Hans G. Feichtinger
NuHAG - University of Vienna
Leverhulm Visiting Professor at Edinburgh
http://www.nuhag.eu

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Mathematical Foundations of Gabor Analysis:
Time-frequency methods for Signal processing

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EE Dept., University of Edinburgh
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May 7, 2008

ABSTRACT

Goals of this presentation

- Explain the background of time-frequency analysis;
- What is Gabor analysis (within this context)?
- What are the mathematical tools and problems?
- How can these methods be used in signal processing?

My Personal Background (from AHA to CHA)

- Trained as an **Abstract Harmonic Analyst** (Adv. H. Reiter, Newcastle)
- working on function spaces on locally compact groups, distribution theory
- turning to applications (signal processing, image processing), wavelets
- doing **numerical work** on scattered data approximation, **Gabor analysis**
- heading nowadays **NuHAG**, a group of some **20 researchers at Univ. Vienna**: Details can be found at: www.nuhag.eu
  MATLAB code, literature, events, papers, etc.
- nowadays propagating **Conceptual Harmonic Analysis**: CHA

What is GABOR ANALYSIS about?

The original idea (D. Gabor, 1946) was to expand every signal into a sum with uniquely determined coefficients, by choosing integer time-frequency shifted copies of the normalized Gauss-function. Unfortunately there are many problems, and one has to resort to TF-lattices of the form $a\mathbb{Z} \times b\mathbb{Z}$, with $ab < 1$, which gives us some "redundancy".
PROBLEMS:

1. we have to work in an infinite dimensional Hilbert space ($\boldsymbol{L}^2(\mathbb{R}^d)$);

2. the system is non-orthogonal
   (hence it is not clear how to get coefficients a priori);

3. when we do computations we have to resort to "finite models"

4. how should one approximate the continuous situation by the finite one?

Key Players for Time-Frequency Analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

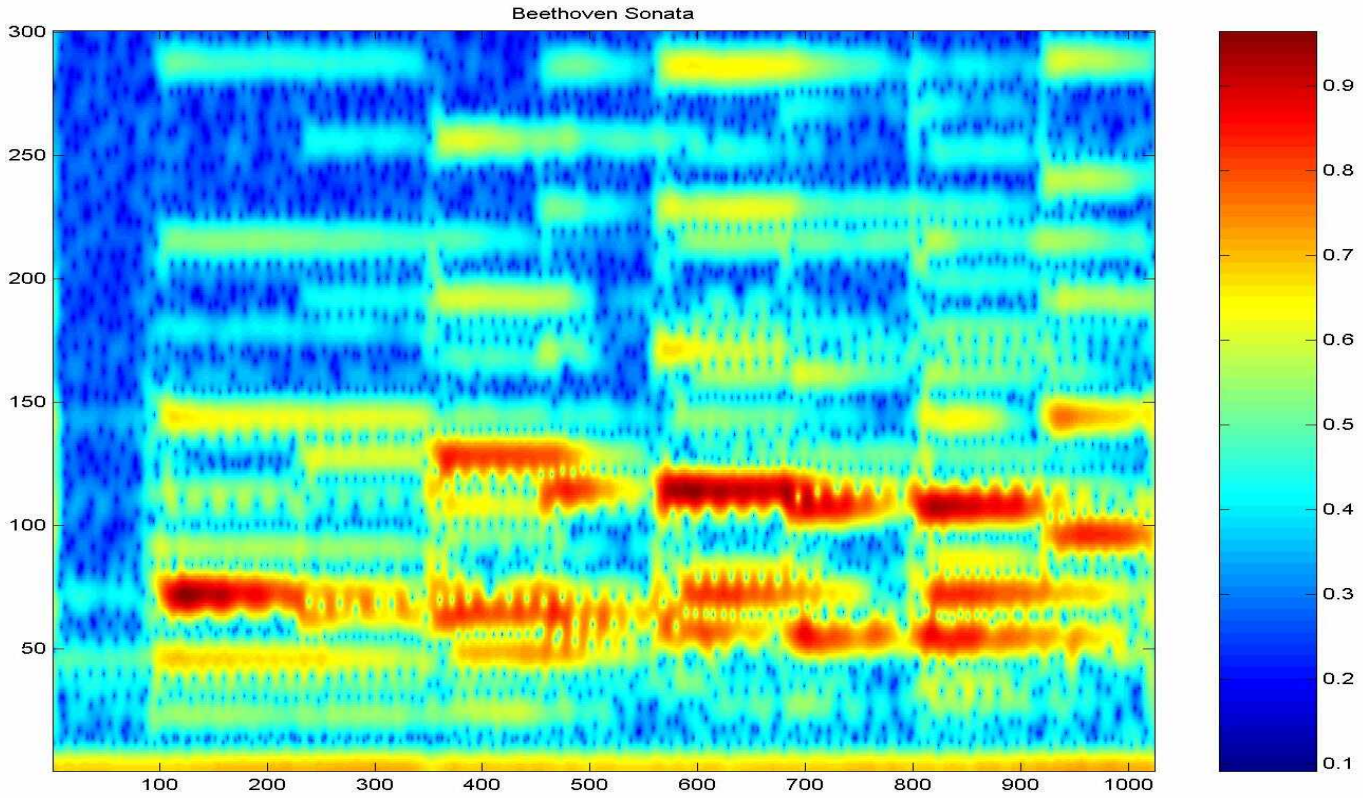
Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

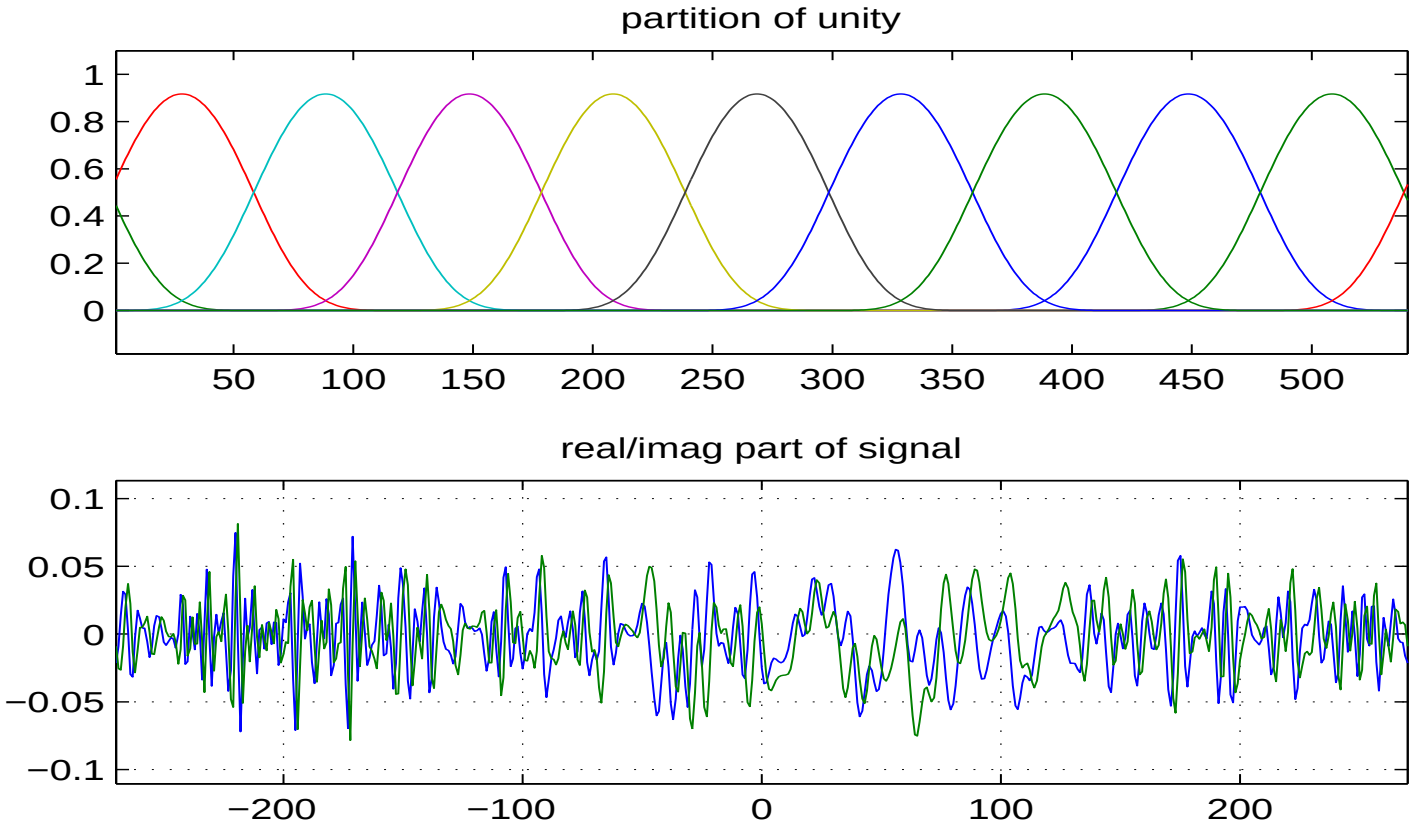
The Short-Time Fourier Transform

$$V_g f(\lambda) = V_g f(t,\omega) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \quad \lambda = (t,\omega);$$

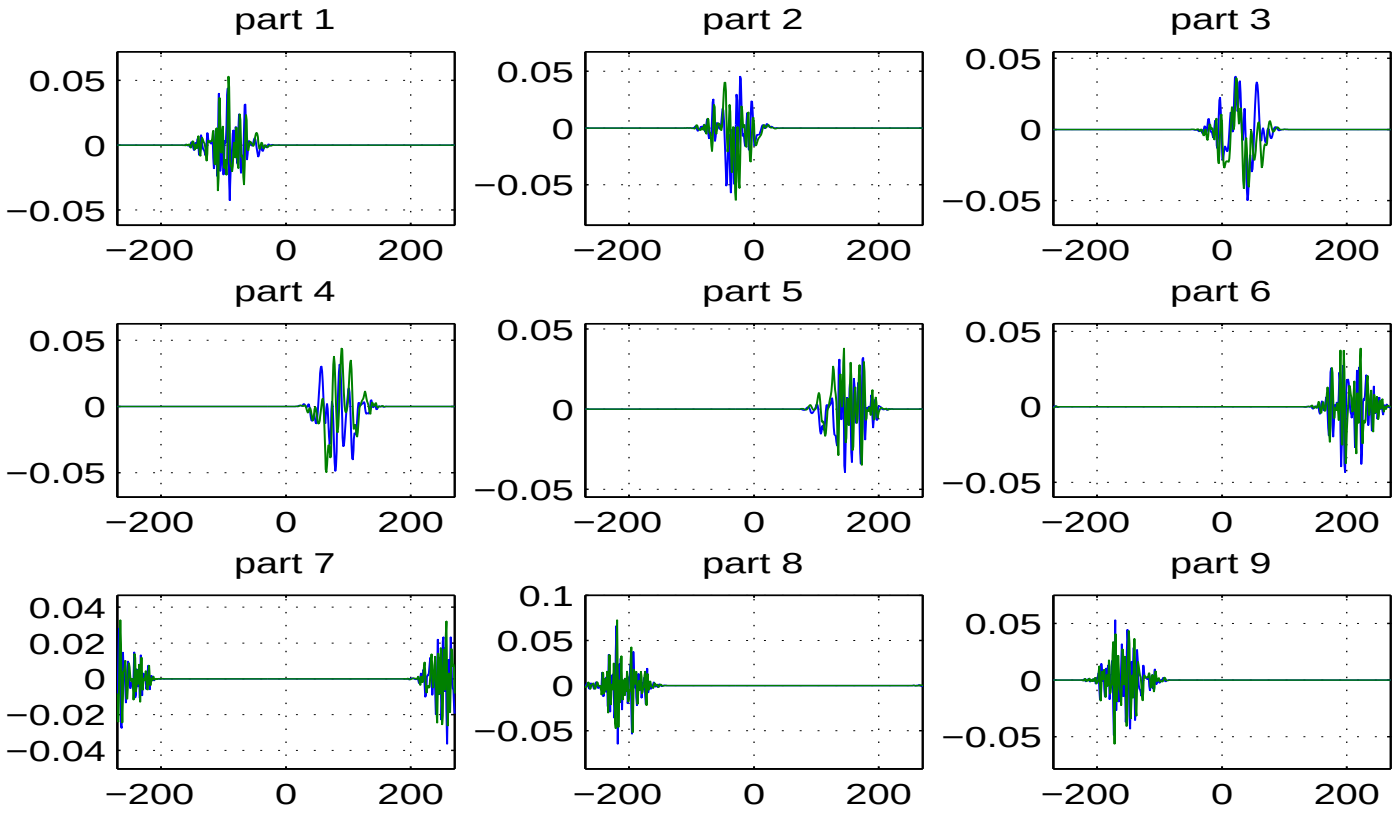
A Typical Musical STFT



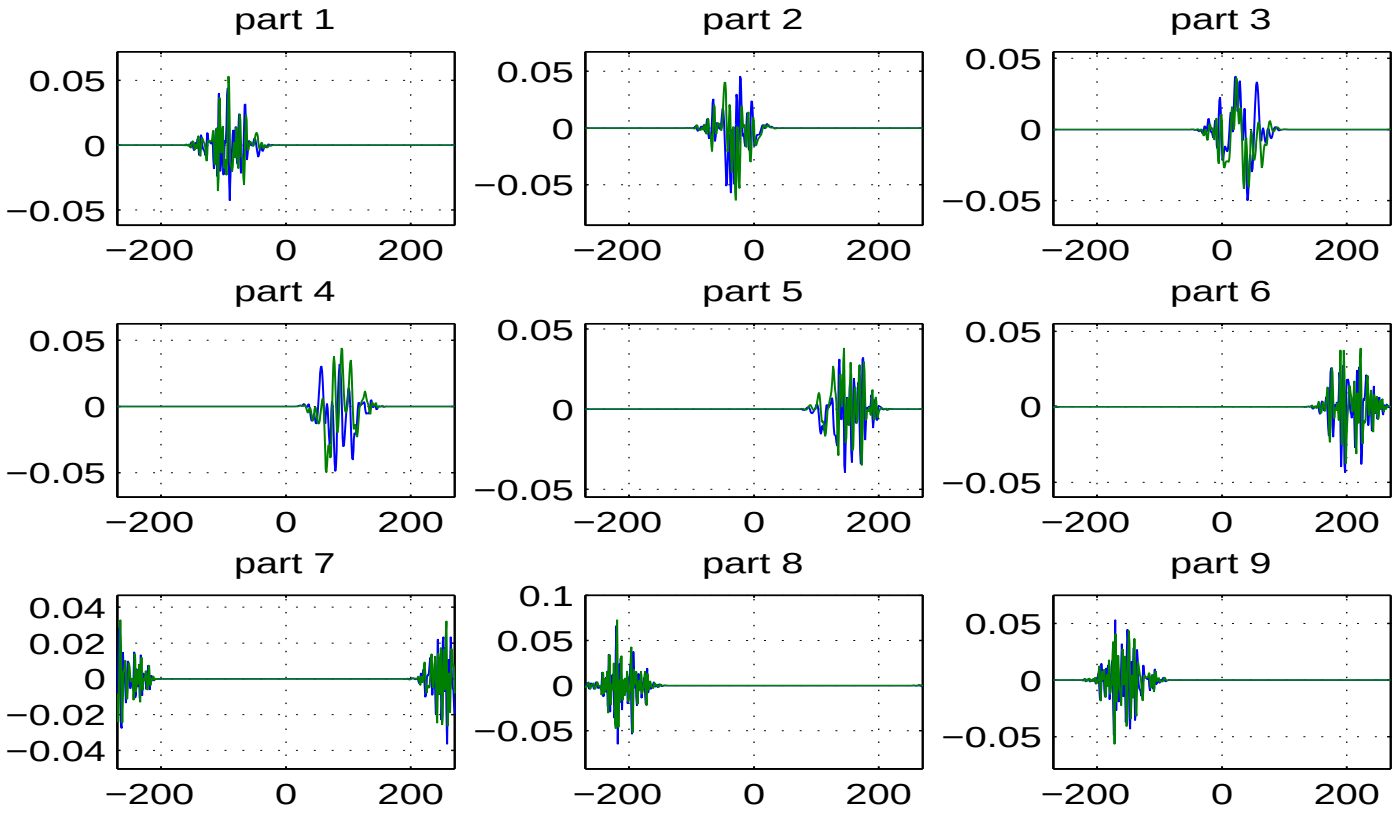
The idea of a "localized Fourier Spectrum"



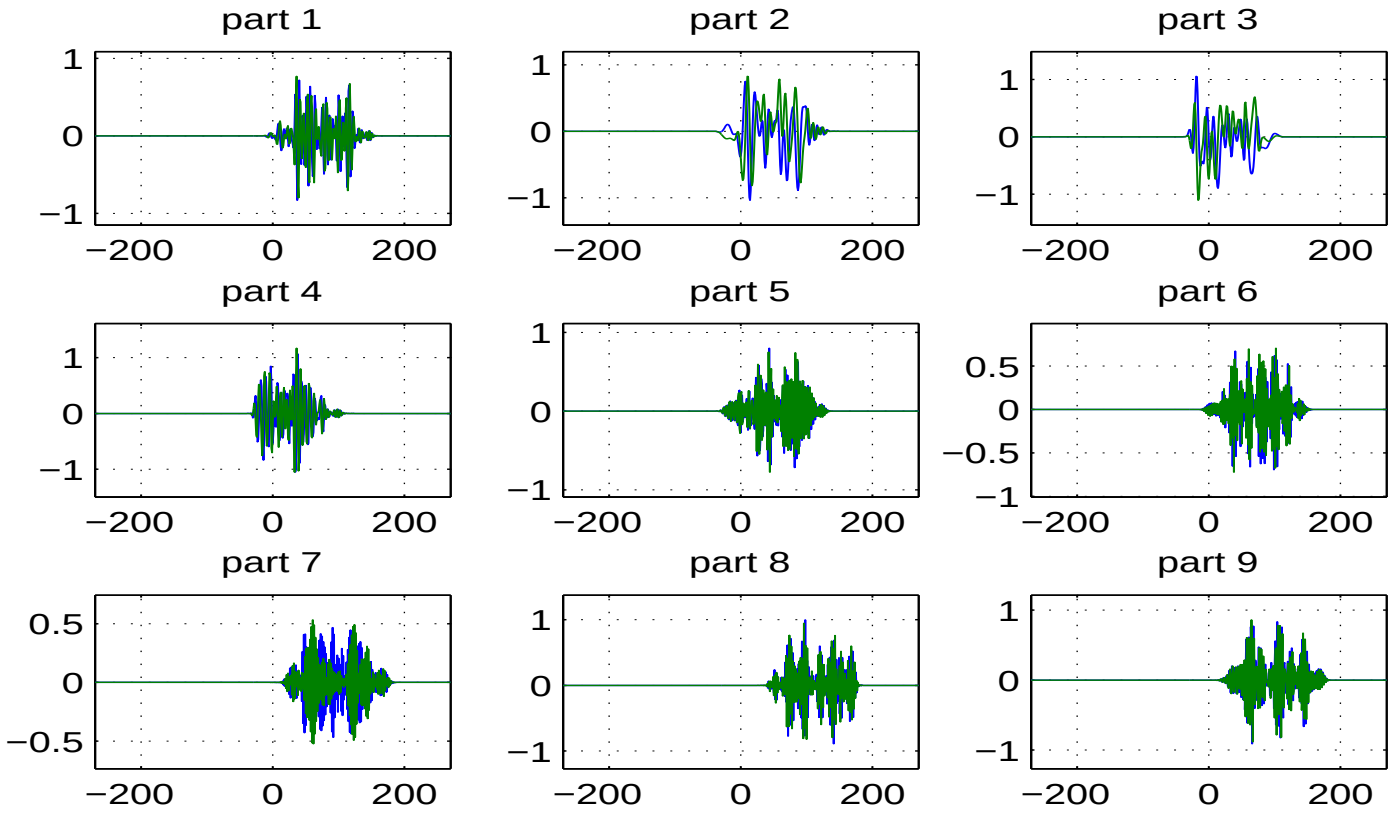
The localized Fourier transform (spectrogram)



The localized Fourier transform (spectrogram)



The 9 individual Fourier spectra



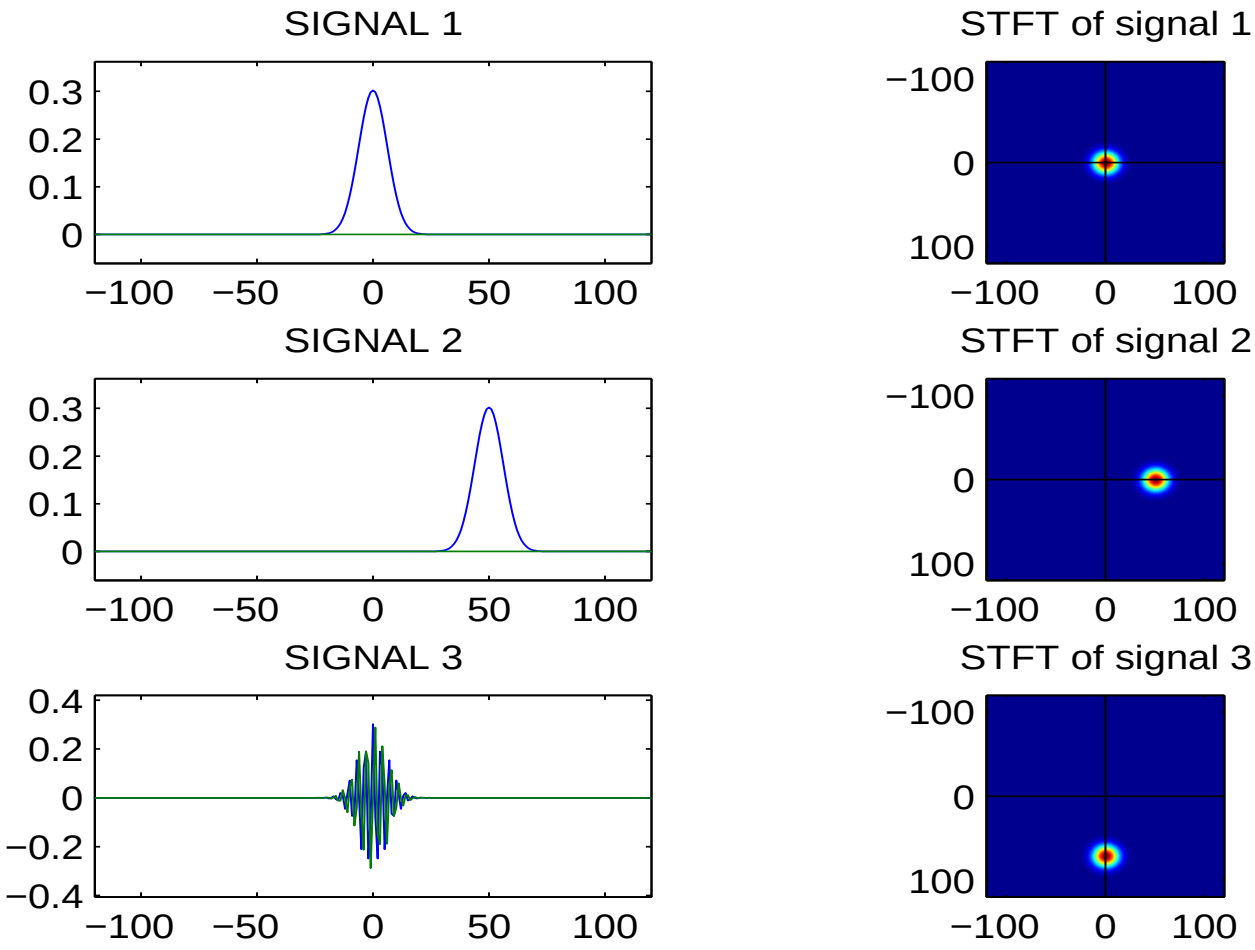
Some algebra in the background: The Heisenberg group

Weyl commutation relation

$$T_x M_\omega = e^{-2\pi i x\cdot\omega} M_\omega T_x, \quad (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

$\{M_\omega T_x : (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$ is a **projective representation** of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ on $L^2(\mathbb{R}^d)$. **Heisenberg group** $\mathbb{H} := \{\tau M_\omega T_x : \tau \in \mathbb{T}, (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$

Schrödinger representation $\{\tau M_\omega T_x : (x,\omega,\tau) \in \mathbb{H}\}$ is a square-integrable (irreducible) group representation of $\mathbb{H}$ on the Hilbert space $L^2(\mathbb{R}^d)$. Then the STFT $V_g f$ is a representation coefficient.


SIGNAL 1
0.3
0.2
0.1
0
−100
−50
0
50
100
STFT of signal 1
−100
0
100
−100
0
100
SIGNAL 2
0.3
0.2
0.1
0
−100
−50
0
50
100
STFT of signal 2
−100
0
100
−100
0
100
SIGNAL 3
0.4
0.2
0
−0.2
−0.4
−100
−50
0
50
100
STFT of signal 3
−100
0
100
−100
0
100

Some geometry: Euclidean Norms are Preserved!

Moyal's formula or orthogonality relations for STFTs:
Let f_1, f_2, g_1, g_2 be in $L^2(\mathbb{R}^d)$. Then

$$\left\langle V_{g_1} f_1, V_{g_2} f_2 \right\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \langle g_2, g_1 \rangle_{L^2(\mathbb{R}^d)}.$$

Reconstruction formula
Let $g, \gamma \in L^2(\mathbb{R}^d)$ with $\langle g, \gamma \rangle \neq 0$. Then for $f \in L^2(\mathbb{R}^d)$ we have

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(x, \omega) \pi(x, \omega) \gamma \, dx \, d\omega.$$

So typically one chooses $\gamma = g$ with $\|g\|_2 = 1$.

Primer on Gabor analysis: The Sampling Viewpoint

Given the fact that the STFT is a smooth function (and indeed satisfies a very nice reproducing kernel relation, with kernel essentially $V_g(g)$, in the case of the Gaussian it is even an analytic function over the complex "phase space"), one may think of the integral as a limit of (regular) Riemannian sums, and try to reproduce from samples on a sufficiently dense lattice only.

So one of the basic question in Gabor analysis is: For which choices of g and the TF-lattice Λ is the function f uniquely determined by the values of $(V_g f(\lambda))_{\lambda\in\Lambda}$?

For the case of finite Abelian groups this means of course that the orthogonal complement of the atoms $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is trivial, or equivalently that every f can be written as a linear combination (cf. next page).

In order to come up with a stable reconstruction process however one has to make the following assumption (which is strictly stronger than uniqueness): There is $A, B > 0$:

$$\|f\|^2_{\boldsymbol{L}^2(\mathbb{R}^d)} \leq \sum |V_g f(\lambda)|^2 \leq \|f\|^2_{\boldsymbol{L}^2(\mathbb{R}^d)} \quad \forall f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Primer on Gabor analysis: Atomic Viewpoint

D.GABOR's suggested to replace the continuous integral representation by a discrete series and still claim that one should have a representation of arbitrary elements of $\boldsymbol{L}^2(\mathbb{R})$!
Let $g \in L^2(\mathbb{R}^d)$ and Λ a lattice in time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

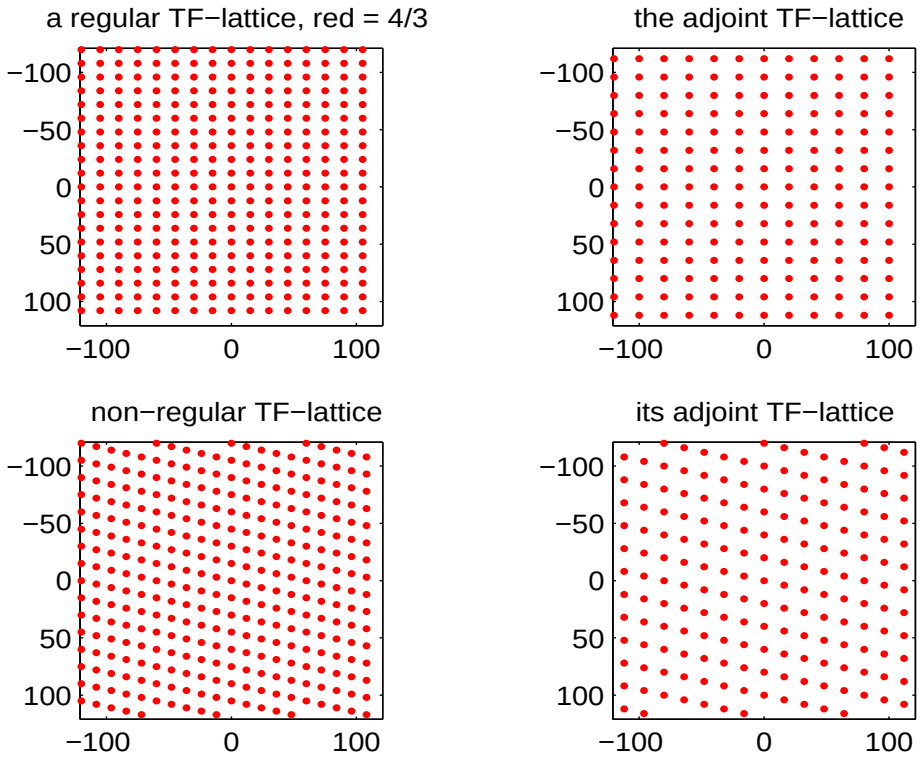
$$\mathbf{f} = \sum_{\lambda \in \mathbf{\Lambda}} \mathbf{a}(\lambda) \pi(\lambda) \mathbf{g}, \quad \text{for some} \quad \mathbf{a} = (\mathbf{a}(\lambda))_{\lambda \in \mathbf{\Lambda}}$$

is a so-called **Gabor expansion** of $f \in L^2(\mathbb{R}^d)$ for the **Gabor atom** g.

1946 - D. Gabor: $\Lambda = \mathbb{Z}^2$ and Gabor atom $g(t) = e^{-\pi t^2}$.
By today we know that the lattice $a\mathbb{Z} \times b\mathbb{Z}$ with $ab > 1$ is "too course" (i.e. the corresponding family is in fact a Riesz basis for a proper subspace of $\boldsymbol{L}^2(\mathbb{R}^d)$, making it in fact suitable for OFDM-like communication), while for $ab < 1$ we have non-uniqueness: Many absolutely convergent Gabor series are giving the same functions (e.g. the zero function). Still, this case does not hurt as we see it now, because one can still use the minimal norm coefficients to enforce uniqueness of coefficients!

Examples of finite Gabor families

Signal length $n = 240$, lattice Λ with $320 = 4/3 * n$ [$180 = 3/4 * n$] points.



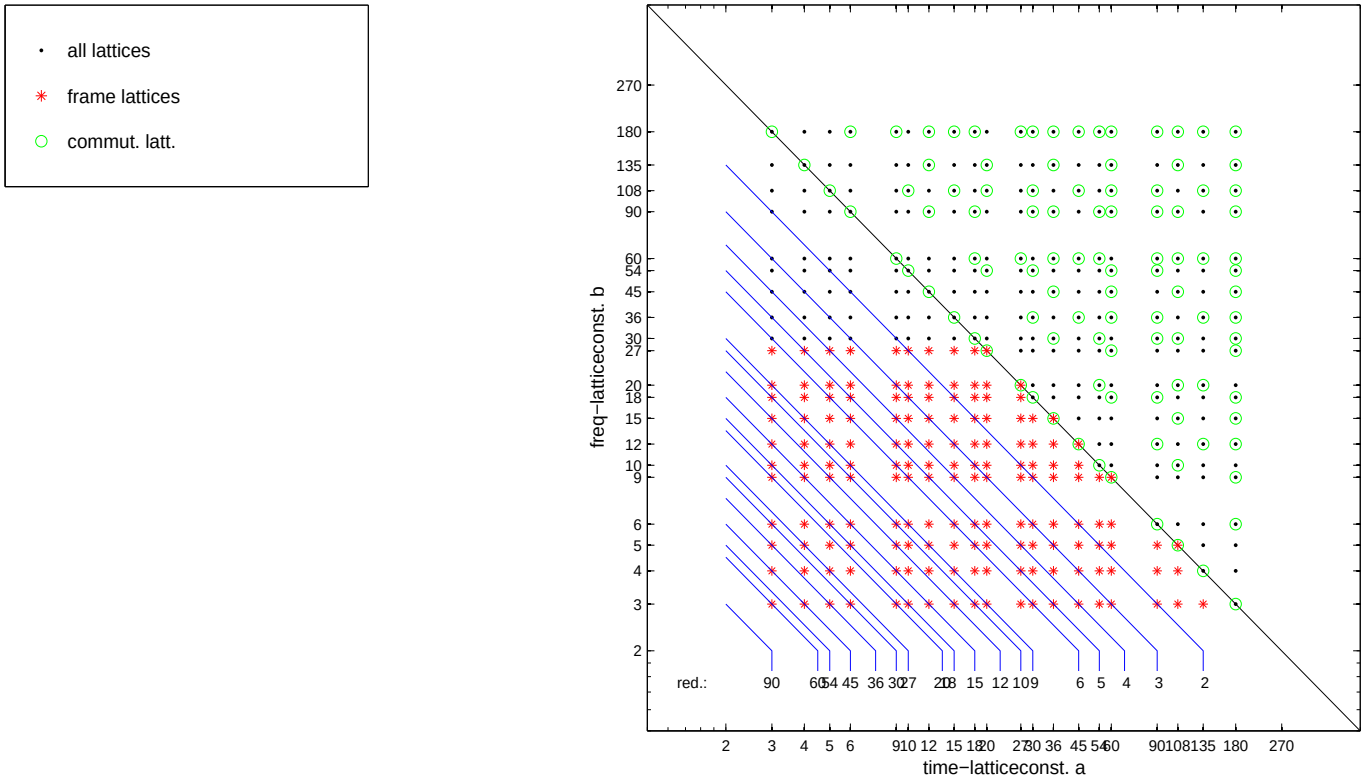
Viewed from a Linear Algebra Point of View

From an linear algebra point of view we simply are looking for expansions of arbitrary (periodic) vectors of length n by elements of a (finite, discrete) Gabor family. Each point in phase space (as in the previous side) represents some unitary TF-shift operator, and so we can apply a family of specific unitary matrices (indexed by the lattice points Λ) to a given Gabor atom g, in order to obtain a family.
Clear enough this family will be OK, i.e. will span the $n-$dim. vector-space only if we have at least n vectors. In this case we talk of a Gaborian Frame. On the other hand they can be linear independent (i.e. a basis for their linear span, which ma be strictly smaller than the full space) only if we have less than n vectors in such a Gaborian Riesz basis.

It is one of the core results of "algebraic Gabor theory" that these two questions have in some way the same answer. It requires the concept of an adjoint TF-lattice: Given the lattice Λ the adjoint lattice Λ° is defined as the lattice of all TF-points which correspond to operators $\pi(\lambda^\circ)$ commuting with all the matrices $\pi(\lambda), \lambda \in \Lambda$. It is easy to check that $\#\Lambda \cdot \#\Lambda^\circ = n^2$.

The previous slide shows in fact pairs of adjoint lattices!

A Discrete Version: Each Point "is" a Lattice, $n = 540$



The Wexler Raz equivalence relation

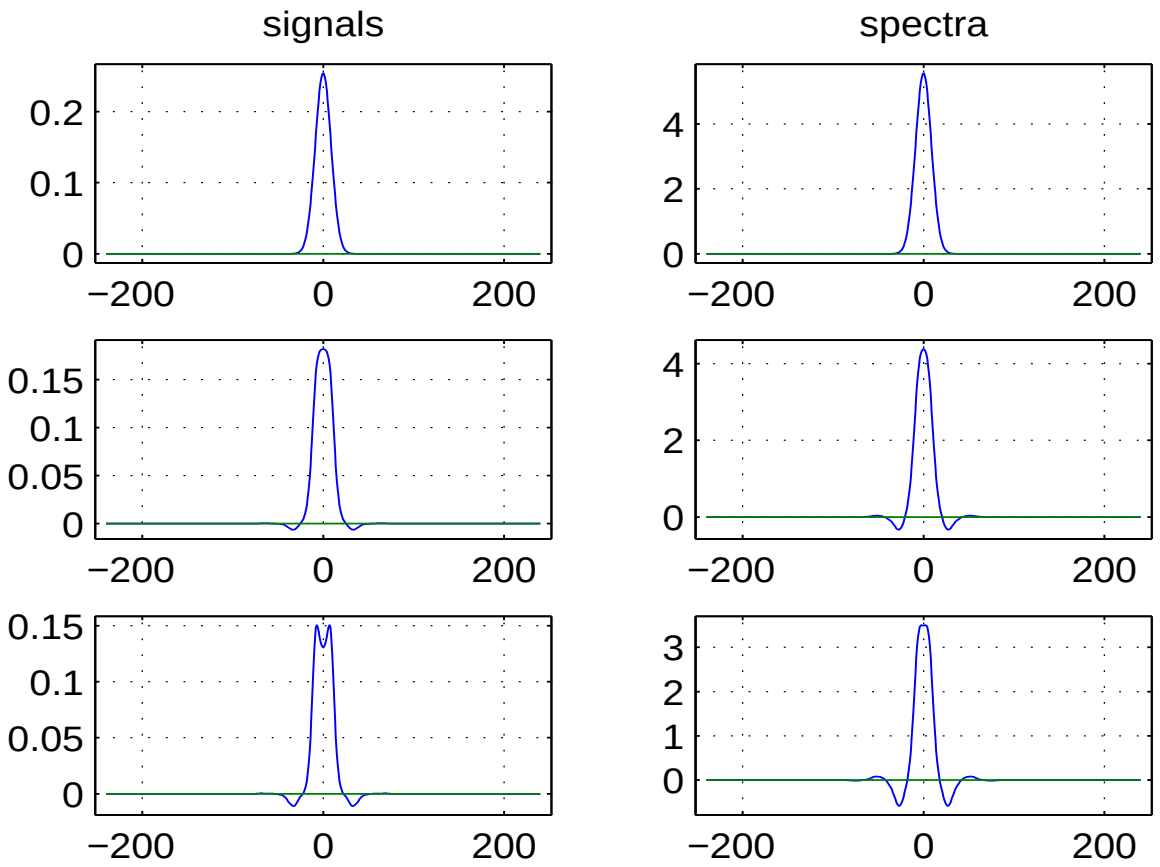
The Wexler-Raz relation shows that $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a Gabor frame *if and only if* the the adjoint family $\pi(\lambda^\circ)_{\lambda^\circ\in\Lambda^\circ}$ is a Gaborian Riesz basis.
Moreover, for every frame we have a dual frame, and for every Riesz basis we have a biorthogonal family (in general). It turns out that the commutation relation (again something algebraic) for the frame operator

$$Sf = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g = \sum \langle f, g_\lambda\rangle g_\lambda :$$

$$S \circ \pi(\lambda) = \pi(\lambda) \circ S \quad \forall \lambda \in \Lambda.$$

This fact has various important (conceptual, computational, and practical) consequences: First of all it implies that the dual frame of a Gabor frame is again a Gabor frame, with the so-called dual atom $\tilde{g}$ as generating atom. The same is true for Riesz basis: the biorthog. family for a Gaborian Riesz basis is another Gaborian Riesz basis, which according to the Wexler-Raz principle has (up to normalization) the SAME generator $\tilde{g}$.!!
Note: if an atom generates $S = Id$ then g is called a tight atom with respect to Λ.

Gabor atom, with canonical tight and dual Gabor atoms



The benefit of having a dual Gabor atom (and duality is a symmetric relationship because the frame operator induced by t $\tilde{g}$ is just the inverse of the frame operator!) is that one can use one for analysis and the other for synthesis as follows:

Seen as a sampling problem, one reconstructs the signal f from the samples of $V_g(f)$ over Λ by the formula $f = S^{-1}S(f) = \sum_\lambda V_g f(\lambda)\pi(\lambda)\tilde{g}$.

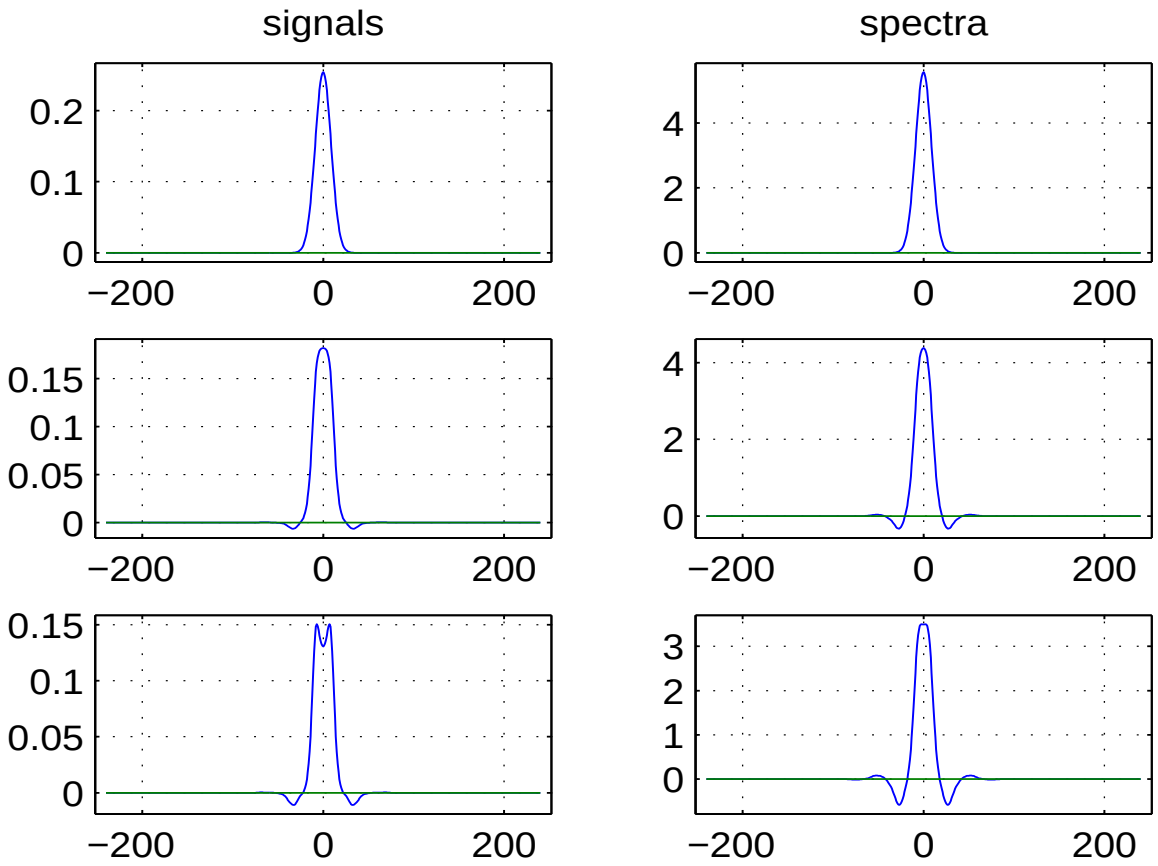
On the other hand, if one takes the atomic point of view, i.e. if one want to fulfill Gabor's wishes by providing in a most efficient ways coefficients for a given function f in order to write it as an (unconditionally convergent) Gabor sum, then one will prefer the formula

$$f = S^{-1}S(f) = \sum_\lambda V_{\tilde{g}} f(\lambda)\pi(\lambda)g.$$

There is also a symmetric way, of modifying both the analysis and synthesis operator in order to (by choosing $h = S^{-1/2}g$)

$$f = \sum_\lambda V_h f(\lambda)\pi(\lambda)h = \sum_\lambda \langle f, h_\lambda\rangle h_\lambda.$$

This looks very much like an orthonormal expansion (although it is not), and h is called a tight Gabor atom Gabor atom, with canonical tight and dual Gabor atoms


signals
spectra
0.2
0.1
0
−200
0
200
4
2
0.15
0.05
3
1

Application areas for Gaborian = Weyl-Heisenberg families

- Gabor families: denoising, separation of signals (defibrillation project);
- Modelling of slowly time-variant channels (MOHAWI project), using the so-called spreading representation
- Mobile (OFMD-like) communication: using Gaborian Riesz bases (because they are good universal approximate eigenvectors to all "underspread systems": hence approximate inversion (decoding) by Gabor multipiers!
- localization (regions of interest) in a TF-representation: to be discussed in detail next:

Understanding tight Gabor frames

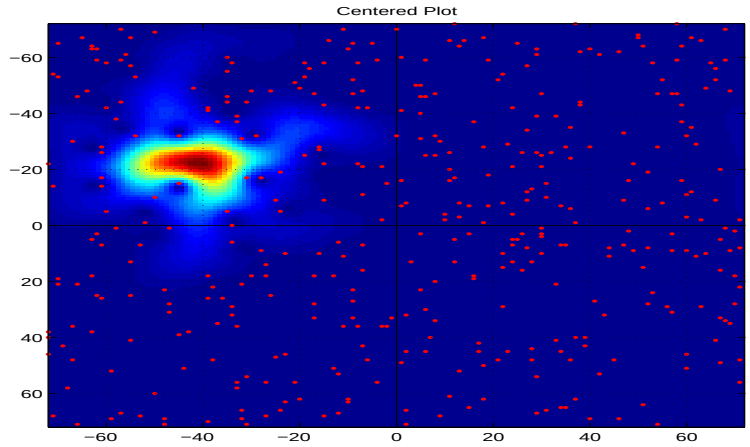
General Gabor frames only satisfy

$$\|f\|^2_{\boldsymbol{L}^2(\mathbb{R}^d)} \leq \sum |V_g f(\lambda)|^2 \leq \|f\|^2_{\boldsymbol{L}^2(\mathbb{R}^d)} \quad \forall f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

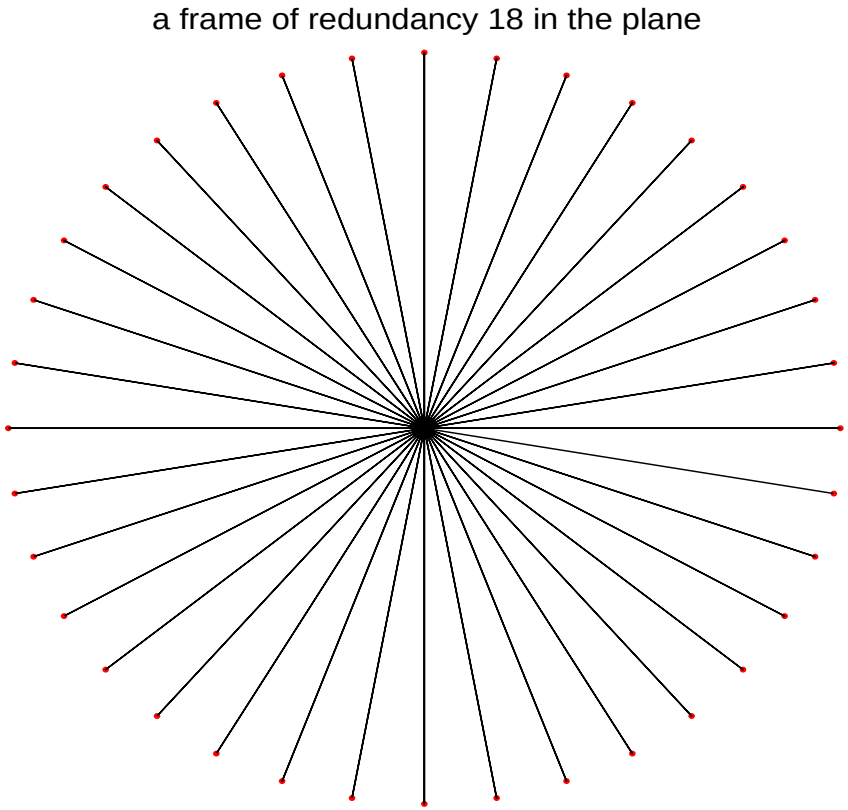
allows (depending on the ration B/A, which is more or less the condition number of the frame operator (squared)) for some variation if $B > A$. In fact, some signals f with $\|f\|_{\boldsymbol{L}^2(\mathbb{R}^d)}$ (concentrating their energy at points far from the lattice Λ) may have sampling energy close to A, while others (typically concentrated near the sampling points) may have sampling energy comparable to B. So proportions in the STFT-domain do not reflect energy proportions in the signal (energy) domain.
Perhaps even more importantly we have that real symbols (multiplier sequences) define self-adjoint operators, hence can be diagonalized etc. . Obviously an operator is called a Gabor multipier if it is the composition of STFT-mapping, followed by a pointwise multiplication and then resynthesis, i.e.

$$G_m(f) = \sum_{\Lambda} m_\lambda V_g f(\lambda) h_\lambda.$$

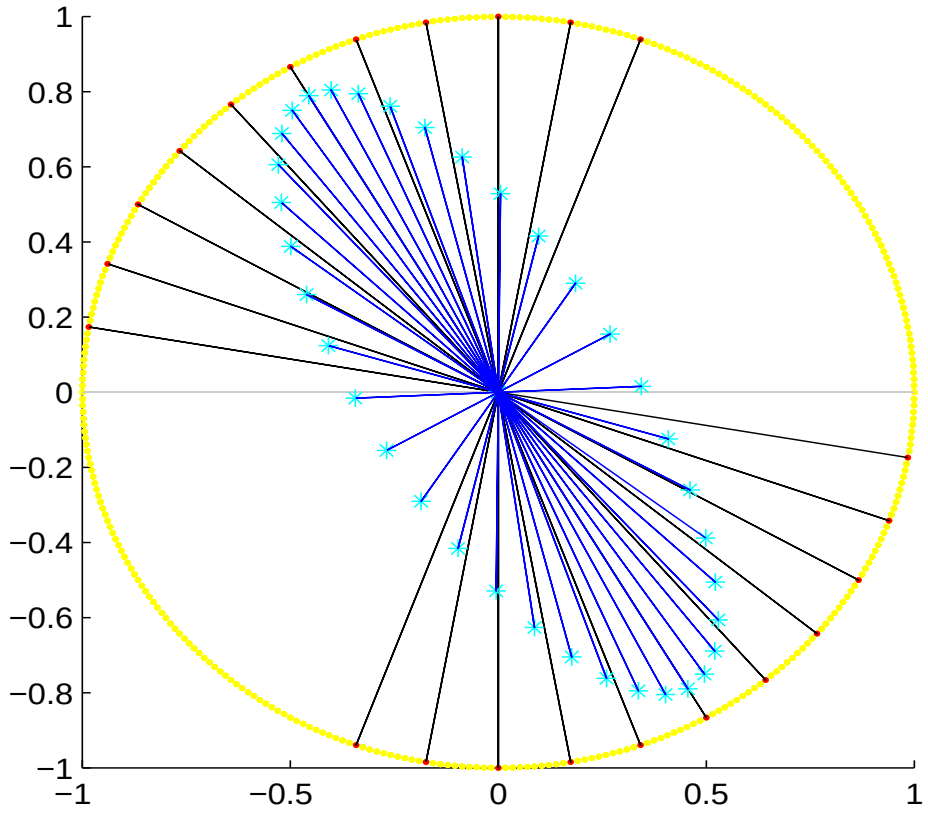
Time-frequency concentration of eigenvectors to frame operators



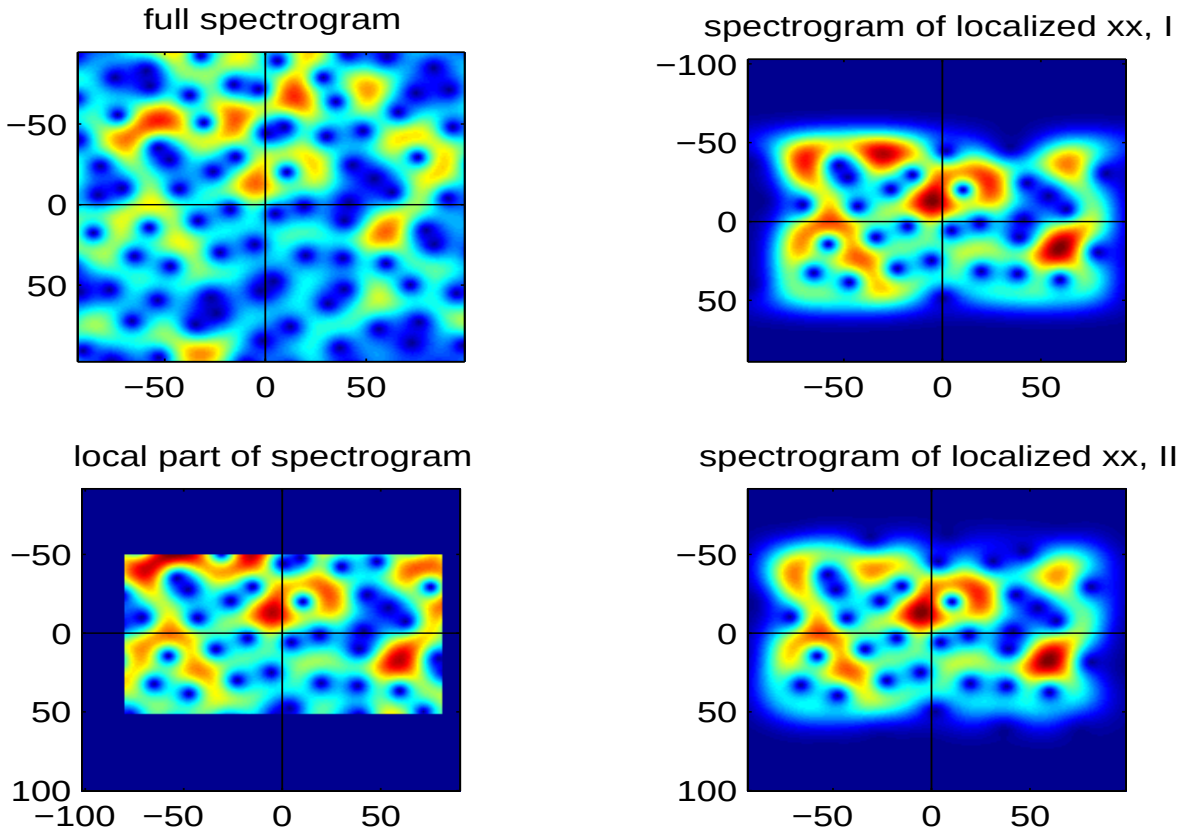
A simple tight frame of redundancy 18 in the plane



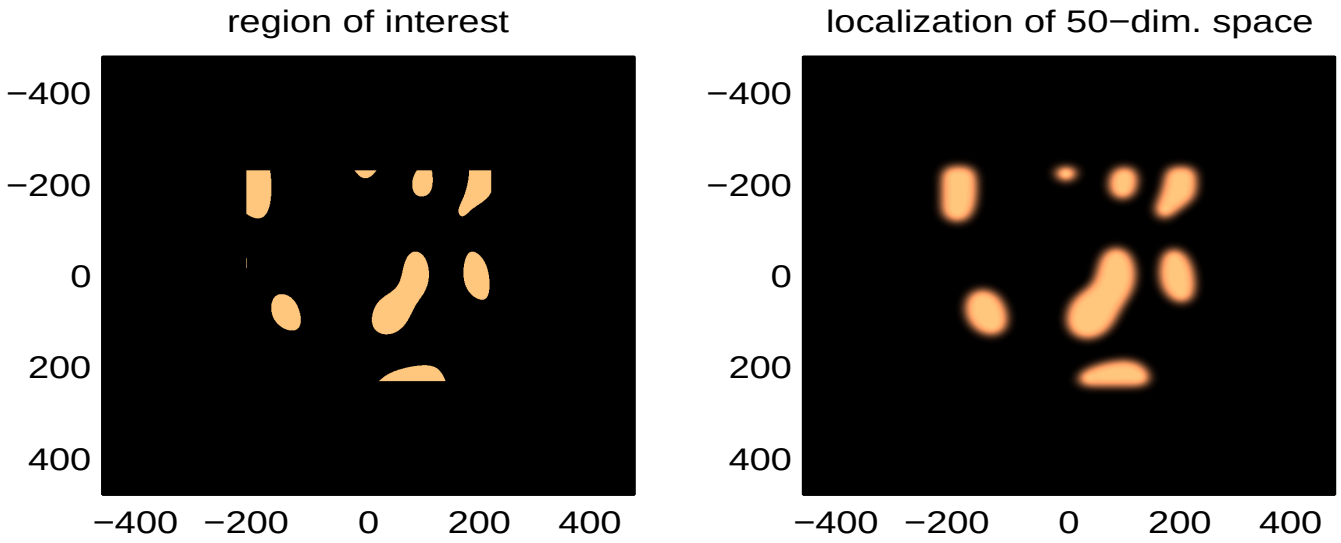
Effect of a "frame multiplier" of redundancy 18 in dimension 2



Gabor multipliers for masking an area of intest



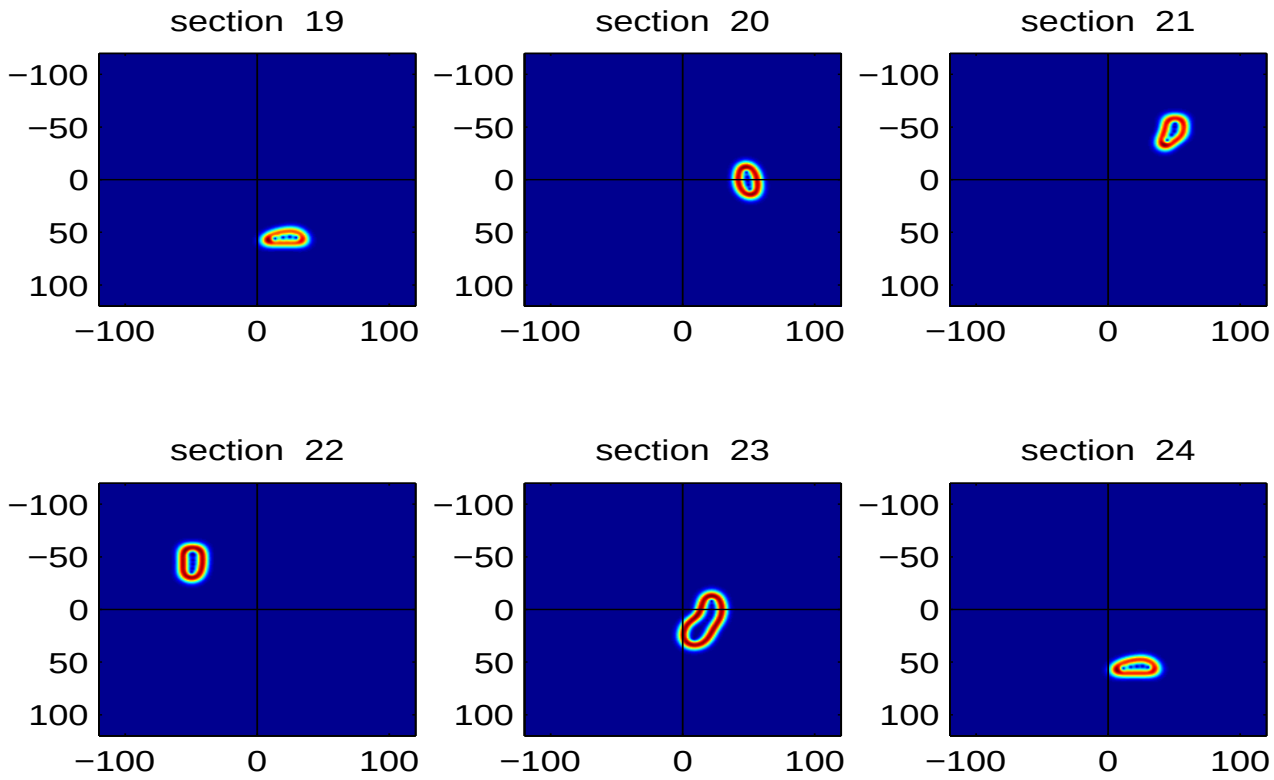
Non-canonical regions of interest



How can one localize a signal to a region of interest?

It is natural to restrict the (regular or irregular) Gabor expansion of a given signal to the region of interest and take this as a kind of projection operator. Alternatively one can take the full STFT and set it to zero outside the region of interest. The disadvantage of such a procedure (which is simple to implement!) is that fact that it does not really give us an (orthogonal) projection operator. In other words, if we apply the same operation twice are a few times there will still be further changes. In fact, the STFT-multipliers (with some $0/1$-mask) all are (mathematically) strict contractions, with a maximal eigenvalue of maybe 0.99. Although iterated application of this denoising procedure (by masking the spectrogram) appears to be useful in many cases it is of interest to find a correct **projection operator**.

Study of the localization operators: eigenvalues and eigenvectors



Best approximation of a given matrix by Gabor multiplier

In many cases, even if the building blocks (g_λ) of a Gabor frame are (of course) a linear dependent set of atoms in our signal space, the corresponding set of projection operators (P_λ), given by $h \mapsto \langle h, g_\lambda \rangle g_\lambda$ has good chances to be a linear independent set (in the continuous case: a Riesz basis within the class of Hilbert Schmidt operators, with the scalar product $\langle A, B \rangle_{\mathcal{HS}} = trace(AB^*)$).
This means that the mapping from the sequence (m_λ) to the operator

$$Th = \sum_\lambda m_\lambda \langle h, g_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda} m_\lambda P_\lambda(h)$$

is one to one, in other words, the "upper symbol" of a Gabor multiplier is uniquely determined, and the set of Gabor multipliers is closed within the space of all Hilbert Schmidt operators.

Consequently every Hilbert Schmidt operator has a best approximation (in the HS-norm) by a Gabor multiplier (with upper symbol $(m_\lambda \in \ell^2(\Lambda)$.

Although Gabor multipliers with respect to "nice atoms" g will have matrices closely concentrated near the main diagonal (both in the time and in the frequency representation!) it is not at all obvious, but still true that the operator T can be identified (and this best approximation can be determined) from the scalar products $(\langle T(g_\lambda), g_\lambda \rangle)$. This sequence is called the "lower symbol" of the operator T.

In fact, this best approximation procedure extends to a much larger class of symbols, including Gabor multipliers with just bounded symbols (m_λ) (which are not Hilbert Schmidt, but may be invertible, for example). In an ongoing project with the EE Dept. (TU Vienna, Franz Hlawatsch) we are studying the approximation of the inverse of a Gabor multiplier (which by itself is NOT! a Gabor multiplier) or more general the inverse of a slowly varying channel by a (generalized) Gabor multiplier. The idea being that the implementation of such operators should be computationally cheap.

THE END!

THANK you for your attention! HGFei

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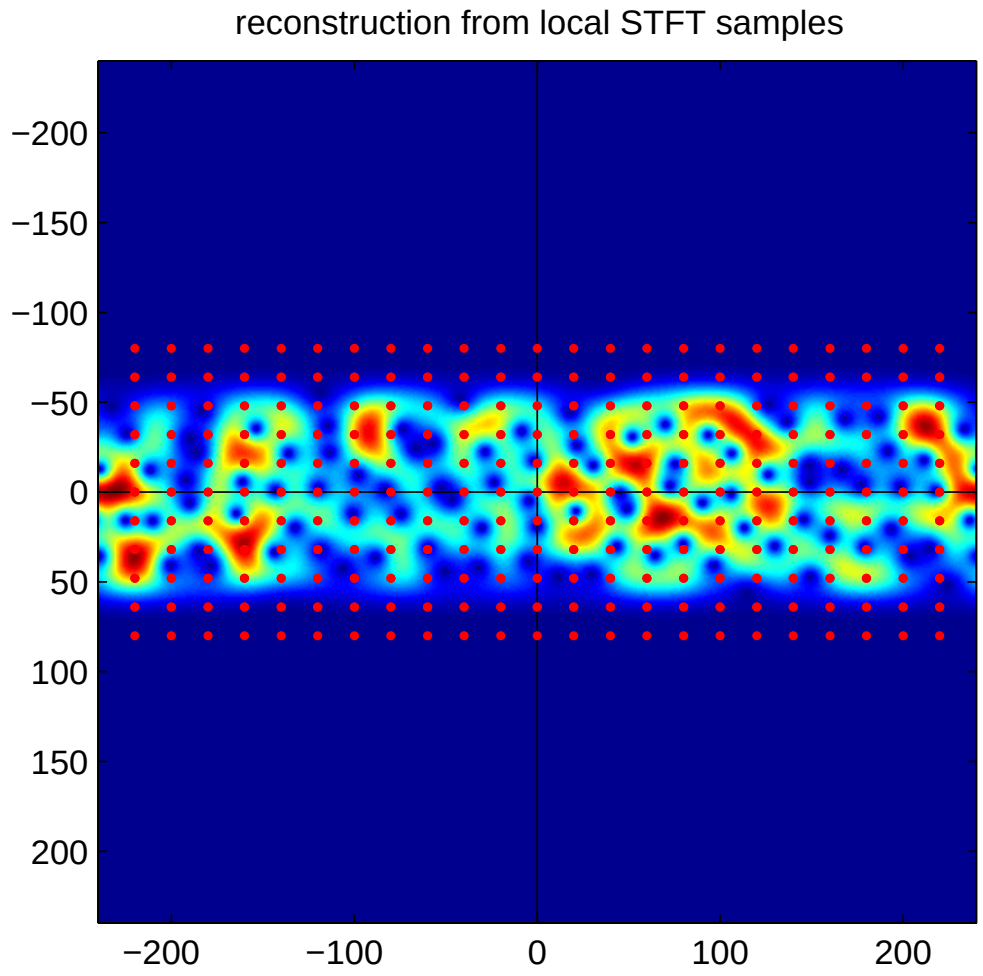
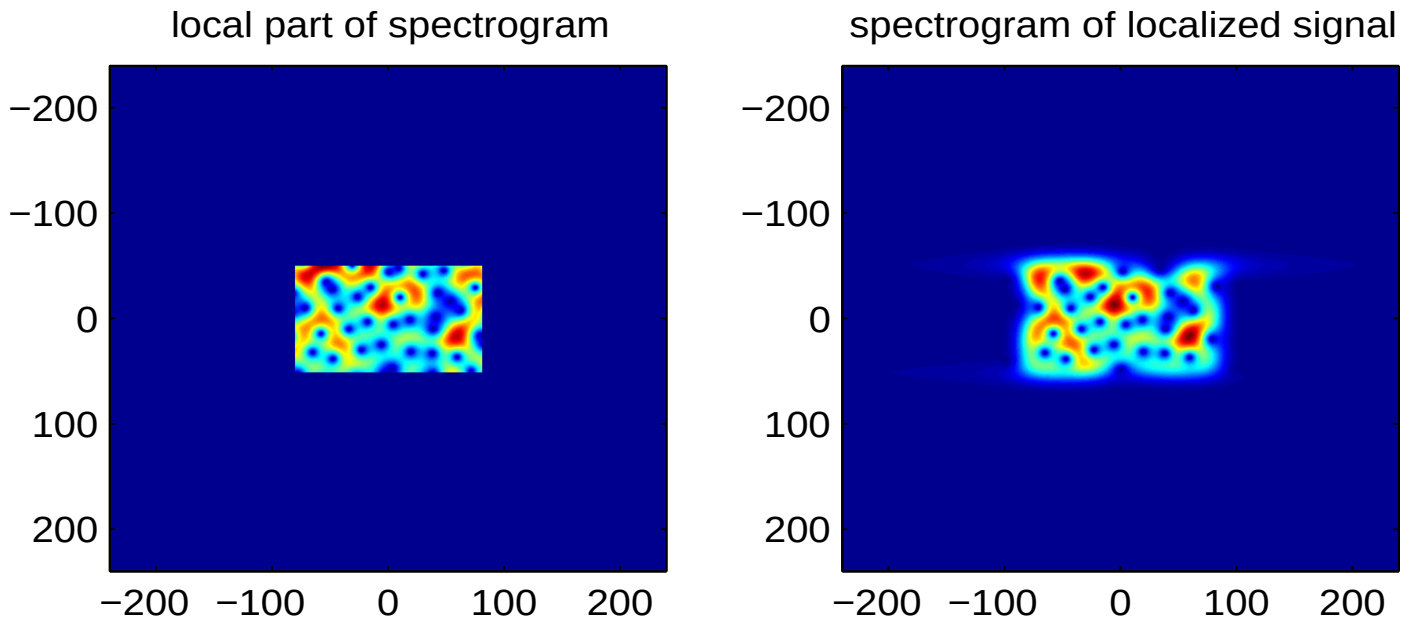
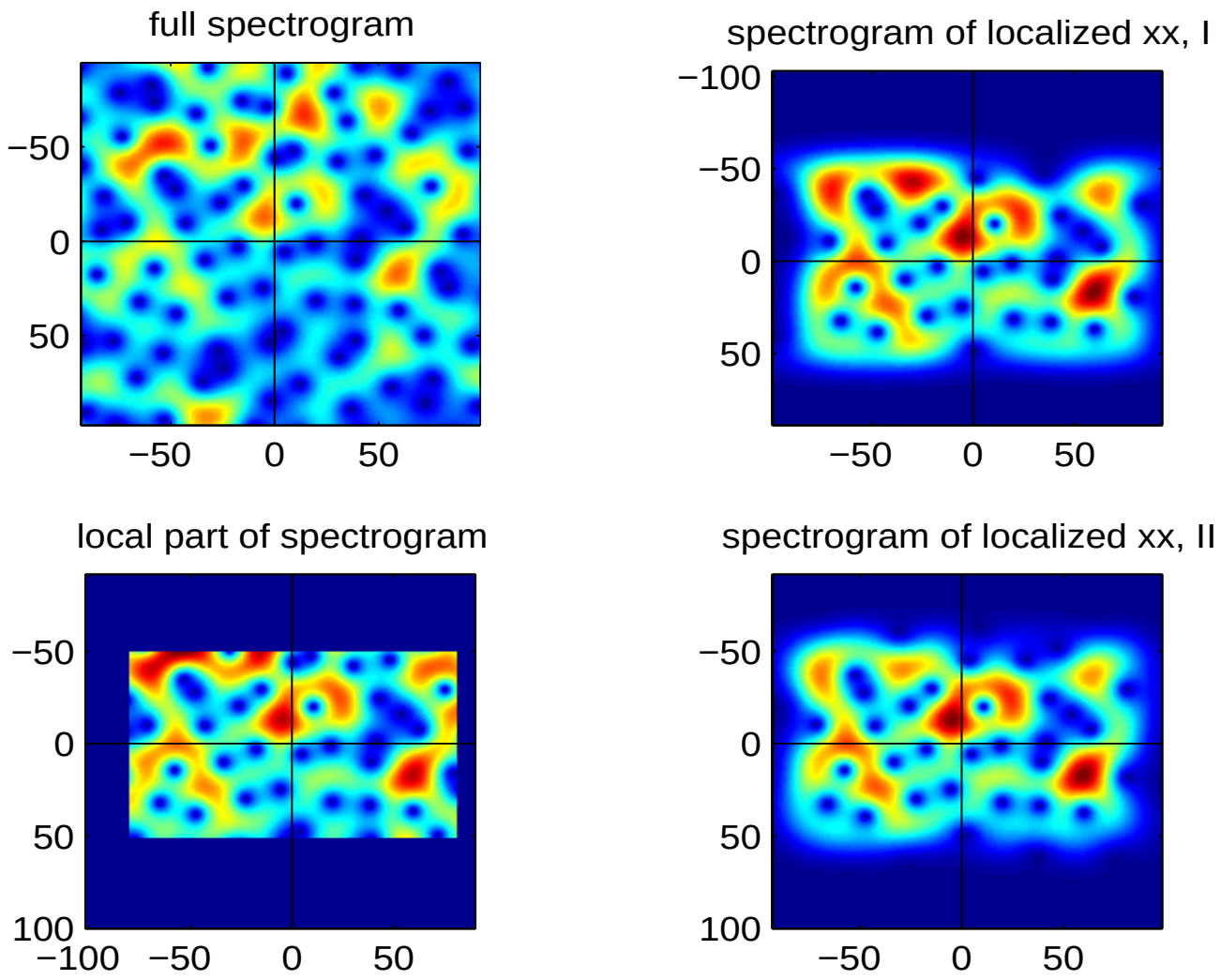
Exact recovery for elements from a subspace

When looking at the above images it is natural to assume that one can have perfect reconstruction of all the signals which are concentrated with the region of interest (looked at from a time-frequency view-point). Unfortunately no single function has its STFT concentrated (for whatever window) in a bounded domain of the time-frequency plane, because that would imply that such a function is both time- and frequency-limited
We are presently investigating (PhD thesis of Roza Acesca) a mathematical clean description for the *idea* of *functions of variable band-width*. The problem with such a concept is that it has to respect the uncertainty principle (which for me implies: one cannot talk about the exact frequency content of a function at a given time, at a precise frequency level!). Also *THERE IS NO SPACE of functions having a their spectrogram in a strip* (of variable width)!


reconstruction from local STFT samples
−200
−150
−100
−50
0
50
100
150
200
−200
−100
0
100
200


local part of spectrogram
spectrogram of localized signal
−200
−100
0
100
200
−200
−100
0
100
200


full spectrogram
−50
0
50
−50
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spectrogram of localized xx, I
−100
−50
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−50
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50
local part of spectrogram
−50
0
50
100
−100
−50
0
50
spectrogram of localized xx, II
−50
0
50
100
−50
0
50

Further references: [2] describes how to get from linear algebra to Gabor analysis; [1] describes the general theory of Gabor families for the case of finite Abelian groups, i.e. gives the details of the algebra, avoiding all the functional analysis.
[3]

References

[1] H. G. Feichtinger, W. Kozek, and F. Luef. Gabor Analysis over finite Abelian groups. *Appl. Comput. Harmon. Anal.*, 2008.

[2] H. G. Feichtinger, F. Luef, and T. Werther. A Guided Tour from Linear Algebra to the Foundations of Gabor Analysis. In *Gabor and Wavelet Frames*, volume 10 of *IMS Lecture Notes Series*. 2007.

[3] K. Gröchenig. Describing functions: atomic decompositions versus frames. *Monatsh. Math.*, 112(3):1–41, 1991.