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From Classical Fourier Analysis to Time-Frequency Analysis and Back

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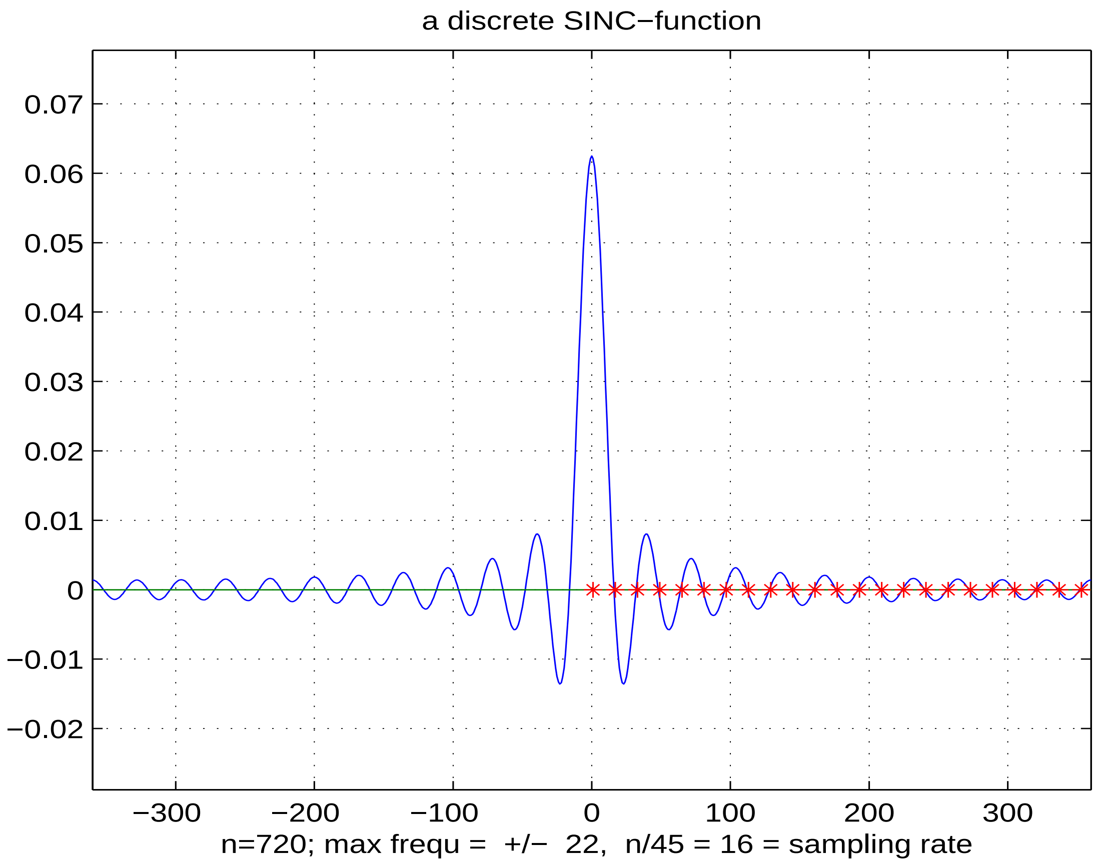
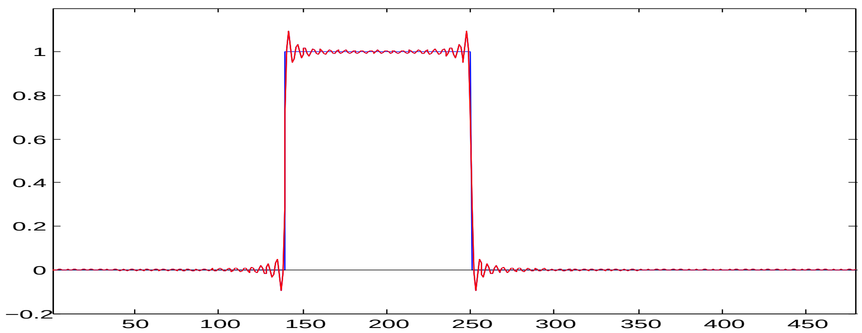
ABSTRACT

Summability methods are an important principle that arise from the classical theory of Fourier series, but are equally important for the inversion of Fourier transforms or the proof of Plancherel's Theorem. The usual "good summability kernels" are also very suitable for signal processing applications, where they are used to localize an ongoing function or distribution in order to define its short-time Fourier transform or sliding window Fourier transform. This opens the way to modern time-frequency analysis and Gabor analysis, its discretized and more practical version. Function spaces defined by the behaviour of their STFT, usually called modulation spaces, are well suited in order to handle e.g. questions about pseudo-differential operators or symbolic calculi for operators. Elements from the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ (also called $\boldsymbol{M}^1(\mathbb{R}^d)$)) are very suitable both in the context of Gabor analysis and as kernels for classical problems. Pertinent results in this direction are obtained in a series of joint papers with Ferenc Weisz.

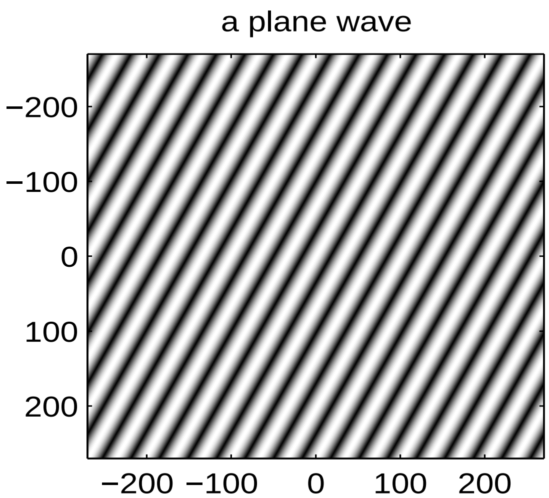
The classical SINC kernel

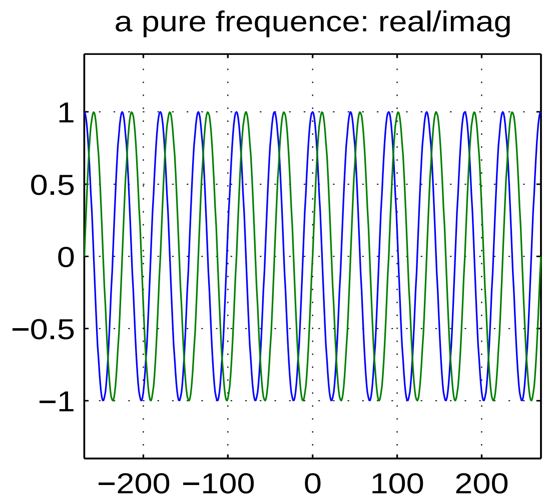


Gibb's Phenomenon



Finite Fourier Transforms





The finite Fourier Transform

The finite Fourier transform is easy to understand from the linear algebra point of view. Up to normalization the building blocks ("pure frequencies") form an orthonormal system, and the coefficients in this orthonormal system are just the "Fourier coefficients" (obtained via FFT2 or FFT efficiently). Hence it maps pure frequencies on unit vectors (and vice versa).

The relevance of the Fourier transform for signal processing applications comes from the so-called "convolution theorem": Translation invariant linear operators are just "convolution operators", and they can be described as pointwise multiplication operator "on the Fourier transform side".

Equivalently, translation invariant operators are just linear combinations of (cyclic) shift operators, and the *common eigenvectors* for this family of operators are just the "pure frequencies", which we will denote by χ_n in the rest.

For the case of periodic functions (classical Fourier series) we have the additional problem of convergence (because there are infinitely many pure frequencies), and in the case of non-periodic functions the additional problem comes in that unit vectors have to be replaced by Dirac measures.

Goals of Fourier Analysis

What should we teach our students in courses on Fourier Analysis?

- Understand the decomposition of an "arbitrary periodic function" into pure frequencies (i.e. to understand the original idea of J. P. Fourier);
- Get familiar with Fourier series and Fourier transforms, and their inversion;
- (A.Weil): Understand Fourier analysis in the "natural context", i.e. over LCA groups (for a modern treatment I may recommend the book of Deitmar, [7]);
- Help them to understand signal processing papers (sampling is the same as periodization of the spectrum of a function, etc.);
- Understand Fourier analysis in the realm of tempered distributions, because this is they way how it is used in the modern theory of PDE and pseudo-differential operators;
- And how much mathematical background do students or applied scientist have to learn in order to make use of the powerful results of Fourier analysis: Lebesgue integration? topological vector spaces? theory of generalized functions?

Fourier Series and Fourier Transforms

If we browse Ruppert Lasser's book on "Fourier Series" (and Fourier transforms) the following terms appear resp. are the essential terms in the description of the subject:

1. convolution, $\left(\boldsymbol{L}^1(\mathbb{T}), \|\cdot\|_1\right)$
2. homogeneous Banach spaces such as $\boldsymbol{L}^1(\mathbb{T}), \boldsymbol{L}^2(\mathbb{T}), \boldsymbol{L}^p(\mathbb{T}), \boldsymbol{A}(\mathbb{T})$, etc.
3. $\boldsymbol{L}^\infty(\mathbb{T})$ is not a homogeneous Banach space;
4. approximate identities (unlike Dirichlet): Fejer, de la Vallee-Poussin or Poisson kernels;
5. $\left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$, Wiener's algebra of absolutely convergent Fourier series;
6. Fourier transforms (and Fourier inversion);
7. Plancherel's theorem;
8. Poisson's formula;
9. Hermite functions as a Fourier invariant ONB for $\boldsymbol{L}^2(\mathbb{R}^d)$;

LET US TAKE a time-frequency view on these topics!

Homogeneous Spaces

Let us first introduce the following definition, see, for instance, [16, 18] .

Definition 1. *A Banach space* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *of locally integrable functions is called a homogeneous Banach space on* $\mathbb{R}^d$ *if it satisfies*

1. $\|T_x f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}} \quad \forall f \in \boldsymbol{B}, x \in \mathbb{R}^d;$
2. $\|T_x f - f\|_{\boldsymbol{B}} \to 0$ *for* $x \to 0, \quad \forall f \in \boldsymbol{B}.$

It is easy to show that any localizable Banach space $\boldsymbol{B}$ of Radon measures for which $\mathcal{D}(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, is a homogeneous Banach space, e.g., $\boldsymbol{B} = \boldsymbol{L}^p(\mathbb{R}^d)$ for $1 \leq p < \infty$.

It is a well known consequence of Definition 1 that the convolution of bounded measures $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ with elements from a homogeneous Banach space exists as a vector-valued integral (improper, Riemannian) and satisfies

$$\|\mu * f\|_{\boldsymbol{B}} \leq \|\mu\|_M \|f\|_{\boldsymbol{B}} \quad \forall \mu \in \boldsymbol{M}(\mathbb{R}^d), f \in \boldsymbol{B}. \tag{1}$$

Segal Algebras

Let us next introduce the following definition, see, the work of Hans Reiter ([20, 21]):

Definition 2. *A Banach space* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *of integrable functions is called a Segal algebra on* $\mathbb{R}^d$ *(or on a LC group* $\mathcal{G}$*, if it satisfies*

1. *it is densely embedded into* $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$*, i.e.* $\|f\|_{\boldsymbol{B}} \leq \|f\|_1 \quad \forall f \in \boldsymbol{B}$ *and* $\boldsymbol{B}$ *is dense in* $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$.
2. $\|T_x f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}} \quad \forall f \in B, x \in \mathbb{R}^d$;
3. $\|T_x f - f\|_{\boldsymbol{B}} \to 0$ *for* $x \to 0, \quad \forall f \in B$.

It is easy to show that any localizable Banach space B of Radon measures for which $\mathcal{D}(\mathbb{R}^d)$ is dense in $(B, \|\cdot\|_B)$, is a homogeneous Banach space, e.g., $B = \boldsymbol{L}^p(\mathbb{R}^d)$ for $1 \leq p < \infty$.

It is a well known consequence of Definition 2 that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a Banach ideal in $MRdN$, and satisfies

$$\|\mu * f\|_{\boldsymbol{B}} \leq \|\mu\|_M \|f\|_{\boldsymbol{B}} \quad \forall \mu \in \boldsymbol{M}(\mathbb{R}^d), f \in \boldsymbol{B}. \tag{2}$$

Homogeneous Banach Spaces on the Torus

It is not hard to check that a homogeneous Banach space on the torus $\mathbb{T}$ is a Segal algebra on $\mathcal{G} = \mathbb{T}$ if and only if it contains the "pure frequencies" $\chi_n, n \in \mathbb{Z}$ belong to $\boldsymbol{B}$.
In fact, if the pure frequencies are contained in $\boldsymbol{B}$, then also all the trigonometric polynomials, and hence one obtains density of $\boldsymbol{B}$ in $\left(\boldsymbol{L}^1(\mathbb{T}), \|\cdot\|_1\right)$.
Conversely, a dense subspace of $\left(\boldsymbol{L}^1(\mathbb{T}), \|\cdot\|_1\right)$ has to contain for each $n \in \mathbb{Z}$ at least one $f \in \boldsymbol{B}$ such that $\hat{f}(n) \neq 0$ (otherwise the pure frequency χ_n labelled by n would be perpendicular to all of $\boldsymbol{B}$. Since also the convolution product of χ_n with f is in $\boldsymbol{L}^1(\mathbb{T}) * \boldsymbol{B} \subseteq \boldsymbol{B}$ and equals $\hat{f}(n)\chi_n$ this implies that $\chi_n \in \boldsymbol{B}$ for any $n \in \mathbb{Z}$.
Hence the homogeneous Banach spaces as described in Lasser's book are exactly the Segal algebras on $\mathbb{T}$.
Among the Segal algebras on the torus WIENER's ALGEBRA of absolutely convergent Fourier series

$$\boldsymbol{A}(\mathbb{T}) := \{f \mid \|f\|_{\boldsymbol{A}(\mathbb{T})} = \sum_{n\in\mathbb{Z}} \hat{f}(n) < \infty\}$$

$\left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$ plays a particular role (e.g. due to Wiener's inversion theorem), and by the fact that it is the smallest among Segal algebras on $\mathbb{T}$ with $\|\chi_n\|_{\boldsymbol{B}} = 1$

Intersection of Segal Algebras

It is easy to check that the intersection of two Segal algebras is again a Segal algebra (with the natural "sum-norm");
The intersection of ALL Segal algebras can be shown to coincide with the set of all band-limited $\boldsymbol{L}^1(\mathbb{R}^d)$-functions. In fact, for every Segal algebra $\boldsymbol{B}$ the linear space

$$BDL_1 := \{f \mid f \in \boldsymbol{L}^1(\mathbb{R}^d),\ \mathrm{supp}(\hat{f}) \text{ is compact } \}$$

is a dense subspace of $(\boldsymbol{B},\ \|\cdot\|_{\boldsymbol{B}})$, while on the other hand for every function $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ for which the support of $\hat{f}$ is not compact (resp. bounded) one can construct Segal algebras not containing such a "bad" functions f. This is also true over the torus. Typical examples are (Sobolev-like) subspaces of $\boldsymbol{L}^2(\mathbb{T})$:

$$\{f \in \boldsymbol{L}^2(\mathbb{T}) \mid (\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 w^2(n))^{-1/2} < \infty\}$$

It was an important observation that there exists a smallest Segal algebra, called $\boldsymbol{S}_0(G)$, with the extra property $\|\chi f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ for all $\chi \in \hat{\mathcal{G}}$ (strong character invariance, [8]).

Problems with classical Fourier analysis

Although the classical Fourier analysis was well established by the second half of the last century it still looked difficult! The last important contribution was certainly L. Carleson's theorem ([5]).

- Inversion from the space of FTs is not via integrals (summability is required);
- Integration (Lebesgue) does not help to establish Plancherel's theorem;
- Although the FT is unitary it is not really a change of basis (except for the Hermite basis);
- For the Poisson formula there is another set of extra conditions required;
- there are very few Fourier invariant function spaces, such as $\boldsymbol{L}^2(\mathbb{R}^d)$ or $\boldsymbol{L}^1(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$;
- the Fourier algebra $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ resp. $\boldsymbol{L}^1(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ do not enjoy the good properties of $\boldsymbol{A}(\mathbb{T}^d)$.

A number of auxiliary spaces have been introduced

- $\boldsymbol{L}^1(\mathbb{R}^d)$ is good, because it is the maximal domain for the *integral definition* of the FT! (using the correct and most general integral, after all, $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ is the prototype of a Banach space);
- $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ is the natural Hilbert space in this context (unfortunately there are no inclusion relations between $\boldsymbol{L}^1(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\mathbb{R}^d)$);
- *Pseudomeasures and Quasimeasures* are used in order to "represent" translation invariant operators between $\boldsymbol{L}^p(\mathbb{R}^d)$ spaces (both on the "time" and on the "frequency"-side, but there is no extension of the Fourier transform to quasi-measures;
- The space $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ of *tempered distributions* is the dual of the *Schwartz space* of rapidly decreasing functions $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, which is only a (nuclear) topological vector space, but both spaces are Fourier invariant!
- The theory of *"transformable" measures* by L. Argabright and J. Gil de Lamadrid provides a theory of (unbounded) measures having a (Radon) measure as a Fourier transform (unfortunately their theory is not Fourier invariant); (see [2, 1]).
- Further complications arise in the consideration of generalized stochastic processes;

Fourier Integral and Inversion Formula

For a Lebesgue integrable function $f \in \boldsymbol{L}^1(\mathbb{R})$ the Fourier transform is defined as

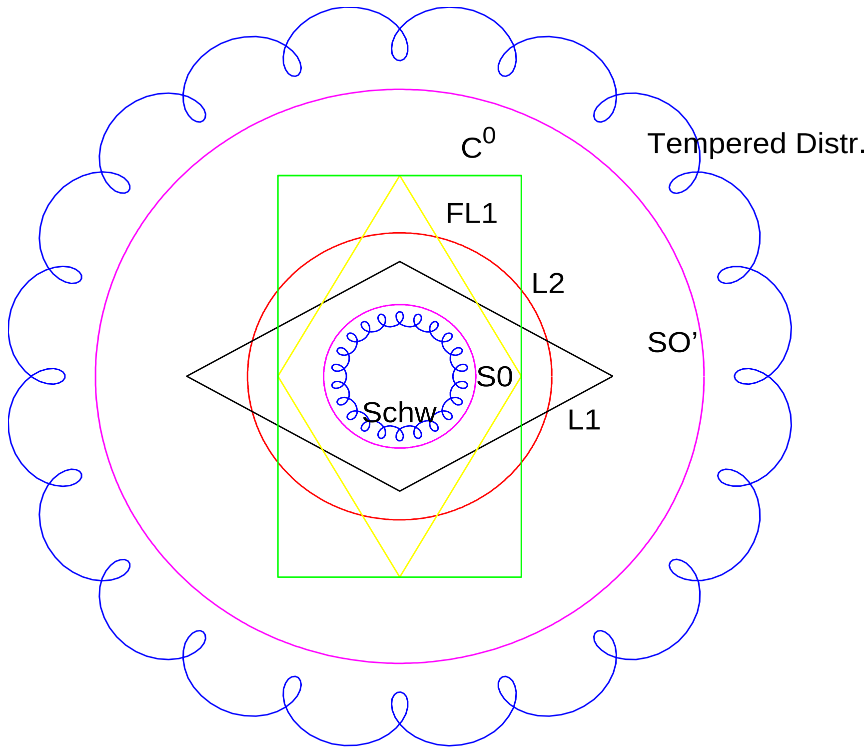
$$\hat{f}(s) = \int_{\mathbb{R}} f(t) e^{-2\pi i s x} dx$$

The range of the Fourier transform is $FLiR$. Due to the Riemann Lebesgue theorem one has $\hat{f} \in \boldsymbol{C}_0(\mathbb{R})$ and $\|\hat{f}\|_\infty \leq \|f\|_1$, but the Fourier transform itself is not necessarily integrable. Hence, the direct inversion formula

$$f(t) = \int_{\mathbb{R}} \hat{f}(s) e^{+2\pi i x s} ds$$

is only valid if $\hat{f} \in \boldsymbol{L}^1(\mathbb{R})$.

Schwartz space, S_0, L^2, S_0', tempered distributions



Summability methods for continuous variables

Instead of Fejér summability we may take a general summability method, the so called θ-summation defined by one single function θ. For $\theta \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ the Uth θ-mean of the Fourier transform of $h \in L_p(\mathbb{R}^d)$ $(1 \leq p \leq 2)$ or $h \in L_p(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ $(2 < p < \infty)$ is introduced by

$$\sigma_U^\theta h(t) := \int_{\mathbb{R}^d} \theta\Big(\frac{-\omega}{U}\Big)\, \hat{h}(\omega) e^{2\pi \imath \omega \cdot t}\, d\omega.$$

For $\theta \in \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d))$ the Uth θ-mean of the Fourier transform of $h \in \boldsymbol{L}^p(\mathbb{R}^d)$ $(1 \leq p \leq 2)$ or $h \in \boldsymbol{L}^p(\mathbb{R}^d) \cap \boldsymbol{L}^2(\mathbb{R}^d)$ $(2 < p < \infty)$ is introduced by

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Key Players for Time-Frequency Analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

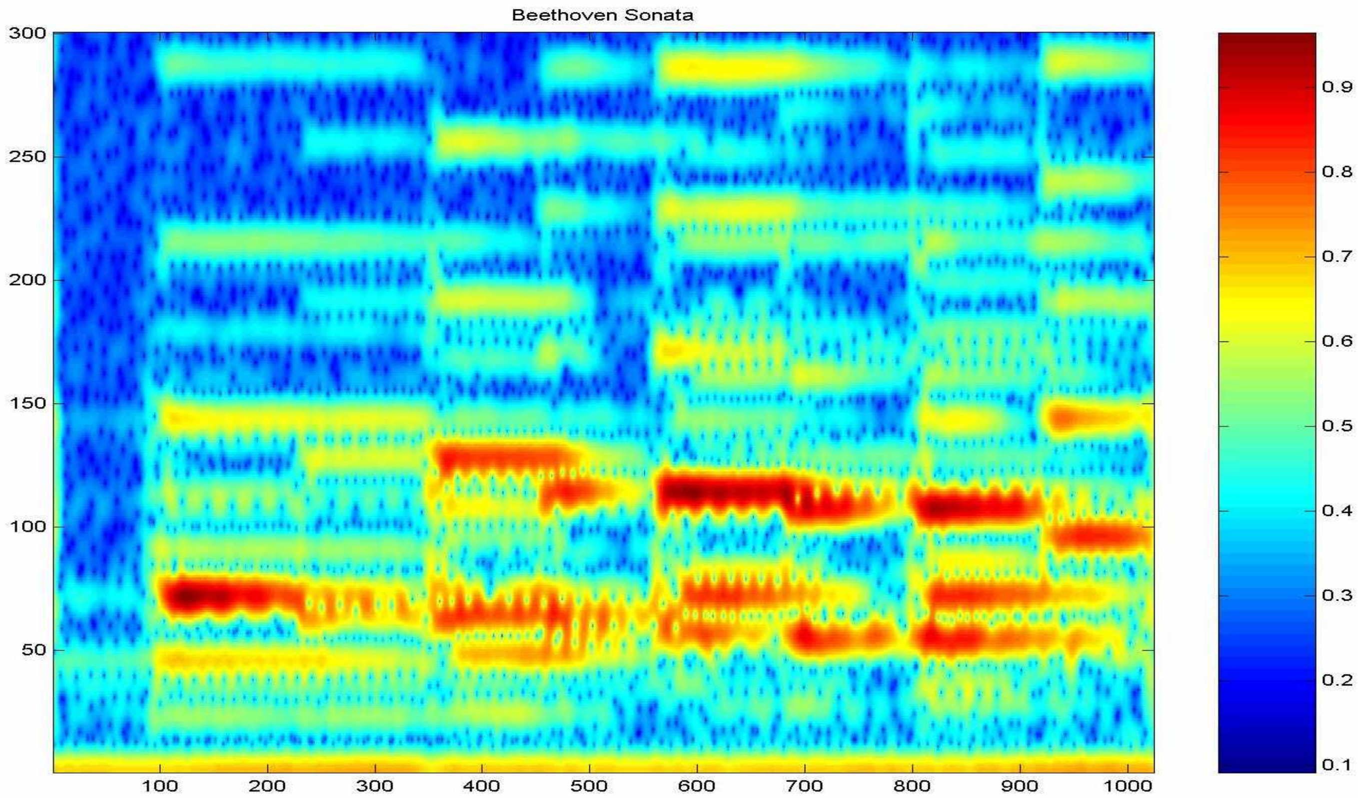
Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = V_g f(t, \omega) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \quad \lambda = (t, \omega);$$

A Typical Musical STFT



$S_0(\mathbb{R}^d) = M^1(\mathbb{R}^d) := M^0_{1,1}(\mathbb{R}^d)$

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is (by definition) in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is a Banach space, for any fixed, non-zero $g \in S_0(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable windows. Often the Gaussian is used as a window. Note that

$$V_g f(x,\omega) = (\widehat{f \cdot T_x g})(\omega), \quad \text{i.e., } g \text{ localizes } f \text{ near } x.$$

Lemma 1. *Let* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, *then the following holds:*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

Remark 2. *Moreover one can show that* $\boldsymbol{S}_0(\mathbb{R}^d)$ *is the smallest non-trivial Banach spaces with this property, i.e., it is continuously embedded into any such Banach space. As a formal argument one can use the continuous inversion formula for the STFT:*

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(\lambda)\pi(\lambda) g \, d\lambda$$

which implies

$$\|f\|_{\boldsymbol{B}} \leq \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(\lambda)| \|\pi(\lambda)g\|_{\boldsymbol{B}} \, d\lambda = \|g\|_{\boldsymbol{B}} \|f\|_{\boldsymbol{S}_0}.$$

Basic properties of $S_0(\mathbb{R}^d)$ resp. $S_0(G)$

THEOREM:

- For any automorphism α of G the mapping $f \mapsto \alpha^*(f)$ is an isomorphism on $\boldsymbol{S}_0(G)$; $[with (\alpha^* f)(x) = f(\alpha(x))], x \in G$.
- $\mathcal{F}\boldsymbol{S}_0(G) = \boldsymbol{S}_0(\hat{G})$; (Invariance under the Fourier Transform);
- $T_H \boldsymbol{S}_0(G) = \boldsymbol{S}_0(G/H)$; (Integration along subgroups);
- $R_H \boldsymbol{S}_0(G) = \boldsymbol{S}_0(H)$; (Restriction to subgroups);
- $\boldsymbol{S}_0(G_1)\hat{\otimes}\boldsymbol{S}_0(G_2) = \boldsymbol{S}_0(G_1 \times G_2)$. (tensor product stability);

Basic properties of $S_0'(\mathbb{R}^d)$

Of course the dual space $\boldsymbol{S}_0'(\mathbb{R}^d)$ is defined as the set of all bounded linear functionals. Since $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is a dense subspace of $\boldsymbol{S}_0(\mathbb{R}^d)$ the continuous linear functionals are are subspace of the space $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ of tempered distributions on $\mathbb{R}^d$.
A tempered distribution $\sigma \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ belongs to $\boldsymbol{S}_0'(\mathbb{R}^d)$ if and only if its short-term Fourier transform is bounded. Norm convergence in $\boldsymbol{S}_0'(\mathbb{R}^d)$ corresponds to uniform convergence of the corresponding STFTs.
There is also a w^*-convergence in $\boldsymbol{S}_0'(\mathbb{R}^d)$, and this is just pointwise convergence (or equivalently uniform convergence over compact sets).

The Banach space $\boldsymbol{S}_0(\mathbb{R}^d)$ is dense in the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, which in turn is (only) w^*-dense in $\boldsymbol{S}_0'(\mathbb{R}^d)$ (not norm dense).
A triple, consisting of a Banach space $\mathbf{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is (hence) contained in B is called a Banach Gelfand triple.

Banach Gelfand Triples

A triple, consisting of a Banach space $\mathbf{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is (hence) contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

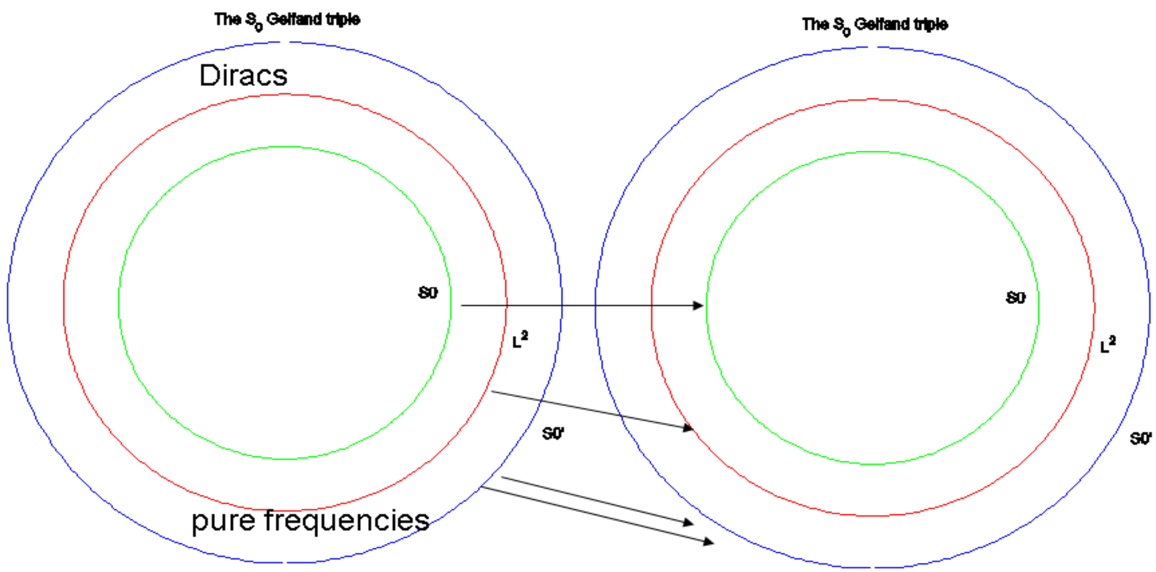
Definition 3. *If* $(B_1, \mathcal{H}_1, B_1')$ *and* $(B_2, \mathcal{H}_2, B_2')$ *are Gelfand triples then an operator* A *is called a [unitary] Gelfand triple isomorphism if*

1. A *is an isomorphism between* B_1 *and* B_2.

2. A *is a [unitary operator resp.] isomorphism between* $\mathcal{H}_1$ *and* $\mathcal{H}_2$.

3. A *extends to a weak** *isomorphism as well as a norm-to-norm continuous isomorphism between* B_1' *and* B_2'.

Gelfand triple mapping



Basic properties of $S_0'(\mathbb{R}^d)$

THEOREM:

- the Generalized Fourier Transforms, defined by transposition
$$\langle \hat{\sigma}, f \rangle = \langle \sigma, \hat{f} \rangle,$$
for $f \in \boldsymbol{S}_0(\hat{G}), \sigma \in \boldsymbol{S}_0{}'(G)$, satisfies $\mathcal{F}(\boldsymbol{S}_0'(G)) = \boldsymbol{S}_0'(\hat{G})$.
- $\sigma \in \boldsymbol{S}_0'(G)$ is H-periodic, i.e. $\sigma(f) = \sigma(T_h f)$ for all $h \in H$, iff there exists $\dot{\sigma} \in \boldsymbol{S}_0'(G/H)$ such that $\langle \sigma, f \rangle = \langle \sigma, T_H f \rangle$.
- $\boldsymbol{S}_0'(H)$ can be identified with a subspace of $\boldsymbol{S}_0'(G)$, the injection i_H being given by
$$\langle i_H \sigma, f \rangle := \langle \sigma, R_H f \rangle.$$
For $\sigma \in \boldsymbol{S}_0'(G)$ one has $\sigma \in i_H(\boldsymbol{S}_0'(H))$ iff $\mathrm{supp}(\sigma) \subseteq H$.

The Usefulness of $S_0(\mathbb{R}^d)$

Theorem 1. *(**Poisson's formula**) For $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ and any discrete subgroup H of $\mathbb{R}^d$ with compact quotient the following holds true: There is a constant $C_H > 0$ such that*

$$\sum_{h \in H} f(h) = C_H \sum_{l \in H^\perp} \hat{f}(l) \tag{3}$$

with absolute convergence of the series on both sides.

By duality one can express this situation as the fact that the Comb-distribution $\mu_{\mathbb{Z}^d} = \sum_{k \in Z^d} \delta_k$, as an element of $\boldsymbol{S}_0'(\mathbb{R}^d)$ is invariant under the (generalized) Fourier transform. Sampling corresponds to the mapping $f \mapsto f \cdot \mu_{\mathbb{Z}^d} = \sum_{k \in \mathbb{Z}^d} f(k)\delta_k$, while it corresponds to convolution with $\mu_{\mathbb{Z}^d}$ on the Fourier transform side = periodization along $(\mathbb{Z}^d)^\perp = \mathbb{Z}^d$ of the Fourier transform $\hat{f}$. For $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ all this makes perfect sense.

Regularizing sequences for (S_0, L^2, S_0')

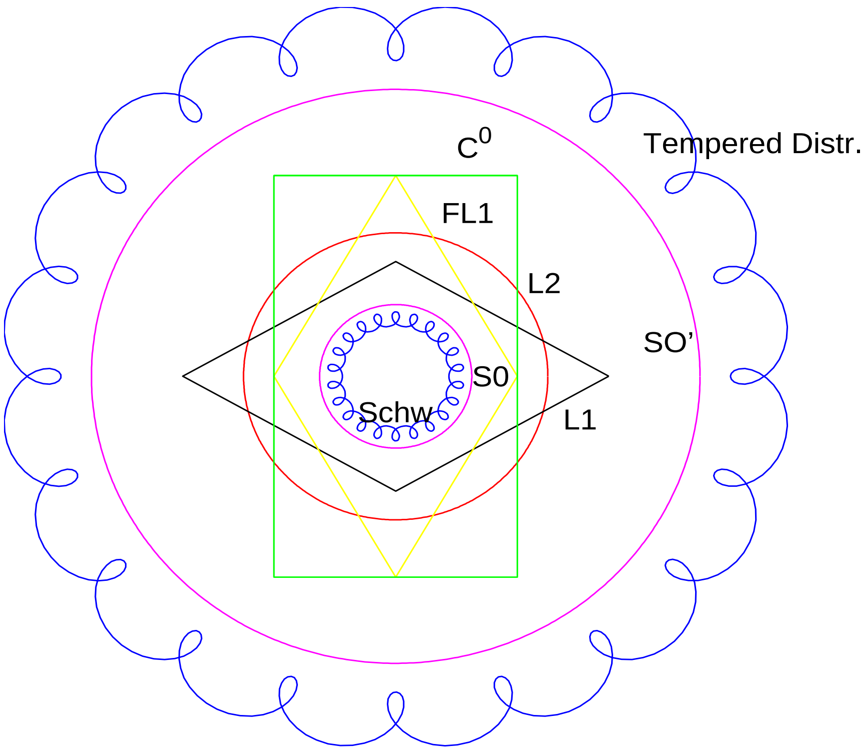
Wiener amalgam convolution and pointwise multiplier results imply that

$$\boldsymbol{S}_0(\mathbb{R}^d) \cdot (\boldsymbol{S}_0'(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d) \quad \boldsymbol{S}_0(\mathbb{R}^d) * (\boldsymbol{S}_0'(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d)$$

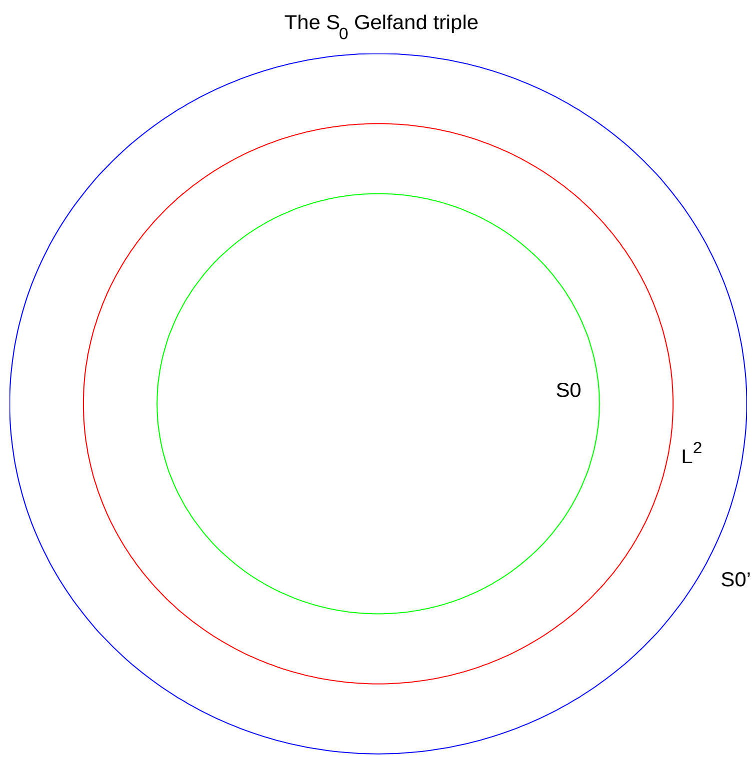
e.g. $\boldsymbol{S}_0(\mathbb{R}^d) * \boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^\infty) \subseteq \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^\infty)$.

Let now $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ be given with $h(0) = 1$. Then the dilated version $h_n(t) = h(t/n)$ are a uniformly bounded family of multipliers on $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$, tending to the identity operator in a suitable way. Similarly, the usual Dirac sequences, obtained by compressing a function $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(x)dx = 1$ are showing a similar behavior: $g_n(t) = n \cdot g(nt)$
Following the above rules the combination of the two procedures, i.e. product-convolution or convolution-product operators of the form provide suitable regularizers: $A_n f = g_n * (h_n \cdot f)$ or $B_n f = h_n \cdot (g_n * f)$.

Schwartz space, S_0, L^2, S_0', tempered distributions



The Gelfand Triple (S_0, L^2, S_0')



The Fourier transform is a prototype of a Gelfand triple isomorphism.

EX1: The Fourier transform as Gelfand Triple Automorphism

Theorem 2. *Fourier transform* $\mathcal{F}$ *on* $\mathbb{R}^d$ *has the following properties:*

(1) $\mathcal{F}$ *is an isomorphism from* $\boldsymbol{S}_0(\mathbb{R}^d)$ *to* $\boldsymbol{S}_0(\widehat{\mathbb{R}^d})$*,*

(2) $\mathcal{F}$ *is a unitary map between* $\boldsymbol{L}^2(\mathbb{R}^d)$ *and* $\boldsymbol{L}^2(\widehat{\mathbb{R}^d})$*,*

(3) $\mathcal{F}$ *is a weak** *(and norm-to-norm) continuous bijection from* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *onto* $\boldsymbol{S}_0'(\widehat{\mathbb{R}^d})$*.*

Furthermore we have that Parseval's formula

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \tag{4}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$*, and therefore on each level of the Gelfand triple* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$*.*

The properties of Fourier transform can be expressed by a Gelfand bracket

$$\langle f, g \rangle_{(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')} = \langle \hat{f}, \hat{g} \rangle_{(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')} \tag{5}$$

which combines the functional brackets of dual pairs of Banach spaces and of the inner-product for the Hilbert space.

One can characterize the Fourier transform as the *uniquely determined* unitary Gelfand triple automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ which maps pure frequencies into the corresponding Dirac measures (and vice versa). [1]
One could equally require that TF-shifted Gaussians are mapped into FT-shifted Gaussians, relying on $\mathcal{F}(M_\omega T_x f) = T_{-\omega} M_x \mathcal{F} f$ and the fact that $\mathcal{F} g_0 = g_0$, with $g_0(t) = e^{-\pi |t|^2}$.

---

[1] as one would expect in the case of a finite Abelian group.

EX.2: The Kernel Theorem for general operators in $\mathcal{L}(S_0, S_0')$

Theorem 3. *If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.*

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)f(y)dyg(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)g(x)f(y)dxdy.$$

This result is the "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.

Again the complete picture can again be best expressed by a unitary Gelfand triple isomorphism. We first describe the innermost shell:

Theorem 4. *The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with* $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ *as scalar product on* $\mathcal{HS}$ *and the usual Hilbert space structure on* $\boldsymbol{L}^2(\mathbb{R}^{2d})$ *on the kernels.*
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ *if and only if the corresponding operator* $\boldsymbol{K}$ *maps* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *into* $\boldsymbol{S}_0(\mathbb{R}^d)$*, but not only in a bounded way, but also continuously from* w^*-*topology into the norm topology of* $\boldsymbol{S}_0(\mathbb{R}^d)$*.*

Remark: Note that for "regularizing" kernels in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ the usual identification (recall that the entry of a matrix $a_{n,k}$ is the coordinate number n of the image of the $n-$th unit vector under that action of the matrix $A = (a_{n,k})$:

$$k(x, y) = K(\delta_y)(x) = \delta_x(K(\delta_y).$$

Since $\delta_y \in \boldsymbol{S}_0'(\mathbb{R}^d)$ and consequently $K(\delta_y) \in \boldsymbol{S}_0(\mathbb{R}^d)$ the pointwise evaluation makes sense.

With this understanding our claim is that the kernel theorem provides a (unitary) isomorphism between the Gelfand triple (of kernels) $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ into the Gelfand triple of operator spaces

$$\left(\mathcal{L}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d)), \, \mathcal{HS}, \, \mathcal{L}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d)) \right).$$

Wilson bases

Lemma 3. *Let* $(\mathbf{\Psi}_{\mathbf{k},\mathbf{n}})$ *be an orthonormal Wilson basis for* $L^2(\mathbb{R}^d)$*. Then the coefficient mapping* $D : f \mapsto \langle f, \mathbf{\Psi}_{k,n} \rangle$ *induces a unitary Gelfand triple isomorphism between*

$$(\boldsymbol{S}_0, L^2, \boldsymbol{S}_0')(\mathbb{R}^d) \quad \textit{and} \quad (\ell^1, \ell^2, \ell^\infty)(\mathbb{Z}^d \times \mathbb{N}^d).$$

From this identification one can get some understanding of the kernel theorem: one can express the w^*-to-norm continuous operators from $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$ with the w^*-to-norm continuous operators from ℓ^∞ into ℓ^1 which turns out to be exactly the space of (bi-infinite) matrices in $\ell^1(\mathbb{Z}^d \times \mathbb{Z}^d)$. Similar statements can be made with respect to

so-called *local Fourier bases* and also *Malvar bases*. For the case of "Gabor frames" one has instead of an isomorphism just a retract property.
However, such is a situation is not completely new, if we recall that in the classical case the Fourier basis establishes a natural Banach Gelfand triple isomorphism between $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(\mathbb{Z})$ and $(\boldsymbol{A}(\mathbb{T}), \boldsymbol{L}^2(\mathbb{T}), \mathcal{P}_M)$.

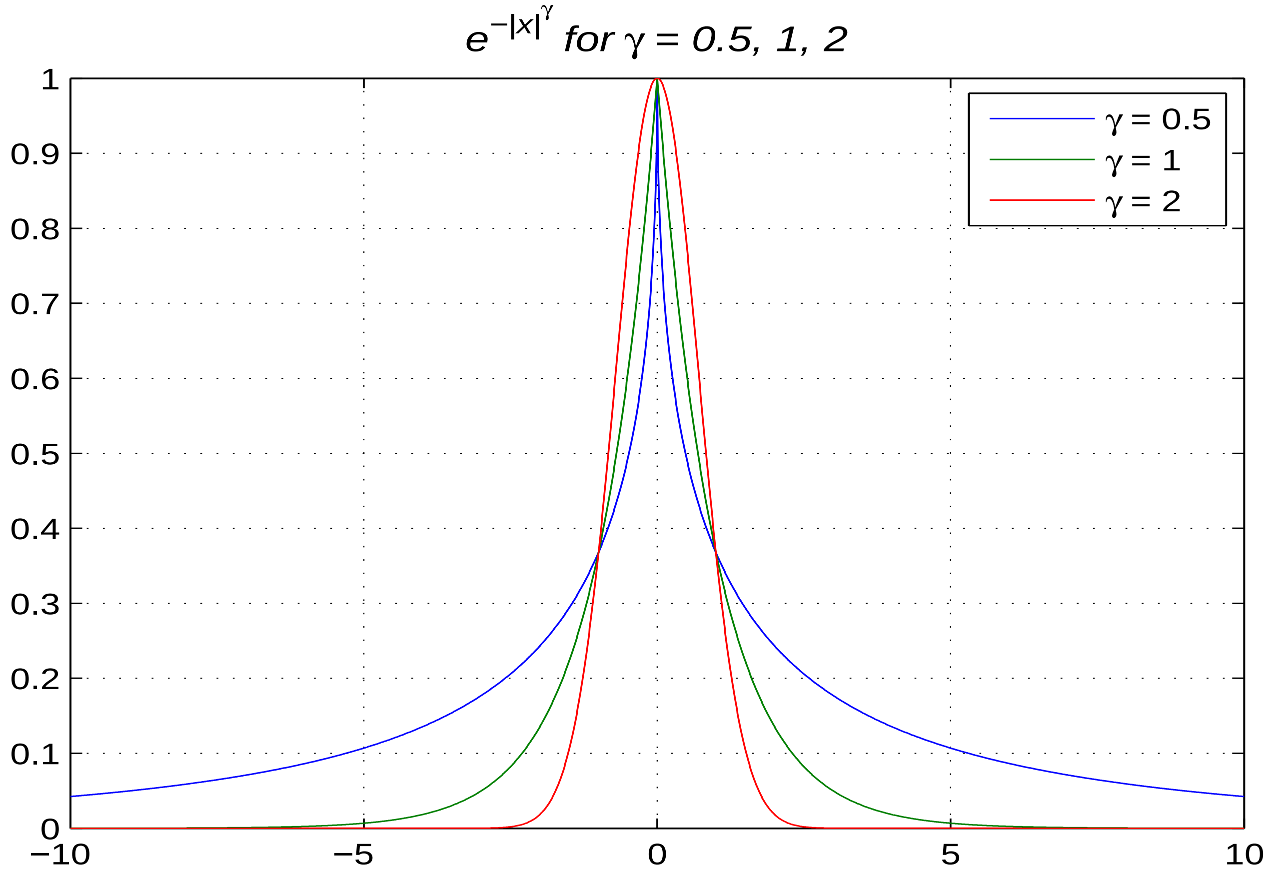
Other talks by HGFei on related topics:

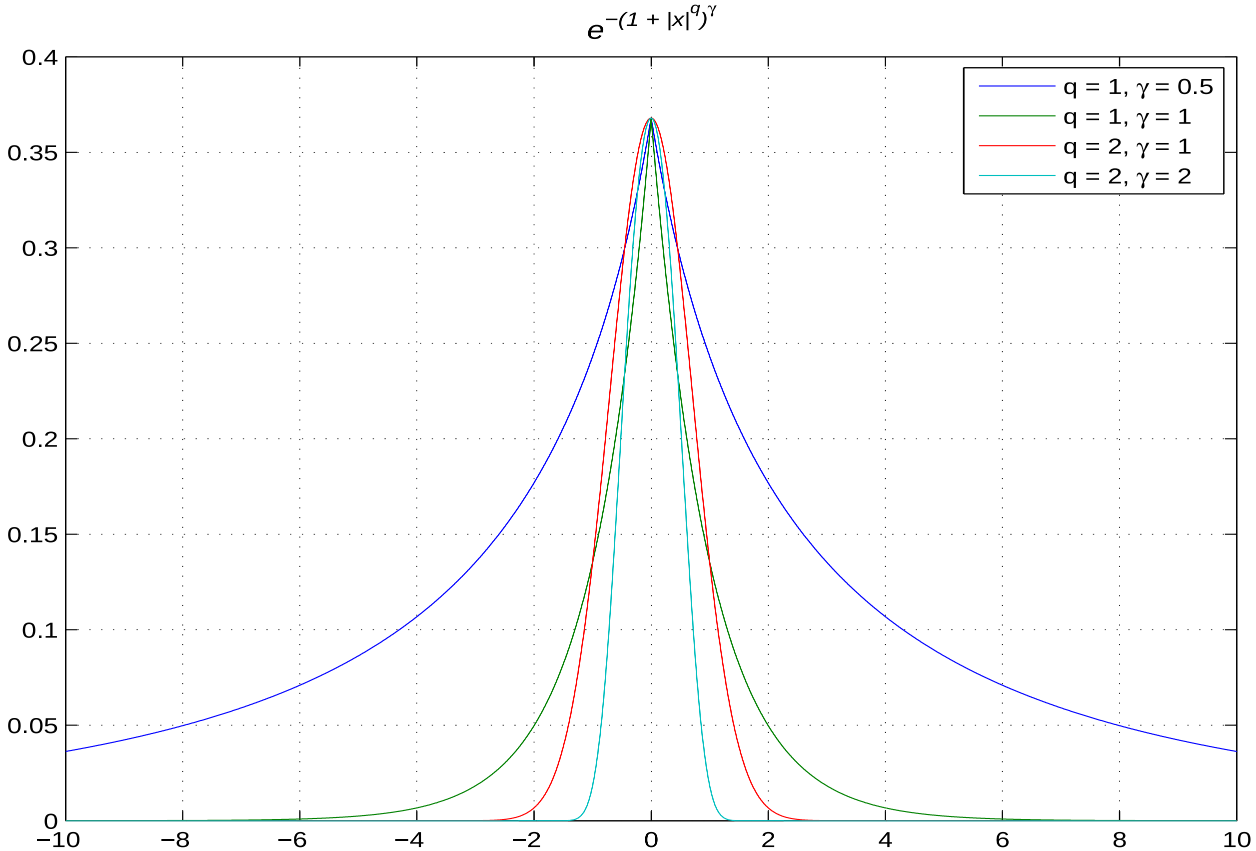
Information about Banach Gelfand Triples can be found in a number of talks available from the NuHAG talk-server, i.e. from www.nuhag.eu
You can inquire e.g.
http://www.univie.ac.at/nuhag-php/program/talks_show.php?name=Feichtinger
or http://www.univie.ac.at/nuhag-php/program/talks_details.php?id=955
or http://www.univie.ac.at/nuhag-php/program/talks_details.php?id=941

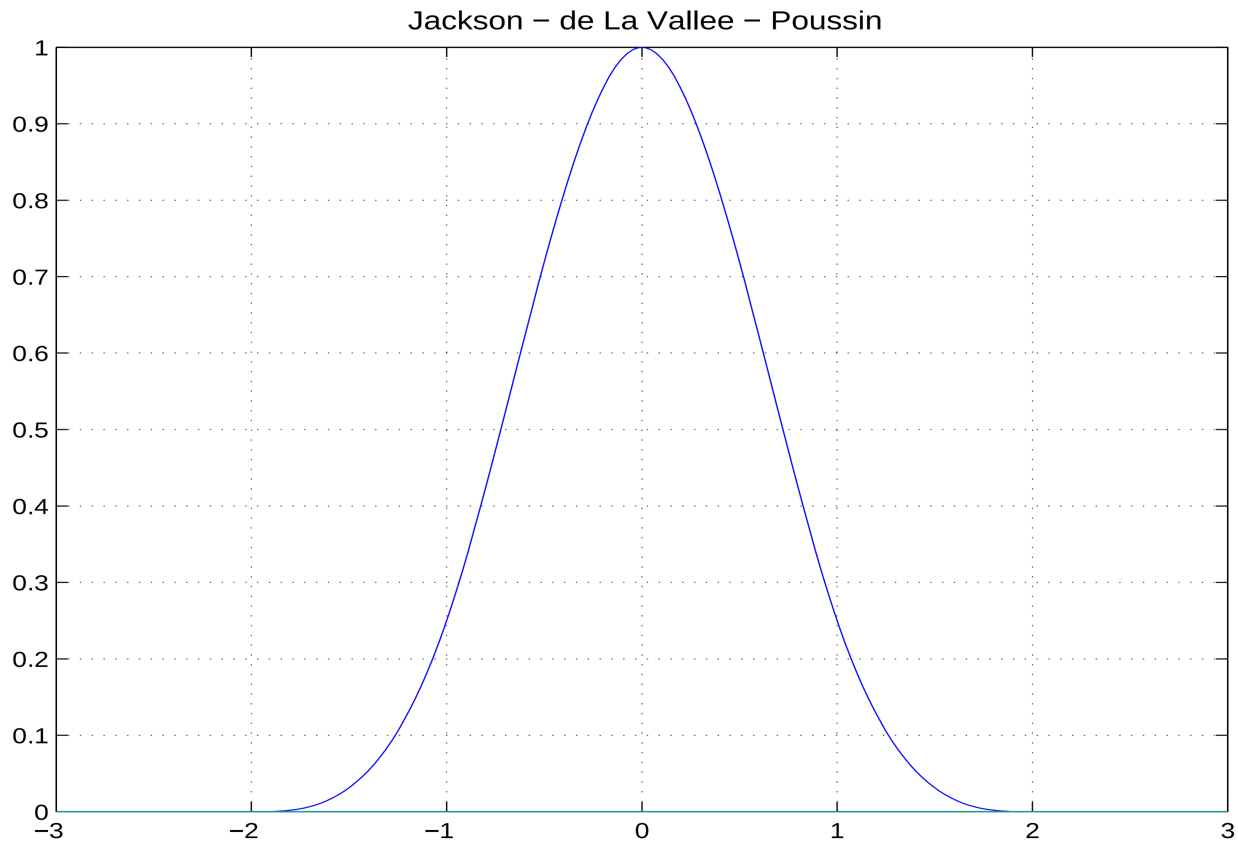
Recent work connecting Gabor and Summability Theory

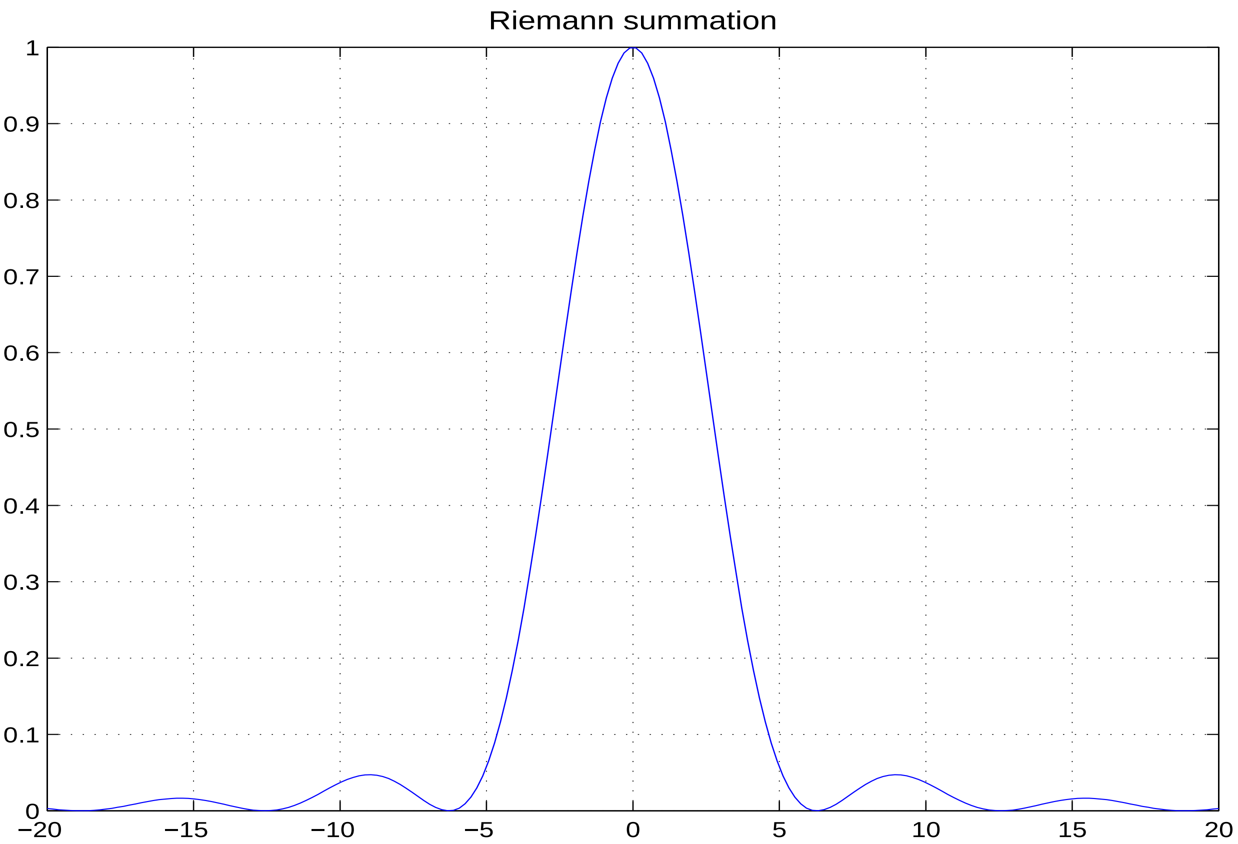
In the years 2004 - 2006 Ferenc Weisz from Budapest spent two years in Vienna, which resulted on a series of papers connecting *classical summability theory* with *Gabor and time-frequency analysis*.
In general one can say: Good kernels are good because they belong to $\boldsymbol{S}_0(\mathbb{R}^d)$, while "bad kernels" (such as the box-function or the SINC-function) simply are *not* in $\boldsymbol{S}_0(\mathbb{R}^d)$. Most sufficient kernels, that imply e.g. that Poisson's formula is valid for them, or makes them good approximate units, simply provide sufficient conditions for membership of functions in $\boldsymbol{S}_0(\mathbb{R})$ or $\boldsymbol{S}_0(\mathbb{R}^d)$ resp. good examples are [14] or a long list of kernels listed in [13] and other papers by F. Weisz (and hgfei), such as [13, 10, 11, 12].
Other places where $\boldsymbol{S}_0(\mathbb{R}^d)$ and other *Wiener amalgam spaces* play a role in order to overcome the (apparent) technical problems in Gabor analysis are [9, 19].
Banach Gelfand triples are described in some detail in [6]. Comments on "Feichtinger's algebra" $\boldsymbol{S}_0(\mathbb{R}^d)$ are also given in [21].

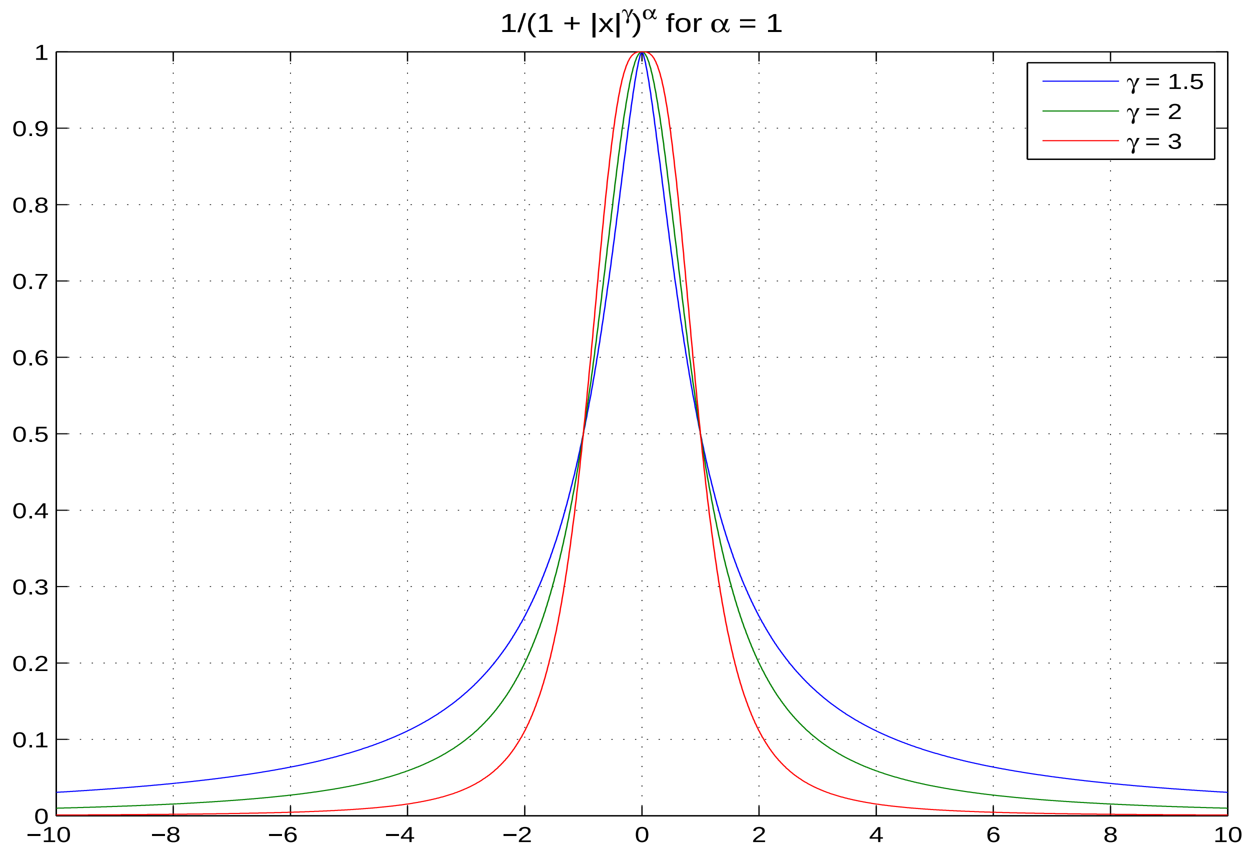
A picture book of kernels in $S_0(\mathbb{R})$:

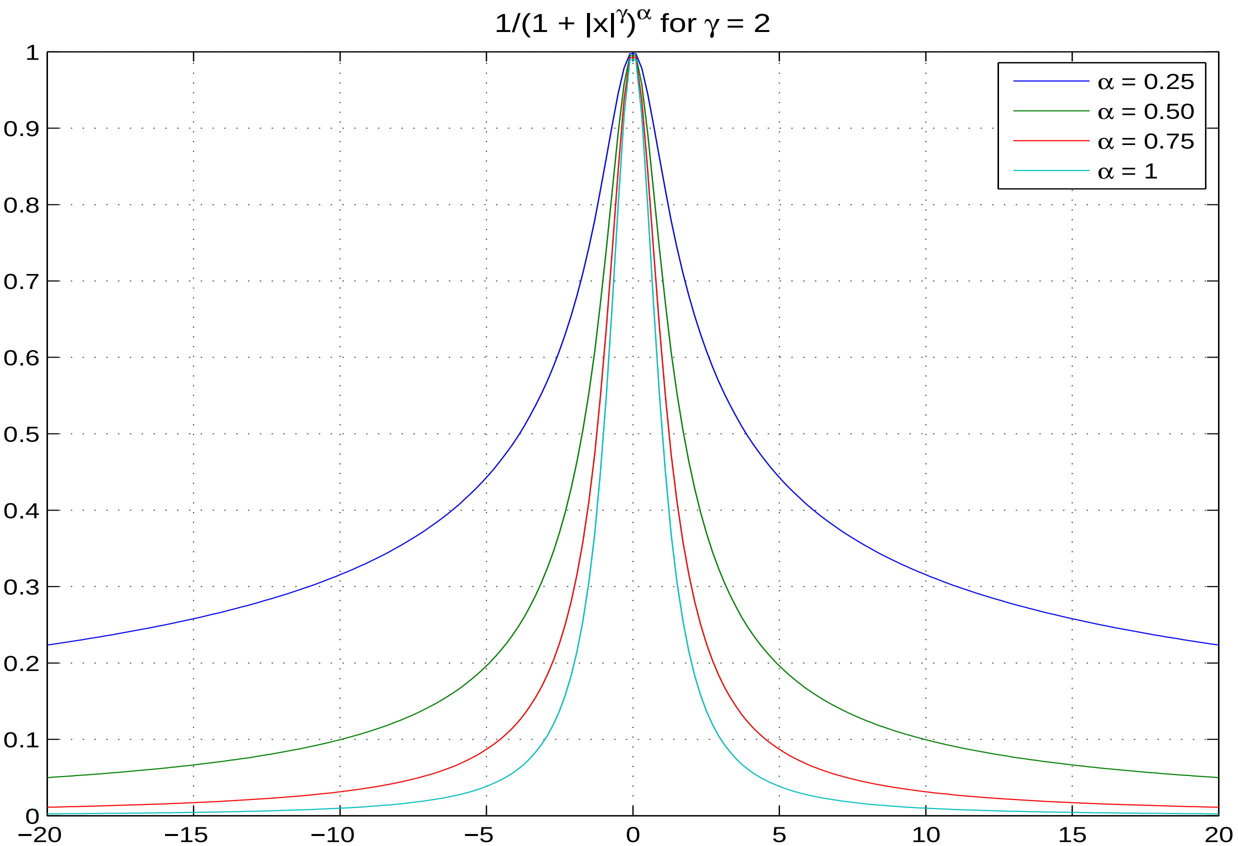


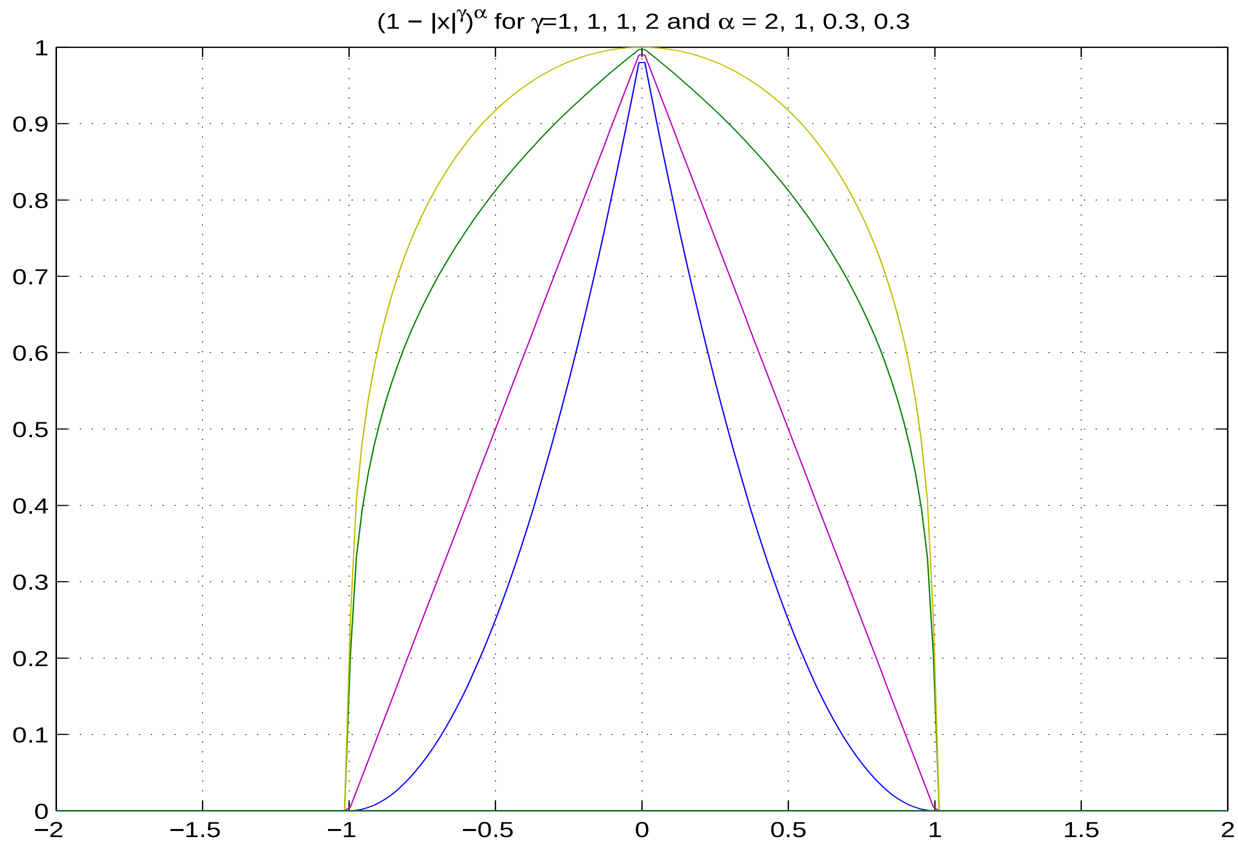

$e^{-(1 + |x|^q)^\gamma}$
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−10
−8
−6
−4
−2
0
2
4
6
8
10
q = 1, γ = 0.5
q = 1, γ = 1
q = 2, γ = 1
q = 2, γ = 2


Jackson – de La Vallee – Poussin
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−3
−2
−1
0
1
2
3

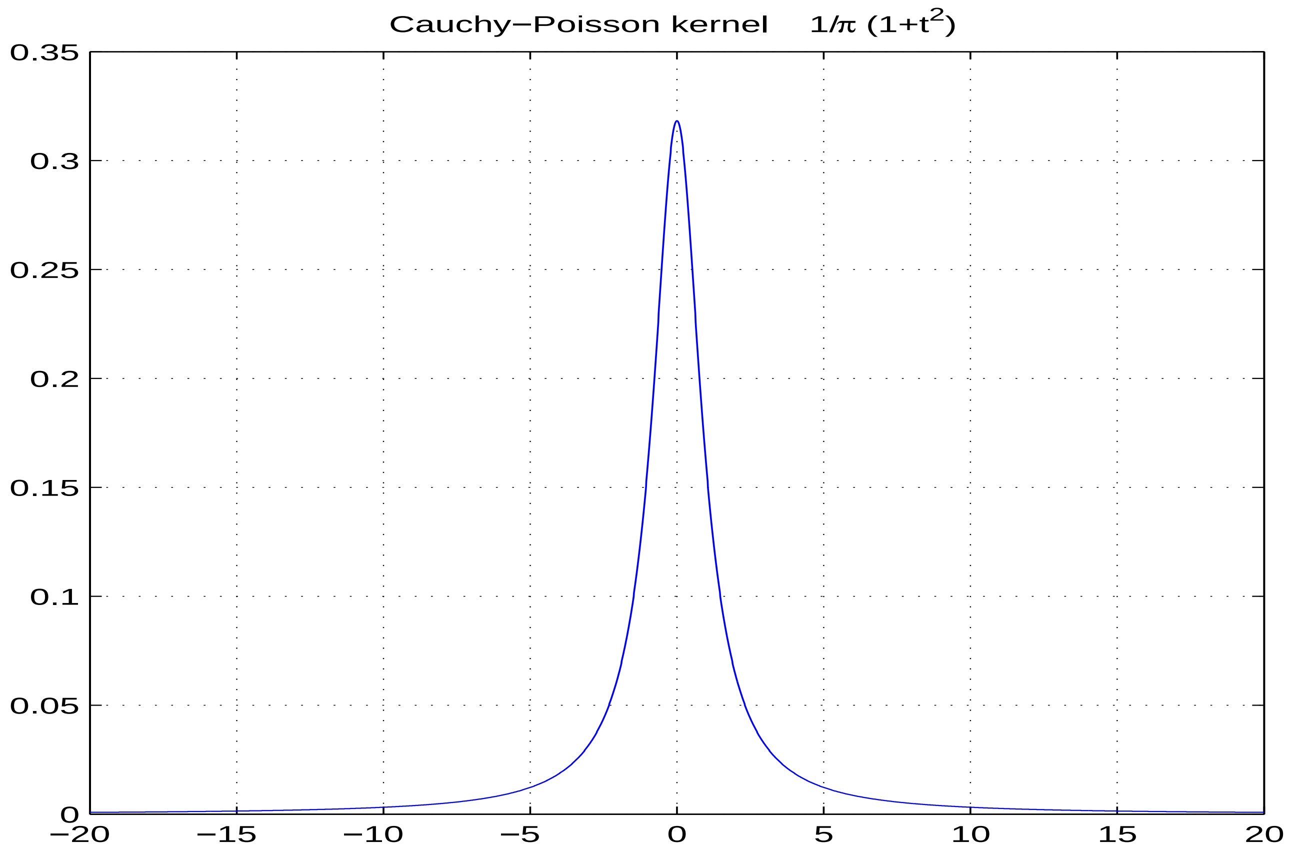

Riemann summation
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−20
−15
−10
−5
0
5
10
15
20


1/(1 + |x|^γ)^α for α = 1
γ = 1.5
γ = 2
γ = 3
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−10
−8
−6
−4
−2
0
2
4
6
8
10

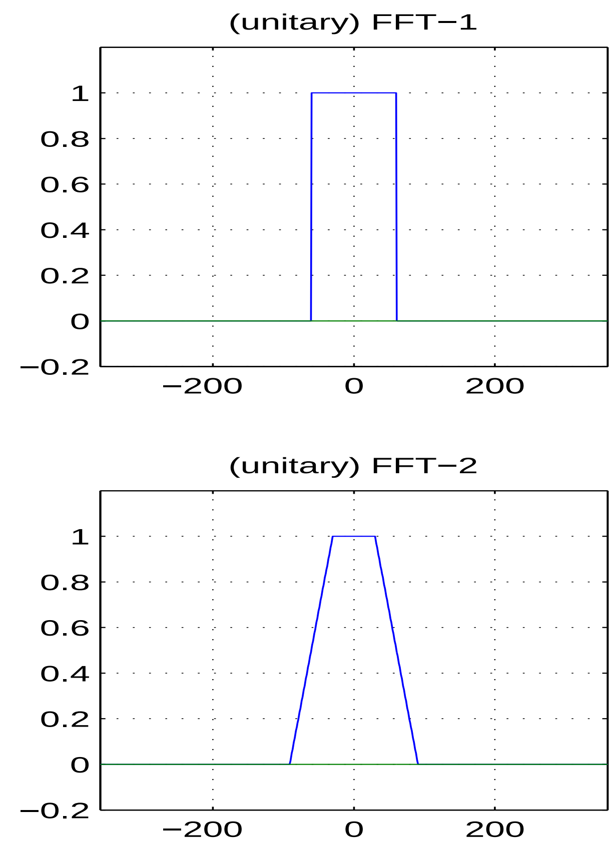

1/(1 + |x|^γ)^α for γ = 2
α = 0.25
α = 0.50
α = 0.75
α = 1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−20
−15
−10
−5
0
5
10
15
20



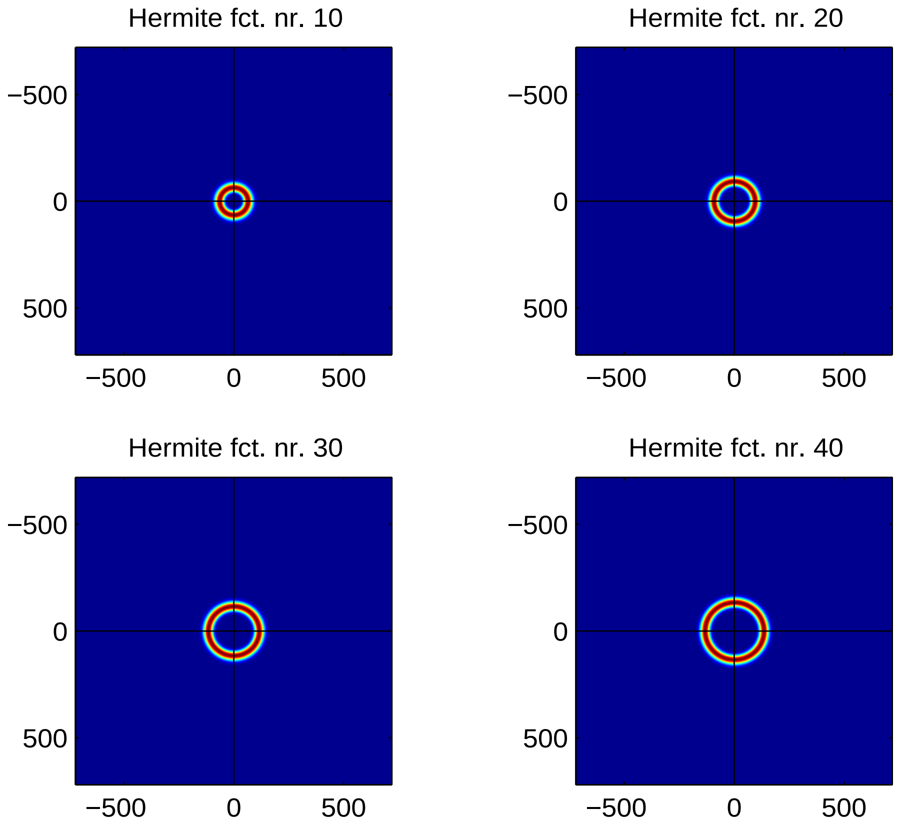
(1 − |x|^γ)^α for γ=1, 1, 1, 2 and α = 2, 1, 0.3, 0.3
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2


Cauchy−Poisson kernel 1/π (1+t2)
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−20
−15
−10
−5
0
5
10
15
20

Comparing Dirichlet kernel and de la Vallee Poussin kernel:



Of course it is also interesting to look at Hermite functions:



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Classical Book by Butzer/Nessel [4]
Yitzhak Katznelson's book (3rd edition): [17]
Benedetto's "Harmonic Analysis and Applications" ([3])
Hans Reiter's book (original and expanded version with Stegeman) [20] and [21]
And of course Charly's book on the "Foundations of TF-analysis" ([15]).

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