Hans G. Feichtinger
Leverhulm Visiting Professor to Edinburgh
NuHAG - University of Vienna - EUCETIFA

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GABOR ANALYSIS:

Sampling Viewpoint versus Atomic Compositions

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The classical Shannon-Whittacker Sampling Theorem tells us that any band-limited function in $\boldsymbol{L}^2(\mathbb{R}^d)$, i.e. every f such that $\mathrm{supp}(f)$ is compact, can be exactly reconstructed from any sufficiently fine regular sampling. In fact, one can show that at a perhaps somewhat higher sampling rate iterative reconstruction of band-limited functions from irregular samples is possible. Moreover, if one has slight $oversampling$ then it is possible to guarantee convergence in functions spaces other than just $\boldsymbol{L}^2(\mathbb{R}^d)$, for example $\boldsymbol{L}^p_w(\mathbb{R}^d)$.
It is not so much that strict band-limitedness, but rather a certain form of global rigidity which makes these algorithms work. In fact, similar properties are shared by other function spaces, e.g. spline (-type) space. Given a sufficiently dense sampling one can reconstruct cubic spline functions from their regular or irregular samples.
The PURPOSE OF THIS TALK is to describe a similar situation for the so-called Short-Time Fourier Transform, which associates to each tempered distribution some continuous and bounded function over the so-called time-frequency plane. These functions, although not band-limited in the strict sense, share many properties with the afore-mentioned spaces. Rather precise results are known for the STFT of functions with respect to a Gaussian window and regular sampling.

Regular Sampling of band-limited functions

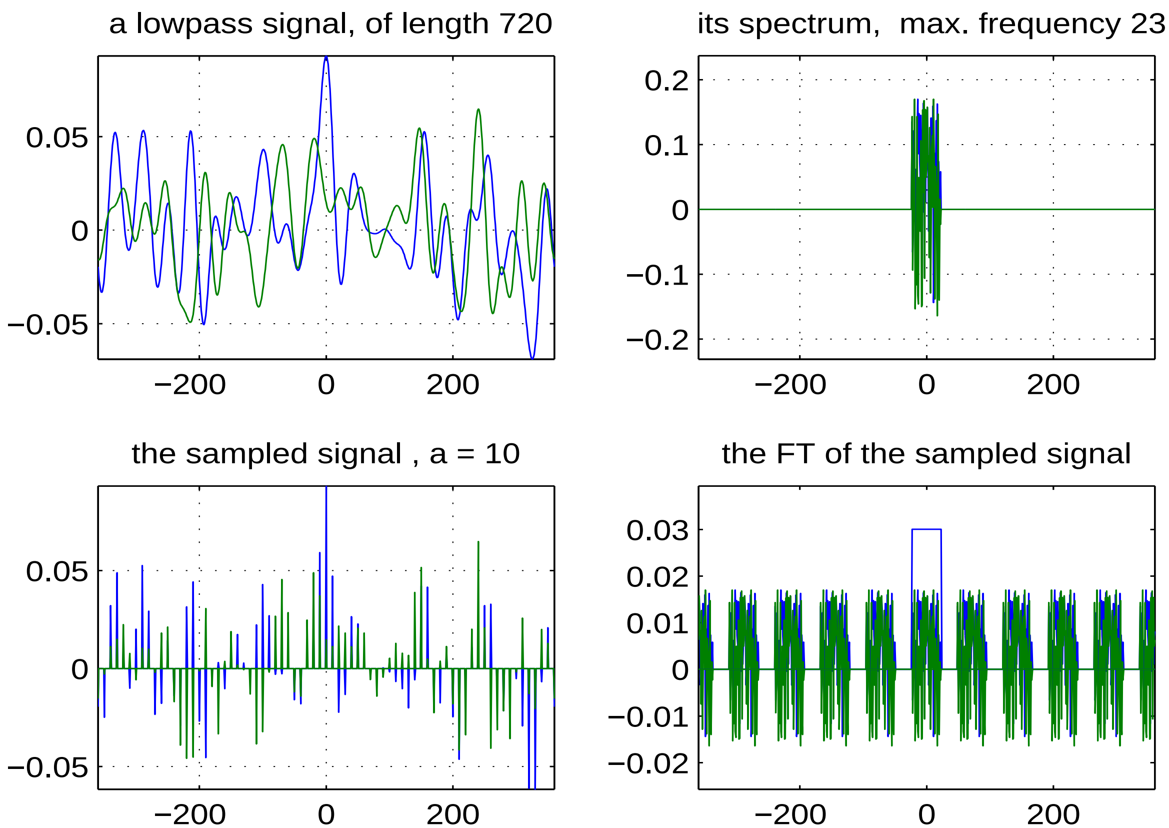
The classical Shannon sampling theorem tells us, that a band-limited function in $\boldsymbol{L}^2(\mathbb{R}^d)$, with spectrum Ω (i.e. with $supp(\hat{f}) \subseteq \Omega$) can be completely reconstructed from a set of regular samples, i.e. from any sequence $(f(\lambda))_{\lambda\in\Lambda}$ or better, one should say from

$$f \cdot ⧢_\Lambda = \sum_{\lambda\in\Lambda} f(\lambda)\delta_\lambda.$$

In fact, one has according to the famous "SHANNON Reconstruction Formula"

$$f(t) = \sum_{\lambda\in\Lambda} f(\lambda) SINC(t-\lambda),$$

with a representation which is pointwise absolutely and uniformly convergent, but also unconditionally norm convergent in the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$.


a lowpass signal, of length 720
its spectrum, max. frequency 23
the sampled signal , a = 10
the FT of the sampled signal
0.05
0
−0.05
−200
0
200
0.2
0.1
0
−0.1
−0.2
0.03
0.02
0.01
−0.01
−0.02

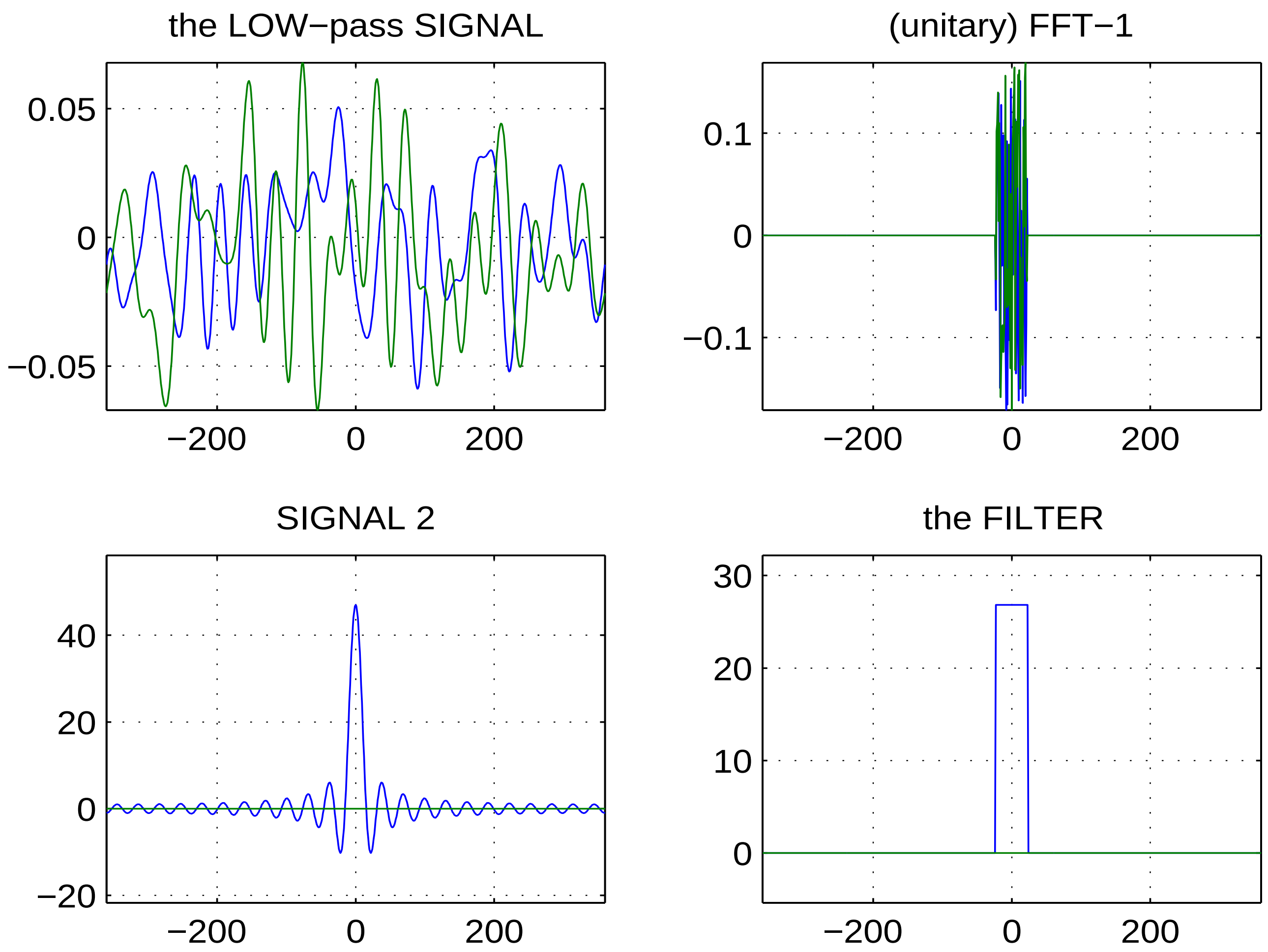
Here SINC is supposed to be the inverse Fourier transform of the indicator function of Ω. The only requirement is that Λ should be sufficiently "fine" or equivalently, that $\Lambda^\perp$ should be sufficiently course, as to guarantee that the translates of Ω along $\Lambda^\perp$ are pairwise disjoint!

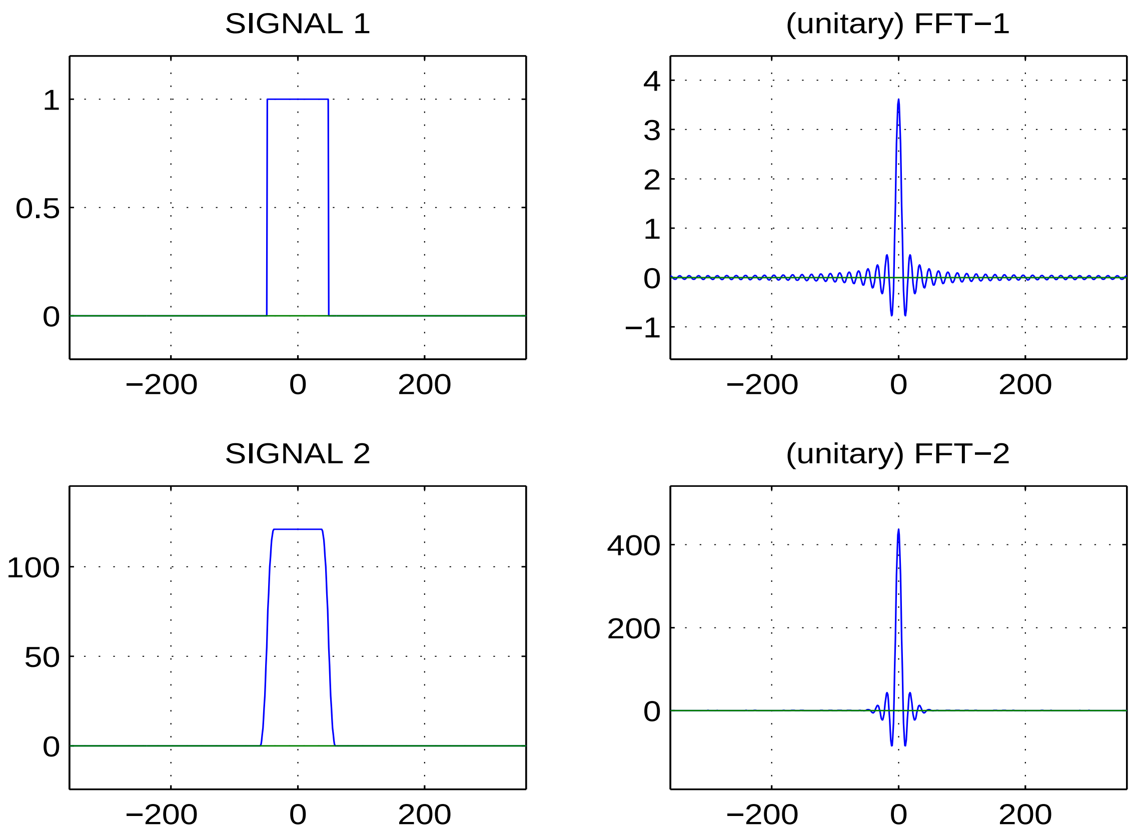
The alternative point of view is the observation that the set of translates of $SINC$ (typically $sin(\pi t)/(\pi t)$) together with its Λ- translates, i.e. the system $(T_\lambda SINC)$ form an orthonormal system of functions generating the whole space of band-limited $\boldsymbol{L}^2$-functions on $\mathbb{R}^d$.

This raises the question, whether similar reconstructions can be done in other spaces of band-limited functions such as

$$\boldsymbol{B}^{\Omega}{}_p := \{ f \in \boldsymbol{L}^p(G) \mid \operatorname{supp}(\hat{f}) \subseteq \Omega \}$$

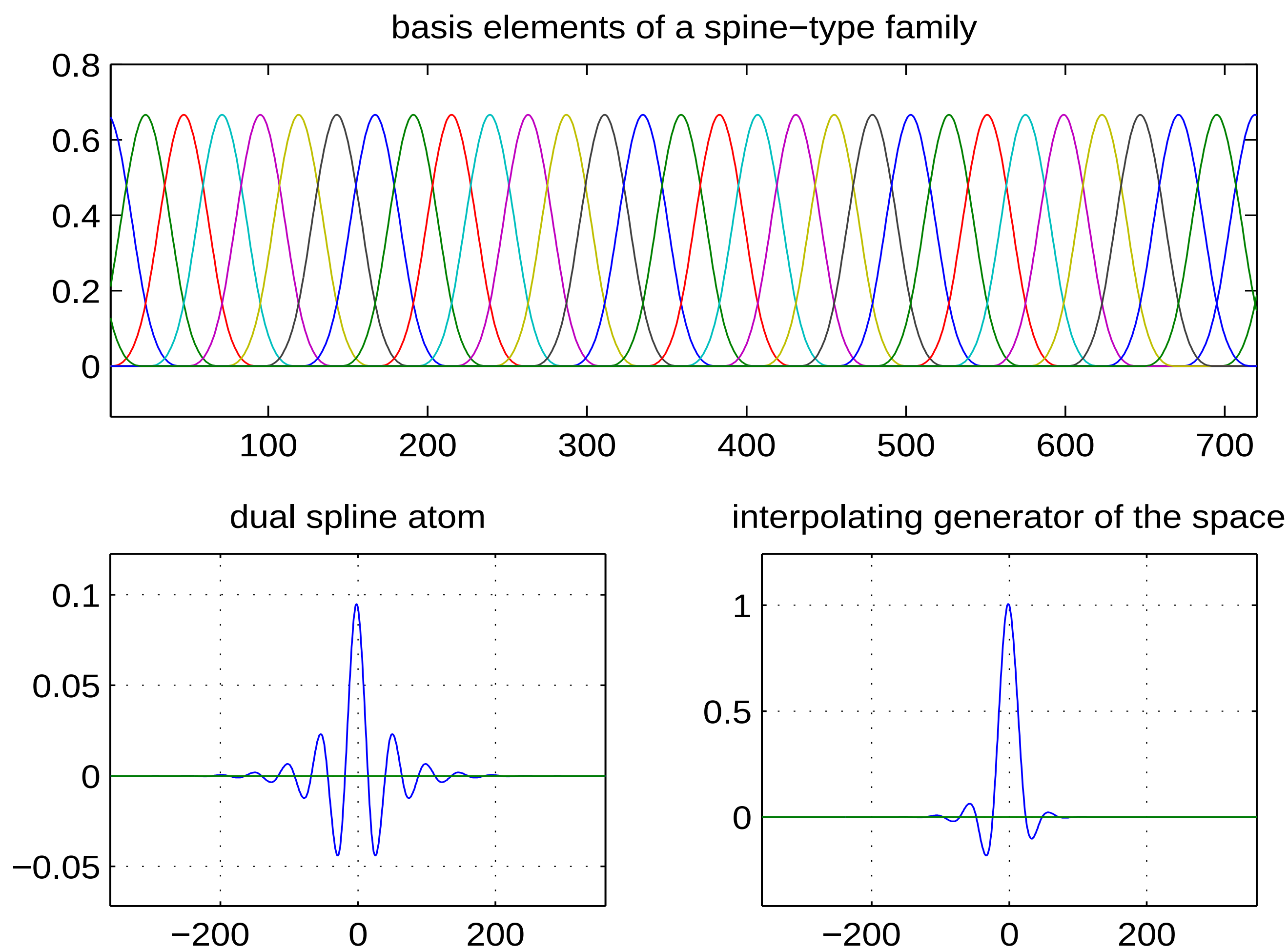
can have a similar representation. Due to the natural inclusion relations $\boldsymbol{B}^{\Omega}{}_p \subseteq \boldsymbol{B}^{\Omega}{}_r$ for $p \leq r$ the question is: Will it converge for $p \neq 2$ in the $\boldsymbol{L}^p$-norm? (and will this question make sense).

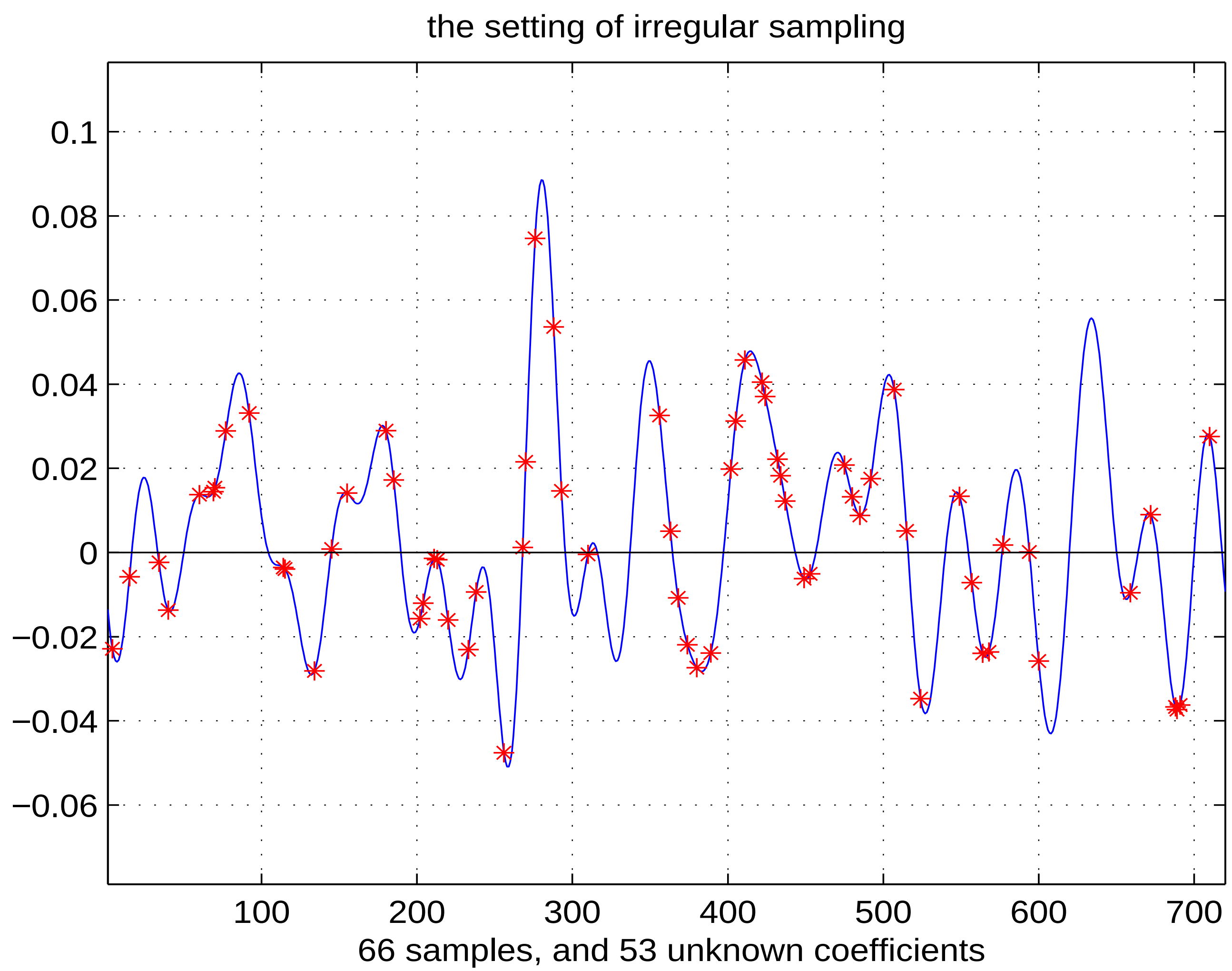

the LOW−pass SIGNAL
(unitary) FFT−1
SIGNAL 2
the FILTER
0.05
0
−0.05
0.1
−0.1
40
20
−20
30
10
−200
200

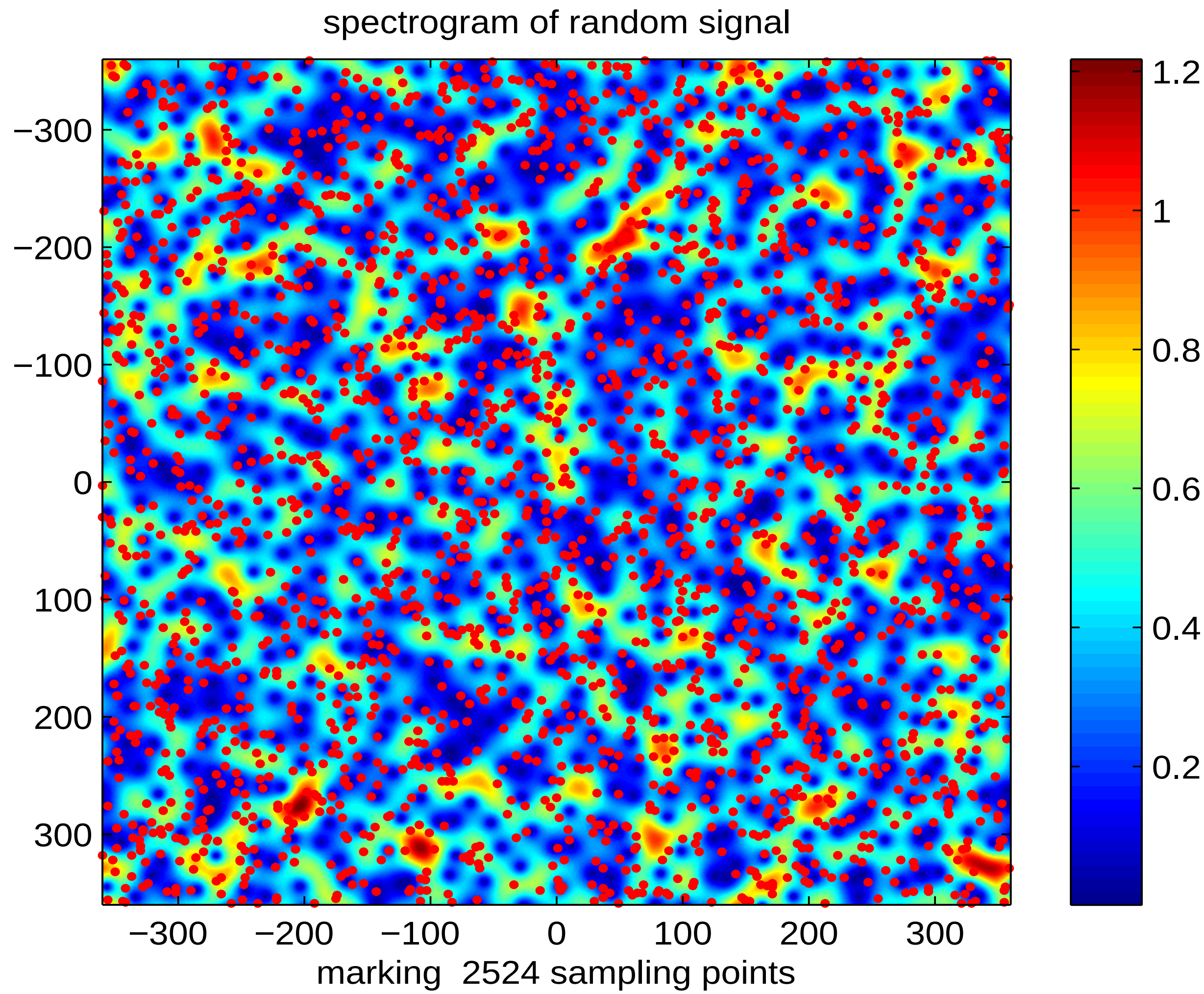

SIGNAL 1
(unitary) FFT−1
SIGNAL 2
(unitary) FFT−2
1
0.5
0
4
3
2
1
0
−1
100
50
0
400
200
0
−200
0
200

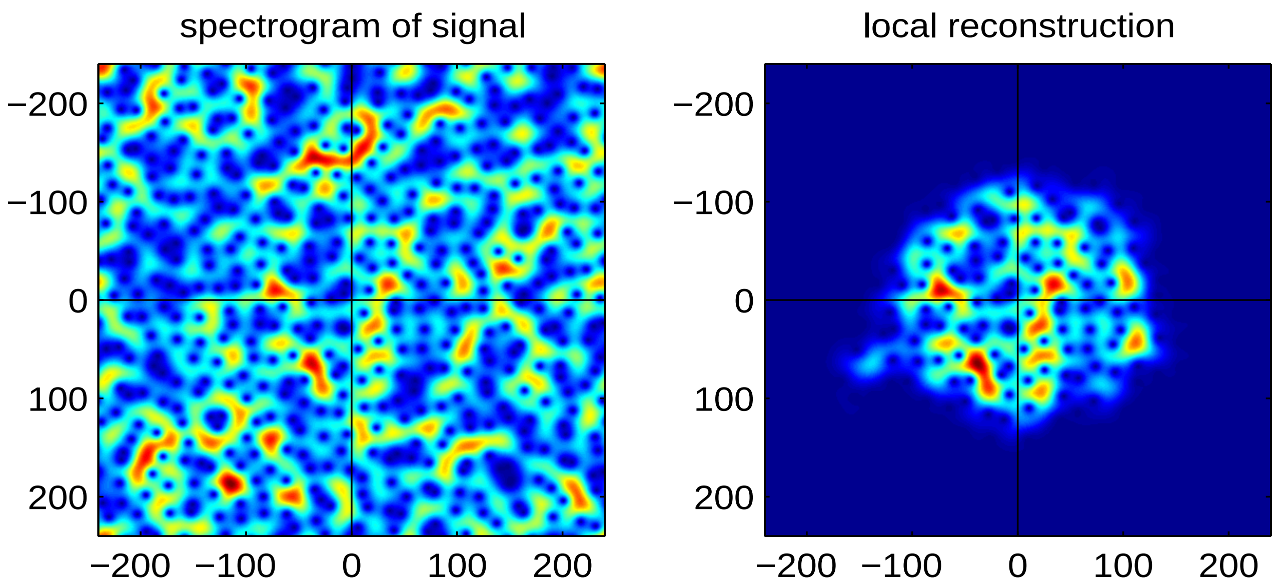
Riesz Basis for Spline-type spaces

Theorem 1. (Spline-type spaces with $\boldsymbol{S}_0(G)$-atom)
Let $g \in \boldsymbol{S}_0(G)$ be given, and let $\Lambda \lhd \widehat{G}$ be a lattice.
(1) The family $(T_\lambda g)_{\lambda \in \Lambda}$ is a Riesz basis (for its closed linear span $V_{g,\Lambda}$) if and only if the $\Lambda^\perp$-periodized version of $\hat{g}$, i.e., $H := \sum_{\lambda^\perp \in \Lambda^\perp} |T_{\lambda^\perp}(\hat{g})|^2$, is free of zeros.
(2) In this case,
(a) the Fourier transform of the function $\tilde{g}$ generating the biorthogonal Riesz basis $(T_\lambda \tilde{g})_{\lambda \in \Lambda}$ is given by $\widehat{\tilde{g}} = \widehat{g}/H$, and $\tilde{g}$ belongs to $\boldsymbol{S}_0(G)$;
(b) there is also some $g_2 \in \boldsymbol{S}_0(G)$ such that $(T_\lambda g_2)_{\lambda \in \Lambda}$ is an orthonormal basis for $V_{g,\Lambda}$, with $\widehat{g_2} = \widehat{g}/\sqrt{H}$; and
(c) if in addition $\hat{g} \geq 0$, then the Lagrange interpolation problem over Λ: $f(0) = 1, f(\lambda) = 0$ for $0 \neq \lambda \in \Lambda$, has a unique solution in $g_L \in V_{g,\Lambda}$, characterized by its Fourier transform $\widehat{g_L} = \hat{g}/\sum_{\lambda^\perp \in \Lambda^\perp}(T_{\lambda^\perp}\hat{g})$. Furthermore $g_L \in \boldsymbol{S}_0(G)$.


basis elements of a spine−type family
dual spline atom
interpolating generator of the space


the setting of irregular sampling
0.1
0.08
0.06
0.04
0.02
0
−0.02
−0.04
−0.06
100
200
300
400
500
600
700
66 samples, and 53 unknown coefficients


spectrogram of random signal
−300
−200
−100
0
100
200
300
marking 2524 sampling points
1.2
1
0.8
0.6
0.4
0.2


spectrogram of signal
local reconstruction
−200
−100
0
100
200
−200 −100 0 100 200

Frames, Banach Frames, Coorbit Spaces

By now the concept of frames in Hilbert spaces $\mathcal{H}$ is well known and widely used. Given a frame $(g_i)_{i\in I}$ there is a well-defined (canonical) dual frame $(\widetilde{g}_i)_{i\in I}$, such that every $f \in \mathcal{H}$ has a representation as

$$f = \sum_i \langle f, g_i\rangle \widetilde{g}_i = \sum_i \langle f, \widetilde{g}_i\rangle g_i$$

Because technically frames can be characterized by the boundedness and invertibility of the frame operator $S : f \mapsto Sf := \sum_i \langle f, g_i\rangle g_i$ the focus of interest is often on the establishment of a simple norm equivalence, namely that of the Hilbert space norm, and the coefficient energy, given by $\sum_{i\in I} |\langle f, \widetilde{g}_i\rangle|^2$.

From this point of view it appears as natural to generalize the concept to a Banach space setting by claiming that for certain Banach spaces (such as Besov or modulation spaces) one has a norm equivalence between the Banach space norm and a corresponding sequence space norm (typically a weighted mixed-norm Banach sequence space): Banach Frames.

Gabor frames and modulation spaces

In this talk we will concentrate on (regular or irregular) **Gabor frames**, hence the corresponding natural family of Banach spaces of (tempered) distributions is the family of modulation spaces. Given a non-zero window $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, a v-moderate weight function m on $\mathbb{R}^{2d}$ of polynomial growth, and $1 \leq p, q \leq \infty$, the **modulation space** $\boldsymbol{M}_m^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ such that $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M_m^{p,q}$ is

$$\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p \, dx \right)^{q/p} d\omega \right)^{1/p} .$$

If $p = q$, we write M_m^p instead of $M_m^{p,p}$, and if $m(z) \equiv 1$ on $\mathbb{R}^{2d}$, then we write $\boldsymbol{M}^{p,q}$ and $\boldsymbol{M}^p$ for $M_m^{p,q}$ and $M_m^{p,p}$.
Then $\boldsymbol{M}_m^{p,q}(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g.

Moreover, if m is v-moderate and $g \in M_v^1 \setminus \{0\}$, then $\|V_g f\|_{L_m^{p,q}}$ is an equivalent norm for $M_m^{p,q}(\mathbb{R}^d)$ We always measure the $\boldsymbol{M}_m^{p,q}$-norm with a fixed non-zero window $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ and repeatedly use the fact that for any $g_1 \in M_v^1(\mathbb{R}^d)$ the norm equivalence

$$\|f\|_{\boldsymbol{M}_m^{p,q}} \asymp \|V_{g_1} f\|_{\boldsymbol{L}_m^{p,q}}$$

holds.

We have Hilbert spaces for the case $p = 2 = q$. For the choice $m(x, \omega) = (1 + |\omega|)^s$, with $s \in \mathbb{R}$ these are just the classical $\boldsymbol{L}^2$-Sobolev space, while we obtain so the called Shubin classes $\mathcal{Q}_s(\mathbb{R}^d)$ by choosing a radial weight $v_s(x, \omega) := (1 + x^2 + \omega^2)^{s/2}$. On the other hand the minimal space among all the (non-trivial) Banach space which are isometrically invariant under TF-shifts is the space $\boldsymbol{M}^1(\mathbb{R}^d)$ which appeared first as a minimal Segal algebra (in 1979), when it was denoted by the symbol $\boldsymbol{S}_0(\mathbb{R}^d)$. The triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$, is a very convenient tool for the description of operators arising in Fourier analysis (e.g., the Fourier transform itself). The space $\boldsymbol{M}_{v_s}^1(\mathbb{R}^d)$, with v_s as above, are important as families of "atoms". Most recently "costumized" modulation spaces are considered (in the PhD thesis of Roza Acesca), which allow to model functions of variable band-width. [2, 1, 3, 8, 9]

General modulation spaces

$\boldsymbol{B} = xxMooY := \{\sigma \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d) \mid V_g\sigma \in \boldsymbol{Y}\}$
where $\boldsymbol{Y}$ is some solid, translation invariant Banach space of (continuous) functions on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, such as a weighted (mixed norm) $\boldsymbol{L}^p$-space over $\mathbb{R}^{2d}$, or a customized function space (in the case of functions of variable band-width), i.e., we require

- $H \in \boldsymbol{Y}, |G(\lambda)| \leq |H(\lambda)| \quad \forall \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d \Rightarrow \|G\|_{\boldsymbol{Y}} \leq \|H\|_{\boldsymbol{Y}}$
- $\|T_\lambda H\|_{\boldsymbol{Y}} \leq w(\lambda)\|H\|_{\boldsymbol{Y}} \quad \forall H \in \boldsymbol{Y};$

The growth of the weight function $w(\lambda)$ will dictate the necessary quality of the Gabor atom g allowed in the Gabor reconstruction theorem.

Better a diagram

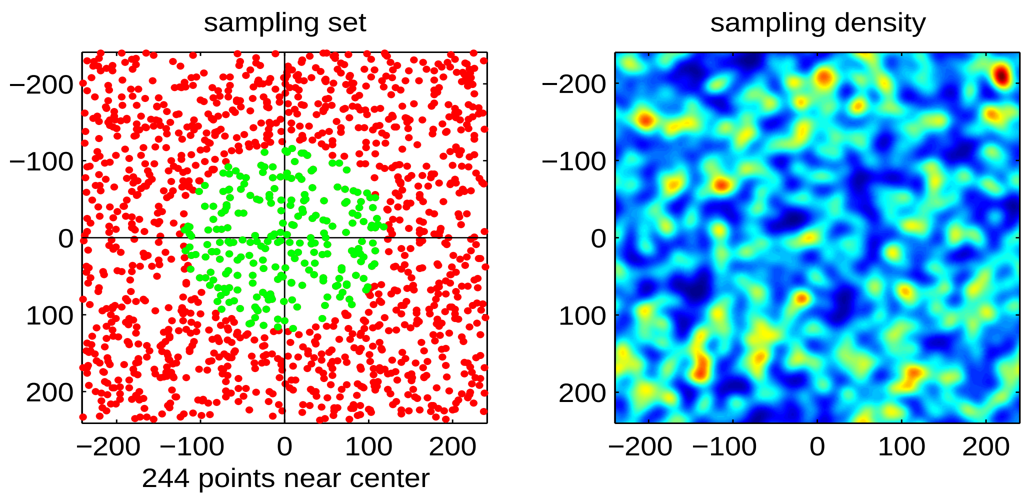
$\mathbf{R}$ being a left inverse of $\mathbf{C}$ implies that $\mathbf{P} = \mathbf{C} \circ \mathbf{R}$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of $\mathbf{C}$, thus we have the following commutative diagram.

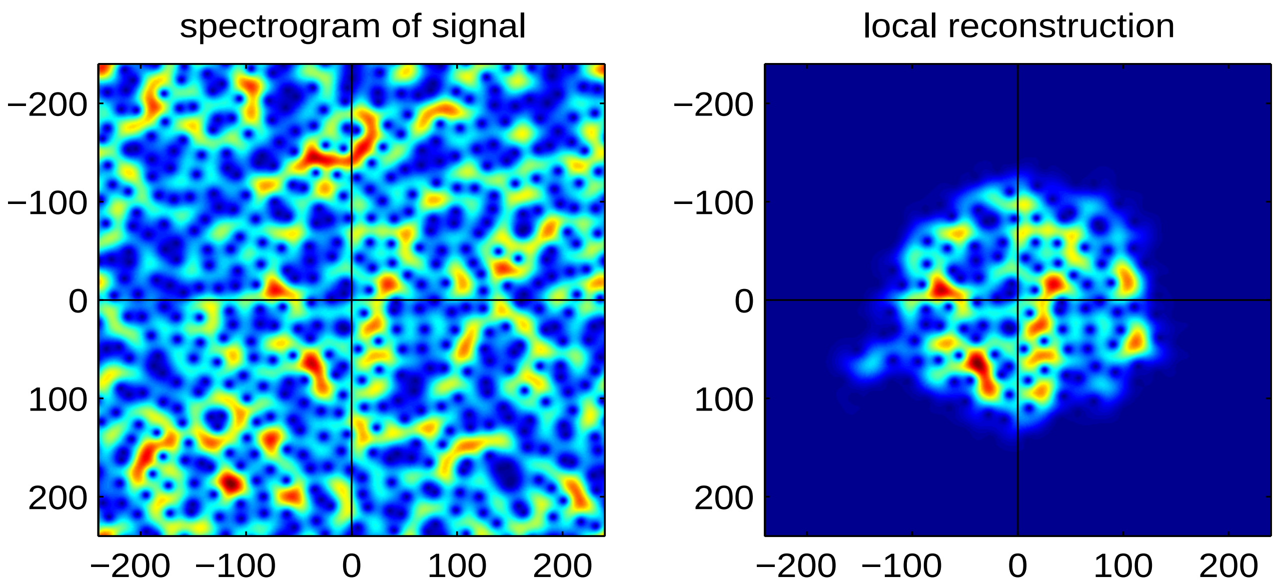
$$
\begin{tikzcd}
 & \boldsymbol{Y} \arrow[dl, "\mathbf{R}"'] \arrow[d, "\mathbf{P}"] \\
\boldsymbol{X} \arrow[r, shift right, "\mathbf{C}"'] & \boldsymbol{Y}_0 \arrow[l, shift right, "\mathbf{R}"']
\end{tikzcd}
$$

Here $\boldsymbol{X}$ is the Banach space under consideration, on which the frame (coefficient or analysis) mapping $\mathbf{C} : f \mapsto (\langle f,\ g_i \rangle)_{i \in I}$, which is injective and has closed range $\boldsymbol{Y}_0$ within the target sequence space $\boldsymbol{Y}$. The reconstruction mapping is then typically given via $\mathbf{R}(\mathbf{e}_i) := \widetilde{g}_i$. It's existence is equivalent to the fact that $\mathbf{P}$ is a projection with range $\boldsymbol{Y}_0$. This larger picture determines so-called Banach frames ([7]).

A typical example

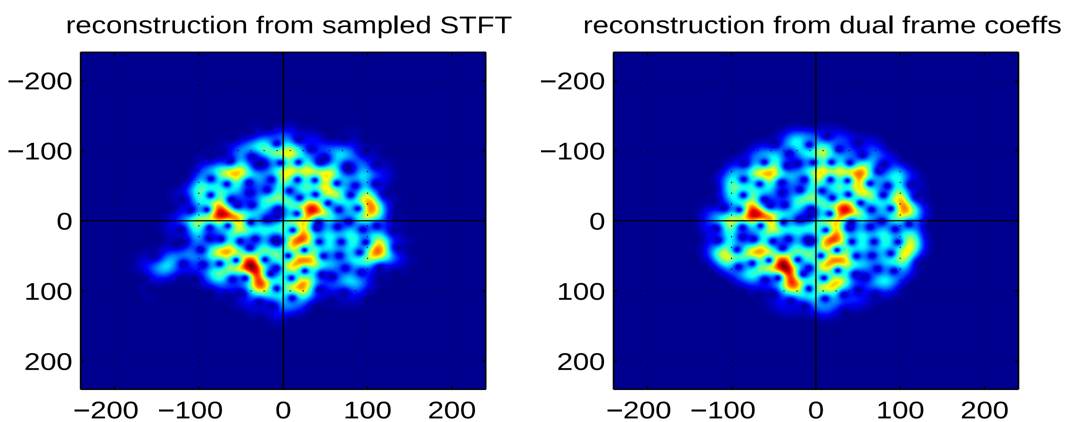
$n = 480$, and an irregular Gabor family (using a Gaussian atom) with 1392 Gabor atoms, out of which 244 $(M \subset I)$ are located near the center (17%).



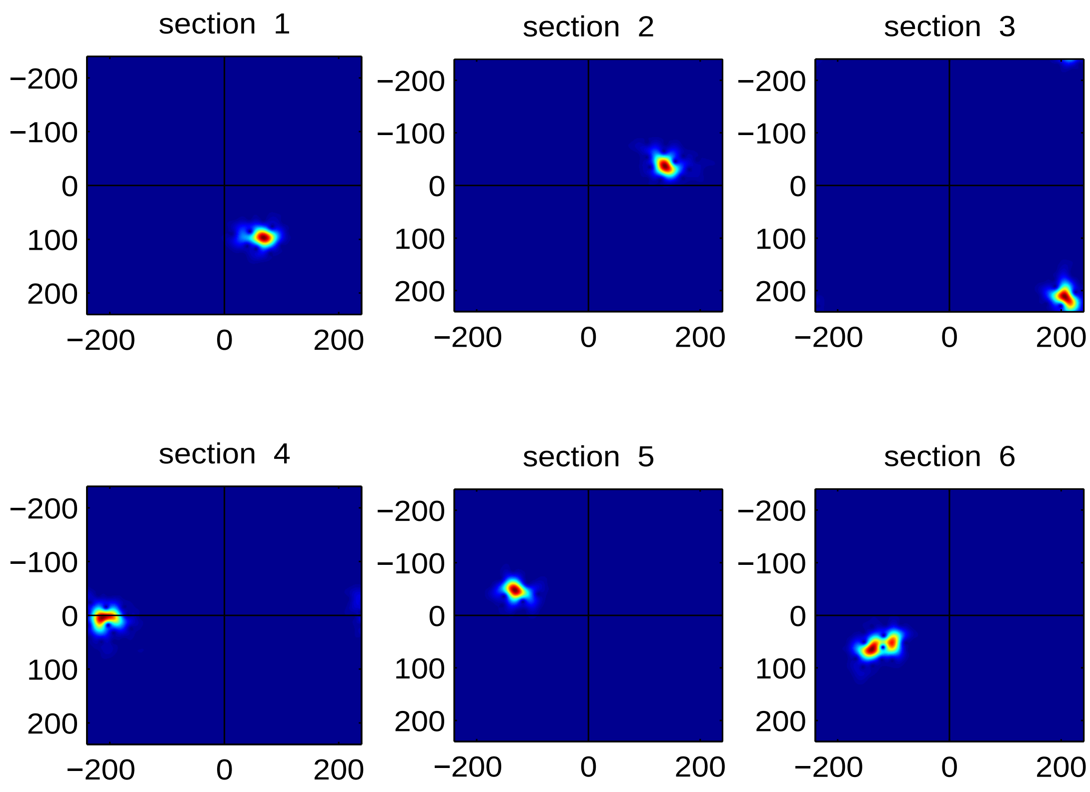

spectrogram of signal
local reconstruction
−200
−100
0
100
200
−200 −100 0 100 200
−200
−100
0
100
200
−200 −100 0 100 200

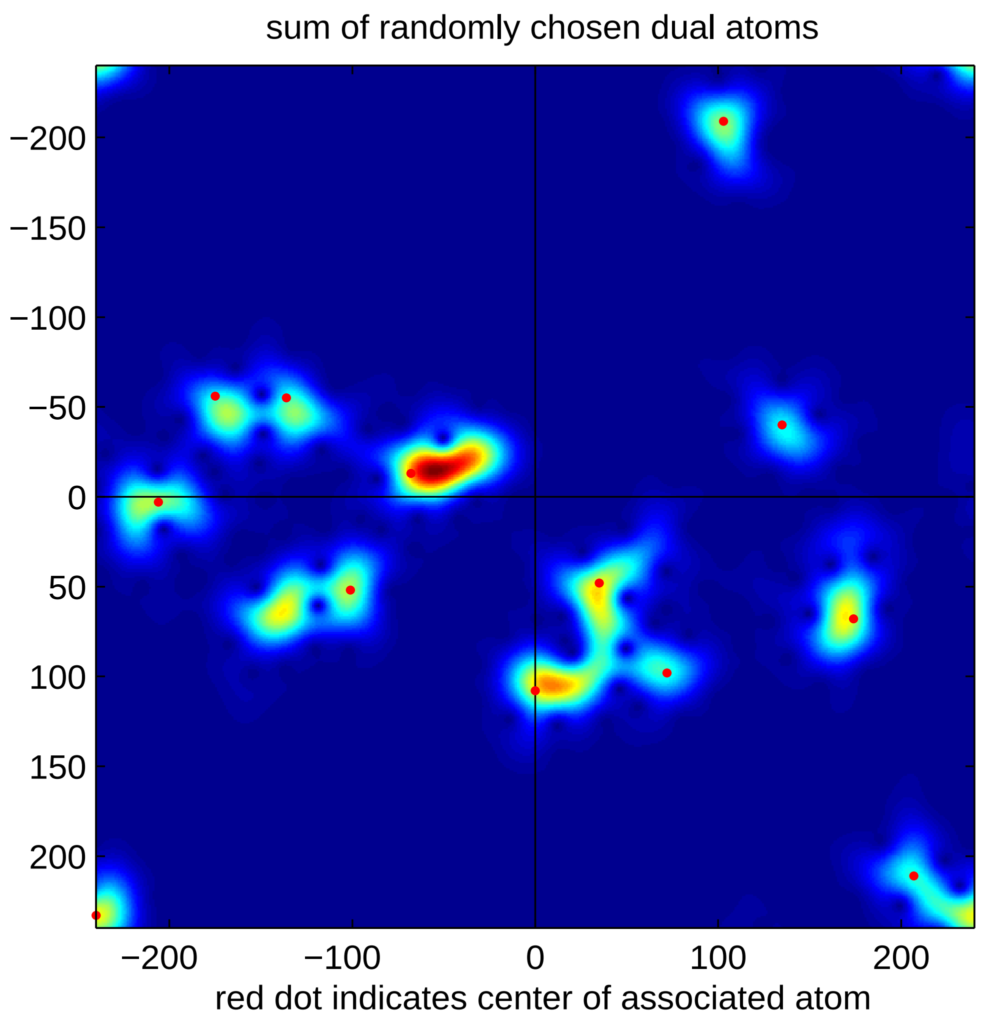
Local reconstructions, using dual frame or frame

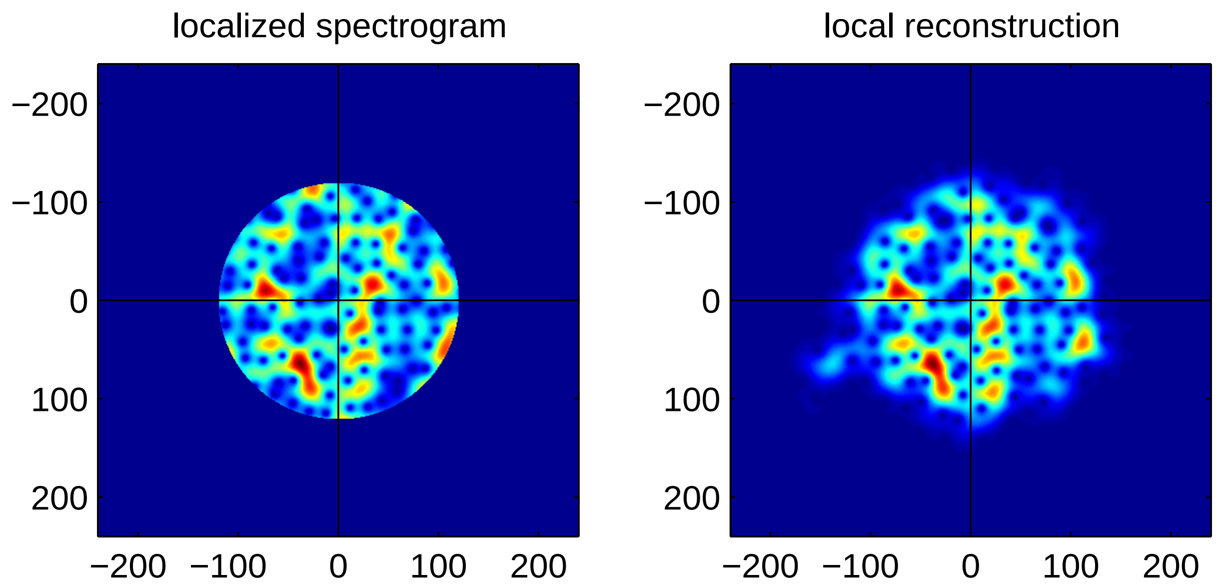
$$f_a = \sum_{i \in M} \langle f, g_i \rangle \widetilde{g}_i \qquad \text{versus} \qquad f_b = \sum_{i \in M} \langle f, \widetilde{g}_i \rangle g_i$$

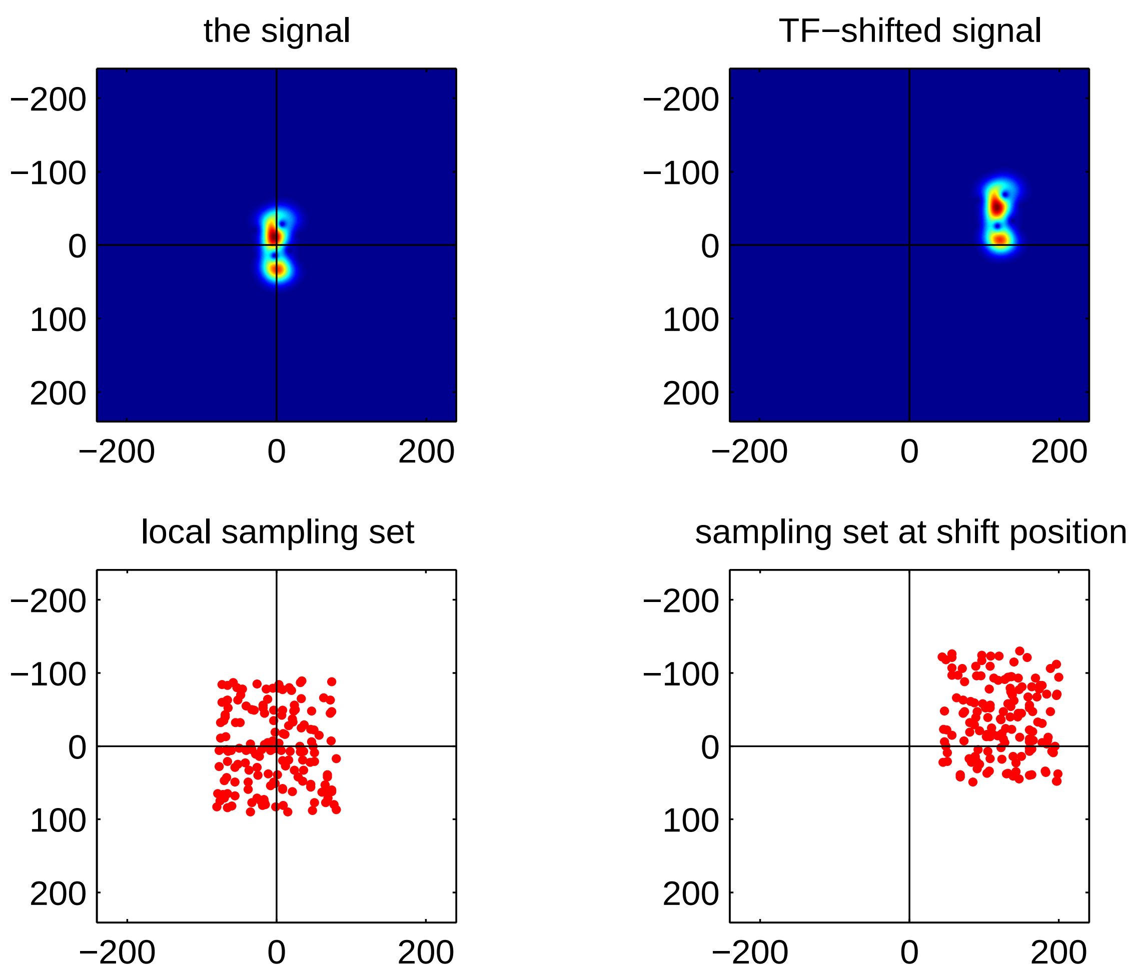


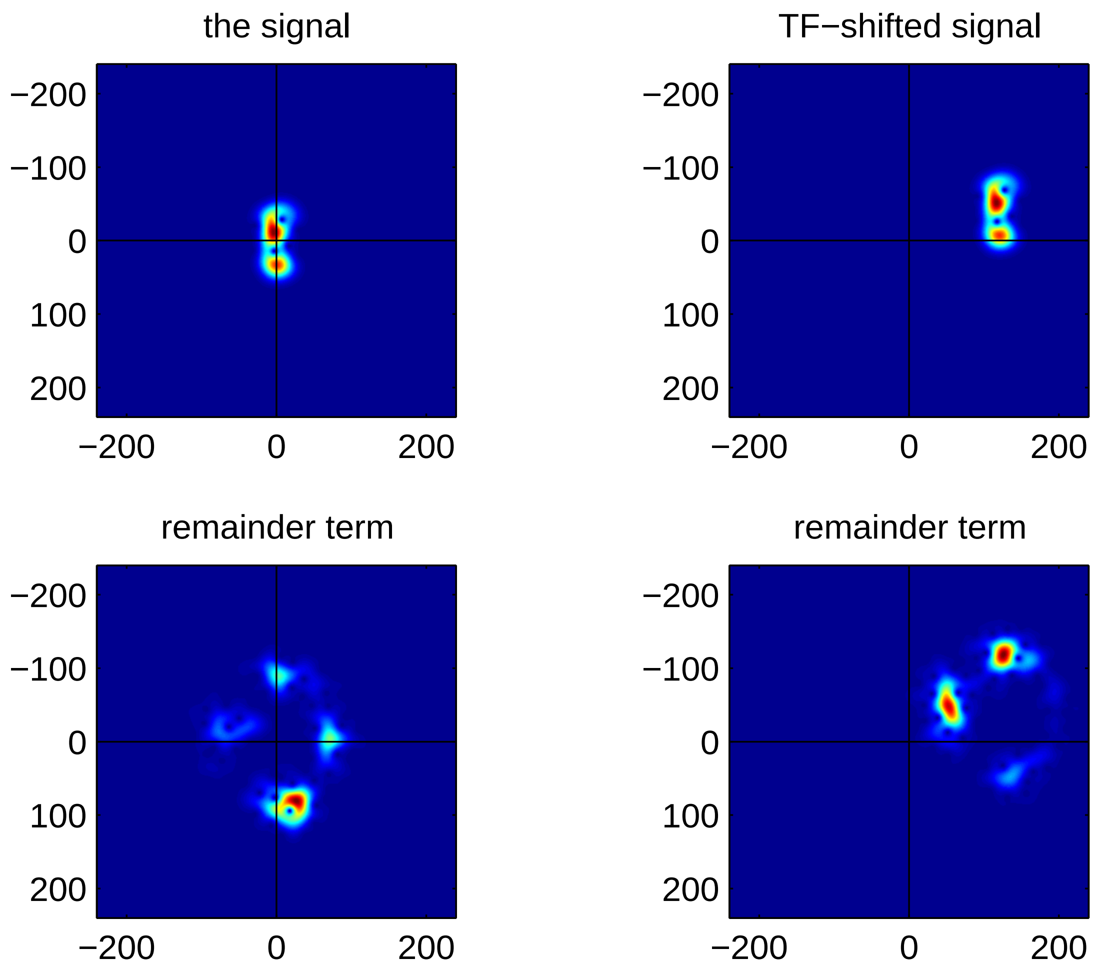
Inspection of dual frame TF-concentrations




sum of randomly chosen dual atoms
−200
−150
−100
−50
0
50
100
150
200
−200
−100
0
100
200
red dot indicates center of associated atom


localized spectrogram
local reconstruction
−200
−100
0
100
200
−200 −100 0 100 200
−200
−100
0
100
200
−200 −100 0 100 200


the signal
TF−shifted signal
local sampling set
sampling set at shift position
−200
−100
0
100
200
−200
0
200



the signal
TF−shifted signal
remainder term
remainder term
−200
−100
0
100
200
−200
0
200

The Uniform Homogeneous Approximation Property

Definition 1. *Given an atomic decomposition of the form*

$$f = \sum_{i \in I} \langle f, g_i \rangle \widetilde{g}_i,$$

with atoms g_i *centered at* λ_i *(e.g.* $g_i = \pi(\lambda_i)g$ *for some nice* g*, and for a family of coorbit space* $\mathbf{B}$*, valid for a family of coorbit spaces, one says that the pair of families* $(g_i, \tilde{g}_i)$ *(*$\widetilde{g}_i$ *which need not be the "canonical dual frame") is said to have the* **UHAP (:= the uniform homogeneous approximation property)** *if for every* $\boldsymbol{B} = Co(Y)$ *in this family, and for every* $f \in B$ *and* $\varepsilon > 0$ *there exists some compact set* $Q \subset \mathcal{G}$ *such that*

$$\| f - \pi(\lambda_0)^{-1} \sum_{i \in I_{\lambda_0 + Q}} \langle \pi(\lambda_0) f, g_i \rangle \widetilde{g}_i \|_{\boldsymbol{B}} < \varepsilon$$

for all $\lambda_0 \in \mathcal{G}$*, where the finite set* I_{λ_0+Q} *is defined as the subset if* I *of indices such that* $\lambda_i \in \lambda_0 + Q$*.*

If $\pi(\lambda)$ acts isometrically on $\boldsymbol{B}$ we are back to the original definition of (HAP).

The “original definition was of the form:

$$\|\pi(\lambda_0)f - \sum_{i \in I_{\lambda_0+Q}} \langle \pi(\lambda_0)f, g_i \rangle \widetilde{g}_i\|_{\boldsymbol{B}} < \varepsilon$$

for all $\lambda_0 \in \mathcal{G}$, where the finite set I_{λ_0+Q} is defined as the subset if I of indices such that $\lambda_i \in \lambda_0 + Q$.
The formulation we have choosen is of course equivalent to

$$\|f - \sum_{i \in I_{\lambda_0+Q}} \langle f, \pi(\lambda_0)^{-1} g_i \rangle [\pi(\lambda_0)^{-1} \widetilde{g}_i]\|_{\boldsymbol{B}} < \varepsilon$$

for all $\lambda_0 \in \mathcal{G}$, where the finite set I_{λ_0+Q} is defined as the subset if I of indices such that $\lambda_i \in \lambda_0 + Q$. But since $\pi(\lambda_0)^{-1} = \gamma \pi(-\lambda_0)$ for some $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ we can also, upon substituting $-\lambda_0 = \lambda_1$ that

$$\|f - \sum_{i \in I_{Q-\lambda_1}} \langle f, \pi(\lambda_1) g_i \rangle [\pi(\lambda_1) \widetilde{g}_i]\|_{\boldsymbol{B}} < \varepsilon$$

Note (!— please check) that the sum is going exactly over those atoms which have their “centers” within some fixed set Q!!
REMARK: Since the centers of an irregular Gabor family have to be relatively separated (as a consequence of the Bessel condition of the frame family) [I think this is found in our old papers, or in Charly's Describing functions [7] one can also conclude, that, independent of λ_1 on can find for a given f and $\varepsilon > 0$ a fixed number of terms in the expansions of f as

$$f = \sum_{i \in I} \langle f, \pi(\lambda_1) g_i \rangle [\pi(\lambda_1) \widetilde{g}_i] = \sum_{i \in I} V_g(\lambda_i - \lambda_1)[\pi(\lambda_1)\widetilde{g}_i]$$

to get an ε-approximation. In contrast, one might have to take an uncontrollable number of terms (depending on the choice of λ_1) in order to achieve such an ε-approximation.

In other words, the size of the set of relevant sampling points for a reconstruction of $\pi(\lambda)f$ (around it's center, which moves with $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$) can be assumed to be essentially of fixed size, for the given level of precision. Of course, if $(g_i)_{i\in I}$ is just a coherent frame the scalar products $(\langle f, g_i\rangle)_{i\in I}$ are just the samples of the continuous transform (e.g. the continuous STFT, or the continuous wavelet transform).
Distinguish between the following views on [U]HAP:

- The HAP, which considers the case $\mathcal{C}o\boldsymbol{L}^2 = \boldsymbol{L}^2(\mathbb{R}^d)$ only;
- The HAP for coorbit spaces which are isometrically $\pi(\lambda)$-invariant, such as the modulation spaces $\boldsymbol{M}^0_{p,q}$.
- the claim that the *canonical* dual frame, together with its original satisfies the HAP (cf. [11, 16])
- the claim that - for a given quality of the frame (expressed in term of "localization" of the frame, resp. low correlation between the elements of the frame elements which are at some distance in phase space (uniformly over the family).
- claims which use (strict) coherence on the side of the analyzing frame (i.e. $g_i = \pi(\lambda_i)g$), or those allowing to switch the roles of $\widetilde{g}_i$ and g_i.
- claims valid for $\widetilde{g}_i$ obtained by a particular (family) of reconstruction method(s).

Equivalent Characterizations of HAP

The following theorem describes a number of equivalent properties:

Theorem 2. *The following conditions for a pair of dual frames* $xxgitgiI$*:*

- *the HAP conditions is satisfied for every element of* $f \in \boldsymbol{B}$*;*
- *the HAP conditions is satisfied simultaneously for every* **finite** *set* $\{f_1, \dots f_k\}$ *of elements from* $\boldsymbol{B}$ *simultaneously;*
- *the HAP is valid for a* **compact** *subset* M *of elements in* $\boldsymbol{B}$*.*
- *For every compact operator* T *one has: There exists some compact set* $Q \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$*, such that for any* $T_\lambda := \pi(\lambda) \circ T \circ \pi(\lambda)^{-1}$ *and any* $f \in \boldsymbol{B}$

$$\|T_\lambda f - \sum_{i \in I_Q} \langle T_\lambda f, g_i \rangle \widetilde{g}_i \|_{\boldsymbol{B}} < \varepsilon \|f\|_{\boldsymbol{B}}$$

Note: It is easy to show that (i) above is (technically) equivalent to the choice of a rank 1 projection operator: $Th = \langle h, f \rangle f$, for some function $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ with $\|f\|_2 = 1$.

UHAP is valid in the context of coorbit theory

Coorbit theory deals with abstract continuous transform, related to integrable group representations on some Hilbert space. It makes use of function spaces of those acting groups and allows a uniformly discrete sampling reconstruction theorem for large families of Banach spaces of functions and distributions.
For the case of the $ax + b$ group we have the continuous wavelet transform, and a detailed study of HAP (for the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$) is given in [11] and [16].
We will stay with the (usual) Schrödinger representation of the reduced Heisenberg group, or in the engineering terminology, work with the STFT, understood as a function over the TF (= time-frequency) plane or phase space (in the context of coherent states). We allow quite general windows, and consider the appropriate class of function spaces, which are the (generalized, ultra-, classical, . . .) **modulation spaces**.
There is the "frame" (or sampling) view-point, which describes the problem as a reconstruction problem from a sampled STFT, and the alternative "atomic decomposition view points", where the Gabor atoms are used as building blocks.

Then the coorbit theory tells us, that for a given "quality" of the Gabor atom g (i.e. knowing the norm in some function space describing smoothness and decay) we can guarantee that every sufficiently dense (and relatively separated) family of sampling points $(\lambda_i)_{i\in I} = (x_i, \omega_i)$ allows to reconstruct functions f from any of those modulation spaces via

$$f = \sum_{i\in I} V_g(\lambda_i)\widetilde{g}_i$$

with unconditional convergence of the infinite sum (in the norm topology of $\boldsymbol{S}(\mathbb{R}^d)$ is dense in $\boldsymbol{B}$, and in the sense of the w^*-topology otherwise, i.e., in the sense of uniform convergence of the STFT over compact sets. In other words, the spectrogram of the partial sums will look more and more similar to the spectrogram of the overall function or distribution, as we take more and more terms (even if the norm of the remainder term does not go to zero).

Once more the diagram

$\mathbf{R}$ being a left inverse of $\mathbf{C}$ implies that $\mathbf{P} = \mathbf{C} \circ \mathbf{R}$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of $\mathbf{C}$, thus we have the following commutative diagram. Think of $\boldsymbol{X}$ as one of the modulation spaces, of $\mathbf{C}$ the operator sending f to the set of sampling values $(V_g f(\lambda_i))_{i \in I}$ which is part of the appropriate (mixed norm, weighted) sequence space $\boldsymbol{Y}$, and $\mathbf{R}$ the synthesis mapping $(c_i)_{i \in I} \mapsto \sum_{i \in I} c_i \widetilde{g}_i$.

$$\begin{tikzcd}
 & \boldsymbol{Y} \arrow[dl, "\mathbf{R}"'] \arrow[d, "\mathbf{P}"] \\
\boldsymbol{X} \arrow[r, shift right, "\mathbf{C}"'] & \boldsymbol{Y}_0 \arrow[l, shift right, "\mathbf{R}"']
\end{tikzcd}$$

Note that it is important for applications that one has not only reconstruction from sequences $(c_i)_{i \in I}$ exactly in the range of $\mathbf{C}$ (which is denoted by $\boldsymbol{Y}_0$ here), but also from the whole (solid) sequence space $\boldsymbol{Y}$, think of noise samples or samples followed by thresholding.

What can Coorbit theory give to you?

The first (*and naive*) view on coorbit theory is to view it as a generalization of the ordinary frame theory to the setting of reconstruction in Hilbert spaces (as a consequence of the norm-equivalence between the Hilbert space norm and the coefficient energy norm) to a Banach space setting. This is also, how **Banach frames** are defined.

Proposition 1. *(A) Given any modulation spaces* $\boldsymbol{B}$ *there is a certain minimal quality required for the Gabor atom* g*, such that for such an atom* g *we can find a density measure* $\delta > 0$ *such that any relatively separated allows stable reconstruction of any* $f \in \boldsymbol{B}$ *in the form*

$$f = \sum_i V_g f(\lambda_i) \widetilde{g}_i \ldots$$

Proposition 2. *(B) Given* **a family of modulation spaces** $\boldsymbol{B}$ *there is a certain minimal quality required for the Gabor atom* g*, such that for such an atom* g *we can find a density measure* $\delta > 0$ *such that any relatively separated ...*

$$f = \sum_i V_g f(\lambda_i) \widetilde{g}_i \qquad \forall f \in \boldsymbol{B}$$

Uniformity with respect to location

Theorem 3. *(exploiting the details of coorbit theory)*
(C) Given **a family of modulation spaces** $\boldsymbol{B}$ *there is a certain minimal quality required for the Gabor atom* g*, and a family of iterative reconstruction methods which allow the reconstruction of any* f *from any of the modulation spaces* $\boldsymbol{B}$ *such that we have UNIFORM bounds on the analysis and synthesis operators for all the* $\delta-$*dense and (uniformly, relatively) separated sampling sets* $(\lambda_i)_{i\in I}$*, (independent of the space* $\boldsymbol{B}$ *choosen, independent of the atom choosen within the family), such that*

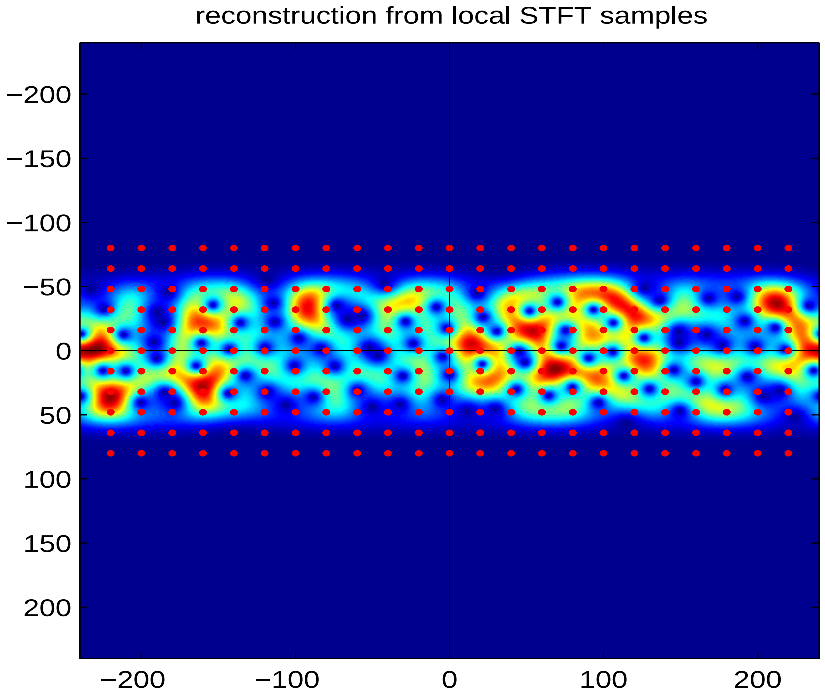
$$f = \pi(\lambda_0)^{-1} \sum_{i\in I} V_g(\pi(\lambda_0) f(\lambda_i) \widetilde{g}_i$$

UHAP then corresponds to the fact that we can find (again uniformly over all these possible choices) for every $\varepsilon > 0$ *some compact set* $Q \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$*, with*

$$\|f - \pi(\lambda_0)^{-1} \Big(\sum_{i\in I_{\lambda_0+Q}} V_g\big[\pi(\lambda_0) f(\lambda_i)\big] \widetilde{g}_i \Big) \|_{\boldsymbol{B}} < \varepsilon$$

Localized reconstruction of band-limited functions

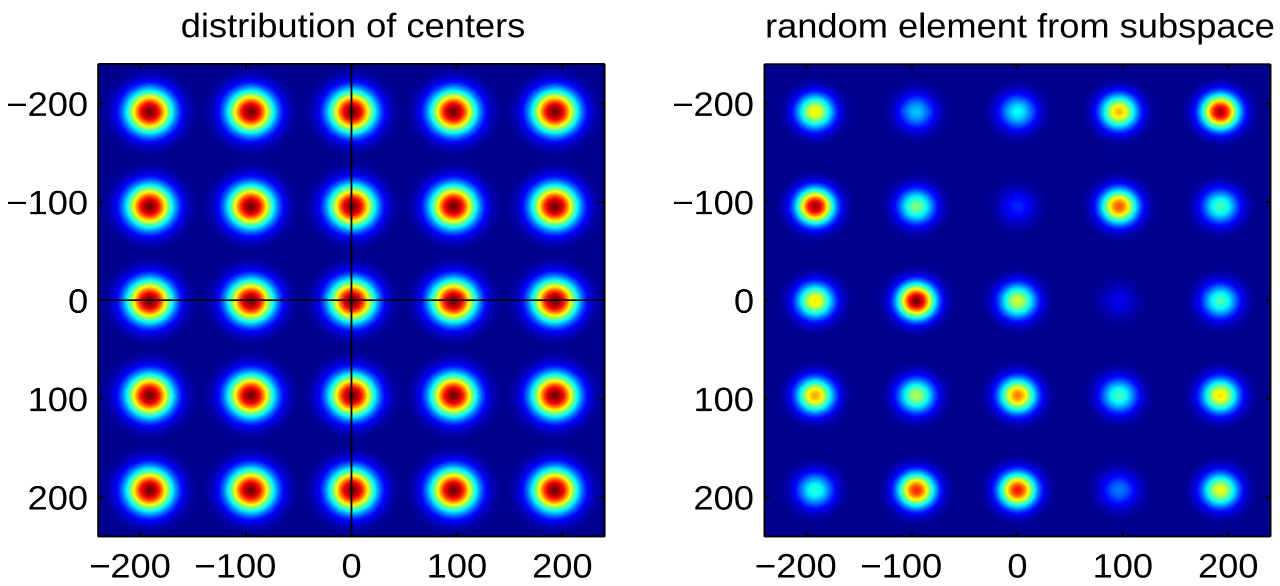
Illustration of a result from [6]

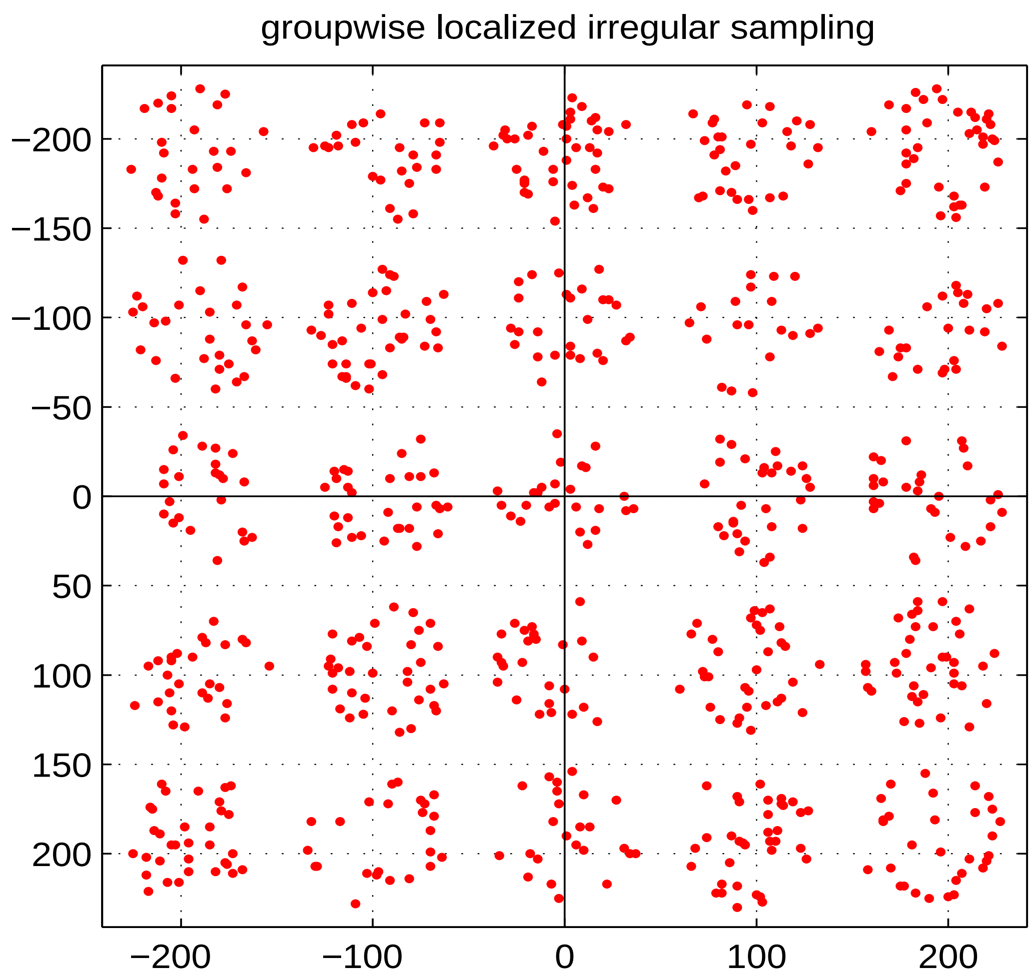


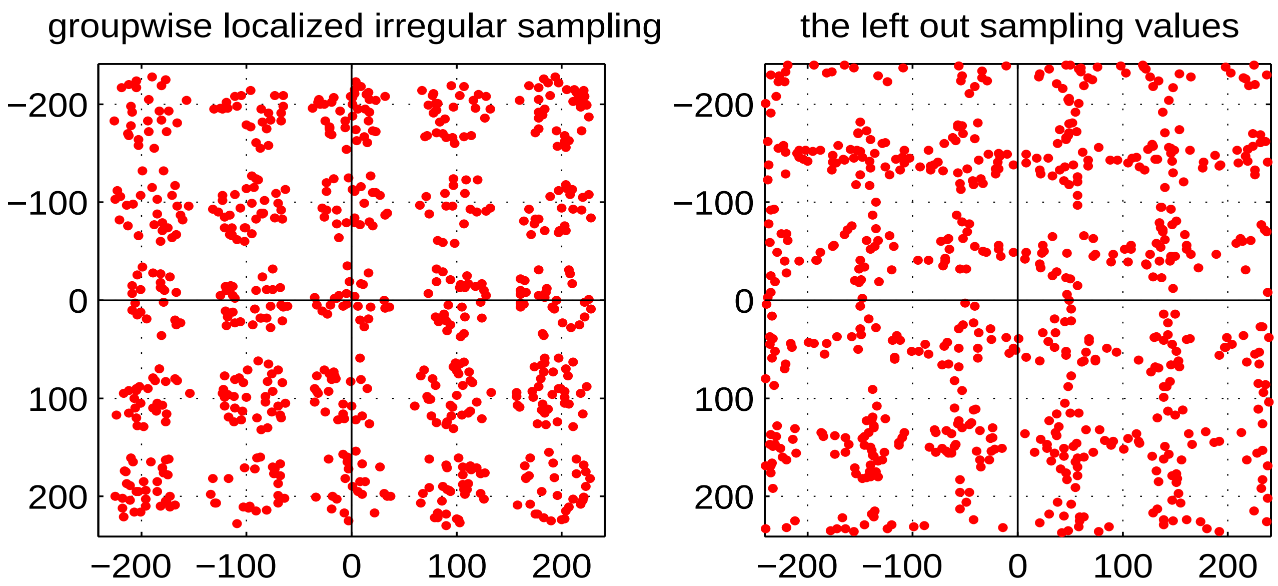
The piano reconstruction theorem revisited

Theorem 4. *Given a class of modulation spaces and some* $g \in \boldsymbol{S}(\mathbb{R}^d)$*, there exists some* $R_0 > 0$ *such that any family of the form* $(\pi(\lambda_j)g)_j inJ$*, with* $dist(\lambda_j, \lambda_j') \geq R_0$ *is a Riesz (projection) basis, with Riesz bounds depending on* g *and* R_0 *only (but not on the individual choice of the family* $(\lambda_j)_{j\in J}$*).*
In particular, we have - uniformly over the class of modulation spaces, equivalence constants between the $\boldsymbol{Y}_d$*-norms of the sequence* $(c_j)_{j\in J}$ *and the* $\boldsymbol{B}$*- modulation space norm* $f = \mathbf{R}(c) = \sum_{j\in J} c_j\pi(\lambda_j)g$*.*

It is therefore natural to ask whether it is enough to know the (irregular) samples of a STFT "around the centers" of such a family (and not in the whole TF-plane) in order to be able to recover the signal f completely. The positive answer to this question is given in the next theorem.


distribution of centers
random element from subspace
−200
−100
0
100
200
−200
−100
0
100
200


groupwise localized irregular sampling
−200
−150
−100
−50
0
50
100
150
200
−200
−100
0
100
200


groupwise localized irregular sampling
−200
−100
0
100
200
−200
−100
0
100
200
the left out sampling values
−200
−100
0
100
200
−200
−100
0
100
200

Theorem 5. *Given a class of modulation spaces and some $\varphi \in \boldsymbol{S}(\mathbb{R}^d)$, there exists some $R_1 > 0$ and $S \geq R_1$ such that any family of the form $(\pi(\lambda_j)\varphi)_j in J$, with $dist(\lambda_j, \lambda_j') \geq R_1$ is a Riesz (projection) basis for its closed linear span in $\boldsymbol{B}$, let us call it V. The Riesz bounds depending on g and R_1 only (but not on the individual choice of the family $(\lambda_j)_{j\in J}$). Furthermore it is enough to know the sampling values of $V_g(f)$ only at the (irregular) sampling values located with the set $\bigcup_{j\in J} B_S(\lambda_j) \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ in order to recover f completely.*

Proof. The idea is similar to the reconstruction of band-limited functions from (regular) samples taken from a horizontal strip in phase space, as given in [6].
The argument makes of course use of the fact that there is a biorthogonal family for the given family $(\pi(\lambda_j)\varphi)_{j\in J} := (\varphi_{\lambda_j})_{j\in J}$, let us call it $(\widetilde{\varphi}_j)_{j\in J}$. The projection onto the closed linear span V of the atoms $(\varphi_{\lambda_j})_{j\in J}$ within $\boldsymbol{B}$ is then obtained by the projection mapping (which is continuous also on $\boldsymbol{B}$):

$$f \mapsto P(f) = \sum_{j\in J} \langle f, \widetilde{\varphi}_j \rangle \varphi_{\lambda_j}$$

and we have (uniform) control over the norm of P (over the whole family, say $\|P(f)\|_{\boldsymbol{B}} \leq C_P \|f\|_{\boldsymbol{B}}$.
It therefore remains to show that we can approximate the identity operator sufficiently well, using only the described sampling values only. In fact, it is enough to have

$$\|f - \sum_{i \in I_S} \langle f, \widetilde{g}_i \rangle g_{\lambda_i}\| \leq \gamma \|f\|_{\boldsymbol{B}},$$

for all $f \in V$, and some $\gamma < 1/C_P$, where

$$I_S = \{i \in I \mid x_i \in \bigcup_{j \in J} B_S(\lambda_j) \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}.$$

Clear enough this allows us to show that the linear mapping

$$f \mapsto P(\sum_{i \in I_S} \langle f \widetilde{g}_i \rangle g_{\lambda_i})$$

which maps V into itself is close enough to the identity operator, and that it is invertible as a consequence. $\square$

Remark 3. *There is an alternative viewpoint: Given the space V and the projection operator P one can modify the original biorthogonal family in such a way that it is not nessesary to know all the sampling values of $(V_g(f)(x_i))_{i \in I}$, but only those belonging to the relevant subset I_S used in the proof. Moreover one can argue (and verify also numerically) that those "perfect reconstructions" for V are quite similar to the original biorthogonal functions in the "interior" of the relevant index sets (i.e., near the centers $(\lambda_j)_{j \in J}$, while they may have to be deformed more seriously near the boundaries.*

Further references: [7], [4, 5],
Maybe [13, 12, 14], [11, 10, 16, 15]

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