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Banach Gelfand Triples for Harmonic Analysis

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What are our goals when doing Fourier analysis?

- find relevant "harmonic components" in [almost] periodic functions;
- define the Fourier transform (first $\boldsymbol{L}^1(\mathbb{R}^d)$, then $\boldsymbol{L}^2(\mathbb{R}^d)$, etc.);
- describe time-invariant linear systems as convolution operators;
- describe such system as Fourier multipliers (via transfer functions);
- deal with (slowly) time-variant channels (communications) ;
- describe changing frequency content ("musical transcription");
- define operators acting on the spectrogram (e.g. for denoising) or perhaps pseudo-differential operators using the Wigner distribution;

What do we have (?) to teach our students?

The typical VIEW that well trained mathematicians working in the field may have, is that ideally a STUDENT have to

- learn about Lebesgue integration (to understand Fourier integrals);
- learn about Hilbert spaces and unitary operators;
- learn perhaps about $\boldsymbol{L}^p$-spaces as Banach spaces;
- learn about topological (nuclear Frechet) spaces like $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$;
- learn about tempered distributions;
- learn quasi-measures, to identify TLIS as convolution operators;

Classical Approach to Fourier Analysis

- Fourier Series (periodic functions), summability methods;
- Fourier Transform on $\mathbb{R}^d$, using Lebesgue integration;
- sometimes: Theory of Almost Periodic Functions;
- Generalized functions, tempered distributions;
- Discrete Fourier transform, FFT (Fast Fourier Transform), e.g. FFTW;
- Abstract ($>>$ Conceptional !)Harmonic Analysis over LCA groups;
- . . . but what are the connections?? What is needed for computations?

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CLAIM: What is really needed!

In contrast to all this the CLAIM is that just a bare-bone version of functional analytic terminology is needed (including basic concepts from Banach space theory, up to w^*-convergence of sequences and basic operator theory), and that the concept of Banach Gelfand triples is maybe quite useful for this purpose. So STUDENTS SHOULD LEARN ABOUT:

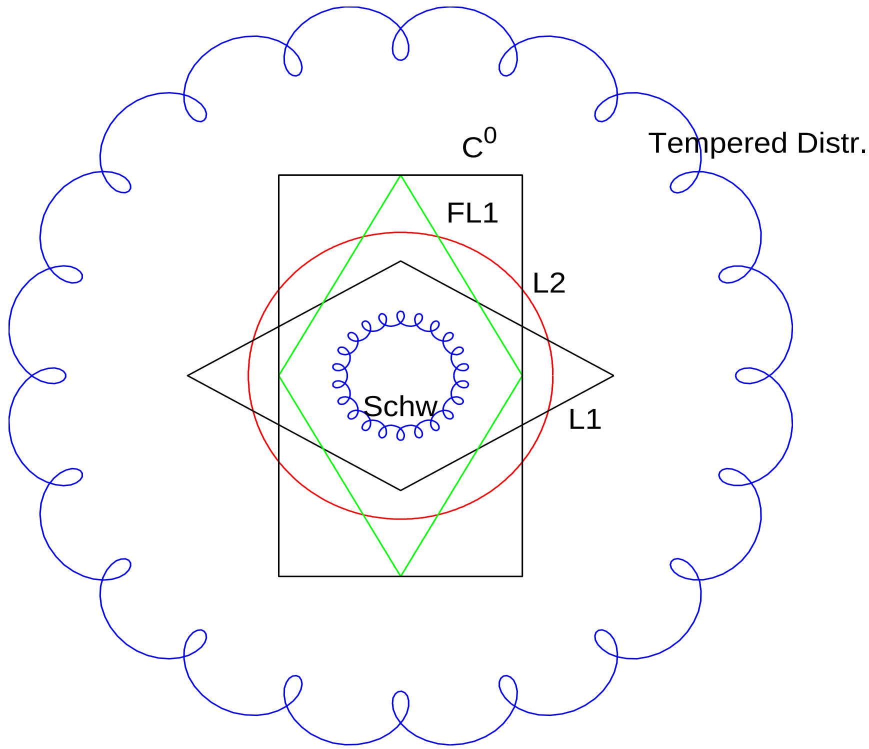
- refresh their linear algebra knowledge (ONB, **SVD**!!!, linear independence, generating set of vectors), and matrix representations of linear mappings between finite dimensional vector spaces;

- Banach spaces, bd. operators, dual spaces norm and w^*-convergence;

- about Hilbert spaces, orthonormal bases and unitary operators;

- about frames and Riesz basis (resp. matrices of maximal rank);

The classical view on the Fourier Transform



There is no time to go through the following list of topics!

- Typical questions of (classical and modern) Fourier analysis
- Fourier transforms, convolution, impulse response, transfer function
- The Gelfand triple $(\boldsymbol{\mathcal{S}}, \boldsymbol{L}^2, \boldsymbol{\mathcal{S}}')(\mathbb{R}^d)$, of Schwartz functions and tempered distributions;
- The Banach Gelfand Triples $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ and its use;
- various (unitary) Gelfand triple isomorphism involving $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$

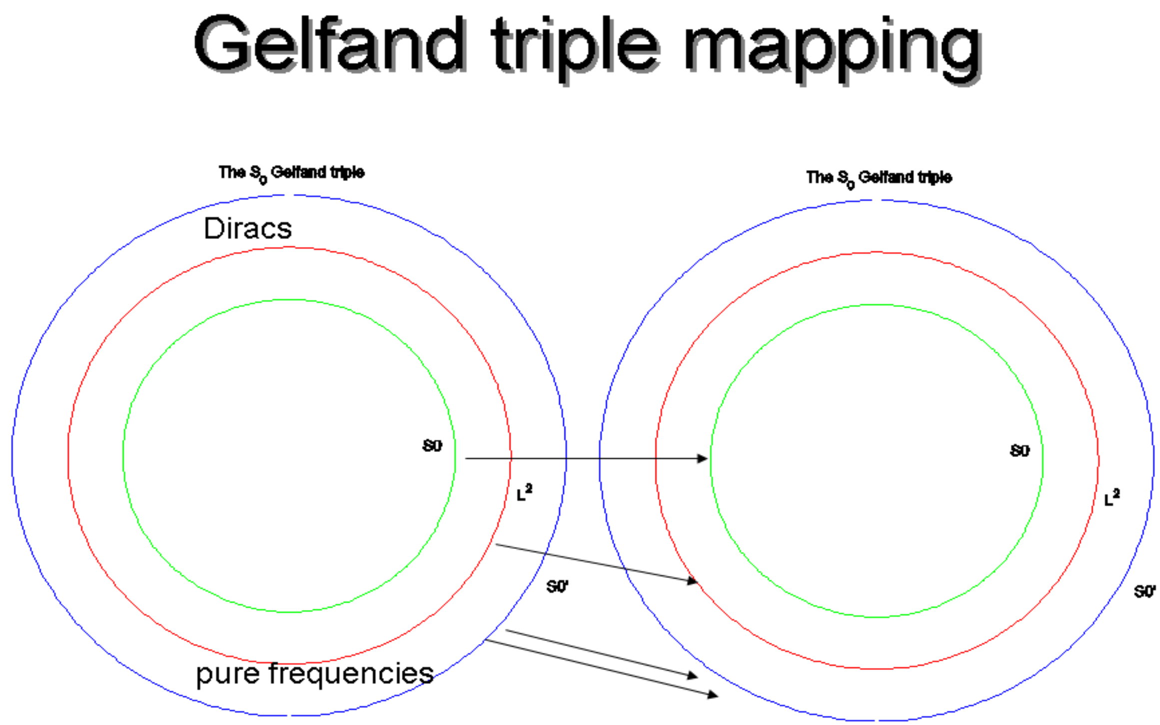
Definition 1. *A triple* $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$*, consisting of a Banach space* $\boldsymbol{B}$*, which is dense in some Hilbert space* $\mathcal{H}$*, which in turn is contained in* $\boldsymbol{B}'$ *is called a Banach Gelfand triple.*

Definition 2. *If* $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}'_1)$ *and* $(\boldsymbol{B}_2, \mathcal{H}_1, \boldsymbol{B}'_2)$ *are Gelfand triples then a linear operator* T *is called a [unitary] Gelfand triple isomorphism if*

1. *A is an isomorphism between* $\boldsymbol{B}_1$ *and* $\boldsymbol{B}_2$*.*
2. *A is a [unitary operator resp.] isomorphism between* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*.*
3. *A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between* $\boldsymbol{B}'_1$ *and* $\boldsymbol{B}'_2$*.*

The prototype is $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$. w^*-convergence corresponds to coordinate convergence in ℓ^∞. It can be transferred to "abstract Hilbert spaces" $\mathcal{H}$. Given any orthonormal basis (h_n) one can relate $\boldsymbol{\ell}^1$ to the set of all elements $f \in \mathcal{H}$ which have an *absolutely convergent* series expansions with respect to this basis. In fact, in the classical case of $\mathcal{H} = \boldsymbol{L}^2(\mathbb{T})$, with the usual

Fourier basis the corresponding spaces are known as Wiener's $\boldsymbol{A}(\mathbb{T})$. The dual space is then $\mathcal{P}_M$, the space of pseudo-measures $= \mathcal{F}^{-1}[\boldsymbol{\ell}^\infty(\mathbb{Z})]$.



Realization of a GT-homomorphism

Very often a Gelfand-Triple homomorphism T can be *realized with the help of some kind of "summability methods"*. In the abstract setting this is a sequence (or more generally a net) A_n, having the following property:

- each of the operators maps $\boldsymbol{B}_1'$ into $\boldsymbol{B}_1$;
- they are a uniformly bounded family of Gelfand-triple homomorphism on $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$;
- $A_n f \to f$ in $\boldsymbol{B}_1$ for any $f \in \boldsymbol{B}_1$;

It then *follows* that the limit $T(A_n f)$ exists in $\mathcal{H}_2$ respectively in $\boldsymbol{B}_2'$ (in the w^*-sense) for $f \in \mathcal{H}_1$ resp. $f \in \boldsymbol{B}_1'$ and thus describes concretely the prolongation to the full Gelfand triple. This continuation is unique due to the w^*-properties assumed for T (and the w^*-density of $\boldsymbol{B}_1$ in $\boldsymbol{B}_1'$).

Typical Philosophy

One may think of $\boldsymbol{B}_1$ as a (Banach) space of test functions, consisting of "decent functions" (continuous and integrable), hence $\boldsymbol{B}_1{}'$ is a space of "generalized functions, containing at least all the $\boldsymbol{L}^p$-spaces as well as all the bounded measures, hence in particular finite discrete measures (linear combinations of Dirac measures).

At the test function level every "transformation" can be carried out very much as if one was in the situation of a finite Abelian group, where sums are convergent, integration order can be interchanged, etc.. At the intermediate level of the Hilbert space on has very often a unitary mapping, while only the out "layer" allows to really describe what is going on in the $ideal\ limit\ case$, because instead of unit vectors for the finite case one has to deal with Dirac measures, which are only found in the big dual spaces (but not in the Hilbert space!).

Using the BGTR-approach one can achieve . . .

- a relative simple minded approach to Fourier analysis (motivated by linear algebra);
- results based on standard functional analysis only;
- provide clear rules, based on basic Banach space theory;
- comparison with extensions $\mathbb{Q} >> \mathbb{R}$ resp. $\mathbb{R} >> \mathbb{C}$;
- provide confidence that "generalized functions" really exist;
- provide simple descriptions to the above list of questions!

Key Players for Time-Frequency Analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

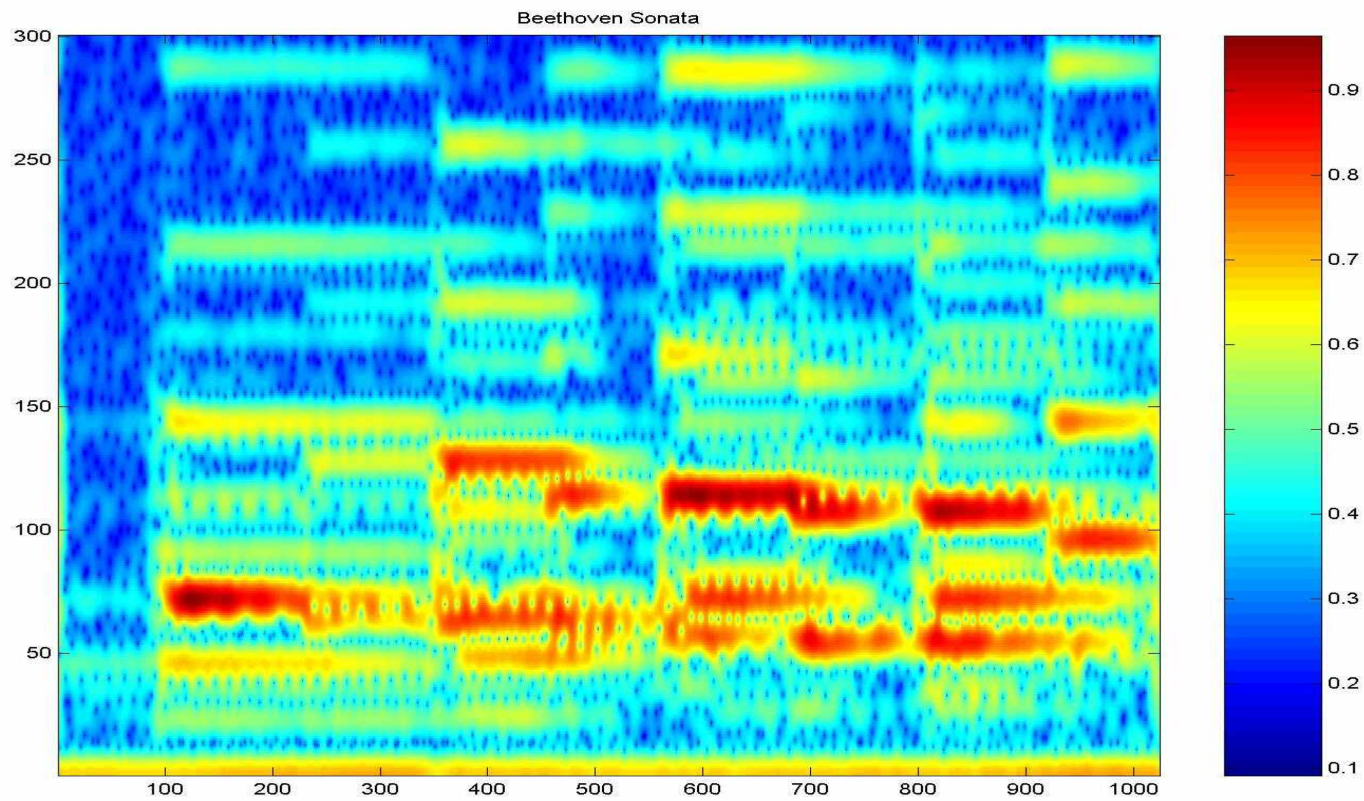
Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = V_g f(t,\omega) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \quad \lambda = (t,\omega);$$

A Typical Musical STFT



$\boldsymbol{S}_0(\mathbb{R}^d) = M^1(\mathbb{R}^d) := M^0_{1,1}(\mathbb{R}^d)$

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is (by definition) in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is a Banach space, for any fixed, non-zero $g \in S_0(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable windows. Often the Gaussian is used as a window. Note that

$$V_g f(x,\omega) = (\widehat{f \cdot T_x g})(\omega), \quad \text{i.e., } g \text{ localizes } f \text{ near } x.$$

Lemma 1. *Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

Remark 2. *Moreover one can show that $\boldsymbol{S}_0(\mathbb{R}^d)$ is the smallest non-trivial Banach spaces with this property, i.e., it is continuously embedded into any such Banach space. As a formal argument one can use the continuous inversion formula for the STFT:*

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(\lambda)\pi(\lambda) g d\lambda$$

which implies

$$\|f\|_{\boldsymbol{B}} \leq \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(\lambda)| \|\pi(\lambda)g\|_{\boldsymbol{B}} \, d\lambda = \|g\|_{\boldsymbol{B}} \|f\|_{\boldsymbol{S}_0}.$$

Basic properties of $S_0(\mathbb{R}^d)$ resp. $S_0(G)$

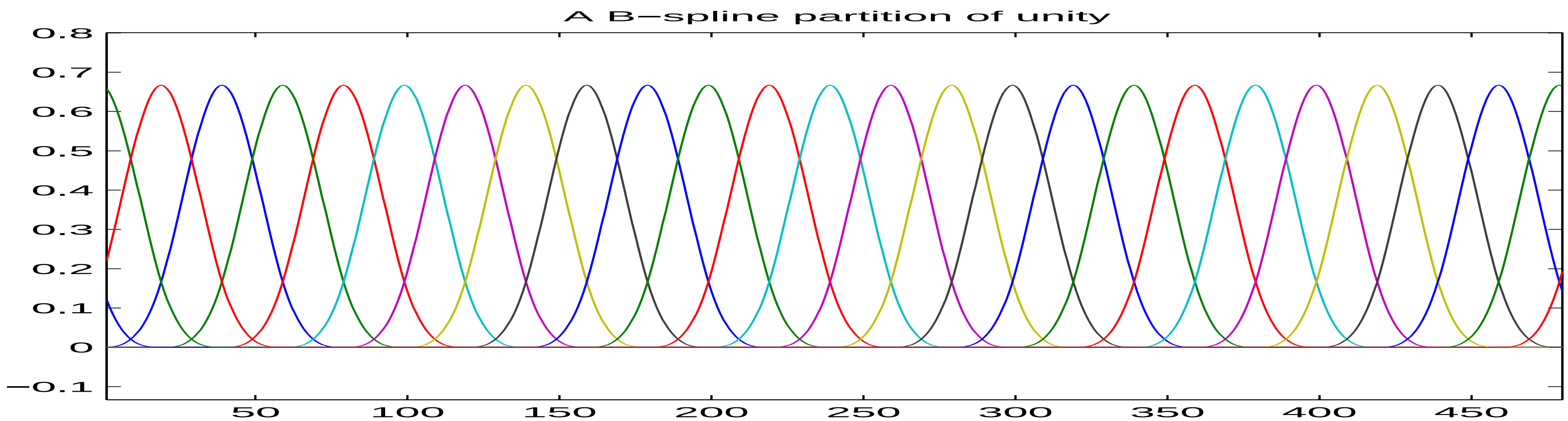
THEOREM:

- For any automorphism α of G the mapping $f \mapsto \alpha^*(f)$ is an isomorphism on $\boldsymbol{S}_0(G)$; $[with(\alpha^* f)(x) = f(\alpha(x))], x \in G$.
- $\mathcal{F}\boldsymbol{S}_0(G) = \boldsymbol{S}_0(\hat{G})$; (Invariance under the Fourier Transform);
- $T_H \boldsymbol{S}_0(G) = \boldsymbol{S}_0(G/H)$; (Integration along subgroups);
- $R_H \boldsymbol{S}_0(G) = \boldsymbol{S}_0(H)$; (Restriction to subgroups);
- $\boldsymbol{S}_0(G_1)\hat{\otimes}\boldsymbol{S}_0(G_2) = \boldsymbol{S}_0(G_1 \times G_2)$. (tensor product stability);

Bounded Uniform Partitions of Unity

Definition 3. *A bounded family* $\Psi = (\psi_n)_{n \in \mathbb{Z}^d}$ *in a Banach algebra* $(\mathbf{A}, \|\cdot\|_A)$ *is a regular A-Bounded Uniform Partition of Unity if*

$$\sum_{n \in \mathbb{Z}^d} \psi(x - n) = 1 \quad \textit{for all} \quad x \in \mathbb{R}^d$$



Wiener Amalgam Characterization of $S_0(\mathbb{R}^d)$

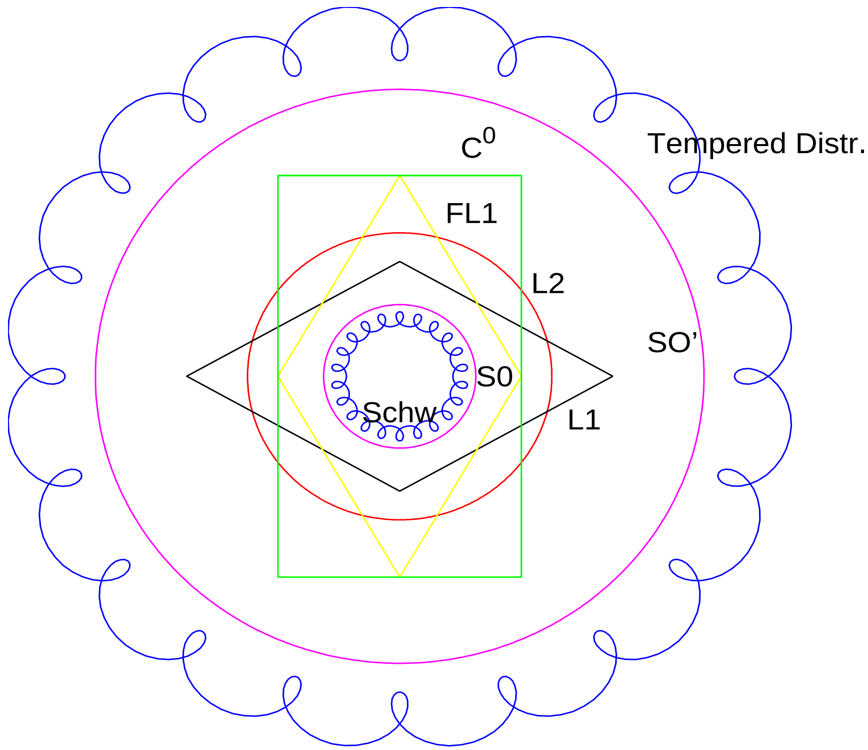
Since the Fourier algebra $\big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)}\big)$ is known to be a Banach algebra, densely embedded into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ (according to the Riemann Lebesgue Lemma), and also is isometrically TF-shift invariant we have $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$. It turns out that it is exactly the space of all elements in $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ such that the (trivial, pointwise) decomposition of f as $f = \sum_n f\psi_n$ is absoutely convergent in the Fourier algebra, or equivalenty, such that the (equivalent) norm $\|f\|_{\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1,\boldsymbol{\ell}^1)} = \sum_{n\in\mathbb{Z}^d} \|\mathcal{F}(f\psi_n)\|_1$ is finite (of course different BUPUs of the same kind define the same space and equivalent norms).
Spaces defined by the behaviour of $\|f\cdot\psi_n\|_{\boldsymbol{B}}$ in some sequence space norm Y are called Wiener amalgam spaces with *local norm* $\boldsymbol{B}$ and *global behaviour* described by Y. As one may expect $\boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \boldsymbol{\ell}^\infty)(\mathbb{R}^d)$. The topic of Wiener amalgam spaces would make a separate topic [3, 4, 5, 9].

Basic properties of $S_0'(\mathbb{R}^d)$ resp. $S_0'(G)$

THEOREM: (Consequences for the dual space)

- $\sigma \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is in $\boldsymbol{S_0'}(\mathbb{R}^d)$ if and only if $V_g\sigma$ is bounded;
- w^*-convergence in $\boldsymbol{S_0'}(\mathbb{R}^d)$ is equivalent to pointwise convergence of $V_g\sigma$;
- $\big(\boldsymbol{S_0'}(G), \|\cdot\|_{\boldsymbol{S_0'}}\big)$ is a Banach space with a translation invariant norm;
- $\boldsymbol{S_0'}(G) \subseteq \mathcal{S}'(G)$, i.e. $\boldsymbol{S_0'}(G)$ consists of tempered distributions;
- $\boldsymbol{P}(G) \subseteq \boldsymbol{S_0'}(G) \subseteq \boldsymbol{Q}(G)$; (sits between pseudo- and quasimeasures)
- $T(G) = W(G)' \subseteq \boldsymbol{S_0'}(G)$; (contains translation bounded measures);

Schwartz space, S_0, L^2, S_0', tempered distributions



Basic properties of $S_0(\mathbb{R}^d)$ continued

THEOREM:

- the Generalized Fourier Transforms, defined by transposition
$$\langle \hat{\sigma}, f \rangle = \langle \sigma, \hat{f} \rangle,$$
for $f \in \boldsymbol{S}_0(\hat{G}), \sigma \in \boldsymbol{S}_0{}'(G)$, satisfies $\mathcal{F}(\boldsymbol{S}_0'(G)) = \boldsymbol{S}_0'(\hat{G})$.

- $\sigma \in \boldsymbol{S}_0'(G)$ is H-periodic, i.e. $\sigma(f) = \sigma(T_h f)$ for all $h \in H$, iff there exists $\dot{\sigma} \in \boldsymbol{S}_0'(G/H)$ such that $\langle \sigma, f \rangle = \langle \sigma, T_H f \rangle$.

- $\boldsymbol{S}_0'(H)$ can be identified with a subspace of $\boldsymbol{S}_0'(G)$, the injection i_H being given by
$$\langle i_H \sigma, f \rangle := \langle \sigma, R_H f \rangle.$$
For $\sigma \in \boldsymbol{S}_0'(G)$ one has $\sigma \in i_H(\boldsymbol{S}_0'(H))$ iff $\mathrm{supp}(\sigma) \subseteq H$.

The Usefulness of $S_0(\mathbb{R}^d)$

Theorem 1. (**Poisson's formula**) *For $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ and any discrete subgroup H of $\mathbb{R}^d$ with compact quotient the following holds true: There is a constant $C_H > 0$ such that*

$$\sum_{h \in H} f(h) = C_H \sum_{l \in H^\perp} \hat{f}(l) \tag{1}$$

with absolute convergence of the series on both sides.

By duality one can express this situation as the fact that the Comb-distribution $\mu_{\mathbb{Z}^d} = \sum_{k \in Z^d} \delta_k$, as an element of $\boldsymbol{S}_0'(\mathbb{R}^d)$ is invariant under the (generalized) Fourier transform. Sampling corresponds to the mapping $f \mapsto f \cdot \mu_{\mathbb{Z}^d} = \sum_{k \in \mathbb{Z}^d} f(k)\delta_k$, while it corresponds to convolution with $\mu_{\mathbb{Z}^d}$ on the Fourier transform side = periodization along $(\mathbb{Z}^d)^\perp = \mathbb{Z}^d$ of the Fourier transform $\hat{f}$. For $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ all this makes perfect sense.

Regularizing sequences for (S_0, L^2, S_0')

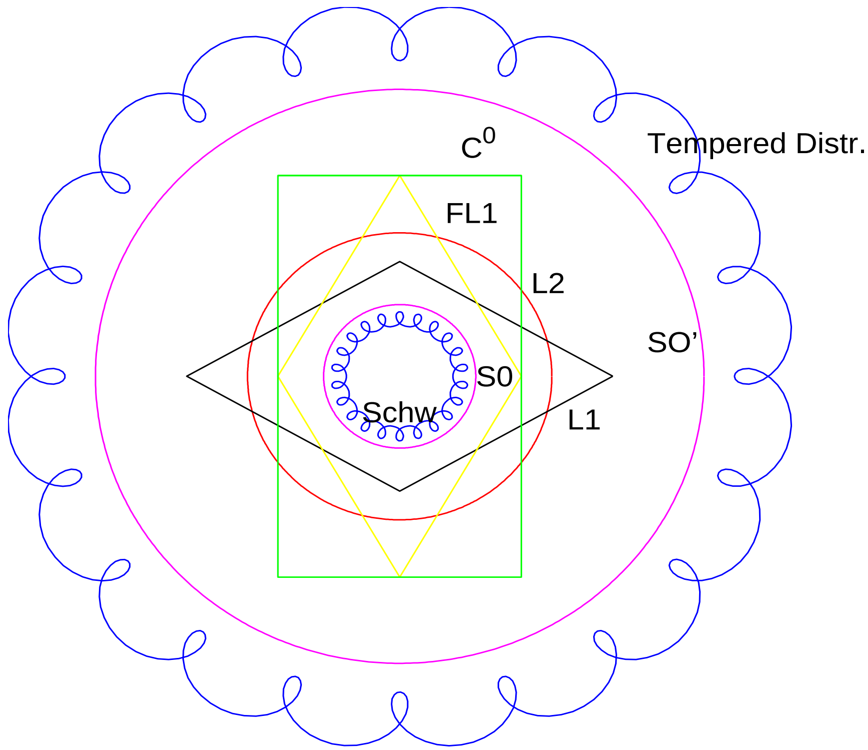
Wiener amalgam convolution and pointwise multiplier results imply that

$$S_0(\mathbb{R}^d) \cdot (S_0'(\mathbb{R}^d) * S_0(\mathbb{R}^d)) \subseteq S_0(\mathbb{R}^d) \quad S_0(\mathbb{R}^d) * (S_0'(\mathbb{R}^d) \cdot S_0(\mathbb{R}^d)) \subseteq S_0(\mathbb{R}^d)$$

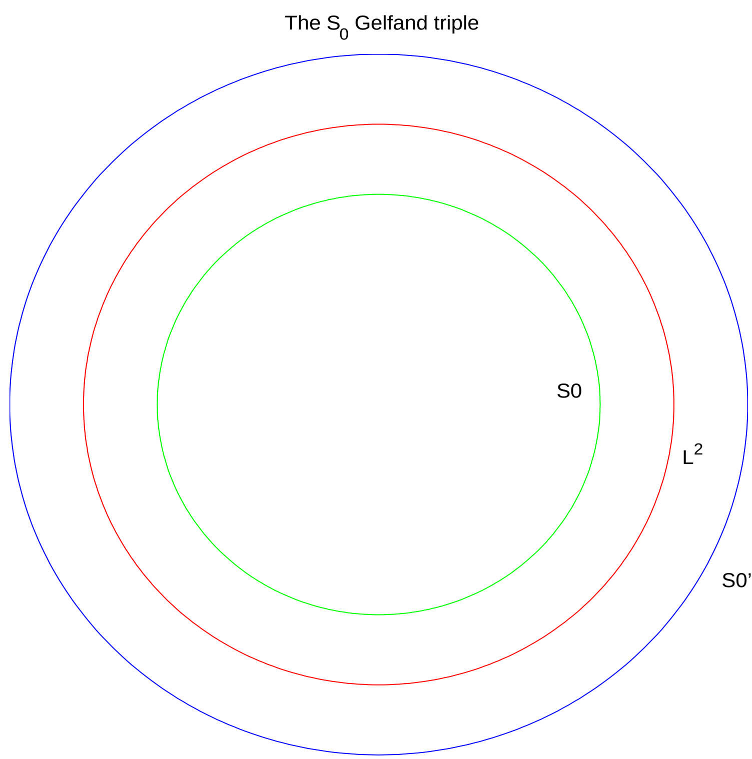
e.g. $S_0(\mathbb{R}^d) * S_0'(\mathbb{R}^d) = W(\mathcal{F}L^1, \ell^1) * W(\mathcal{F}L^\infty, \ell^\infty) \subseteq W(\mathcal{F}L^1, \ell^\infty)$.

Let now $h \in \mathcal{F}L^1(\mathbb{R}^d)$ be given with $h(0) = 1$. Then the dilated version $h_n(t) = h(t/n)$ are a uniformly bounded family of multipliers on (S_0, L^2, S_0'), tending to the identity operator in a suitable way. Similarly, the usual Dirac sequences, obtained by compressing a function $g \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(x)dx = 1$ are showing a similar behavior: $g_n(t) = n \cdot g(nt)$ Following the above rules the combination of the two procedures, i.e. product-convolution or convolution-product operators of the form provide suitable regularizers: $A_n f = g_n * (h_n \cdot f)$ or $B_n f = h_n \cdot (g_n * f)$.

Schwartz space, S_0, L^2, S_0', tempered distributions



The Gelfand Triple (S_0, L^2, S_0')



The Fourier transform is a prototype of a Gelfand triple isomorphism.

EX1: The Fourier transform as Gelfand Triple Automorphism

Theorem 2. *Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:*

(1) *$\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}^d})$,*

(2) *$\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}^d})$,*

(3) *$\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}^d})$.*

Furthermore we have that Parseval's formula

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \tag{2}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

The properties of Fourier transform can be expressed by a Gelfand bracket

$$\langle f, g \rangle_{(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')} = \langle \hat{f}, \hat{g} \rangle_{(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')} \tag{3}$$

which combines the functional brackets of dual pairs of Banach spaces and of the inner-product for the Hilbert space.

One can characterize the Fourier transform as the *uniquely determined* unitary Gelfand triple automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ which maps pure frequencies into the corresponding Dirac measures (and vice versa). [1]

One could equally require that TF-shifted Gaussians are mapped into FT-shifted Gaussians, relying on $\mathcal{F}(M_\omega T_x f) = T_{-\omega} M_x \mathcal{F} f$ and the fact that $\mathcal{F} g_0 = g_0$, with $g_0(t) = e^{-\pi |t|^2}$.

[1] as one would expect in the case of a finite Abelian group.

EX.2: The Kernel Theorem for general operators in $\mathcal{L}(S_0, S_0')$

Theorem 3. *If* K *is a bounded operator from* $\boldsymbol{S}_0(\mathbb{R}^d)$ *to* $\boldsymbol{S}_0'(\mathbb{R}^d)$, *then there exists a unique kernel* $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ *such that* $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ *for* $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, *where* $g \otimes f(x, y) = g(x)f(y)$.

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Formally sometimes one writes by “abuse of language”

$$Kf(x) = \int_{\mathbb{R}^d} k(x, y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)f(y)dyg(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)g(x)f(y)dxdy.$$

This result is the "outer shell of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.

Again the complete picture can again be best expressed by a unitary Gelfand triple isomorphism. We first describe the innermost shell:

Theorem 4. *The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ on the kernels.*
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if the corresponding operator K maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, but not only in a bounded way, but also continuously from w^*-topology into the norm topology of $\boldsymbol{S}_0(\mathbb{R}^d)$.

Remark: Note that for "regularizing" kernels in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ the usual identification (recall that the entry of a matrix $a_{n,k}$ is the coordinate number n of the image of the $n-$th unit vector under that action of the matrix $A = (a_{n,k})$:

$$k(x,y) = K(\delta_y)(x) = \delta_x(K(\delta_y).$$

Since $\delta_y \in \boldsymbol{S}_0'(\mathbb{R}^d)$ and consequently $K(\delta_y) \in \boldsymbol{S}_0(\mathbb{R}^d)$ the pointwise evaluation makes sense.

With this understanding our claim is that the kernel theorem provides a (unitary) isomorphism between the Gelfand triple (of kernels) $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ into the Gelfand triple of operator spaces

$$\left(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)),\, \mathcal{HS},\, \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\right).$$

The Kohn Nirenberg Symbol and Spreading Function

The Kohn-Nirenberg symbol $\sigma(T)$ of an operator T (respectively its *symplectic* Fourier transform, the *spreading distribution* $\eta(T)$ of T) can be obtained from the kernel using some automorphism and a partial Fourier transform, which again provide unitary Gelfand isomorphisms. In fact, the symplectic Fourier transform is another unitary Gelfand Triple (involutive) automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

Theorem 5. *The correspondence between an operator T with kernel K from the Banach Gelfand triple $\big(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\big)$ and the corresponding spreading distribution $\eta(T) = \eta(K)$ in $\boldsymbol{S}_0'(\mathbb{R}^{2d})$ is the uniquely defined Gelfand triple isomorphism between $\big(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d))\big)$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ which maps the time-frequency shift operators $M_y \circ T_x$ onto the Dirac measure $\delta_{(x,y)}$.*

Kohn-Nirenberg and Spreading Symbols of Operators

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x+\frac{y}{2}, x-\frac{y}{2})$
- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y-x)$
- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$
- *partial Fourier transform in the first variable:* $\mathcal{F}_1$
- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

Kohn-Nirenberg correspondence

1. Let σ be a tempered distribution on $\mathbb{R}^d$ then the operator with *symbol* σ

$$K_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}d\omega$$

is the *pseudodifferential operator* with Kohn-Nirenberg symbol σ.

$$\begin{aligned} K_\sigma f(x) &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\sigma(x,\omega)e^{-2\pi i(y-x)\cdot\omega}d\omega\Big)f(y)dy \\ &= \int_{\mathbb{R}^d} k(x,y)f(y)dy. \end{aligned}$$

2. Formulas for the (integral) kernel k: $k = \mathcal{T}_a\mathcal{F}_2\sigma$

$$\begin{aligned} k(x,y) &= \mathcal{F}_2\sigma(\eta, y-x) = \mathcal{F}_1^{-1}\widehat{\sigma}(x, y-x) \\ &= \widehat{\sigma}(\eta, y-x)e^{2\pi i\eta\cdot x}d\eta. \end{aligned}$$

3. The *spreading representation* of the same operator arises from the identity

$$K_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta, u) M_\eta T_{-u} f(x) du d\eta.$$

$\widehat{\sigma}$ is called the spreading function of the operator K_σ.

If $f, g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, then the so-called *Rihaczek distribution* is defined by

$$R(f,g)(x,\omega) = e^{-2\pi i x \cdot \omega} \overline{\widehat{f}(\omega)} g(x).$$

and belongs to $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, for any $\sigma \in \mathcal{S}'(\mathbb{R}^d)$

$$\langle \sigma, R(f,g) \rangle = \langle K_\sigma f, g \rangle$$

is well-defined and describes a uniquely defined operator from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

Weyl correspondence

1. Let σ be a tempered distribution on $\mathbb{R}^d$ then the operator

$$L_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\xi, u) e^{-\pi i \xi \cdot u} f(x) du d\xi$$

is called the *pseudodifferential operator* with *symbol* σ. The map $\sigma \mapsto L_\sigma$ is called the *Weyl transform* and σ the Weyl symbol of the operator L_σ.

$$\begin{aligned} L_\sigma f(x) &= \iint_{\mathbb{R}^{2d}} \widehat{\sigma} e^{-\pi i u \cdot \xi} T_{-u} M_\xi f(x) du d\xi \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \widehat{\sigma}(\xi, y - x) e^{-2\pi i \xi \frac{x+y}{2}} \right) f(y) dy. \end{aligned}$$

2. Formulas for the kernel k from the KN-symbol: $k = \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma$

$$\begin{aligned} k(x,y) &= \mathcal{F}_1^{-1}\widehat{\sigma}\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{F}_2\sigma\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{F}_2^{-1}\sigma\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma. \end{aligned}$$

3. $\langle L_\sigma f, g\rangle = \langle k, g \otimes \overline{f}\rangle$. (Weyl operator vs. kernel)

If $f, g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, then the *cross Wigner distribution* of f, g is defined by

$$W(f,g)(x,y) = \int_{\mathbb{R}^d} f(x+t/2)\overline{g}(x-t/2)e^{-2\pi i \omega\cdot t}dt = \mathcal{F}_2\mathcal{T}_s(f \otimes \overline{g})(x,\omega).$$

and belongs to $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, for any $\sigma \in \mathcal{S}'(\mathbb{R}^d)$

$$\langle \sigma, W(f,g) \rangle = \langle L_\sigma f, g \rangle$$

is well-defined and describes a uniquely defined operator L_σ from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

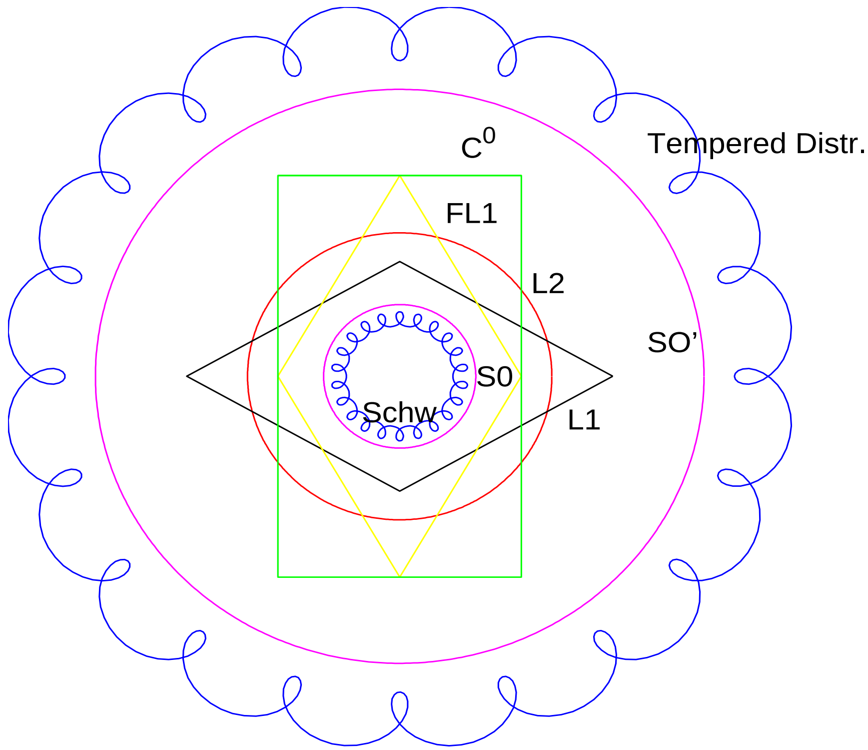
$$(\mathcal{U}\sigma)(\xi, u) = \mathcal{F}^{-1}(e^{\pi i u \cdot \xi} \widehat{\sigma}(\xi, u)).$$

$$K_{\mathcal{U}\sigma} = L_\sigma$$

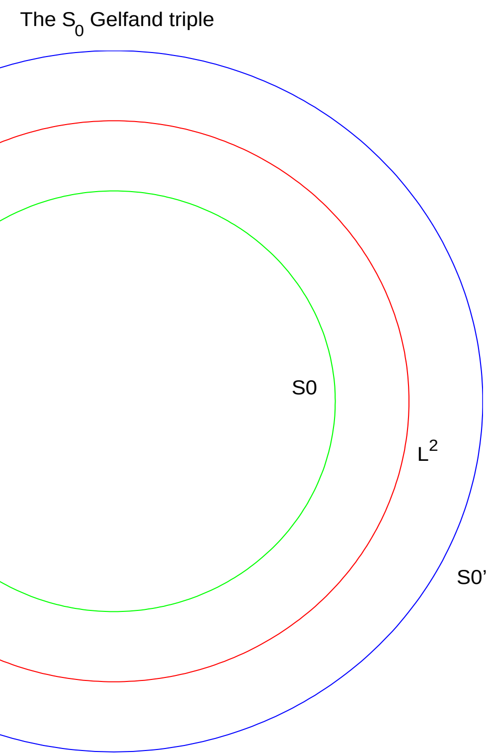
describes the connection between the Weyl symbol and the operator kernel.

In all these considerations the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ can be correctly replaced by $\boldsymbol{S}_0(\mathbb{R}^d)$ and the tempered distributions by $\boldsymbol{S}_0'(\mathbb{R}^d)$.

Schwartz space, S_0, L^2, S_0', tempered distributions



The Gelfand Triple (S_0, L^2, S_0')



Fourier transform is a prototype of a unitary Gelfand triple isomorphism.

Back to the Classical Problems: Fourier Inversion

We want to give an interpretation of the usual summability methods showing the relevance of $\boldsymbol{S}_0(\mathbb{R}^d)$ in this business (see [7, 6] for a number of sufficient conditions for f to belong to $\boldsymbol{S}_0(\mathbb{R}^d)$: in the case of $d = 1$ a sufficient condition is that f is an integrable and piecewise linear function with not too irregular nodes, or $f, f', f'' \in \boldsymbol{L}^1(\mathbb{R})$. Recall $\boldsymbol{S}_0 \cdot \mathcal{F}\boldsymbol{L}^1 \subseteq \boldsymbol{S}_0$.
The typical reasoning where summability methods are applied is in order to give the usual inversion formula $f(t) = \int_{\mathbb{R}^d} \hat{f}(s) e^{2\pi i s t} ds$ a meaning, even if $\hat{f} \notin \boldsymbol{L}^1(\mathbb{R}^d)$. This is done by multiplying it with some integrable and continuous function h, with $h(0) = 1$, which is then dilated. In other words, one replaces the integrand $f(s)$ by $f(s)h(\varrho s)$, for some small value of ϱ.
It can be shown for all the "good classical kernels" that they are of this form, for some $h \in \boldsymbol{S}_0(\mathbb{R}^d)$. This means of course that $s \mapsto h(\varrho s)$ is the Fourier transform of some compressed $\boldsymbol{S}_0(\mathbb{R}^d)$ version $St_\varrho g$ of some function g (with $\hat{g} = h$) and hence $St_\varrho g * f$ converges to f in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$.

Characterize translation invariant operators as convolution operators

Let us start by citing the introduction of Larsen's book [10]: Given any pair of Banach space of [equivalence classes] of functions on a locally compact Abelian group one may ask: "what are the bounded linear operators between them which are also commuting with translations". Let us call those spaces $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ and $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$, and ask for $H_{\mathcal{G}}(\boldsymbol{B}_1, \boldsymbol{B}_2)$:

$$H_{\mathcal{G}}(\boldsymbol{B}_1, \boldsymbol{B}_2) = \{T : \boldsymbol{B}_1 \mapsto \boldsymbol{B}_2, \text{bd. and linear}, \, T_x \circ T = T \circ T_x, \, \forall x \in \mathcal{G}\,\}. \tag{4}$$

In most cases one shows that it equals $H_{\boldsymbol{L}^1}(\boldsymbol{B}_1, \boldsymbol{B}_2)$, defined as follows:

$$H_{\boldsymbol{L}^1}(\boldsymbol{B}_1, \boldsymbol{B}_2) = \{T : \boldsymbol{B}_1 \mapsto \boldsymbol{B}_2, \text{bd. and linear}, \, T(g*f) = g*Tf\,, \, \forall g \in \boldsymbol{L}^1\,\}, \tag{5}$$

which will be called the space of all $\boldsymbol{L}^1$-module homomorphism between $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ and $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ (cf. Rieffel! [11]). There are not too many cases where this space can be identified in an easy and complete way:

1. Wendel's Theorem, $p = 1$, ([10])

$$\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1, \boldsymbol{L}^1)(G) \approx \boldsymbol{M}_b(G)$$

   or in words: The bounded operator on $\boldsymbol{L}^1$ commuting with $\boldsymbol{L}^1$-convolution are exactly the convolution operators with bounded measures $\mu \in \boldsymbol{M}_b(G)$.

2. $p = 2$:

$$\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^2, \boldsymbol{L}^2)(G) \approx \mathcal{F}\boldsymbol{L}^\infty$$

   i.e. the bounded $\boldsymbol{L}^1$-homomorphism on $\boldsymbol{L}^2(G)$ are exactly the operators of the form $f \mapsto \mathcal{F}^{-1}(h\hat{f})$, for some $h \in \boldsymbol{L}^\infty$. By a suitable interpretation of $\mathcal{F}\boldsymbol{L}^\infty$ it can is called the space $\boldsymbol{P}(G)$ of pseudo-measures, and T is represented as convolution with a pseudo-measure.

3. For general $p \in (1,\infty)$ one can show that $H_{\boldsymbol{L}^1}(\boldsymbol{L}^p,\boldsymbol{L}^p)$ equals $H_{\boldsymbol{L}^1}(\boldsymbol{L}^{p'},\boldsymbol{L}^{p'})$ for $1/p+1/p'=1$. It follows therefrom via complex interpolation (with the choice $\theta=0.5$) that

$$H_{\boldsymbol{L}^1}(\boldsymbol{L}^p,\boldsymbol{L}^p) \subseteq \mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^2,\boldsymbol{L}^2) = \mathcal{F}\boldsymbol{L}^\infty.$$

   This implies that in the context of $\boldsymbol{L}^p$-spaces (except for $p=\infty$) one can describe $\boldsymbol{L}^1$-homomorphism as convolution operators with a pseudo-measure.

4. As soon as one wants to generalize this characterization of $\boldsymbol{L}^1$-homomorphism to the case where the two spaces are not equal anymore, i.e. when one is interested in the characterization of $H_{\boldsymbol{L}^1}(\boldsymbol{L}^p,\boldsymbol{L}^q)$, for some pair of values p and q one finds that pseudo-measures are not sufficient anymore! Just note that obviously any $\boldsymbol{L}^2$-function h defines a

bounded linear operator from $\boldsymbol{L}^1$ into $\boldsymbol{L}^2$ via convolution, since obviously $\widehat{h * f} = \widehat{h} \cdot \widehat{f} \in \mathcal{F}\boldsymbol{L}^2 \cdot \mathcal{F}\boldsymbol{L}^1 \subseteq \mathcal{F}\boldsymbol{L}^2$. [2]

The theory of quasi-measures was a vehicle providing a way out of this dilemma. From todays view-point quasi-measures are exactly the (tempered) distributions which equal locally pseudo-measures, but the original definition was much more involved (going back to Gaudry, see [8] the equivalence was established by Cowling in [2]).

In contrast, from the point of view of the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ this question has a fairly simple answer, however. Since $\boldsymbol{S}_0(G) \subseteq \boldsymbol{L}^p(G) \subseteq \boldsymbol{S}_0'(G)$ for any value of $p \in [1, \infty]$ (due to the minimality of $\boldsymbol{S}_0(G)$, hence the maximility of $\boldsymbol{S}_0'(G)$) it is easy to observe the following natural embeddings:

$$H_{\boldsymbol{L}^1}(\boldsymbol{L}^p, \boldsymbol{L}^q) \hookrightarrow H_{\boldsymbol{L}^1}(\boldsymbol{S}_0, \boldsymbol{S}_0') \approx \boldsymbol{S}_0'(G). \tag{6}$$

[2] We only need $\boldsymbol{L}^1 * \boldsymbol{L}^2 \subseteq \boldsymbol{L}^2$!?

We only have to recall the definition of the convolution of $\sigma \in \boldsymbol{S}_0'(G)$ with $f \in \boldsymbol{S}_0(G)$, indeed the standard interpretation of the convolution of a bounded linear functional with a test function applies:

$$\sigma * f(x) = \sigma(T_x \check{f}).$$

This also implies that $\boldsymbol{S}_0'(G) * \boldsymbol{S}_0(G) \subseteq \boldsymbol{C}_b(G)$. Hence it in fact possible to recover σ, given the operator $T : f \mapsto \sigma * f$, by means of the identity $\sigma(f) = T(\check{\sigma})(0)$.

It is of course not difficult to show that the generalized FT on $\boldsymbol{S}_0'(G)$ allows to describe T as a "multiplication operator on the FT side", by giving a meaning to the formula: $T(f) = \mathcal{F}^{-1}(\hat{\sigma} \cdot \hat{f})$. The transfer function $\hat{\sigma}$ is therefore an element of $\boldsymbol{S}_0'$, hence a quasi-measure. This fact has to be proven separately in the book of Larsen ([**?**]), because the space $\boldsymbol{Q}(\mathbb{R}^d)$ of quasi-measures is too large in order to be invariant with respect to the Fourier transform (leave alone the fact that the original definition of the space of quasi-measures was a quite complicated one).

5. Sometimes unbounded measures still have measures as Fourier transforms. The so-called chirp function $x \mapsto e^{\pi i|x|^2}$ is an excellent example, because it is even invariant under the Fourier transform. Dilated version therefore are mapped onto correspondingly inversly dilated chirp functions. The most general theory in this direction has been developed by Argabright and Gil de Lamadrid ([1]) in the 1970-th. It can also be subsumed in the S_0'-context.

Examples of Gelfand Triple Isomorphisms

1. The standard Gelfand triple $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$.

2. The family of orthonormal Wilson bases (obtained from Gabor families by suitable pairwise linear-combinations of terms with the same absolute frequency) extends the natural unitary identification of $\boldsymbol{L}^2(\mathbb{R}^d)$ with $\ell^1(I)$ to a unitary Banach Gelfand Triple isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ and $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(I)$.

   This isomorphism leeds to the observation that essentially the identification of $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ boils down to the identification of the bounded linear mappings from $\ell^1(I)$ to $\ell^\infty(I)$, which are of course easily recognized as $\ell^\infty(I \times I)$ (the bounded matrices). The fact that tensor products of $1D$-Wilson bases gives a characterization of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ over $\mathbb{R}^{2d}$ then gives the kernel theorem.

Automatic Gelfand-triple invertibility

Gröchenig and Leinert have shown (J. Amer. Math. Soc., 2004):
Theorem 6. *Assume that for* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *the Gabor frame operator*

$$S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g$$

is invertible as an operator on $\boldsymbol{L}^2(\mathbb{R}^d)$*, then it is also invertible on* $\boldsymbol{S}_0(\mathbb{R}^d)$ *and in fact on* $\boldsymbol{S}_0'(\mathbb{R}^d)$*.*
In other words: Invertibility at the level of the Hilbert space automatically !! implies that S *is (resp. extends to) an isomorphism of the Gelfand triple automorphism for* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$*.*
In a recent preprint K. Gröchenig shows among others, that invertibility of S follows already from a dense range of $S(\boldsymbol{S}_0(\mathbb{R}^d))$ in $\boldsymbol{S}_0(\mathbb{R}^d)$.

Robustness resulting from those three layers:

In the present situation one has also (in contrast to the "pure Hilbert space case") various robustness effects:
1) One has robustness against jitter error. Depending (only) on Λ and $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ one can find some $\delta_0 > 0$ such that the frame property is preserved (with uniform bounds on the new families) if any point $\lambda \in \Lambda$ is not moved more than by a distance of δ_0.
2) One even can replace the lattice generated by some non-invertible matrix $\mathbf{A}$ (applied to $\mathbb{Z}^{2d}$) by some "sufficiently similar matrix $\mathcal{B}$ and also preserve the Gabor frame property (with continuous dependence of the dual Gabor atom $\tilde{g}$ on the matrix $\mathbf{B}$) (joint work with N. Kaiblinger, Trans. Amer. Math. Soc.).

Stability of Gabor Frames with respect to Dilation (F/Kaibl.)

For a subspace $X \subseteq \boldsymbol{L}^2(\mathbb{R}^d)$ define the set

$$F_g = \big\{(g, L) \in X \times \mathrm{GL}(\mathbb{R}^{2d}) \text{ which generate a Gabor frame } \{\pi(Lk)g\}_{k\in\mathbb{Z}^{2d}}\big\}. \tag{7}$$

The set $F_{\boldsymbol{L}^2}$ need not be open (even for good ONBs!). But we have:

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Theorem 7. (i) *The set $F_{\boldsymbol{S}_0(\mathbb{R}^d)}$ is open in $\boldsymbol{S}_0(\mathbb{R}^d) \times \mathrm{GL}(\mathbb{R}^{2d})$.*
(ii) *$(g, L) \mapsto \widetilde{g}$ is continuous mapping from $F_{\boldsymbol{S}_0(\mathbb{R}^d)}$ into $\boldsymbol{S}_0(\mathbb{R}^d)$.*

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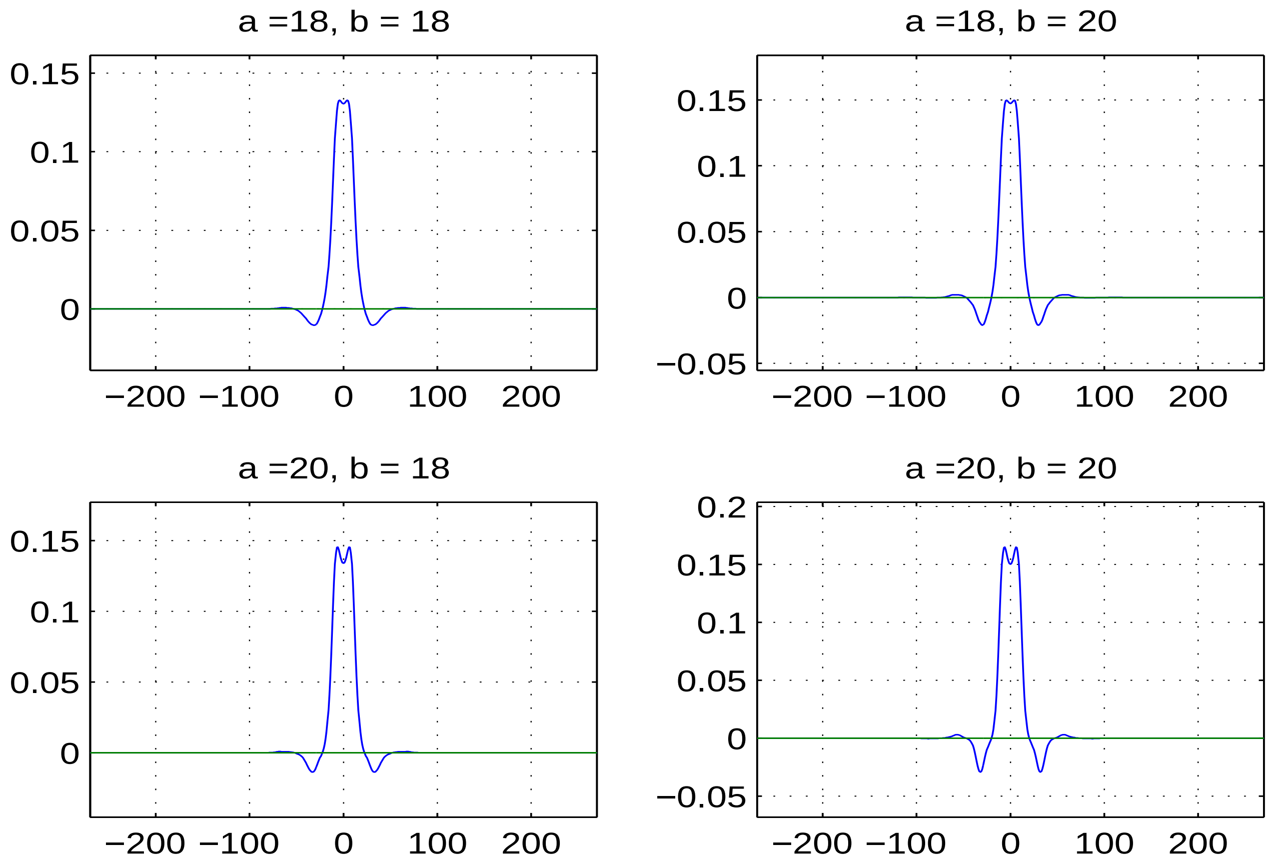
There is an analogous result for the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.
Corollary 3. (i) *The set* $F_{\mathcal{S}}$ *is open in* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \times \mathrm{GL}(\mathbb{R}^{2d})$.
(ii) *The mapping* $(g, L) \mapsto \widetilde{g}$ *is continuous from* $F_{\mathcal{S}}$ *into* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

On the continuous dependence of dual atoms on the TF-lattice



Quasi-interpolation and discretization

Together with N. Kaiblinger a paper on quasi-interpolation has just been published (J. Approx. Th.). We show that piecewise linear interpolation resp. quasi-interpolation (using for example cubic splines), i.e. operators of the form

$$Q_h f = \sum_{k\in\mathbb{Z}^d} f(hk) T_{hk} D_h \psi$$

are norm convergent to $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ in the $\boldsymbol{S}_0$-norm.
This is an important step for his work on the approximation of "continuous Gabor problems" by finite ones (handled computationally using MATLAB, for example), a subject which has been driven further to the context of Gabor Analysis (using code for the determination of dual Gabor atoms over finite Abelian groups in order to determine approximately solutions to the continuous question).

Quasi-interpolation and wavelet approximation

Since one has $\|e_n * f - f\|_{\mathbf{S}_0} \to 0$ for any $f \in \mathbf{S}_0(\mathbb{R}^d)$ for any Dirac sequence (e_n) and because sampling is a bounded operation from $\mathbf{S}_0(\mathbb{R}^d)$ into $\boldsymbol{\ell}^1(\mathbb{Z}^d)$ one also derives from this fact that (at least for suitable wavelets)

$$\|P_n f - f\|_{\mathbf{S}_0} \to 0 \quad \text{for} \quad f \in \mathbf{S}_0(\mathbb{R}^d).$$

where P_n is the projection onto the scale-space V_n (which - as we know is created by dilating the space V_0 appropriately. The result therefore follows from the explicit form the projection operator onto V_0 which is of the form $P_0 f = \sum_{\lambda \in \Lambda} f * \widetilde{\varphi}(\lambda) T_\lambda \varphi$.

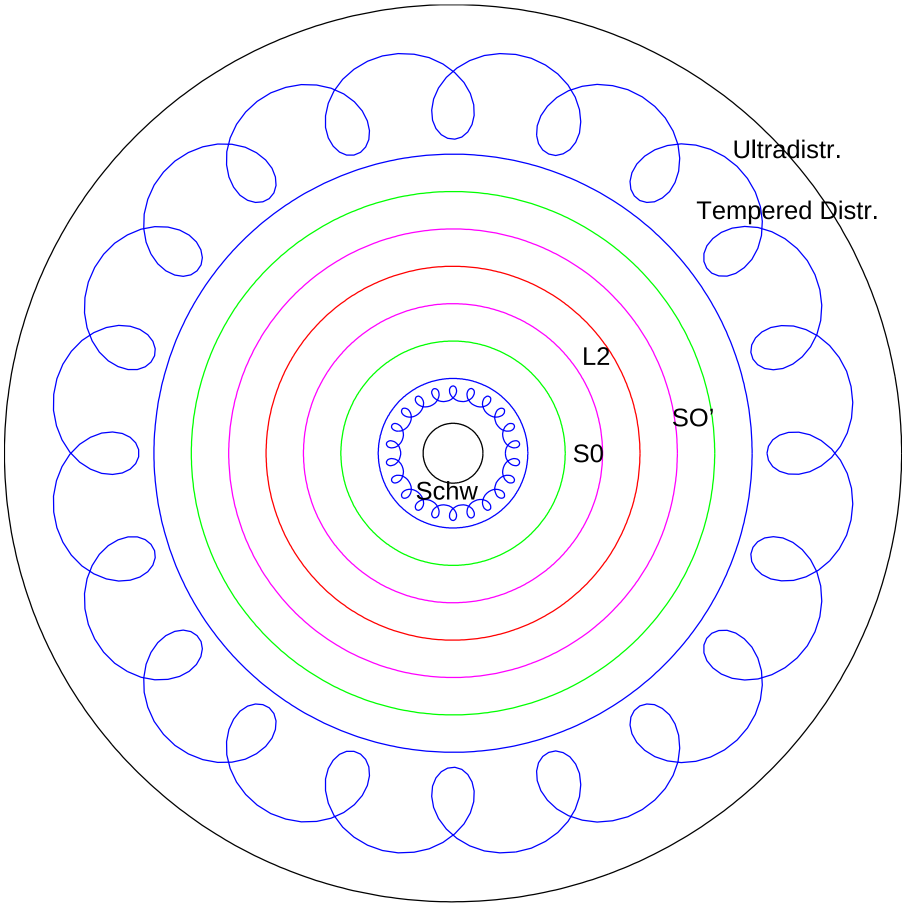
Note: Since the adjoint operator to the quasi-interpolation operator is of the form $Q^*(\sigma) = \sum_{\lambda \in \Lambda} \sigma(T_\lambda \psi) \delta_\lambda$ we also see (constructively) that (finite) discrete measures are w^*-dense in $\mathbf{S}_0'(\mathbb{R}^d)$.

OUTLOOK and SUMMARY

Of course both the Banach Gelfand Triple setting in general as well as the specific choice, with $\boldsymbol{B} = \boldsymbol{S}_0$ are of course "ubiquitous" in time-frequency analysis and Gabor analysis, in considerations about stability of Gabor systems, but also in the context of projective modules and non-communtantive tori ("noncommutative geometry") as treated by M. Rieffel or A. Connes (as discovered by Franz Luef, cf. his PhD thesis).
Returning to "classical Fourier analysis" it is not just the flexibility of the concept which allows to treat most of the basic results in complete generality, but also the fact that $\boldsymbol{S}_0(\mathbb{R}^d)$ appears to be a very suitable replacement for the Schwartz (-Bruhat) space [!except for PDE], e.g. in the context of abstract/conceptual harmonic analysis or for signal processing.
On the other hand $\boldsymbol{S}_0(\mathbb{R}^d)$ is "just the right reservoir" of kernels, when it comes to applications in classical summability theory, and practically all the "good kernels" belong to $\boldsymbol{S}_0(\mathbb{R}^d)$ (cf. joint work with Ferenc Weisz).

A Collection of Fourier Invariant Spaces

It is also possible to make use of radial weights of sub-exponential growth, and obtain in this way a family of Fourier invariant Banach spaces of test functions and corresponding spaces of ultra-distributions.
In this way one can give a time-frequency perspective on space of test functions and the corresponding spaces of so-called ultra-distributions, Shilov-classes and similar objects (such as variants of Shubin classes) of relevance in the theory of pseudo-differential operators.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Tempered Distr.
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The END!

THANK you for your attention! HGFei

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