Definition 1.

$$\boldsymbol{C}_c(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C}, \text{ continuous and with compact support}\}^1$$

Ccdef **Definition 2.** The *support* of a continuous (!) function is defined as the closure of the set of "relevant points":

$$\text{supp}(f) := \{x, f(x) \neq 0\}^-$$

Definition 3. (1) translation $T_x : T_x f(z) = [T_x f](z) = f(z - x)$
(the graph is preserved but moved by the vector x to another position);
(2) dilation D_ρ (value preserving) and St_ρ (mass preserving);
(3) involution $f \mapsto \check{f}$ with $\check{f}(z) = f(-z)$;
(4) modulation M_ω: Multiplication with the character $x \mapsto exp(2\pi i\omega x)$, i.e. $[M_\omega f](z) := e^{2\pi i\omega \cdot z} f(z)$
(5) Fourier transform $\mathcal{F}, \mathcal{F}^{-1}$, to be discussed only later: our normalization will be that given for $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ by the integral

our-intdef (1)
$$\mathcal{F} : f \mapsto \hat{f} : \quad \hat{f}(s) = \int_{\mathbb{R}^d} f(t) e^{2\pi i s \cdot t} dt$$

Cbdef **Definition 4.**

$\boldsymbol{C}_b(\mathbb{R}^d) := \{f : \mathbb{R}^d \mapsto \mathbb{C}$, continuous and bounded with the norm $\|f\|_\infty = sup_{x\in\mathbb{R}^d}|f(x)|$ $\}$

The spaces $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ and $\boldsymbol{C}_0(\mathbb{R}^d)$ are defined as the subspaces of functions which are uniformly continuous resp. decaying at infinity, i.e.

$$f \in \boldsymbol{C}_0(\mathbb{R}^d) \quad \text{if and only if} \quad \lim_{|x|\to\infty} |f(x)| = 0.$$

Lemma 1. *Characterization of* $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ *within* $\boldsymbol{C}_b(\mathbb{R}^d)$ *(or even* $\boldsymbol{L}^\infty(\mathbb{R}^d)$*).*

$$\|T_x f - f\|_\infty \to 0 \quad \textit{for} \quad x \to 0.$$

if and only if $f \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$*.*

We will use the symbol $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ for the Banach space of all bounded and linear operators on the Banach space $\boldsymbol{C}_0(\mathbb{R}^d)$, endowed with the operator norm $\|T\|_\infty := sup_{\|f\|_\infty \leq 1}\|Tf\|_\infty$.

Definition 5. An directed family (a net or sequence) $(h_\alpha)_{\alpha\in I}$ in a Banach algebra $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a **BAI** (= bounded approximate identity or "approximate unit" for $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$) if

$$\lim_\alpha \|h_\alpha \cdot h - h\|_{\boldsymbol{B}} = 0 \quad \forall h \in \boldsymbol{B}.$$

Definition 6. Definition of value preserving **dilation operators**:

Drho1 (2)
$$D_\rho f(z) = f(\rho \cdot z),\ \rho > 0,\ z \in \mathbb{R}^d$$

Definition 7. We denote the dual space of $\big(\boldsymbol{C}_0(\mathbb{R}^d),\ \|\cdot\|_\infty\big)$ with $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Sometimes the symbl $\boldsymbol{M}_b(\mathbb{R}^d)$ is used in order to emphasize that one has "bounded" (regular Borel) measures.

[1]The capital "C" stands for *continuous*, the subscript for *compact* support

Definition 8.

$$\boldsymbol{M}_d(\mathbb{R}^d) = \{\mu \in \boldsymbol{M}(\mathbb{R}^d) : \mu = \sum_{k=1}^{\infty} c_k \delta_{t_k} \; s.t. \; \sum_{k=1}^{\infty} |c_k| < \infty\}$$

Definition 9. A sequence $\Phi = (T_\lambda \varphi)_{\lambda \in \Lambda}$, where φ is a compactly supported function (i.e. $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$), and $\Lambda = A(\mathbb{Z}^d)$ a lattice in $\mathbb{R}^d$ (for some non-singular $d \times d$-matrix) is called a **regular BUPU** if

$$\sum_{\lambda} \varphi(x - \lambda) \equiv 1.$$

Definition 10. For any BUPU Φ the *Spline-type Quasi-Interpolation operator* Sp_Φ is given by:

$$f \mapsto Sp_\Phi(f) := \sum_{\lambda \in \Lambda} f(\lambda) \phi_\lambda$$

Definition 11. Definition of **dilation operators**:

$$D_\rho f(z) = f(\rho \cdot z), \; \rho > 0, \; z \in \mathbb{R}^d$$

f-BIBOTLIS **Definition 12.** The Banach space of all "translation invariant linear systems" on $\boldsymbol{C}_0(\mathbb{R}^d)$ is given by [2]

$$\mathcal{H}_G(\boldsymbol{C}_0(\mathbb{R}^d)) = \{T : \boldsymbol{C}_0(\mathbb{R}^d) \to \boldsymbol{C}_0(\mathbb{R}^d), \text{ bounded, linear} : T \circ T_z = T_z \circ T, \forall z \in \mathbb{R}^d\}$$

Definition 13. recall the notion of a FLIP operator: $\check{f}(z) = f(-z)$

Given $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ we define the *convolution operator* C_μ by: $C_\mu(f)(z) := \mu(T_z \check{f})$.

The reverse mapping R recovers a measure $\mu = \mu_T$ from a given translation invariant system T via $\mu(f) := T(\check{f})(0)$.

Strhodef **Definition 14.** The *adjoint* action of the group $\mathbb{R}^+$ on $\boldsymbol{M}(\mathbb{R}^d)$ is defined as the family of adjoint operators on $\boldsymbol{M}(\mathbb{R}^d)$ via:

def-Stroh (3) $$St_\rho \mu(f) := \mu(D_\rho f), \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d), \; \rho > 0.$$

Definition 15. ALTERNATIVE DESCRIPTION (replaced later >> theorem): Given $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ the uniquely determined measure corresponding to the operator $D_\rho^{-1} \circ C_\mu \circ D_\rho$ will be denoted by $St_\rho \mu$.

BanMod **Definition 16.** A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach module* over a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ if one has a bilinear mapping $(a, b) \mapsto a \bullet b$, from $\boldsymbol{A} \times \boldsymbol{B}$ into $\boldsymbol{B}$ with

$$\|a \bullet b\|_{\boldsymbol{B}} \leq \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{B}} \quad \forall \, a \in \boldsymbol{A}, b \in \boldsymbol{B}$$

which behaves like an ordinary multiplication, i.e. is associative, distributive, etc.:

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}.$$

[2] The letter $\mathcal{H}$ in the definition refers to *homomorphism* [between normed spaces], while the subscript G in the symbol refers to "commuting with the action of the underlying group $G = \mathbb{R}^d$ realized by the so-called regular representation, i.e. via ordinary translations

dual-alg **Definition 17.** For any Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ the dual space can be turned naturally into a $\boldsymbol{A}$-Banach module via the action

dual-alg
$$[a_1 \bullet \sigma](a) := \sigma(a_1 \cdot a), \quad \forall a, a_1 \in \boldsymbol{A}, \sigma \in \boldsymbol{A}^d. \tag{4}$$

Definition 18. A *character* is a continuous function from a topological group into the torus group $\mathbb{T} = \{z \in \mathbb{C} \,|\, |z| = 1\}$. In other words, χ is a character if $\chi(x+y) = \chi(x)\cdot\chi(y)$ for all $x, y \in G$. Moreover, since $|\chi(x)| = 1$ one has $\overline{\chi(x)} = 1/\chi(x)$ for all $x \in G$.

Definition 19. The set of all character is called the *dual group*, because those characters form an Abelian group under pointwise multiplication (!Exercise!). We write $\hat{G}$ for the dual group corresponding to G (the group law written as addition in this case).

Definition 20. The Fourier transform of $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ is defined by

eas-FT-def
$$\hat{\mu}(s) = \mu(\chi_s) \tag{5}$$

Li-Def **Definition 21.** We define $\boldsymbol{L}^1(\mathbb{R}^d)$ as the closure of $\boldsymbol{M}_{C_c}$ within $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

as-support **Definition 22.** A point x does *not* belongs to the support of a measure $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ if there exists some $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $k(x) = 1$, but nevertheless $k \cdot \mu \neq 0$. The complement of this set is denoted by $\mathrm{supp}(\mu)$.

Definition 23. A bounded subset $H \subset \boldsymbol{M}(\mathbb{R}^d)$ is called (uniformly) tight if for every $\varepsilon > 0$ there exists $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|\mu - k \cdot \mu\|_{\boldsymbol{M}} \leq \varepsilon$ for al $\mu \in H$.

Definition 24. A bounded subset $H \subset \boldsymbol{C}_0(\mathbb{R}^d)$ is called (uniformly) tight if for every $\varepsilon > 0$ there exists $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|h - k \cdot h\|_{\infty} \leq \varepsilon$ for al $h \in H$.

WCdefphi **Definition 25.** Let φ be any non-zero, non-negative function on $\mathbb{R}^d$.

WCdefphi1
$$\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1) = \{\, f \in \boldsymbol{C}_0(\mathbb{R}^d) \,|\, \exists (c_k)_{k\in\mathbb{N}} \in \boldsymbol{\ell}^1, (x_k)_{k\in\mathbb{N}} \,\mathrm{in}\, \mathbb{R}^d, |f(x)| \leq \sum_{k\in\mathbb{N}} c_k \varphi(x - x_k) \,\} \tag{6}$$

We define

$$\|f \,|\, \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)\| := \inf \{ \|\mathbf{c}\|_{\boldsymbol{\ell}^1} = \sum_{k\in\mathbb{N}} |c_k| \,\}$$

where the infimum is taken over all "admissible dominations" of f as in (6).

Faculty of Mathematics, Nordbergstrasse 15, 1090 Vienna, Austria
E-mail address: hans.feichtinger@univie.ac.at