Frame Theory and its Acoustical Applications

Peter Balazs
(joint work with a lot of people, many of them are in the room!)

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Austrian Academy of Sciences, Vienna

Modern Methods of Time-Frequency Analysis II, ESI
Time-frequency methods for the applied sciences,
04.12.2012

Overview:

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1. Acoustics Research Institute
2. Frame Theory
   - Time-Frequency representation
   - Semi-Frames
   - Non-stationary Gabor Transform
3. Frame Multipliers
4. Applications
   - Perceptual Sparsity by Irrelevance
   - Acoustic System Estimation
5. Conclusions

Österreichische Akademie der Wissenschaften

Overview:

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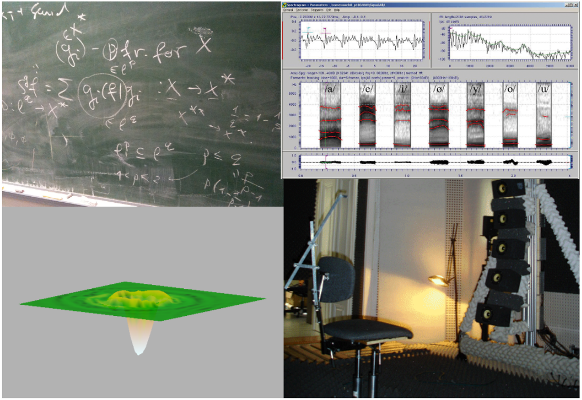
Acoustics Research Institute

Part of the Austrian Academy of Sciences, approx. 30 employees.
Work-Groups:



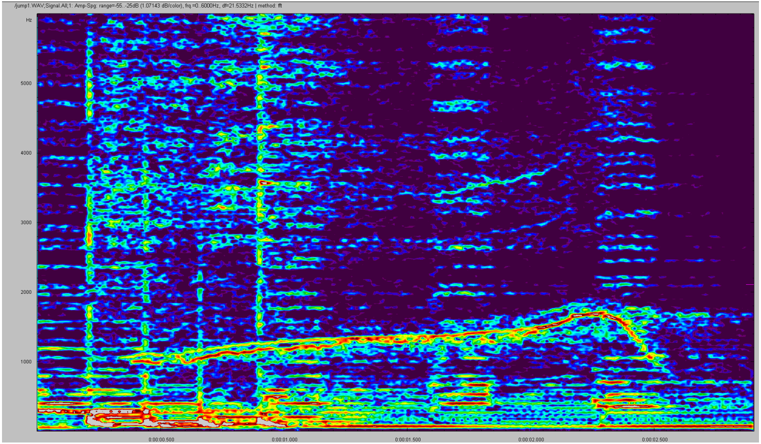
Acoustics Research Institute

Part of the Austrian Academy of Sciences, approx. 30 employees.
Work-Groups:

- Acoustic Phonetics
- Physical and Computational Acoustics
- Psychoacoustics and Experimental Audiology
- Mathematics and Signal Processing in Acoustics

Signal Representations: Time-Frequency Analysis and Filterbanks

Spectogram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1000
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Short Time Fourier Transformation (STFT)

Definition (see e.g [Gröchenig, 2001])

Let $f,g \neq 0$ in $L^2\left(\mathbb{R}^d\right)$, then we call

$$\mathcal{V}_g f(\tau,\omega) = \int\limits_{\mathbb{R}^d} f(x)\overline{g(x-\tau)}e^{-2\pi i \omega x}dx\,.$$

the Short Time Fourier Transformation (STFT) of the signal f with the window g.

$$\mathcal{V}_g(f)(\tau,\omega) = \mathcal{F}\left(f\cdot\overline{T_\tau g}\right).$$

Sampled Version: Gabor transform : $\tau = a\cdot k$, $\omega = b \cdot l$ with $k,l \in \mathbb{Z}$.

$$f \mapsto \mathcal{V}_g(f)(a\cdot k, b\cdot l).$$

When is perfect reconstruction possible?

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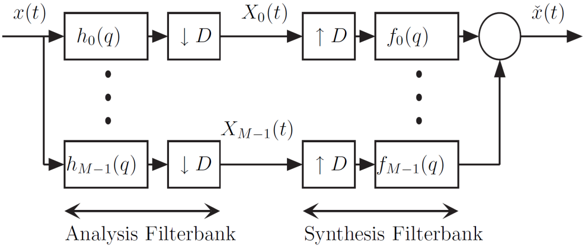
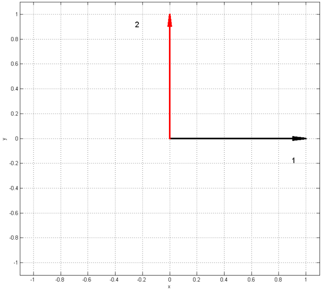
Filterbank



When is perfect reconstruction possible?

Orthonormal Basis (ONB)

Standard aproach: orthonormal basis.

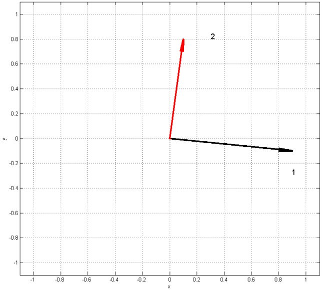


Problems:

- Perturbation
- Construction
- Error Robustness

Riesz bases

Riesz bases

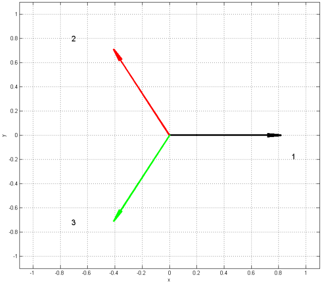


Problems:

- ~~Perturbation~~
- Construction
- Error Robustness

Frames

Alternate approach: introduce redundancy.



Problems:

- ~~Perturbation~~
- ~~Construction~~
- ~~Error Robustness~~

Frames

Definition

The (countable) sequence $\Psi = (\psi_k | k \in K)$ is called a frame for the Hilbert space $\mathcal{H}$ if constants $A > 0$ and $B < \infty$ exist such that

$$A \cdot \|f\|_{\mathcal{H}}^2 \leq \sum_k |\langle f, \psi_k \rangle|^2 \leq B \cdot \|f\|_{\mathcal{H}}^2 \ \forall \ f \in \mathcal{H}.$$

[Duffin and Schaeffer, 1952, Daubechies et al., 1986]

Frames

Beautiful abstract mathematical setting:

- Frames = generalization of bases; can be overcomplete, allowing redundant representations.
- Frame inequality = generalization of Parseval's condition.
- Active field of research in mathematics!

Interesting for applications:

- **Much more freedom.** Finding and constructing frames can be easier and faster.
  Some advantageous side constraints can **only** be fulfilled for frames.
- Perfect reconstruction is guaranteed with the 'canonical dual frame' $\tilde{\psi}_k = S^{-1}\psi_k$

$$f = \sum_k < f, \psi_k > \tilde{\psi}_k = \sum_k < f, \tilde{\psi}_k > \psi_k.$$

Semi-Frames

Sometimes even the concept of frames is too restrictive, in the sense that one cannot satisfy both frame bounds simultaneously.

Ψ is an upper semi-frame [Antoine and Balazs, 2011] if there exists a constant $\mathsf{B} < \infty$ such that

$$0 < \sum_{k \in \Gamma} |\langle \psi_k, f \rangle|^2 \leq \mathsf{B} \|f\|^2, \forall f \in \mathcal{H}, \ f \neq 0.$$

Theorem

For all $f \in R_D$, we have the reconstruction formula

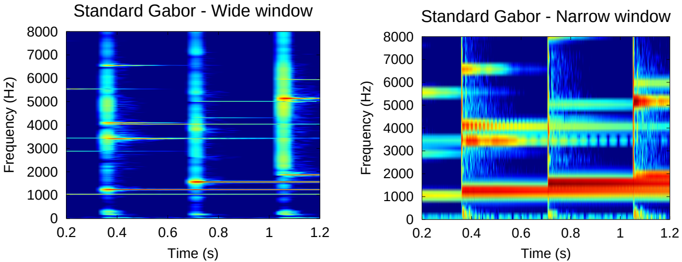
$$f = \sum_k \left[G^{-1} \left(\langle f, \psi_k \rangle_{\mathcal{H}} \right) \right] \psi_k \tag{1}$$

with unconditional convergence.

Non-stationary Gabor transform

Non-stationary Gabor transform

Limitations of Standard Gabor analysis: Quality of representation highly depends on window choice, but optimal window choice is different for different signal components



Our proposition [Jaillet, 2005, Balazs et al., 2011]: simple extension to reduce this limitation by using window evolving over time.

Non-stationary Gabor transform

Given a sequence of windows $(\gamma_n)_{n\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and sequences of real numbers $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$, the non-stationary Gabor transform (NSGT) elements are defined, for $(m,n)\in\mathbb{Z}^2$, by:

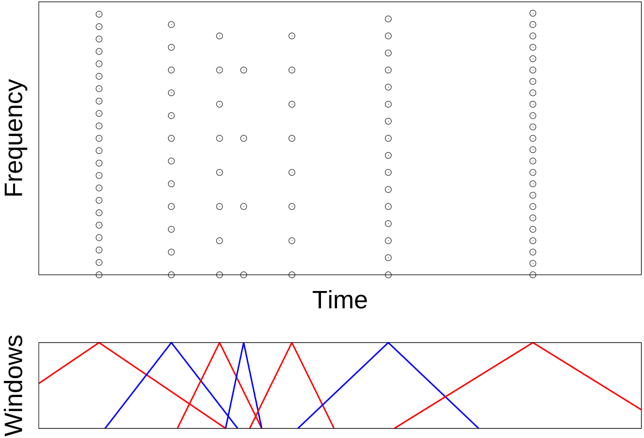
$$\gamma_{m,n}(t) = \gamma_n(t - na_n)e^{i2\pi mb_n t} = M_{mb_n} T_{na_n} \gamma_n.$$

Regular structure in frequency allows FFT implementation.

A analogue construction in the frequency domain allows easy implementation of, e.g. wavelet frames; an invertible CQT [Velasco et al., 2011]; or a filterbank adapted to human auditory perception (see talk by Thibaud Necciari).

Non-stationary Gabor transform

Sampling grid example:



Non-stationary Gabor transform

Frame theory allows perfect reconstruction. Especially easy and fast in the 'painless' case:

Theorem

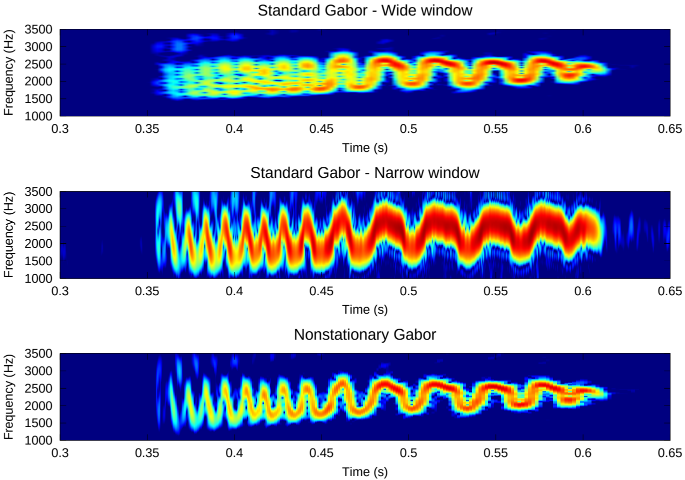
For every $n \in \mathbb{Z}$, let the function $\gamma_n \in L^2(\mathbb{R})$ be compactly supported with $\operatorname{supp}(\gamma_n) \subseteq [c_n + na_n, d_n + na_n]$ such that $d_n - c_n \leq \frac{1}{b_n}$. the system of functions $g_{m,n}$ forms a frame for $L^2(\mathbb{R})$ if and only if there exists $A > 0$ and $B < \infty$, such that $A \leq \sum_n \frac{1}{b_n} |\gamma_n(t - na_n)|^2 \leq B$. In this case, the canonical dual frame has the same structure and is given by:

$$\tilde{\gamma}_{m,n}(t) = \frac{\gamma_n(t)}{\sum_k \frac{1}{b_k} |\gamma_k(t - ka_k)|^2} e^{2\pi i m b_n t}. \quad (2)$$

For approaches and algorithm beyond the painless case see [Necciari et al., 2013] (and talk by Thibaud).

Non-stationary Gabor transform

Bird vocalization example:



What is a

Frame Multiplier:

Analysis

$\Downarrow$

Multiplication

$\Downarrow$

Synthesis

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Synthesis

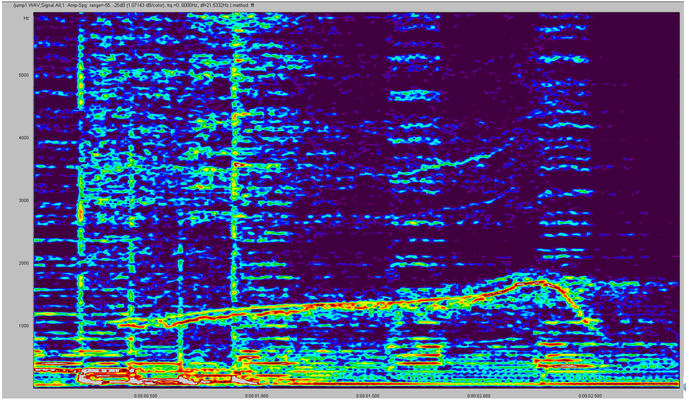
Multipliers

Those are operators, that are of utmost importance in

- Mathematics, where they are used for the diagonalization of operators [Schatten, 1960].
- Physics, where they are a link between classical and quantum mechanics, so called quantization operators [Ali et al., 2000].
- Signal Processing, where they are a particular way to implement time-variant filters [Matz and Hlawatsch, 2002].
- Acoustics, where those time-frequency filters are used in several fields, for example in Computational Auditory Scene Analysis [Wang and Brown, 2006].

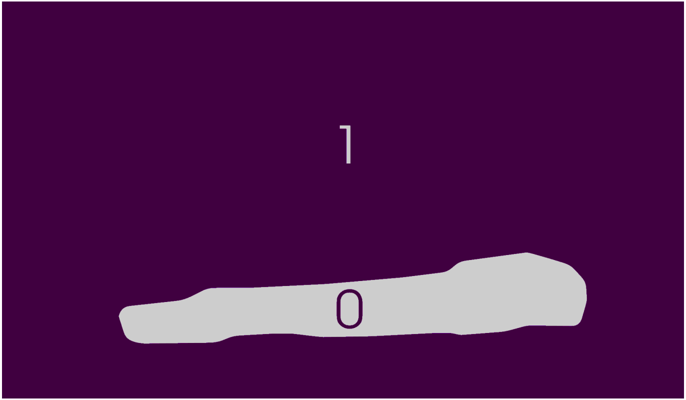
Example for a Multiplier

Original audio file:



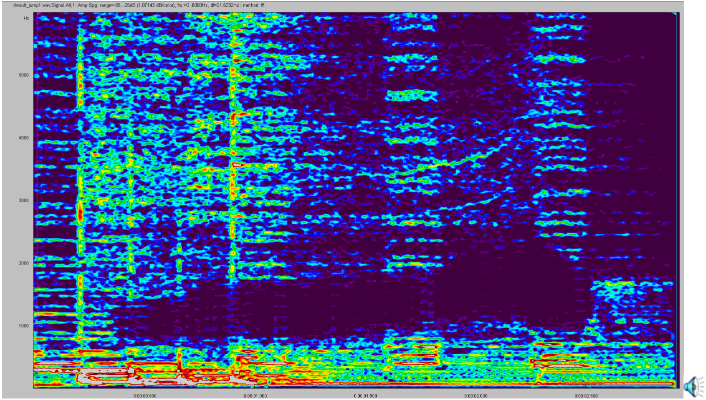
Example for a Multiplier

Symbol:



Example for a Multiplier

Result of Gabor Multiplier.



Frame Multipliers: Definition

Definition

Let $(\psi_k)_{k \in K}$, $(\phi_k)_{k \in K}$ be frames in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Define the operator $\mathbf{M}_{m,(\phi_k),(\psi_k)} : \mathcal{H}_1 \to \mathcal{H}_2$, the frame multiplier, as the operator

$$\mathbf{M}_{m,(\phi_k),(\psi_k)} f = \sum_k m_k \langle f, \psi_k \rangle \phi_k$$

where $m \in l^\infty(K)$ is called the symbol.

Generalization of Gabor multipliers [Feichtinger and Nowak, 2003] to the general frame case [Balazs, 2007].

Fundamental Research in the Theory of Multipliers

Theorem ([Balazs, 2007])

Let $\mathbf{M} = \mathbf{M}_{m,(\phi_k),(\psi_k)}$ *be a frame multiplier for* (ψ_k) *and* (ϕ_k) *with the upper frame bounds* B *and* B' *respectively. Then*

1. *If* $m \in l^\infty$, *then* $\mathbf{M}$ *is a well defined bounded operator.* $\|\mathbf{M}\|_{Op} \leq \sqrt{B'}\sqrt{B} \cdot \|m\|_\infty$.
2. $\mathbf{M}^*_{m,(\phi_k),(\psi_k)} = \mathbf{M}_{\overline{m},(\psi_k),(\phi_k)}$. *Therefore if* m *is real-valued and* $\phi_k = \psi_k$ *for all* k, $\mathbf{M}$ *is self-adjoint.*
3. *If* $m \in c_0$, $\mathbf{M}$ *is compact.*
4. *If* $m \in l^1$, $\mathbf{M}$ *is a trace class operator with* $\|\mathbf{M}\|_{trace} \leq \sqrt{B'}\sqrt{B} \cdot \|m\|_1$, *and* $tr(\mathbf{M}) = \sum_k m_k \langle \phi_k, \psi_k \rangle$.
5. *If* $m \in l^2$, $\mathbf{M}$ *is a Hilbert Schmidt operator with* $\|\mathbf{M}\|_{\mathcal{HS}} \leq \sqrt{B'}\sqrt{B} \cdot \|m\|_2$.

Unconditionally Convergence and Invertibility of Frame Multipliers

We put quite some effort into the investigation of invertible multipliers. A quite surprising result is the following:

Theorem (B., Stoeva; submitted)

Let Φ and Ψ be frames for $\mathcal{H}$, and let m be semi-normalized. Let $\mathbf{M}_{m,\Phi,\Psi}$ be invertible. Then there exist a dual frame $\Phi^\dagger$ of Φ and a dual frame $\Psi^\dagger$ of Ψ, so that for any dual frame Φ^d of Φ and any dual frame Ψ^d of Ψ we have

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^\dagger,\Phi^d} = M_{1/m,\Psi^d,\Phi^\dagger}. \tag{3}$$

The frames $\Psi^\dagger$ are uniquely determined.

Unconditionally Convergence and Invertibility of Frame Multipliers

We also found sufficient conditions, when multipliers are invertible. In this case, formulas for $\mathbf{M}^{-1}_{(m_n),(\phi_n),(\psi_n)}$ are determined. For example in the following case:

Proposition ([Stoeva and Balazs, 2012])

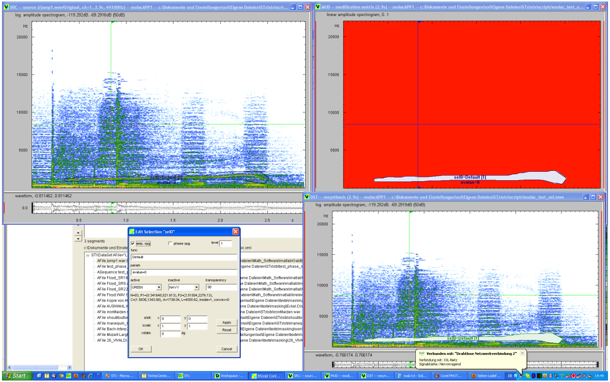
Let $\Phi = (\phi_k)$ *be a frame for* $\mathcal{H}$. *Assume that* $\exists\, \mu \in [0, \frac{A_\Phi^2}{B_\Phi})$ *such that* $\sum |\langle f, m_n\psi_n - \phi_n\rangle|^2 \leq \mu \|f\|^2, \quad \forall\ f \in \mathcal{H}$. *Then* $m\Psi$ *is a frame for* $\mathcal{H}$, *the multipliers* $\mathbf{M}_{\overline{m},\Phi,\Psi}$ *and* $\mathbf{M}_{m,\Psi,\Phi}$ *are invertible on* $\mathcal{H}$ *and*

$$\frac{1}{B_\Phi + \sqrt{\mu B_\Phi}} \|h\| \leq \|\mathbf{M}^{-1} h\| \leq \frac{1}{A_\Phi - \sqrt{\mu B_\Phi}} \|h\|, \ \forall h \in \mathcal{H}, \qquad (4)$$

$$\mathbf{M}^{-1} = \sum_{k=0}^{\infty} [S_\Phi^{-1}(S_\Phi - \mathbf{M})]^k S_\Phi^{-1} \qquad (5)$$

where $\mathbf{M}$ *denotes any one of* $\mathbf{M}_{\overline{m},\Phi,\Psi}$ *and* $\mathbf{M}_{m,\Psi,\Phi}$.

Numerical results and algorithms



Implementation in STx and MATLAB, in the Linear Time-Frequency Analysis Toolbox (LTFAT) [Soendergaard et al., 2012] (available at Sourceforge, see talk by Peter Søndergaard.) .

Applications in Acoustics: Perceptual Sparsity by Irrelevance

MP3-Player

MP3:

- encoding / decoding scheme
- MPEG1/MPEG2 (Layer 3)
- signal processing
- psychoacoustical masking model

Psychoacoustic Masking: introduction

Masking:
presence of one stimulus, the masker, decreases the response to another stimulus, the target.

Irrelevance Filter: searches (and deletes) perceptional irrelevant data (in complex signals) using a masking model.

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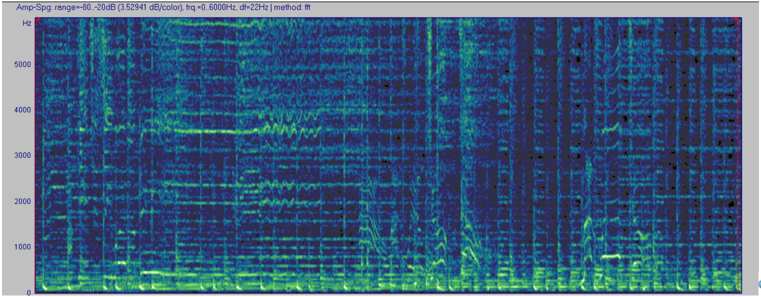
Perceptual Sparsity by Irrelevance

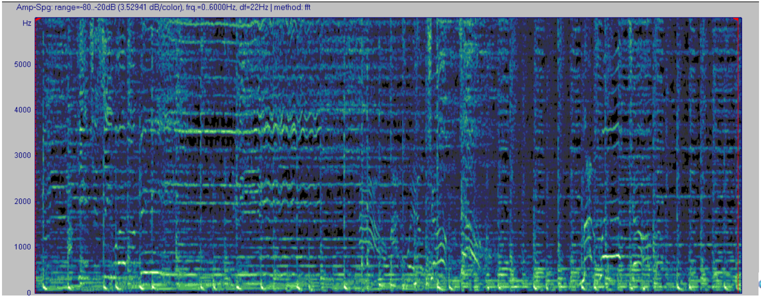
Algorithm in STx:
Original audio file



Psychoacoustic Masking

Algorithm in STx:

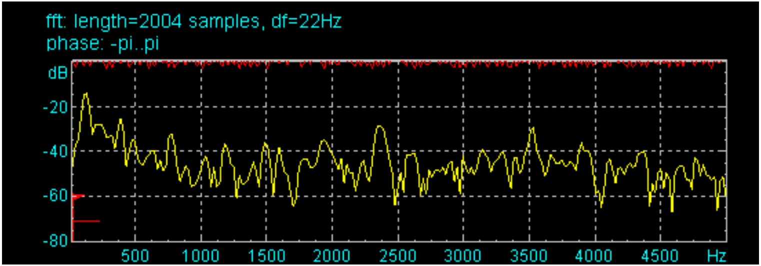
Filtered signal



"Lossy Coding"

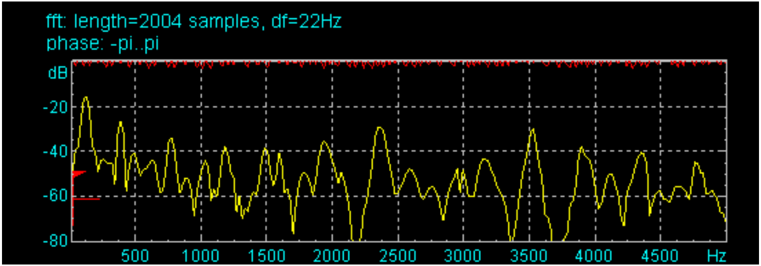
Irrelevance Filter

Existing algorithm in ST^X: Original audio file (Spectrum)



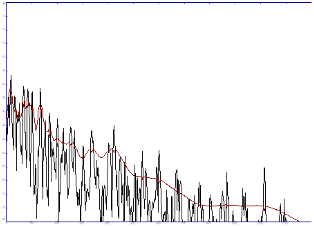
Irrelevance Filter

Existing algorithm in STx: Masked signal (Spectrum)



Psychoacoustic Masking

[Balazs et al., 2010] models simultaneous frequency masking and uses single spectra.



The irrelevance method calculates an adaptive threshold function for each spectra of a Gabor transform. This corresponds to an adaptive Gabor frame multiplier with coefficients in $\{0, 1\}$.
With frame theory some properties are explained:

- perfect reconstruction,
- time frequency concentration (spread),
- smoothness and
- numerical efficiency.

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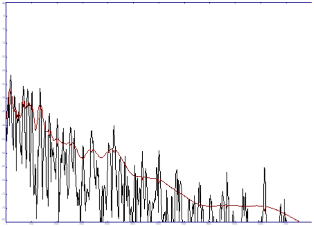

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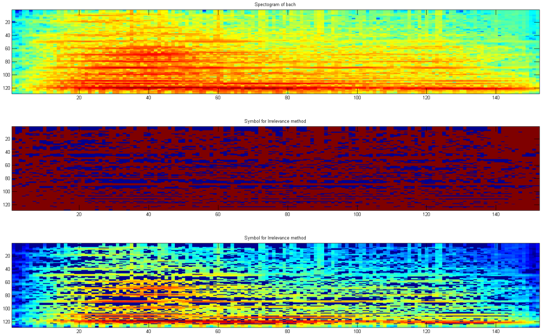
Perceptual Sparsity by Irrelevance

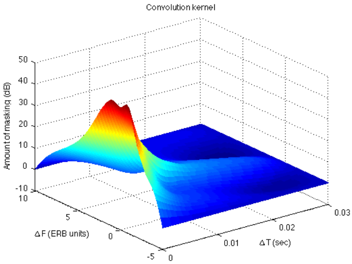
Interpreted as adaptive Gabor frame multiplier:



Irrelevance by Time Frequency Masking

Extend to True Time-Frequency Model using Multipliers:

- Base it on ERBlets (NSGT adapted to perception) (see talb by Thibaud).
- Use new psychoacoustical data on time-frequency masking [Laback et al., 2011].



Applications in Acoustics:

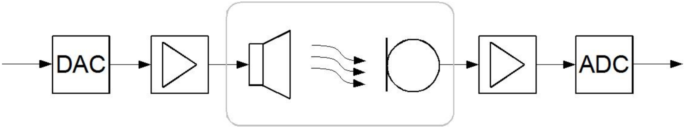
Acoustic System Estimation

Acoustic System Estimation

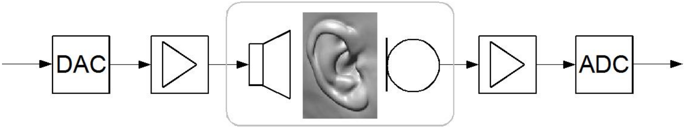
Measurement of Head Related Transfer Functions (HRTFs)

Acoustic System Estimation

Electro-acoustic signal path:
weakly non-linear, time invariant systems (PA, Speakers)

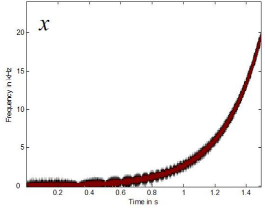


with head-movement weakly non-linear, time variant system
But the interesting part is the HRTF: an LTI system!

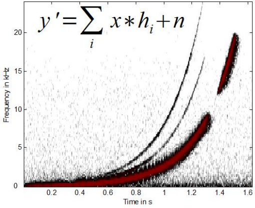


Acoustic System Estimation

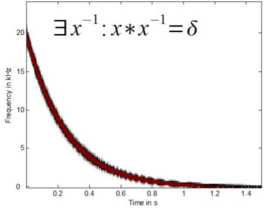
Input

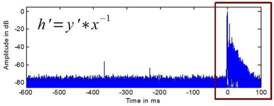


Output



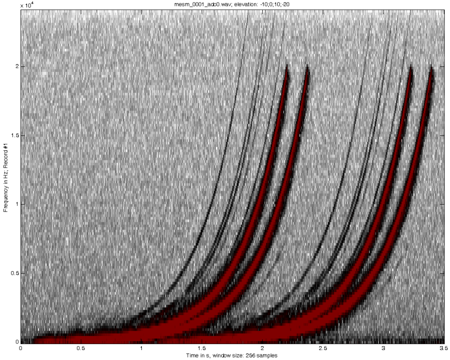
Deconvolution





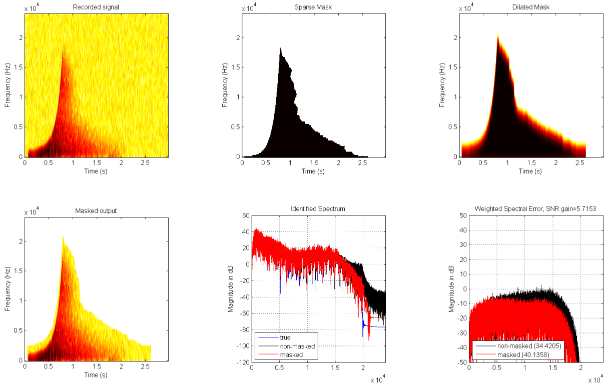
Acoustic System Estimation

Measurement of Head Related Transfer Functions (HRTFs) by the Multiple Exponential Sweeps Method (MESM) [Majdak et al., 2007]



Speed up measurement by factor of four.

Time-Frequency Denoising [Majdak et al., 2011]:



Conclusions

Frames and Frame Multipliers allow

- interesting mathematical results, as well as
- provide new methods and models for acoustics, as well as their implementation.

From Theory to Applications!

Thank you for your attention!

Questions? Comments?

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