The α-Modulation Transform: Admissibility, Coorbit Theory, and Frames of Compactly Supported Functions

Stephan Dahlke

Fachbereich Mathematik und Informatik
Philipps–Universität Marburg

Time–Frequency Analysis and Related Topics,
Strobl, June 6–10, 2016
(joint work with P. Balazs, D. Bayer, M. Fornasier, H. Rauhut, M. Speckbacher, G. Steidl, and G. Teschke)

Motivation

Admissibility

Coorbit Spaces

Generalized Coorbit Spaces

Discretization

Outline

Motivation

Admissibility

Coorbit Spaces

Generalized Coorbit Spaces

Discretization

Outline

Motivation

Admissibility

Coorbit Spaces
- Generalized Coorbit Spaces
- Discretization

Classical Problem:

- analyze/decompose/approximate... a given signal/function

  Basic step: decomposition into suitable building blocks

  Examples:

  - Fourier transform
  - windowed Fourier transform, Gabor transform
  - wavelet transform....

Classical Problem:

- analyze/decompose/approximate... a given signal/function

  Basic step: decomposition into suitable building blocks

  Examples:

  - Fourier transform
  - windowed Fourier transform, Gabor transform
  - wavelet transform....

- Piecewise smooth signals $\rightsquigarrow$ sparse representation by wavelet transforms/wavelet frames
- Sharp instantaneous frequencies $\rightsquigarrow$ sparse representation by Gabor transforms/Gabor frames

Classical Problem:

- analyze/decompose/approximate... a given signal/function

  Basic step: decomposition into suitable building blocks

  Examples:

  - Fourier transform
  - windowed Fourier transform, Gabor transform
  - wavelet transform....

- Piecewise smooth signals $\rightsquigarrow$ sparse representation by wavelet transforms/wavelet frames
- Sharp instantaneous frequencies $\rightsquigarrow$ sparse representation by Gabor transforms/Gabor frames
- What happens if both features are present?

Possible Solutions:

- Combine a Gabor and a wavelet frame to one huge frame [B. Torresani]

Possible Solutions:

- Combine a Gabor and a wavelet frame to one huge frame [B. Torresani]
- Start from group theory:
  - Wavelet transform $\leftrightarrow$ square-integrable representation of the affine group

$$\pi(a,x)f(t) = |a|^{-1/2} f\left(\frac{t-x}{a}\right)$$

  - Gabor transform $\leftrightarrow$ representation of the Weyl–Heisenberg group

$$\pi(x,\omega,\tau)f(t) = e^{2\pi i\tau} e^{-i\pi x\omega} e^{2\pi i\omega t} f(t-x]$$

Use the Affine Weyl–Heisenberg Group!

The Affine Weyl–Heisenberg Group:

Definition

$\mathbb{R}^{2+1} \times \mathbb{R}_+$ with group law

$$(x,\omega,a,\tau)\circ(x',\omega',a',\tau')=(x+ax',\omega+a^{-1}\omega',aa',\tau+\tau'+\omega ax')$$

is a locally compact, unimodular topological group called *affine Weyl–Heisenberg group* G_{aWH}.

$$g^{-1} = (-a^{-1}x, -a\omega, a^{-1}, -\tau + x\omega)$$

Haar measure: $d\mu(x,\omega,a,\tau) = dxd\omega\frac{da}{a}d\tau$

Stone-von-Neumann representation on $L_2(\mathbb{R})$:

$$\pi(x,\omega,a,\tau)f(t) = a^{-1/2}e^{2\pi i(\omega(t-x)+\tau)}f\left(\frac{t-x}{a}\right)$$

Representations:

- Problem: π is not square integrable!

Representations:

- Problem: π is not square integrable!
- Way out: Representations modulo quotients!

$$H = \{(0,0,a,\tau)\} \subseteq G_{aWH}. \quad X := G_{aWH}/H \simeq \mathbb{R}^2.$$

X is a *homogeneous space* with invariant measure $dxd\omega$
For $0 \leq \alpha < 1$, choose the *section (or lifting)*

$$\begin{aligned} \sigma : X &\rightarrow G_{aWH}, \\ (x,\omega) &\mapsto (x,\omega,\beta(\omega),0) \quad \beta(\omega) = (1+|\omega|)^{-\alpha}. \end{aligned}$$

Representations:

- Problem: π is not square integrable!
- Way out: Representations modulo quotients!

$$H = \{(0,0,a,\tau)\} \subseteq G_{aWH}. \quad X := G_{aWH}/H \simeq \mathbb{R}^2.$$

X is a *homogeneous space* with invariant measure $dxd\omega$
For $0 \leq \alpha < 1$, choose the *section (or lifting)*

$$\begin{aligned} \sigma : X &\rightarrow G_{aWH}, \\ (x,\omega) &\mapsto (x,\omega,\beta(\omega),0) \quad \beta(\omega) = (1+|\omega|)^{-\alpha}. \end{aligned}$$

- α–Modulation Transform:

$$\langle f, (\pi \circ \sigma)(x,\omega)\psi \rangle = \int_{\mathbb{R}} f(t)(1+|\omega|)^{\alpha/2} e^{2\pi i(\omega(t-x)} \psi\left(\frac{t-x}{(1+|\omega|)^{-\alpha}}\right) dt$$

$\alpha \longrightarrow 0$: Gabor
$\alpha \longrightarrow 1$: Wavelet

α–Modulation Transform:

- [D./Fornasier/Rauhut/Steidl/Teschke (2008)] It works! Square-integrability, coorbit space theory, atomic decompositions via α–modulation frames...

α–Modulation Transform:

- [D./Fornasier/Rauhut/Steidl/Teschke (2008)] It works! Square-integrability, coorbit space theory, atomic decompositions via α–modulation frames...
- [D./Teschke/Stingl (2008)] Applicable in data analysis, Prenatal diagnosis, Medical multichannel superconducting quantum interference divice

Problems...

- Applications in numerical analysis, operator equations?

  D-A-CH Project "Adaptive Wavelet and Frame Techniques for Acoustic BEM"
  (P. Balazs/W. Kreuzer (ARI, Vienna), H. Harbrecht (Basel), S. Dahlke (Marburg))

  One goal: Adaptive numerical schemes based on α-modulation frames for integral equations on manifolds

Problems...

- Applications in numerical analysis, operator equations?

  D-A-CH Project "Adaptive Wavelet and Frame Techniques for Acoustic BEM"
  (P. Balazs/W. Kreuzer (ARI, Vienna), H. Harbrecht (Basel), S. Dahlke (Marburg))

  One goal: Adaptive numerical schemes based on α-modulation frames for integral equations on manifolds

- Problem: Atoms not compactly supported!

  First step: Derive compactly supported atoms!

Problems...

- Applications in numerical analysis, operator equations?

  D-A-CH Project "Adaptive Wavelet and Frame Techniques for Acoustic BEM"
  (P. Balazs/W. Kreuzer (ARI, Vienna), H. Harbrecht (Basel), S. Dahlke (Marburg))

  One goal: Adaptive numerical schemes based on α-modulation frames for integral equations on manifolds

- Problem: Atoms not compactly supported!

  First step: Derive compactly supported atoms!

- [M. Speckbacher/D. Bayer/S. Dahlke/P. Balazs (2016)]
  It works!

Outline

Motivation

Admissibility

Coorbit Spaces

Generalized Coorbit Spaces

Discretization

Admissibility:

Definition

Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a representation, $\sigma : X = G/H \to G$ a section and $\psi \in \mathcal{H} \setminus \{0\}$. Define

$$A_\sigma f := \int_X \langle f, \pi(\sigma(x))\psi\rangle \pi(\sigma(x))\psi \, d\mu(x), \quad f \in \mathcal{H}.$$

If A_σ is boundedly invertible, then ψ is called *admissible*, and π is called *square-integrable modulo* (H, σ).

Definition

Let ψ be admissible. Then the *voice transforms* are defined by

$$V_\psi f(x) := \langle f, \pi(\sigma(x))\psi\rangle, \quad x \in X.$$

and

$$W_\psi f(x) := V_\psi(A_\sigma^{-1} f)(x) = \langle f, A_\sigma^{-1} \pi(\sigma(x))\psi\rangle, \quad x \in X.$$

Voice Transforms:

Theorem

Let ψ be admissible. Then, for all $f_1, f_2 \in \mathcal{H}$,

$$\langle f_1, f_2 \rangle = \langle W_\psi f_1, V_\psi f_2 \rangle_{L^2(G,\mu)} = \langle V_\psi f_1, W_\psi f_2 \rangle_{L^2(G,\mu)}.$$

In our setting:

$$\widehat{A_\sigma f} = m_\psi \cdot \hat{f}, \qquad m_\psi(\xi) = \int_{\mathbb{R}} |\hat{\psi}(\beta(\omega)(\xi - \omega))|^2 \beta(\omega)\, d\omega$$

Thus: $\psi \in L^2(\mathbb{R})$ admissible $\iff \exists A, B$ s.t.

$$0 < A \leq m_\psi(\xi) \leq B < \infty$$

Admissibility for G_{aWH}:

Theorem

Let $0 \leq \alpha < 1$ and $\psi \in L^2(\mathbb{R}) \setminus \{0\}$ be such that $\operatorname{supp} \hat{\psi}$ is compact, then ψ is admissible.

Theorem

Let $0 \leq \alpha < 1$ and $\psi \in L^2(\mathbb{R}) \setminus \{0\}$ be such that $\hat{\psi}$ is continuous and

$$|\hat{\psi}(\xi)| \leq C(1 + |\xi|)^{-r} \quad r > \max\left\{1, \frac{\alpha}{2(1-\alpha)}\right\},$$

then ψ is admissible.

Outline

Motivation

Admissibility

Coorbit Spaces

Generalized Coorbit Spaces

Discretization

Another Goal:

Try to apply the coorbit theory of Feichtinger and Gröchenig. It provides

- Natural families of smoothness spaces, smoothness measured by the decay of the voice transform.

Another Goal:

Try to apply the coorbit theory of Feichtinger and Gröchenig. It provides

- Natural families of smoothness spaces, smoothness measured by the decay of the voice transform.
- Frames for the scales of associated coorbit spaces, even Banach frames.

Problem: Generalizations to representations modulo quotients is necessary!

General Setting:

- Basic conditions:

$$L^p_v(X) := \{ F \text{ measurable}, \; Fv \in L^p(X) \} \quad v \geq 1$$

$$\mathcal{R}(x,y) := \langle A_\sigma^{-1} \pi(\sigma(x))\psi, \pi(\sigma(y))\psi \rangle.$$

$$w(x,y) := \max \left\{ \frac{v(x)}{v(y)}, \frac{v(y)}{v(x)} \right\}$$

$$\rho := \operatorname*{ess\,sup}_{y \in X} \int_X |\mathcal{R}(x,y)| w(x,y) d\mu(x) < \infty \qquad (*)$$

General Setting:

- Basic conditions:

$$L^p_v(X) := \left\{ F \text{ measurable}, \; Fv \in L^p(X) \right\} \quad v \geq 1$$

$$\mathcal{R}(x,y) := \langle A_\sigma^{-1} \pi(\sigma(x))\psi, \pi(\sigma(y))\psi \rangle.$$

$$w(x,y) := \max \left\{ \frac{v(x)}{v(y)}, \frac{v(y)}{v(x)} \right\}$$

$$\rho := \operatorname*{ess\,sup}_{y \in X} \int_X |\mathcal{R}(x,y)| w(x,y) d\mu(x) < \infty \qquad (*)$$

- The reservoir spaces:

$$\mathcal{H}_{1,v} := \{ f \in \mathcal{H} : \; W_\psi f \in L^1_v(X) \}$$
$$\mathcal{K}_{1,v} := \{ f \in \mathcal{H} : \; V_\psi f \in L^1_v(X) \},$$
$$\|f\|_{\mathcal{H}_{1,v}} := \|W_\psi f\|_{L^1_v}, \quad \|f\|_{\mathcal{K}_{1,v}} := \|V_\psi f\|_{L^1_v}$$

The Coorbit Spaces:

- Gelfand triple:

$$\mathcal{H}_{1,v} \subset \mathcal{H} \subset \mathcal{H}'_{1,v}$$
$$\mathcal{K}_{1,v} \subset \mathcal{H} \subset \mathcal{K}'_{1,v}$$

The Coorbit Spaces:

- Gelfand triple:

$$\mathcal{H}_{1,v} \subset \mathcal{H} \subset \mathcal{H}'_{1,v}$$
$$\mathcal{K}_{1,v} \subset \mathcal{H} \subset \mathcal{K}'_{1,v}$$

- $(*) \Longrightarrow$ extended voice transform:

$$V_\psi f(x) := \langle f, \pi(\sigma(x))\psi \rangle_{\mathcal{H}'_{1,v} \times \mathcal{H}_{1,v}}$$
$$W_\psi f(x) := \langle f, A_\sigma^{-1}\pi(\sigma(x))\psi \rangle_{\mathcal{K}'_{1,v} \times \mathcal{K}_{1,v}}$$

The Coorbit Spaces:

- Gelfand triple:

$$\mathcal{H}_{1,v} \subset \mathcal{H} \subset \mathcal{H}'_{1,v}$$
$$\mathcal{K}_{1,v} \subset \mathcal{H} \subset \mathcal{K}'_{1,v}$$

- $(*) \Longrightarrow$ extended voice transform:

$$V_\psi f(x) := \langle f, \pi(\sigma(x))\psi \rangle_{\mathcal{H}'_{1,v} \times \mathcal{H}_{1,v}}$$
$$W_\psi f(x) := \langle f, A_\sigma^{-1}\pi(\sigma(x))\psi \rangle_{\mathcal{K}'_{1,v} \times \mathcal{K}_{1,v}}$$

- The coorbit spaces:

$$\mathcal{H}_{p,v} := \{f \in \mathcal{K}'_{1,v} : \ W_\psi f \in L^p_v(X)\},$$
$$\mathcal{K}_{p,v} := \{f \in \mathcal{H}'_{1,v} : \ V_\psi f \in L^p_v(X)\},$$
$$\|f\|_{\mathcal{H}_{p,v}} := \|W_\psi f\|_{L^p_v}, \quad \|f\|_{\mathcal{K}_{p,v}} := \|V_\psi f\|_{L^p_v}$$

In Our Case:

polynomial weight functions:

$$v_s(x,\omega) := v_s(\omega) := (1+|\omega|)^s, \quad s \in \mathbb{R}$$

Theorem

Let $s \geq 0$, $\psi, \varphi \in L^2(\mathbb{R})$, *s.t.* $\hat{\psi}, \hat{\varphi} \in C^2(\mathbb{R})$, $|\hat{\psi}^{(l)}(\xi)| \leq C(1+|\xi|)^{-r}$, $l = 0,1,2$ *(same for* φ*), where*

$$r > \frac{2+2s+7\alpha-4\alpha^2}{2(1-\alpha)^2}.$$

Then

- *Coorbit spaces well–defined!*
- $\mathcal{H}_{p,v} = \mathcal{K}_{p,v}$
- φ *and* ψ *generate the same coorbits!*

Before We Go On....

classical α–modulation spaces are of course well–known [Feichtinger/Gröbner (1985/92)]

$$V_\psi^\alpha(f)(x,\omega) = \langle f, T_x M_\omega D_{\beta(\omega)}\psi\rangle.$$

$$M_{p,q}^{s+\alpha(1/q-1/2),\alpha}(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R}) : V_\psi^\alpha(f) \in L_{p,q}^s(\mathbb{R}^2)\},$$

$$\|f\|_{M_{p,q}^{s+\alpha(1/q-1/2),\alpha}} \asymp \|V_\psi^\alpha(f)\|_{L_{p,q}^s},$$

Theorem

The coorbit spaces $\mathcal{H}_{p,v_{s-\alpha(1/p-1/2)},\alpha}$ can be identified with the α–modulation spaces $M_{p,p}^{s,\alpha}$.

Discretization:

Definition

A family $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ is an *admissible covering* of X if

1. Covering property: $X = \bigcup_{i\in\mathcal{I}} U_i$,
2. Finite overlap:
   $\sup_{j\in\mathcal{I}} \#\{i \in \mathcal{I} : \; U_i \cap U_j \neq \emptyset\} \leq N < \infty$.

$$\mathrm{osc}_{\mathcal{U},\Gamma}(x,y) = \sup_{z\in Q_y} |\mathcal{R}(x,y) - \Gamma(y,z)\mathcal{R}(x,z)|$$

$$|\Gamma| \equiv 1, \; Q_y := \bigcup_{i\in\mathcal{I}(y)} U_i, \quad \mathcal{I}(y) := \{i \in \mathcal{I} : y \in U_i\}$$

$$\gamma_1 := \operatorname*{ess\,sup}_{x\in X} \int_X |\mathrm{osc}_{\mathcal{U},\Gamma}(x,y)| w(x,y) d\mu(y)$$

$$\gamma_2 := \operatorname*{ess\,sup}_{y\in X} \int_X |\mathrm{osc}_{\mathcal{U},\Gamma}(x,y)| w(x,y) d\mu(x).$$

Atomic Decomposition:

Theorem

Same conditions as before. Let γ_1, γ_2 sufficiently small.

- *If $f \in \mathcal{H}_{p,v}$, $1 \leq p \leq \infty$, then*

$$f = \sum_{i \in \mathcal{I}} c_i(\pi \circ \sigma(x_i))\psi, \qquad x_i \in U_i,$$

where

$$||(c_i)_{i\in\mathcal{I}}||_{\ell_{p,va^{1/p-1}}} \leq A||f||_{\mathcal{H}_{p,v}}, \qquad a_i := \mu(U_i)$$

- $(d_i)_{i\in I} \in \ell_{p,va^{1/p-1}}$, *then* $f = \sum_{i\in\mathcal{I}} d_i(\pi \circ \sigma(x_i))\psi \in \mathcal{H}_{p,v}$

$$||f||_{\mathcal{H}_{p,v}} \leq B||(d_i)_{i\in\mathcal{I}}||_{\ell_{p,va^{1/p-1}}}.$$

Reconstruction:

Theorem

Let γ_1, γ_2 sufficiently small. Then the set

$$\{\psi_i := A_\sigma^{-1}(\pi \circ \sigma(x_i))\psi : i \in \mathcal{I}\}$$

is a Banach frame for $\mathcal{H}_{p,v}$, i.e.,

i) $f \in \mathcal{H}_{p,v} \iff (\langle f, \psi_i \rangle_{\mathcal{H}'_{1,w} \times \mathcal{H}_{1,w}})_{i \in \mathcal{I}} \in \ell_{p,va^{1/p}}$;

ii) $\exists 0 < A' \leq B' < \infty$ *such that*

$$A' \, \|f\|_{\mathcal{H}_{p,v}} \leq \|(\langle f, \psi_i \rangle_{\mathcal{H}'_{1,w} \times \mathcal{H}_{1,w}})_{i \in \mathcal{I}}\|_{\ell_{p,va^{1/p}}} \leq B' \, \|f\|_{\mathcal{H}_{p,v}};$$

iii) $\exists$ *a bounded, linear reconstruction operator $\mathcal{S}$ from $\ell_{p,va^{1/p}}$ to $\mathcal{H}_{p,v}$ such that*
$\mathcal{S}\left((\langle f, \psi_i \rangle_{\mathcal{H}'_{1,w} \times \mathcal{H}_{1,w}})_{i \in \mathcal{I}}\right) = f.$

It Works!

Theorem

Let $\psi \in L^2(\mathbb{R})$, *such that* $\hat{\psi} \in C^3(\mathbb{R})$ *fulfills* $|\hat{\psi}^{(n)}(\xi)| \leq C(1+|\xi|)^{-r}$, *for* $n = 0, 1, 2, 3$, *with*

$$r > \frac{2 + 2s + 7\alpha - 4\alpha^2}{2(1-\alpha)^2} + 1.$$

Then there exist admissible coverings $\mathcal{U}^{\varepsilon}$ *such that*

$$\gamma(\varepsilon) = \max\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\} \to 0, \ \textit{for } \varepsilon \to 0,$$

Consequently: atomic decompositions and Banach frames for the α–modulation transform are established!

References:

1 S. Dahlke, M. Fornasier, H. Rauhut, G. Steidl, and G. Teschke, *Generalized coorbit theory, Banach frames, and the relations to α-modulation spaces*, Proc. Lond. Math. Soc. **96** (2008), 464–506.

2 S. Dahlke and G. Teschke, *Coorbit theory, multi-α-modulation frames and the concept of joint sparsity for medical multi-channel analysis*, EURASIP Journal on Advances in Signal Processing, ID 471601.

3 L. Daudet and B. Torresani, *Hybrid representation for audiophonic signal encoding*, Signal Processing **82** (2002), 1595–1617.

4 H. Feichtinger and P. Gröbner, *Banach spaces of distributions defined by decomposition methods*, Math. Nachr. **123** (1985), 97–120.

5. H. G. Feichtinger and K. Gröchenig, *A unified approach to atomic decompositions via integrable group representations*, Proc. Conf. "Function Spaces and Applications", Lund 1986, Lecture Notes in Math. **1302** (1988), 52–73.

6 H. G. Feichtinger and K. Gröchenig, *Banach spaces related to integrable group representations and their atomic decomposition I,* J. Funct. Anal. **86** (1989), 307–340.

7 H. G. Feichtinger and K. Gröchenig, *Banach spaces related to integrable group representations and their atomic decomposition II,* Monatsh. Math. **108** (1989), 129–148.

8. M. Fornasier and H. Rauhut, *Continuous frames, function spaces, and the discretization problem*, J. Fourier Anal. Appl. **11** (2005), 245–287.

9 P. Gröbner, *Banachräume glatter Funktionen und Zerlegungs-Methoden*, Ph.D. thesis, University of Vienna (1992).

10 M. Speckbacher, D. Bayer, S. Dahlke, and P. Balazs *The α-modulation transform: Admissibility, coorbit theory, and frames of compactly supported functions*, Preprint Nr. 2016–1, Philipps–University of Marburg.

References:

1 S. Dahlke, M. Fornasier, H. Rauhut, G. Steidl, and G. Teschke, *Generalized coorbit theory, Banach frames, and the relations to α-modulation spaces*, Proc. Lond. Math. Soc. **96** (2008), 464–506.

2 S. Dahlke and G. Teschke, *Coorbit theory, multi-α-modulation frames and the concept of joint sparsity for medical multi-channel analysis*, EURASIP Journal on Advances in Signal Processing, ID 471601.

3 L. Daudet and B. Torresani, *Hybrid representation for audiophonic signal encoding*, Signal Processing **82** (2002), 1595–1617.

4 H. Feichtinger and P. Gröbner, *Banach spaces of distributions defined by decomposition methods*, Math. Nachr. **123** (1985), 97–120.

5. H. G. Feichtinger and K. Gröchenig, *A unified approach to atomic decompositions via integrable group representations*, Proc. Conf. "Function Spaces and Applications", Lund 1986, Lecture Notes in Math. **1302** (1988), 52–73.

6 H. G. Feichtinger and K. Gröchenig, *Banach spaces related to integrable group representations and their atomic decomposition I,* J. Funct. Anal. **86** (1989), 307–340.

7 H. G. Feichtinger and K. Gröchenig, *Banach spaces related to integrable group representations and their atomic decomposition II,* Monatsh. Math. **108** (1989), 129–148.

8. M. Fornasier and H. Rauhut, *Continuous frames, function spaces, and the discretization problem*, J. Fourier Anal. Appl. **11** (2005), 245–287.

9 P. Gröbner, *Banachräume glatter Funktionen und Zerlegungs-Methoden*, Ph.D. thesis, University of Vienna (1992).

10 M. Speckbacher, D. Bayer, S. Dahlke, and P. Balazs *The α-modulation transform: Admissibility, coorbit theory, and frames of compactly supported functions*, Preprint Nr. 2016–1, Philipps–University of Marburg.

Thanks a lot for your attention!