, non-zero $g \in S_0(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable windows. Often the Gaussian is used as a window. Note that

$$V_g f(x, \omega) = (\widehat{f \cdot T_x g})(\omega), \quad \text{i.e., } g \text{ localizes } f \text{ near } x.$$

Lemma 1. *Let* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$*, then the following holds:*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$*, and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$*, and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

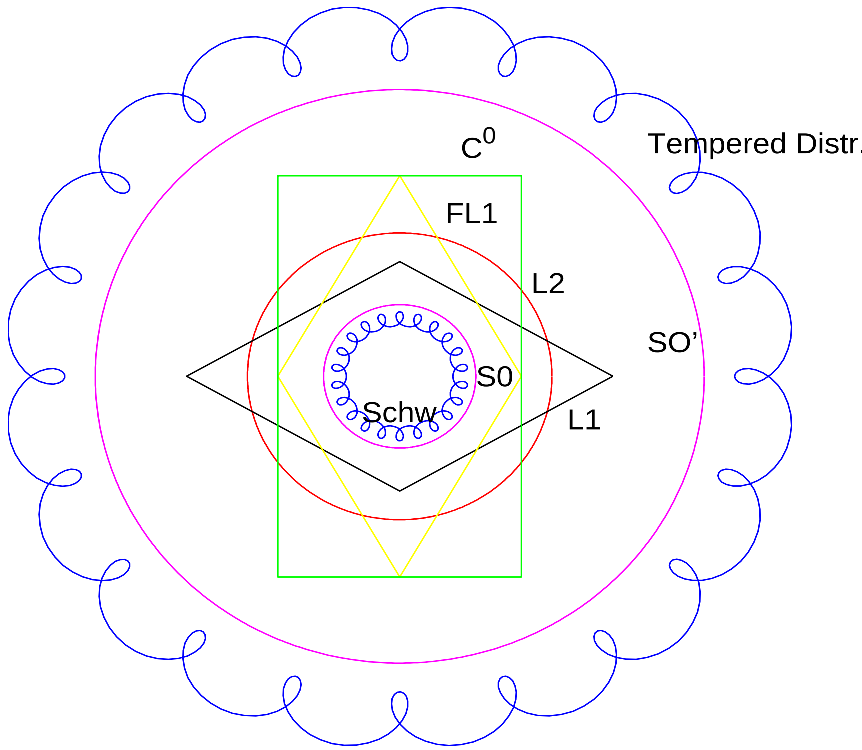
Remark 2. *Moreover one can show that* $\boldsymbol{S}_0(\mathbb{R}^d)$ *is the smallest non-trivial Banach spaces with this property, i.e., it is continuously embedded into any such Banach space. As a formal argument one can use the continuous inversion formula for the STFT:*

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(\lambda)\pi(\lambda) g d\lambda$$

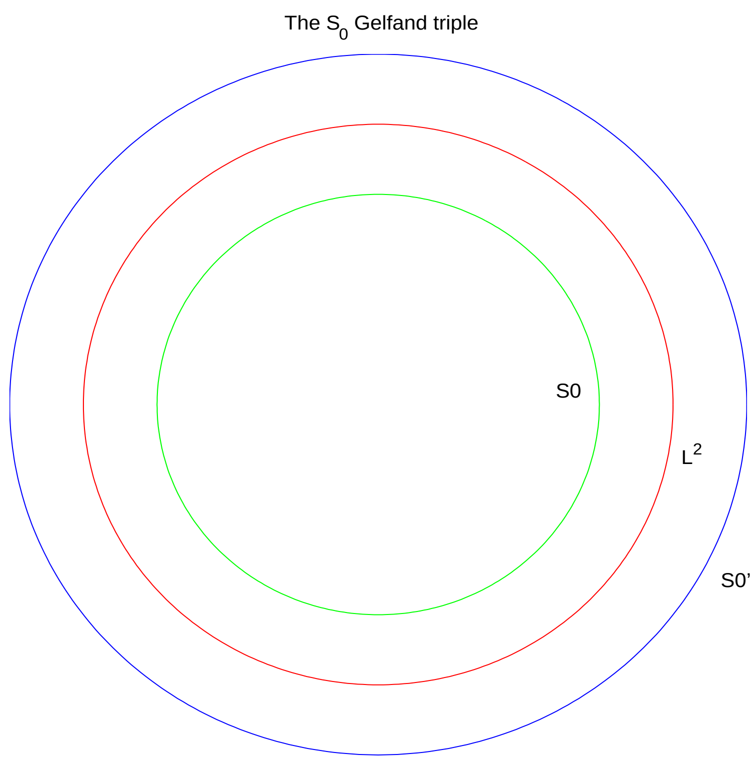
which implies

$$\|f\|_{\boldsymbol{B}} \leq \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(\lambda)| \|\pi(\lambda)g\|_{\boldsymbol{B}} \, d\lambda = \|g\|_{\boldsymbol{B}} \|f\|_{\boldsymbol{S}_0}.$$

Schwartz space, S_0, L^2, S_0', tempered distributions



The Gelfand Triple (S_0, L^2, S_0')



The Fourier transform is a prototype of a Gelfand triple isomorphism.

EX1: The Fourier transform as Gelfand Triple Automorphism

Theorem 1. *Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:*

(1) *$\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}^d})$,*

(2) *$\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}^d})$,*

(3) *$\mathcal{F}$ is a weak*-weak* (and norm-to-norm) continuous isomorphism between $\boldsymbol{S}_0'(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\widehat{\mathbb{R}^d})$.*

Furthermore we have that Parseval's formula

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \tag{1}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, or $(f, g) \in \boldsymbol{L}^2(\mathbb{R}^d) \times \boldsymbol{L}^2(\mathbb{R}^d)$ or other pairings from the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

The properties of Fourier transform can be expressed by a Gelfand bracket

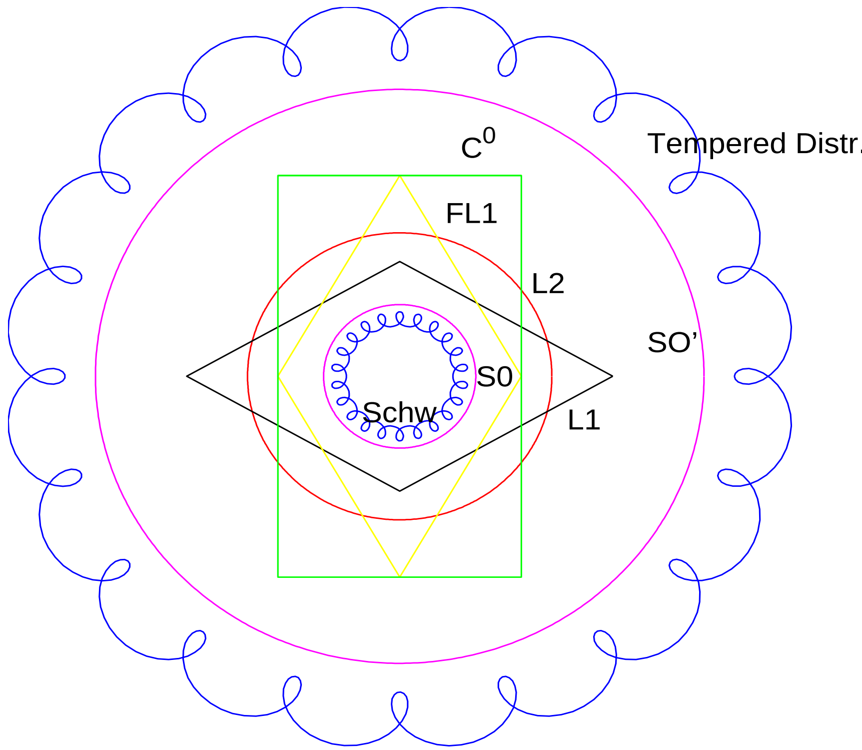
$$\langle f, g \rangle_{(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')} = \langle \hat{f}, \hat{g} \rangle_{(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')} \quad (2)$$

which combines the functional brackets of dual pairs of Banach spaces and of the inner-product for the Hilbert space.

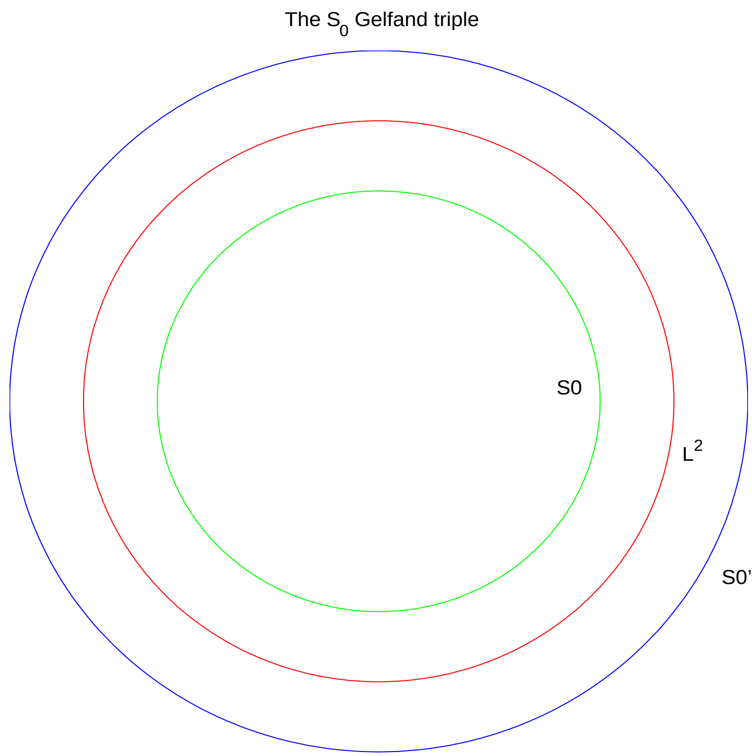
One can characterize the Fourier transform as the $uniquely$ $determined$ unitary Gelfand triple automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ which maps pure frequencies into the corresponding Dirac measures (and vice versa). [1]
One could equally require that TF-shifted Gaussians are mapped into FT-shifted Gaussians, relying on $\mathcal{F}(M_\omega T_x f) = T_{-\omega} M_x (\mathcal{F} f)$ and the fact that $\mathcal{F} g_0 = g_0$, with $g_0(t) = e^{-\pi |t|^2}$.

[1] as one would expect in the case of a finite Abelian group.

Schwartz space, S_0, L^2, S_0', tempered distributions



The Gelfand Triple (S_0, L^2, S_0')



Fourier transform is a prototype of a unitary Gelfand triple isomorphism.

Examples of Gelfand Triple Isomorphisms

1. The standard Gelfand triple $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$.

2. The family of orthonormal Wilson bases (obtained from Gabor families by suitable pairwise linear-combinations of terms with the same absolute frequency) extends the natural unitary identification of $\boldsymbol{L}^2(\mathbb{R}^d)$ with $\ell^1(I)$ to a unitary Banach Gelfand Triple isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ and $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(I)$.

   This isomorphism leeds to the observation that essentially the identification of $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ boils down to the identification of the bounded linear mappings from $\ell^1(I)$ to $\ell^\infty(I)$, which are of course easily recognized as $\ell^\infty(I \times I)$ (the bounded matrices). The fact that tensor products of $1D$-Wilson bases gives a characterization of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ over $\mathbb{R}^{2d}$ then gives the kernel theorem.

Automatic Gelfand-triple invertibility

Gröchenig and Leinert have shown (J. Amer. Math. Soc., 2004):
Theorem 2. *Assume that for* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *the Gabor frame operator*

$$S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g$$

is invertible as an operator on $\boldsymbol{L}^2(\mathbb{R}^d)$*, then it is also invertible on* $\boldsymbol{S}_0(\mathbb{R}^d)$ *and in fact on* $\boldsymbol{S}_0'(\mathbb{R}^d)$*.*
In other words: Invertibility at the level of the Hilbert space automatically !! implies that S *is (resp. extends to) an isomorphism of the Gelfand triple automorphism for* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$*.*
In a recent paper K. Gröchenig shows among others, that invertibility of S follows already from a dense range of $S(\boldsymbol{S}_0(\mathbb{R}^d))$ in $\boldsymbol{S}_0(\mathbb{R}^d)$.

Robustness resulting from those three layers:

In the present situation one has also (in contrast to the "pure Hilbert space case") various robustness effects:

1) One has robustness against jitter error. Depending (only) on Λ and $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ one can find some $\delta_0 > 0$ such that the frame property is preserved (with uniform bounds on the new families) if any point $\lambda \in \Lambda$ is not moved more than by a distance of δ_0.

2) One even can replace the lattice generated by some non-invertible matrix $\mathbf{A}$ (applied to $\mathbb{Z}^{2d}$) by some "sufficiently similar matrix $\mathcal{B}$ and also preserve the Gabor frame property (with continuous dependence of the dual Gabor atom $\tilde{g}$ on the matrix $\mathbf{B}$) (joint work with N. Kaiblinger, Trans. Amer. Math. Soc.).

Stability of Gabor Frames with respect to Dilation (F/Kaibl.)

For a subspace $X \subseteq \boldsymbol{L}^2(\mathbb{R}^d)$ define the set

$$F_g = \big\{(g, L) \in X \times \mathrm{GL}(\mathbb{R}^{2d}) \text{ which generate a Gabor frame } \{\pi(Lk)g\}_{k \in \mathbb{Z}^{2d}}\big\}. \tag{3}$$

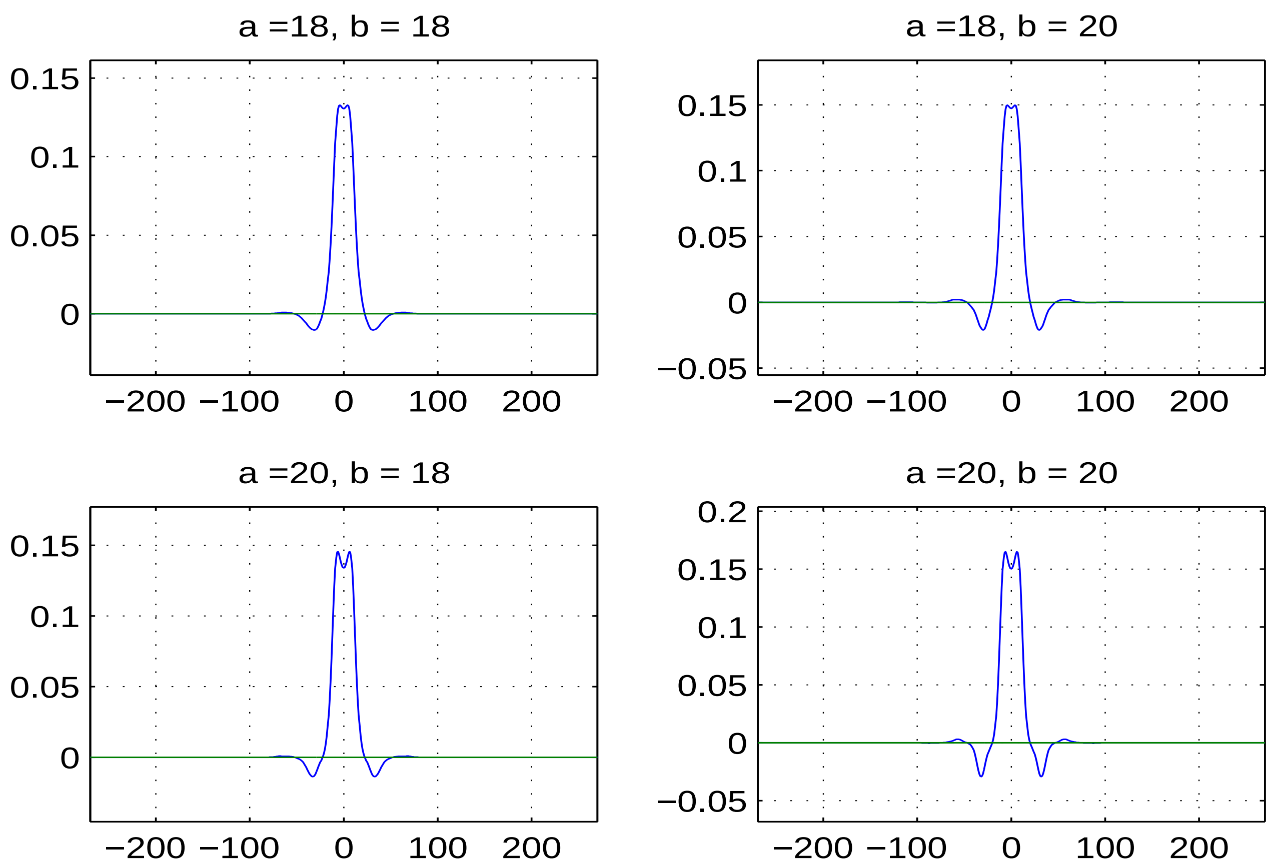
The set $F_{\boldsymbol{L}^2}$ need not be open (even for good ONBs!). But we have:

Theorem 3. (i) *The set* $F_{\boldsymbol{S}_0(\mathbb{R}^d)}$ *is open in* $\boldsymbol{S}_0(\mathbb{R}^d) \times \mathrm{GL}(\mathbb{R}^{2d})$.
(ii) $(g, L) \mapsto \widetilde{g}$ *is continuous mapping from* $F_{\boldsymbol{S}_0(\mathbb{R}^d)}$ *into* $\boldsymbol{S}_0(\mathbb{R}^d)$.

There is an analogous result for the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

Corollary 3. (i) *The set* $F_{\mathcal{S}}$ *is open in* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \times \mathrm{GL}(\mathbb{R}^{2d})$.
(ii) *The mapping* $(g, L) \mapsto \widetilde{g}$ *is continuous from* $F_{\mathcal{S}}$ *into* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

On the continuous dependence of dual atoms on the TF-lattice



THE END!

THANK you for your attention! HGFei

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