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A FUNCTIONAL ANALYTIC APPROACH TO APPLIED ANALYSIS: NUHAG: VERSION DATED AUTUMN 2012

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Abstract. We present a functional analytic approach to harmonic analysis, avoiding heavy measure theoretic tools. The results are non-trivial even for the real line $\mathbb{R}$, but are formulated for the $d-$dimensional Euclidean space $\mathbb{R}^d$, viewed as a prototype for a locally compact Abelian group $\mathscr{G}$ (with the usual addition of vectors and the topology provided by the Euclidian metric). [1] We start with the description of $\boldsymbol{M}_b(G)$, the space of bounded linear measures, as the dual space of $\boldsymbol{C}_0$, which is naturally endowed with a convolution structure.

sec:intro

1. Introduction

We start with a few symbols. First note that we will permanently make use of the fact that $\mathbb{R}^d$ is a locally compact Abelian group with respect to addition (dilation will come in only as a convenient but not crucial side aspect). Such $\mathbb{R}^d$-specific parts are marked by the symbol [$\mathbf{Rd-specific}$!]!

First we define the most simple algebra (pointwise, later on with respect to convolution) of continuous *"test functions"*. [1] Because of the *local compactness* of $\mathbb{R}^d$ the following space of *continuous functions with compact support* is a non-trivial (i.e. *not just* the zero-space) linear space of complex-valued functions on $\mathscr{G}$:

Definition 1.

$$\boldsymbol{C}_c(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C}, \text{ continuous and with compact support}\} \quad {}^{2,3}\ .$$

Here we make use of the standard definition of the support of a function:

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[1] They can be defined on any topological group, and the space is interesting and non-trivial for any locally compact group. So much of the material given below extends without difficulty to the setting of locally compact, or at least locally compact Abelian group, except for the statements which involve dilations

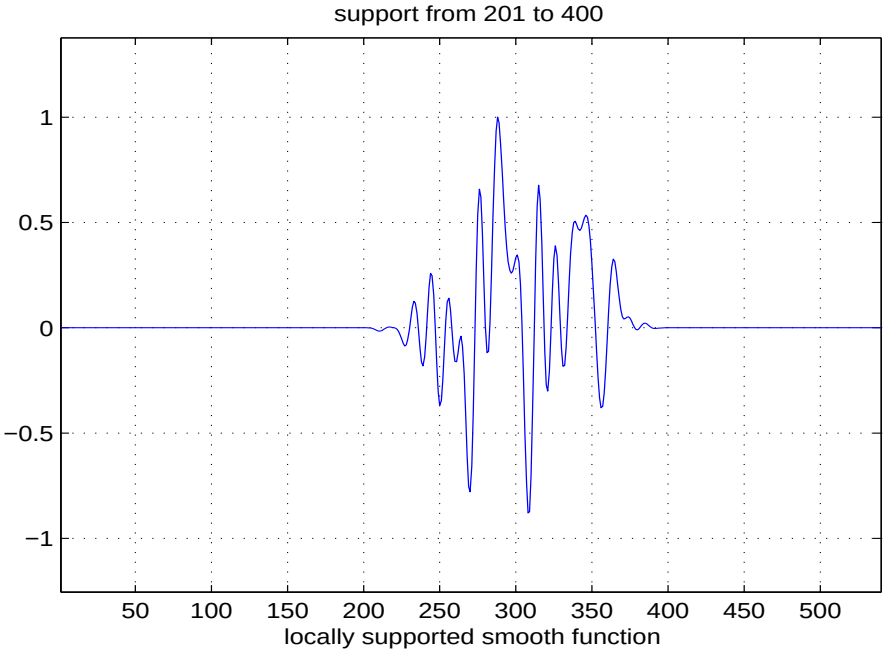
[2] Why one takes the closure in the above definition will become more clear later on, when the support of a generalized function or distribution will be defined.

[3] The capital "C" stands for *continuous*, and the subscript "c"stands for *compact* support; Hans Reiter uses the symbol $\mathcal{K}(\mathscr{G})$ for the space $\boldsymbol{C}_c(G)$.

Ccdef **Definition 2.** The *support* of a continuous (!) function is defined as the closure of the set of "relevant points": [4]

$$\mathrm{supp}(f) := \{x \,|\, f(x) \neq 0\}^-$$

Lemma 1. *(Ex to Def. Ccdef 2) A continuous, complex-valued function on $\mathbb{R}^d$ is in $\boldsymbol{C}_c(\mathbb{R}^d)$ if and only if there exists $R = R(f) > 0$ such that $f(x) = 0$ for all x with $|x| \geq R$.*



A list of symbols: Operators, names of operators, and their definitions:

Definition 3 (Operators defined on functions). .

(1) translation $T_x : T_x f(z) = [T_x f](z) = f(z - x)$
(the graph is preserved but moved by the vector x to another position);
(2) involution $f \mapsto \check{f}$ with $\check{f}(z) = f(-z)$;
We may switch to the notation $\checkmark f(z)$ instead later on. r
(3) modulation M_ω: Multiplication with the character $x \mapsto exp(2\pi i\omega x)$, i.e.

modopdef1 (1) $$[M_\omega f](z) := e^{2\pi i\omega \cdot z} f(z)$$

(4) **[Rd-specific!]**dilation D_ρ (value preserving) and St_ρ (mass preserving); cf. below
(5) Fourier transform $\mathcal{F}, \mathcal{F}^{-1}$, to be discussed only later: will be given for $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ by the integral (with the *normalization* 2π in the exponent!)

our-intdef (2) $$\mathcal{F}\colon f \mapsto \hat{f} : \quad \hat{f}(s) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i s\cdot t} dt$$

Cbdef **Definition 4 (Banach spaces of continuous functions on $\mathscr{G}$).**

$\boldsymbol{C}_b(\mathbb{R}^d) := \{f : \mathbb{R}^d \mapsto \mathbb{C}$, continuous and bounded with the norm $\|f\|_\infty = sup_{x\in\mathbb{R}^d} |f(x)|\ \}$

[4]The superscript bar stands for "closure" of a set. Hence the supp(f) is by definition a closed set.

The spaces $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ and $\boldsymbol{C}_0(\mathbb{R}^d)$ are defined as the subspaces of $\boldsymbol{C}_b(\mathbb{R}^d)$ consisting of functions which are *uniformly continuous* (and bounded) resp. *decaying at infinity*, i.e.,

$$f \in \boldsymbol{C}_0(\mathbb{R}^d) \quad \text{if and only if} \quad \lim_{|x|\to\infty} |f(x)| = 0.$$

Cu-char **Lemma 2.** *Characterization of $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ within $\boldsymbol{C}_b(\mathbb{R}^d)$ (or even $\boldsymbol{L}^\infty(\mathbb{R}^d)$).*

Cu-char0 (3) $$\|T_x f - f\|_\infty \to 0 \quad \textit{for} \quad x \to 0.$$

if and only if $f \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$ (characterization within $\boldsymbol{C}_b(\mathbb{R}^d)$).

Proof. The reformulation of the usual $\varepsilon - \delta$-definition into the format of (3) is left to the interested reader. □

We will use the symbol $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ for the Banach space of all bounded and linear operators on the Banach space $\boldsymbol{C}_0(\mathbb{R}^d)$, endowed with the operator norm

$$\|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} = \|T\|_\infty := sup_{\|f\|_\infty \le 1} \|Tf\|_\infty.$$

2. Banach algebras of bounded and continuous functions

banalg-de **Definition 5. [Banach algebra]** A Banach space $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a *Banach algebra* if it has a *bilinear multiplication* $(a, b) \to a \bullet b$ (or simply $a \cdot b$ or just ab, this means that it is also associative and distributive) with the extra property that for some constant $C > 0$[5]

Banalgsubm (4) $$\|a \bullet b\|_{\boldsymbol{A}} \le C \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{A}} \quad \forall a, b \in \boldsymbol{A}$$

Theorem 1. *(Banach algebras of continuous functions)*

1. $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *is a Banach algebra with respect to pointwise multiplication, even a B^*-algebra, with involution $f \mapsto \bar{f}$ (i.e., $\|\bar{f}\|_\infty = \|f\|_\infty$ and $\bar{\bar{f}} = f$).*
2. $\big(\boldsymbol{C}_{ub}(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *is a closed subalgebra of* $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$.
3. $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *is a closed ideal within* $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$.

Proof. First we show that $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ is a closed subspace of $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$. In fact, let (f_n) be a uniformly convergent sequence in $\boldsymbol{C}_{ub}(\mathbb{R}^d)$, convergent to $f \in \boldsymbol{C}_b(\mathbb{R}^d)$. Then for given $\varepsilon > 0$ there exists n_0 such that $\|f - f_{n_0}\|_\infty < \varepsilon/3$. Since $f_{n_0} \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$ we can find $\delta > 0$ such that $\|T_z f_{n_0} - f_{n_0}\|_\infty < \varepsilon/3$ for $|z| < \delta$. This implies of course for $|z| < \delta$:

if-uniflim (5) $$\|T_z f - f\|_\infty \le \|T_z(f - f_{n_0})\|_\infty + \|T_z f_{n_0} - f_{n_0}\|_\infty + \|f - f_{n_0}\|_\infty < 3\varepsilon/3 = \varepsilon.$$

We have to estimate $\|T_x(gf) - gf\|_\infty$, for $|x| \to 0$. If we choose $\delta > 0$ such that both $\|T_x f - f\|_\infty \le \varepsilon'$ and $\|T_x g - g\|_\infty \le \varepsilon'$ for $|x| \le \delta$ we have

$$\|T_x(g \cdot f) - g \cdot f\|_\infty \le \|T_x g \cdot (T_x f - f)\|_\infty + \|(T_x g - g) \cdot f\|_\infty \le 2(\|f\|_\infty + \|g\|_\infty)\varepsilon'.$$

□

Lemma 3. *Characterization of $\boldsymbol{C}_0(\mathbb{R}^d)$ within $\boldsymbol{C}_b(\mathbb{R}^d)$: $\boldsymbol{C}_0(\mathbb{R}^d)$ coincides with the closure of $\boldsymbol{C}_c(\mathbb{R}^d)$ within $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$.*

[5]Without loss of generality one can assume $C = 1$, because for the case that $C \ge 1$ one moves on to the equivalent norm $\|a\|'_{\boldsymbol{A}} := C \cdot \|a\|_{\boldsymbol{A}}$.

Definition 6. A directed family (a net or sequence) $(h_\alpha)_{\alpha\in I}$ in a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is called a **BAI** (= bounded approximate identity or "approximate unit" for $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$) if

$$\lim_\alpha \|h_\alpha \cdot h - h\|_{\boldsymbol{A}} = 0 \quad \forall h \in \boldsymbol{A}.$$

We need the following definition:

Definition 7. [**Rd-specific!**]Definition of value preserving **dilation operators**:

Drho1 (6) $$D_\rho f(z) = f(\rho \cdot z),\ \rho > 0,\ z \in \mathbb{R}^d$$

It is easy to verify that

Drho2 (7) $$\|D_\rho f\|_\infty = \|f\|_\infty \quad \text{and} \quad D_\rho(f \cdot g) = D_\rho(f) \cdot D_\rho(g)$$

For later use let us mention that the dilation and translation operators satisfy the following commutation relation [**Rd-specific!**]:

trans-comm (8) $$D_\rho \circ T_x \ = \ T_{x/\rho} \circ D_\rho, \quad x \in \mathbb{R}^d, \rho > 0.$$

Proof. For any $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ one has, for any $z \in \mathbb{R}^d$:

$$[D_\rho(T_x f)](z) = (T_x f)(\rho z) = f(\rho z - x) = f(\rho(z - x/\rho)) = (D_\rho f)(z - x/\rho) = [T_{x/\rho}(D_\rho f)](z).$$

□

Theorem 2. $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *is a Banach algebra with bounded approximate units. In fact, any family of functions* $(h_\alpha)_{\alpha\in I}$ *which is uniformly bounded, i.e. with*

$$|h_\alpha(x)| \le C < \infty \qquad \forall x \in \mathbb{R}^d, \forall \alpha \in I$$

and satisfies

$$\lim_\alpha h_\alpha(x) = 1 \quad \textit{uniformly over compact sets}$$

constitutes a BAI (the converse is true as well). [**Rd-specific!**]*In particular, one obtains a BAI by stretching any function* $h_0 \in \boldsymbol{C}_0(\mathbb{R}^d)$ *with* $h_0(0) = 1$*, i.e. by considering the family* $D_\rho h_0(g)(t) := h_0(\rho \cdot t)$*, for* $\rho \to 0$*.*

The following elementary lemma is quite standard and could in principle be left to the reader. Probably it should be placed in the appendix. However, it is quite typical for arguments to be used repeatedly throughout these notes and therefore we state it explicitly.

Lemma 4. *Assume that one has a bounded net* $(T_\alpha)_{\alpha\in I}$ *of operators from a Banach space* $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ *to another normed space* $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$*, such that* $T_\alpha \to T_0$ *strongly, i.e. for any finite set* $F \subset \boldsymbol{B}^{(1)}$ *and* $\varepsilon > 0$ *there exists an index* α_0 *such that for* $\alpha \succeq \alpha_0$ *one has* $\|T_\alpha f - T_0 f\|_{\boldsymbol{B}^2}$*, then one has uniform convergence over compact subsets* $M \subset \boldsymbol{B}^1$*.*

Proof. The argument is based on the usual compactness argument. Assuming that $\big\|\!\big|T_\alpha\big|\!\big\| \le C_1$ for all $\alpha \in I$ we can find some finite set $f_1, \dots f_K \in M$ such that balls of radius $\delta = \varepsilon/(3C_1)$ around these points cover M. According to the assumption (and the general properties of nets) one finds α_0 such that

$$\|T_\alpha f_j - T_0 f_j\|_{\boldsymbol{B}^{(2)}} < \varepsilon/3, \quad \text{for } j = 1, 2, \dots K.$$

Consequently on has for any $f \in M$ and suitable chosen index j (with $\|f - f_j\|_{\boldsymbol{B}^{(1)}} < \delta$):

comp-convg (9) $$\|T_\alpha f - T_0 f\|_{\boldsymbol{B}^{(2)}} < \|T_\alpha(f - f_j)\|_{\boldsymbol{B}^{(2)}} + \|T_\alpha f_j - T_0 f_j\|_{\boldsymbol{B}^{(2)}} + \|T_0(f_j - f)\|_{\boldsymbol{B}^{(2)}}.$$

Since the operator norm of T_0 is also not larger than C_1 (! easy exercise) this implies

omp-convg2 (10) $$\|T_\alpha f - T_0 f\|_{\boldsymbol{B}^{(2)}} \leq 2C_1\|f - f_j\|_{\boldsymbol{B}^{(1)}} + \|T_\alpha f_j - T_0 f_j\|_{\boldsymbol{B}^{(2)}} \leq 2C_1\delta + \varepsilon/3 < \varepsilon.$$

□

otal-test1 *Remark* 1. In fact, a similar argument can be used to verify that the w^*-convergence of a bounded net of operators by verifying the convergence only for f from a total subset within its domain. In fact, using linearity implies that convergence is true for linear combinations of the elements for such a set, and by going to a limit one obtains convergence for arbitrary elements in $\boldsymbol{B}^{(1)}$.

Obviously the above argument applies to nets of bounded linear functionals as well (choose $\boldsymbol{B}^{(2)} = \mathbb{C}$).

Lemma 5. [Rd-specific!]*[Group of dilation operators] The family of operators $(D_\rho)_{\rho>0}$ is a family of isometric isomorphisms on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. Moreover, the mapping $\rho \mapsto D_\rho, \mathbb{R}^+ \to \mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ is a group homomorphism from the multiplicative group of positive reals into the isometric linear operators on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. In particular on has*

dilcomut1 (11) $$D_{\rho_2} \circ D_{\rho_1} = D_{\rho_1 \cdot \rho_2} = D_{\rho_1} \circ D_{\rho_2}$$

approxunit **Lemma 6.** *Let I be the family of all compact subsets of $\mathbb{R}^d$, and define $K \succeq K'$ if $K \supseteq K'$. If we choose for every such $K \subset\subset \mathbb{R}^d$ [6] a plateau function p_K such that $0 \leq p(x) \leq 1$ on $\mathbb{R}^d$ and $p_K(x) \equiv 1$ on K. Then $(p_K)_{K\in I}$ constitutes a BAI for $\boldsymbol{C}_0(\mathbb{R}^d)$.*

Proof. First we have to show that the "direction" of I is reflexive ($K \supseteq K$ and transitive, i.e. $K_1 \supseteq K_2$ and $K_2 \supseteq K_3$ obviously implies $K_1 \supseteq K_3$. Finally the key property for the index set of a net is easily verified: Given two "indices" K_1, K_2 the set $K_0 := K_1 \cup K_2$ is a element of I, i.e. a compact set, with $K_0 \succeq K_i$ for $i = 1, 2$. Hence $(p_K)_{K\in I}$ is in fact a net.

In order to verify the BAI property let $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ and $\varepsilon > 0$ be given. Then - by definition of $\boldsymbol{C}_0(\mathbb{R}^d)$ there exists some $R > 0$ such that $|x| > R$ implies $|f(x)| \leq \varepsilon$. Then one has $|f(x) \cdot p(x) - f(x)| = |(1 - p(x))||f(x)| \leq |f(x)| \leq \varepsilon$ for $|x| \geq R$. On the other hand $|f(x) \cdot p(x) - f(x)| = |f(x) \cdot 1 - f(x)| = 0$ for $|x| \leq R$, as long as p is a plateau function equal to 1 on the compact set $K_0 = B_R(0)$ [7], i.e. as long as $p = p_K$ has an index with $K \succeq K_0$. Altogether $\|f \cdot p - f\|_\infty \leq \varepsilon$ □

The family of all partial sums of BUPUs (defined below) will satisfy the conditions described in Lemma 6. plat-approxunit

har-COinCb **Lemma 7.** *Another characterization of $\boldsymbol{C}_0(\mathbb{R}^d)$ within $\boldsymbol{C}_b(\mathbb{R}^d)$:*
$h \in \boldsymbol{C}_b(\mathbb{R}^d)$ belongs to $\boldsymbol{C}_0(\mathbb{R}^d)$ if and only if

char-CO1 (12) $$\lim_\alpha \|h_\alpha \cdot h - h\|_\infty = 0$$

for one (hence all) BAIs for $\boldsymbol{C}_0(\mathbb{R}^d)$ (as described above).

[6] We write $K \subset\subset \mathbb{R}^d$ to indicate that K is a *compact* subset of $\mathbb{R}^d$.
[7] $B_R(0)$ denotes the ball of radius R around 0.

Remark 2. It is a good exercise that such a BAI acts *uniformly* on compact subsets, i.e. for any relatively compact set M in $\big(\boldsymbol{C}_0(G), \|\cdot\|_\infty\big)$ one finds: Given $\varepsilon > 0$ one can find some α_0 such that for $\alpha \succeq \alpha_0$ implies

char-C02

$$\|h_\alpha \cdot h - h\|_{\boldsymbol{B}} \leq \varepsilon \quad \forall h \in M. \tag{13}$$

The following simple lemma will be useful later on (characterization of LTIS, i.e. *linear time-invariant systems*, for $d = 2$ we can talk about linear space-invariant operators, as they are used for image processing applications):

homog-BF1

Lemma 8. *A function $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ belongs to $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ if and only if the (non-linear) mapping $z \mapsto T_z f$ is continuous from $\mathbb{R}^d$ into $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$. In fact, such a mapping is continuous at zero if and only if it is uniformly continuous.*

Proof. It is clear that continuity is a necessary condition (and we have already seen that it is equivalent to uniform continuity for $f \in \boldsymbol{C}_b(\mathbb{R}^d)$). Conversely assume continuity at zero, i.e. $\|T_z f - f\|_\infty < \varepsilon$ for sufficiently small z ($|z| < \delta$). Then one derives continuity at x as follows:

$$\|T_{x+z} f - T_x f\|_\infty = \|T_x(T_z f - f)\|_\infty = \|T_z f - f\|_\infty < \varepsilon$$

implying uniform continuity of the discussed mapping. [8] □

Remark 3. On can say, that the mapping $x \mapsto T_x$ from $\mathbb{R}^d$ into $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ is a representation of the Abelian group $\mathbb{R}^d$ on the Banach space $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, which by definition means that the mapping is a homomorphism between the additive group $\mathbb{R}^d$ and the group of invertible (even isometric) operators on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. The additional property that $z \mapsto T_z f$ is continuous for $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ is referred to as the *strong continuity* of this representation. The same mapping into the larger space $\mathcal{L}(\boldsymbol{C}_b(\mathbb{R}^d))$ would not be strongly continuous, because for $f \in \boldsymbol{C}_b(\mathbb{R}^d) \setminus \boldsymbol{C}_{ub}(\mathbb{R}^d)$ this mapping fails to be continuous (cf. below)

Remark 4. One can derive from the above definition that the mapping $(x, f) \to T_x f$, which maps $\mathbb{R}^d \times \boldsymbol{C}_0(\mathbb{R}^d)$ into $\boldsymbol{C}_0(\mathbb{R}^d)$ is continuous with respect to the product topology (Exercise).

Definition 8. We denote the dual space of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ with $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Sometimes the symbol $\boldsymbol{M}_b(\mathbb{R}^d)$ is used in order to emphasize that one has "bounded" (regular Borel) measures.

The *Riesz Representation Theorem* (see e.g. [2], p.112, as well as many other sources, probably also [3], or [1]) provides the justification for this definition and establishes the

[8] The proof gives an argument, that a homomorphism from an Abelian topological group G ($\mathbb{R}^d$ in our case) into a group of operators on a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (here $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$), is strongly continuous, i.e. satisfies the discussed continuity property analogue to $f \mapsto T_z f$, is continuous from G into $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ if and only it is continuous at zero (the identity element of G).

link to the concept explained in measure theory. In that case the action of μ on the test function is of course written in the form

meas-act (14) $$\mu(f) = \int_{\mathbb{R}^d} f(t) d\mu(x).$$

In the classical case of functionals on $\boldsymbol{C}(I)$, where $I = [a, b]$ is some interval, on can describe these integrals using Riemann-Stieltjes integrals. They make use of functions F of bounded variation. The distribution function [9] F is connected with the *measure* μ (defined on the σ-algebra of Borel sets in I via

dist-meas (15) $$F(x) = \int_a^x d\mu(x); \quad \int_I f(x) d\mu(x) = lim_{\delta \to 0} \sum_i f(\xi_i)[F(x_i) - F(x_{i-1})])$$

The original approach to $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$ was within this context: A closed subspace of the linear space of all functions of bounded variation $BV(\mathbb{R})$ is the space of *absolutely continuous functions* (cf. historical notes, in particular see the article of 1929 by Plessner [32], where he shows that these are exactly the elements within BV which have continuous translation, hence are approximated using e.g. convolution by nice summability kernels.[10]

Of course the norm (measurement of the *size of total variation*), the so called BV-norm of F and the norm of the linear functional turn out to be the same. Moreover, there is natural way to split a function F of bounded variation into a difference of two non-descreasing functions, intuitively the "increasing" and the "decreasing part"[11]. The sum of these two parts (as opposed to the difference) is a monotonuous function which induces therefore a non-negative measure of the same total variation, which is called $|\mu|$ in the general measure theory (Bochner's decomposition), the absolute value of the measure μ (just a symbolic operation). Normally the (non-linear) mapping $\mu \mapsto |\mu|$ requires quite some measure theory (see e.g. [1]).

Remark 5. As a compromise between the simple Riemannian integral and the high-level Lebesgue integral (which is *complete* with respect to the $\boldsymbol{L}^1$-norm!) people sometimes restrict their attention to the closure of the step functions in the sup-norm (this is a space containing both the step functions and the continuous functions), if we work over an interval or bounded subset of $\mathbb{R}^d$. One could use the closure of the simple (i.e. defined by indicator functions of $d-$dimensional boxes or cubes) in the sense of the Wiener norm of $\boldsymbol{W}(\boldsymbol{L}^1, \boldsymbol{\ell}^\infty)(\mathbb{R}^d)$.

EXAMPLES: point measures $\delta_0 : f \mapsto f(0)$, $\delta_x : f \mapsto f(x)$, or integrals over bounded sets: $f \mapsto \int_a^b f(x)dx$, or more generally $f \mapsto \int_{\mathbb{R}^d} f(x)k(x)dx$, where integration is taken in

[9] This use of the word distribution is quite different from the use of distributions in the sense of generalized functions, as it is used in the rest of these notes.

[10] It is also interesting to see how most of the still relevant facts concerning Fourier series, Lebesgue integral and so on appear already in a rather clear form in those early papers and books on the subject. However, this was before the age of Banach space theory and in fact not at all referring to abstract Hilbert space theory, as it is seen nowadays as a corner stone. For further comments see the [forthcoming] section on historical notes.

[11] Think of a hiker how walks up and down in the mountains and reports on the total height of ascent and descent part.

the sense of Riemannian (or Lebesgue) integrals, for $k \in \boldsymbol{C}_c(\mathbb{R}^d)$. For the setting of locally compact groups $\mathscr{G}$ one will use the *Haar measure* for the definition of such integrals. Note: the norm of $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ is of course just the functional norm, i.e.

$$\|\mu\|_{\boldsymbol{M}} := \sup_{\|f\|_\infty \le 1} |\mu(f)| = \sup_{\|f\|_\infty = 1} |\mu(f)|$$

EXERCISE: $\|\delta_t\|_{\boldsymbol{M}} = 1$, and more generally:

Theorem 3. *For any finite linear combinations of Dirac-measures, i.e. for $\mu = \sum_{k\in F} c_k \delta_{t_k}$ (where F is some finite index set and we assume that one has the natural representation, with $t_k \neq t_{k'}$) one has $\|\mu\|_{\boldsymbol{M}_b} = \sum_{k\in F} |c_k|$*

A simple lemma helps us to identify the closed linear subspace of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ generated from the (linear) subspace of finite-discrete measures. We will call it the space of *discrete measures*.

Lemma 9. *Let V be a linear subspace of a normed space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Then the closure of V coincides with the space obtained by taking the absolutely convergent sequences in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with elements from V:*

s-convser1

$$Abs(\boldsymbol{B}) := \{x = \sum_n v_n, \ with \quad \sum_n \|v_n\|_{\boldsymbol{B}} < \infty\} \tag{16}$$

Proof. It is obvious that $Abs(\boldsymbol{B}) \subseteq V^-$, the closure of V in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Conversely, any element $x = lim_{k\to\infty} v_k$ can also be written as a limit of a sequence with $\|x - v_k\|_{\boldsymbol{B}} < 2^{-n}$ and may therefore be rewritten as a telescope sum, with $y_1 = v_1$, $y_{n+1} = v_{n+1} - v_n$. .... □

conv-dense

Remark 6. A simple but powerful variant of the above lemma is obtained if one allows the elements v_n being only taken from a dense subset of V (density of course in the sense of the $\boldsymbol{B}-$norm. Since such a set has the same closure this is an immediate corollary from the above lemma.

As a consequence (of the last two results) one finds that the closed linear subspace generated by the finite discrete measures coincides with absolutely convergent series of Dirac measures:

Definition 9 (Bounded discrete measures)**.**

$$\boldsymbol{M}_d(\mathbb{R}^d) = \{\mu \in \boldsymbol{M}(\mathbb{R}^d) : \mu = \sum_{k=1}^{\infty} c_k \delta_{t_k} \ s.t. \ \sum_{k=1}^{\infty} |c_k| < \infty\}$$

The elements of $\boldsymbol{M}_d(\mathbb{R}^d)$ are called the discrete measures, and thus we claim that they form a (proper) subspace of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

Definition 10. A sequence of measures $(\mu_n)_{n\ge1}$ is *Bernoulli convergent* to some $\mu_0 \in \boldsymbol{M}(\mathbb{R}^d)$ according to Bochner ([4], p.15), if it is bounded in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ and if

$$\mu_n(f) \to \mu_0(f) \quad \text{for all} \quad f \in \boldsymbol{C}_b(\mathbb{R}^d).$$

Lemma 10 (Bernoulli convergence for tight sequences)**.** *Assume that $(\mu_n)_{n\ge1}$ is a bounded and tight sequence in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, such that $\mu_0 = w^* - lim_n \mu_n$. Then μ_n is Bernoulli convergent to μ_0.*

Comment (May 31st, 2010) The definition of tightness is coming later. In short it means that there is uniform concentration of a whole family over the same compact set. More precisely

-tightdef1 **Definition 11.** A bounded set of measures $M_1 \subseteq \boldsymbol{M}_b(G)$ is called tight, if for every $\varepsilon > 0$ there exists some $k \in \boldsymbol{C}_c(G)$ such that $\|p\mu - \mu\|_{\boldsymbol{M}} < \varepsilon$ for **all** $\mu \in M_1$.

We claim, however, that the finite discrete measures form a w^*-dense subspace of $\boldsymbol{M}(\mathbb{R}^d)$. Before doing this we will define the adjoint of the pointwise multiplication within $\boldsymbol{C}_0(\mathbb{R}^d)$ on its dual space, i.e. how to define $h \cdot \mu$ (for $h \in \boldsymbol{C}_0(\mathbb{R}^d)$ or even $h \in \boldsymbol{C}_b(\mathbb{R}^d)$).

ctimesmeas **Definition 12.** $h \cdot \mu(f) := \mu(h \cdot f), h \in \boldsymbol{C}_b(\mathbb{R}^d), f \in \boldsymbol{C}_0(\mathbb{R}^d)$.

asmodlemma **Lemma 11.** *For $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ and $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ one has*

cmeasnorms (17) $$\|h \cdot \mu\|_{\boldsymbol{M}} \leq \|h\|_\infty \|\mu\|_{\boldsymbol{M}}.$$

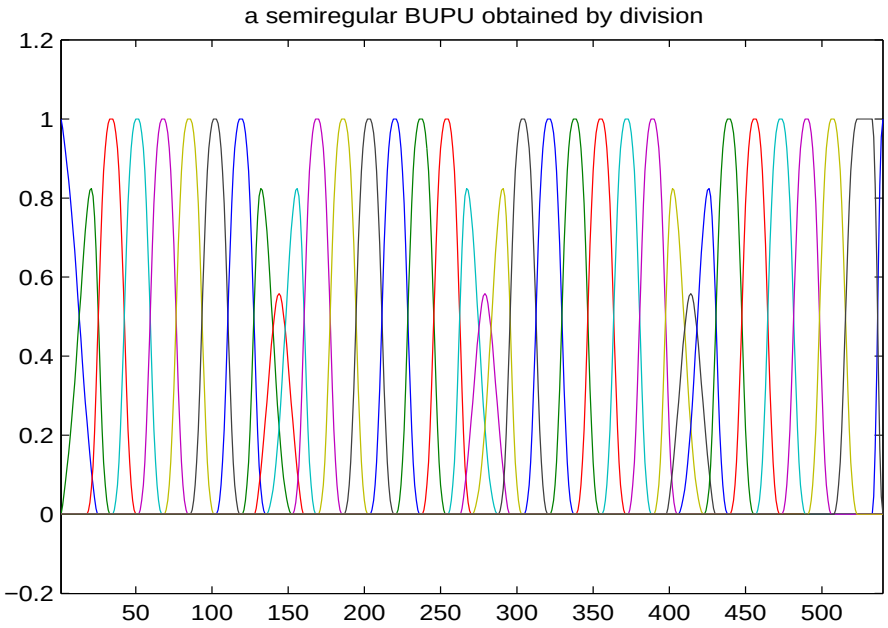
In fact, the mapping from the pointwise algebra $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ into the algebra of pointwise multiplication operators $M_h : \mu \mapsto h \cdot \mu$ is in fact an isometric embedding.

Proof. The statements are more an exercise in terminology than a deep mathematical statement and are therefore left to the interested reader. □

For this reason we will introduce a simple tool, the so-called BUPUs, the "bounded uniform partitions of unity". For simplicity we only consider the regular case, i.e. BUPUs which are obtained as translates of a single function:

Definition 13. A sequence $\Phi = (T_\lambda \varphi)_{\lambda \in \Lambda}$, where φ is a compactly supported function (i.e. $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$), and $\Lambda = A(\mathbb{Z}^d)$ a lattice in $\mathbb{R}^d$ (for some non-singular $d \times d$-matrix) is called a **regular BUPU** if

$$\sum_\lambda \varphi(x - \lambda) \equiv 1.$$



The regular BUPUs are sufficient for our purposes. They are a special case for a more general concept of (unrestricted) BUPUs:

BUPU-gen0

Definition 14. A BUPU, a so-called **b**ounded **u**niform **p**artition of **u**nity in some Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ of continuous functions on $\mathscr{G}$ is a family $\Psi = (\psi_i)_{i\in I}$ of non-negative functions on $\mathscr{G}$, if the following set of conditions is satisfied:

(1) There exists some neighborhood U of the identity element of the group $\mathscr{G}$ that for each $i \in I$ there exists $x_i \in \mathscr{G}$ such that $\mathrm{supp}(\psi_i) \subseteq x_i + U$ for all $i \in I$;
(2) The family Ψ is bounded in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, i.e. there exists $C_A > 0$ such that $\|\psi\|_{\boldsymbol{A}} \leq C_A$ for all $i \in I$;
(3) The family of supports $(x_i + U)_{i\in I}$ is *relatively separated*, i.e. for each $i \in I$ the number of intersecting neighbors is uniformly bounded in the following sense

$$\#\{j \,|\, (x_i + U) \cap (x_j + U) \neq \varnothing\} \leq C_0;$$

(4) $\sum_{i\in I} \psi_i(x) \equiv 1$.

Occasionally we will refer to U as the *size of the* BUPU. The constant C_A is the norm of the family Ψ in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, and C_0 is a kind of *overlapping constant* of the family.

Let $\Psi = (\psi_i)_{i\in I}$ be a BUPU, i.e. a bounded uniform partition of unity.

sabsdecomp

Theorem 4. *Let $\Psi = (\psi_i)_{i\in I}$ be a non-negative Then $\|\mu\|_{\boldsymbol{M}} = \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}}$, i.e. $\mu = \sum_{i\in I} \mu\psi_i$ is absolutely convergent for $\mu \in \boldsymbol{M}(\mathbb{R}^d)$.*

Proof. The estimate $\|\mu\|_{\boldsymbol{M}} \leq \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}}$ is obvious, by the triangular inequality of the norm and the completeness of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. In order to prove the opposite inequality (and in fact the finiteness of the series on the right hand side) we can argue as follows. Given $\varepsilon > 0$ we can choose a sequence $\varepsilon_i > 0$ such that $\sum_{i\in I} \varepsilon_i < \varepsilon$. By the definition of $\|\mu\psi_i\|_{\boldsymbol{M}}$ we can find $f_i \in \boldsymbol{C}_0(\mathbb{R}^d)$ with $\|f_i\|_\infty$, such that $|\mu\psi_i(f_i)| = |\mu(\psi_i f_i)| > \|\mu\psi_i\|_{\boldsymbol{M}} -$

ε_i. Without loss of generality (by changing the phase of f_i if necessary) we can assume that $\mu(\psi_i f_i)$ is real-valued and in fact non-negative, i.e. absolute values can be omitted.

Putting together the function $f := \sum_{i\in I} f_i\psi_i$ we have $|f(t)| \leq \sum_{i\in I} \|f_i\|_\infty |\psi_i(t)| \leq \sum_{i\in I} \psi_i(t) = 1$, and in fact norm convergence of the sum $f = \sum_{i\in I} \psi_i f_i$ is norm convergent in $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$. Since μ is a continuous linear functional we have therefore:

$$|\mu(f)| = |\sum_{i\in I} \mu(\psi_i f_i)| = \sum_{i\in I} \mu(\psi_i f_i) > \sum_{i\in I} (\|\mu\psi_i\|_{\boldsymbol{M}} - \varepsilon_i) \geq \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}} - \varepsilon,$$

thus completing our argument. □

Corollary 1. *Every measure μ is a limit of its finite partial sums. Hence the compactly supported measures are dense in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. In particular, $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is an essential Banach module over $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ with respect to pointwise multiplications. One possible approximate unit consists of the families Ψ_J, where the index J is running through the finite subsets of I and is given as $\Psi_J = \sum_{i\in J} \psi_i$.*

Definition 15. For any BUPU Φ the *Spline-type Quasi-Interpolation operator* Sp_Φ is given by:

$$f \mapsto Sp_\Phi(f) := \sum_{\lambda\in\Lambda} f(\lambda)\phi_\lambda$$

PERHAPS BETTER/MORE CONSISTENT:
For any BUPU Ψ the *Spline-type Quasi-Interpolation operator* Sp_Ψ is given by:

$$f \mapsto \mathrm{Sp}_\Psi(f) := \sum_{i\in I} f(x_i)\psi_i$$

Its adjoint operator, which we will call *discretization* operator, which maps bounded measures into discrete measures, denoted by D_Ψ.

Lemma 12. *The operators Sp_Ψ are uniformly bounded on $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ or $\big(\boldsymbol{C}_b(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ respectively. Moreover $\|\mathrm{Sp}_\Psi(f) - f\|_\infty \to 0$ for $|U| \to 0$.*

The adjoint operator can be obtained from the following reasoning:

Ppsi-dual0 (18) $$\mathrm{Sp}^*_\Psi(\mu)(f) = \mu(\mathrm{Sp}_\Psi(f)) = \mu(\sum_{i\in I} f(x_i)\psi_i) = \sum_{i\in I} \mu(\psi_i) f(x_i) = \sum_{i\in I} \mu(\psi_i)\delta_{x_i}(f),$$

hence we can make the following definition

DPsi-def1 **Definition 16.** $D_\Psi(\mu) \quad (or) \quad D_\Psi\mu = \sum_{i\in I} \mu(\psi_i)\delta_{x_i}$.

It is a good exercise to verify the following statements:

Lemma 13. *For $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ the sum defining $Sp_\Phi f$ is (unconditionally) norm convergent in $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ (even finite at each point), and $\|Sp_\Phi(f)\|_\infty \leq \|f\|_\infty$, i.e. Sp_Φ is a linear and non-expansive mapping on $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$. In particular, the family of operators $(Sp_\Phi)_\Phi$, where Φ is running through the family of all (regular) BUPUs of uniform size (that means that the support size of ϕ with $\|\phi\|_\infty$ is limited) is uniformly bounded on $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$.* [12] *Moreover, these spline-type quasi-interpolants are norm convergent*

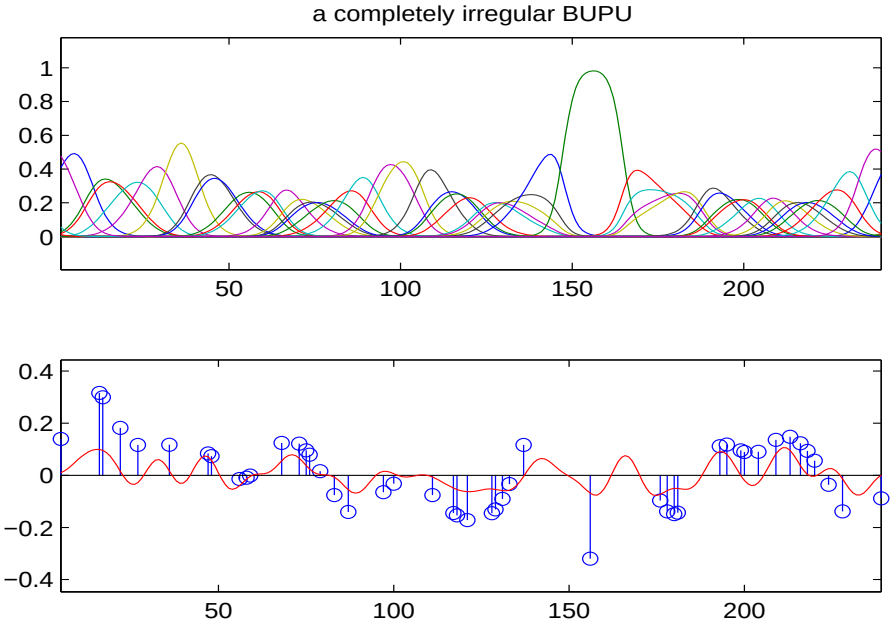
[12] these statements have to be simplified

to f as $|\Phi|$ (the maximal diameter of members of Φ) tends to zero. In other words, we claim: For every $\varepsilon > 0$ and any finite subset $F \subset \boldsymbol{C}_0(\mathbb{R}^d)$ there exists $\delta > 0$ such that $\|Sp_\Phi f - f\|_\infty < \varepsilon$ if only $|\Phi| \leq \delta_0$.

[13]

For every BUPU Φ we denote the adjoint mapping to Sp_Φ by D_Φ: Discretization operator to the partition of unity Φ. It maps $\boldsymbol{M}(\mathbb{R}^d)$ into itself (in a linear way), and since Sp_Φ is non-expansive the same is true for D_Φ

AGAIN: ONE MIGHT PREFER TO WRITE D_Ψ !



Definition 17. [**Rd-specific!**]Definition of **dilation operators**:

$$D_\rho f(z) = f(\rho \cdot z),\ \rho > 0,\ z \in \mathbb{R}^d$$

We will allow to apply operators of this kind of operators also to families, i.e. we will shortly write $D_\rho \Phi$ for the family $(D_\rho(T_\lambda \varphi))_{\lambda \in \Lambda}$. Since D_ρ preserves values of functions (it moves the values via stretching or compression to "other places"), hence $D_\rho \mathbf{1}_{\mathbb{R}^d} = \mathbf{1}_{\mathbb{R}^d}$[14]. Consequently Φ is a (regular) BUPU if and only if $D_\rho \Phi$ is a BUPU (for some, hence all) $\rho > 0$.

In order to understand the engineering terminology of an "impulse response" uniquely describing the bevaviour of a linear time-*invariant* system let us take a quick look at the situation over the group $G = \mathbb{Z}_n$, the cyclic group (of complex unit roots of order n).

[13] In some cases it might be of interest to look at BUPUs which preserve uniform continuity, i.e. which have the property that $SpPsif \in \boldsymbol{C}_{ub}(G)$ for any $f \in CbG$. This is certainly the case if one has a regular BUPU, i.e. a BUPU which is generated from translate of a single, or perhaps a finite collection of "building blocks".

[14] We use the symbol $\mathbf{1}_M$ to indicate the *indicator function* of a set, which equals 1 on M and zeros elsewhere.

Obviously $c_0(\mathbb{Z}_n)$ is just $\mathbb{C}^n$, and the (group-) translation is just cyclic index shift (mod n).

Cn-TILS **Lemma 14.** *A matrix A represents a "translation-invariant" linear mapping on $\mathbb{C}^n$ if and only if it is circulant, i.e. if it is constant along side-diagonals (we will also call such matrices "convolution matrices");*

Proof. We can start with the "first unit vector", which is mapped onto some column vector in $\mathbb{C}^n$ by the linear mapping $x \mapsto A * x$. Since we can interpret all further vectors as the image of the other unit vectors, but these are obtained from the first unit vector by cyclic shift, we see immediately that the columns of the matrix are obtained by cyclic shift of the first column, hence the matrix A has to be circulant (in the cyclic sense).

Usually engineers call the first column of this matrix, which is the output corresponding to an "impulse like" input (the first unit vector) the *impulse response of the translation invariant linear system A*.

Since the converse is easily verified we leave it to the interested reader. In fact, it may be interesting to verify the translation invariance by deriving first for a general matrix action A the action of $T_{-1} \circ A \circ T_1$ and to observe subsequently that this operation does not change circulant matrices. □

For the "continuous domain" (i.e., for linear systems over $\mathbb{R}$ resp. over $\mathbb{R}^d$) one has to invoke functionals in order to be able to "represent" the translation invariant systems.

mpact-CORd **Theorem 5.** *A bounded and closed subset $M \subset \boldsymbol{C}_0(\mathbb{R}^d)$ is compact if and only if it is uniformly tight and equicontinuous, i.e. if the following conditions are satisfied:*

- *for $\varepsilon > 0$ there exists δ_0 such that $|y| \leq \delta \Rightarrow \|T_y f - f\|_\infty \leq \varepsilon \quad \forall f \in M$;*
- *for $\varepsilon > 0$ there exists some $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|f - kf\|_\infty \leq \varepsilon, \quad \forall f \in M$;*

Proof. It is obvious that finite subsets $M \subset \boldsymbol{C}_0(\mathbb{R}^d)$ have these two properties, and it is easy to derive them for compact sets by the usual approximation argument.

So we have to show the converse. We observe first that $\|pf - f\|_\infty \to 0$, if p is a sufficiently large plateau-function. The set $\{pf \mid f \in M\}$ is still equicontinuous (cf. the proof, that $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ is a Banach algebra with respect to pointwise multiplication). We may assume that p has compact support. Then we apply a (sufficiently) fine BUPU to ensure that $\|pf - Sp_\Psi(pf)\|_\infty \leq \varepsilon$. Since p has compact support only finitely many terms make up $Sp_\Psi(pf)$, i.e. one can approximate by finite linear combinations of the elements of Ψ, and the proof is complete (?more details)? □

f-BIBOTLIS **Definition 18.** The Banach space of all "translation invariant linear systems" on $\boldsymbol{C}_0(\mathbb{R}^d)$ is given by [15]

$$\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d)) = \{T : \boldsymbol{C}_0(\mathbb{R}^d) \to \boldsymbol{C}_0(\mathbb{R}^d), \text{ bounded, linear} : T \circ T_z = T_z \circ T, \forall z \in \mathbb{R}^d\}$$

[15] The letter $\mathcal{H}$ in the definition refers to *homomorphism* [between normed spaces], while the subscript G in the symbol refers to "commuting with the action of the underlying group $G = \mathbb{R}^d$ realized by the so-called regular representation, i.e. via ordinary translations

Remark 7. [16] It is easy to show that $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ is a closed subalgebra of the Banach algebra of $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ (in fact it is even closed with respect to the strong operator topology), hence it is a Banach algebra of its own right (with respect to composition as multiplication). We will see later that it is in fact a commutative Banach algebra.

Definition 19. recall the notion of a FLIP operator: $\check{f}(z) = f(-z)$

Given $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ we define the *convolution operator* C_μ by: $C_\mu(f)(z) := \mu(T_z\check{f})$.

The reverse mapping R recovers a measure $\mu = \mu_T$ from a given translation invariant system T via $\mu(f) := T(\check{f})(0)$.

ation-TLIS **Theorem 6.** *[Characterization of LTISs on $\boldsymbol{C}_0(\mathbb{R}^d)$]*
There is a natural isometric isomorphism between the Banach space $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$, endowed with the operator norm, and $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, the dual of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, by means of the following pair of mappings:

(1) *Given a bounded measure $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ we define the operator C_μ (to be called convolution operator with convolution kernel μ later on) via:*

nv-measdef (19) $$C_\mu f(x) = \mu(T_x\check{f}).$$

(2) *Conversely we define $T \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ the linear functional $\mu = \mu_T$ by*

impuls-def (20) $$\mu_T(f) = [T\check{f}](0).$$

The claim is that both of these mappings: $C : \mu \mapsto C_\mu$ and the mapping $T \mapsto \mu_T$ are linear, non-expansive, and inverse to each other. Consequently they establish an isometric isomorphism between the two Banach spaces with

TIS-isomet (21) $$\|\mu_T\|_{\boldsymbol{M}} = \|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} \quad \textit{and} \quad \|C_\mu\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} = \|\mu\|_{\boldsymbol{M}}.$$

Proof. The proof has a number of partial steps carried out in the course. Some of them are quite elementary (e.g. that the resulting objects are really linear systems resp. linear functionals, or that the mappings C and its inverse are linear mappings) and are left to the interested reader. We concentrate on the most interesting parts.

First of all it is easy to check that the definition of C_μ really defines an operator on $\boldsymbol{C}_0(\mathbb{R}^d)$, with the output being a bounded function.[17] Since

conv-bd1 (22) $$|C_\mu(f)(x)| \le \|\mu\|_{\boldsymbol{M}}\|T_x\check{f}\|_\infty = \|\mu\|_{\boldsymbol{M}}\|f\|_\infty,$$

and hence the operator norm of C_μ[18] satisfies $\|C_\mu\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} \le \|\mu\|_{\boldsymbol{M}(\mathbb{R}^d)}$.

Furthermore it commutes with translations, since $D_{-1}(T_z f) = T_{-z}D_{-1}f = T_{-z}\check{f}$

nv-commut1 (23) $$C_\mu(T_z f)(x) = \mu(T_x T_{-z}\check{f}) = \mu(T_{x-z}\check{f}) = C_\mu(f)(x-z) = T_z C_\mu f(x).$$

It is also easy to check - using this fact - that (uniform) continuity is preserved, since

conv-cont (24) $$\|T_z(C_\mu(f)) - C_\mu(f)\|_\infty = \|C\mu(T_z f - f)\| \le \|\mu\|_{\boldsymbol{M}}\|T_z f - f\|_\infty \to 0 \quad \text{for} \quad |z| \to 0.$$

[16] Sometimes we will write $[T, T_z] \equiv 0$ in order to express the commutation formula using the commutator symbol, and the "$\equiv$"–symbol to express that this relation holds true $\forall z \in \mathbb{R}^d$.

[17] Engineers talk of BIBOS, i.e. Bounded Input [gives] Bounded Output Systems.

[18] Of course we think of the "convolution by the measure μ, in conventional terms, and this is also the reason for using the symbol C.

Finally we have to verify that the output of $f \to C_\mu(f)$ is not just continuous but also decaying at infinity. Due to the closedness of $\boldsymbol{C}_0(\mathbb{R}^d)$ within $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ it is sufficient to approximate $C_\mu(f)$ for $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ by compactly supported (continuous) functions. This is achieved using Theorem 4 resp. Corollary 2. Given $\varepsilon > 0$ one can choose $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\|\mu - h\mu\|_{\boldsymbol{M}} < \varepsilon$. Then the difference

compapprox
$$\|C_{h\mu}f - C_\mu f\|_\infty \leq \|h \cdot \mu - \mu\|_{\boldsymbol{M}} \|f\|_\infty \tag{25}$$

can be made arbitrarily small, while on the other hand

mpsuppconv
$$C_{h\mu}f(x) = h\mu(T_x f^\vee) = \mu(h \cdot T_x f^\vee) = 0 \tag{26}$$

if $\operatorname{supp}(h) \cap \operatorname{supp}(T_x f^\vee) = \operatorname{supp}(h) + (x - \operatorname{supp}(f)) \neq \emptyset$, i.e. if $x \notin \operatorname{supp}(h) + \operatorname{supp}(f)$, since we have assumed that f has compact support. Thus altogether we have found that $f \mapsto C_\mu(f)$ is in $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$, with a control of the operator norm on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ by $\|\mu\|_{\boldsymbol{M}}$.

Let us verify next that the two mappings are inverse to each other. Let us verify first that the system associated with μ_T is the original system, in other words we have to show that $C_{\mu_T} = T$ for all $T \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$. Using the relevant definitions this follows from the following chain of equalities

ysmeassys1
$$[C_{\mu_T}(f)](x) = \mu_T(T_x \check{f}) = T(\check{(}T_x \check{f}))(0) = T(T_{-x}f)(0) \tag{27}$$

Since T commutes with all the translations we continue by

sysmeassys
$$[C_{\mu_T}(f)](x) = T(T_{-x}f)(0) = T_{-x}Tf(0) = Tf(0 - -x) = Tf(x). \tag{28}$$

Since this is valid for all $x \in \mathbb{R}^d$ and $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ one direction is shown. It also shows that *every* $TG \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ is a convolution operator by a uniquely determined measure $\mu_T \in \boldsymbol{M}_b(\mathbb{R}^d)$, and that $\|\mu_T\|_{\boldsymbol{M}} = \|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))}$. In fact, since we know already that $\|\mu_T\|_{\boldsymbol{M}} \leq \|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))}$ we only have to verify that a strict inequality would imply a contradiction. Thus using (28) we come to the following exact isometry:

$$\|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} = \|C_{\mu_T}\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} \leq \|\mu_T\| \leq \|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))}.$$

Let us now look for the converse identity: the measure associated to the convolution operator C_μ associated with some given $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ is just the original measure, since

easysmeas1
$$\mu_{C_\mu}(f) = C_\mu((\check{f}))(0) == \mu(T_0\check{(}\check{f})) = \mu(f). \tag{29}$$

which in turn implies that the inverse mapping is surjective, i.e. that the (already known to be isometric) bijection between bounded measures and systems is in fact surjective onto the bounded measures. Thus our proof is complete. □

Note also that there is a matter of consistency to be verified. For "ordinary functions" $g \in \boldsymbol{C}_c(\mathbb{R}^d)$ (resp. $g \in \boldsymbol{L}^1(\mathbb{R}^d)$) one can define a unique bounded measure

regbdmeas1 **Definition 20.** $\mu_g(f) = \int_{\mathbb{R}^d} f(x)g(x)dx$,

Many ways to introduce convolutions!

The previous characterization allows to introduce in a natural way a Banach algebra structure on $\boldsymbol{M}(\mathbb{R}^d)$. In fact, given μ_1 and μ_2 the translation invariant system $C_{\mu_1} \circ C_{\mu_2}$ is represented by a bounded measure μ. In other words, we can define a new (so-called) *convolution product* $\mu = \mu_1 * \mu_2$ of the two bounded measures such that the relation (completely characterizing the measure $\mu_1 * \mu_2$)

CONVO1 (30) $$C_{\mu_1 * \mu_2} = C_{\mu_1} \circ C_{\mu_2}$$

It is immediately clear from this definition that $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is a **Banach algebra with respect to convolution**. ! Associativity is given for free, but *commutativity of the new convolution is not so obvious* (and will follow only later, clearly as a consequence of the commutativity of the underlying group).

The translation operators themselves, i.e. T_z are elements of $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$, which correspond exactly to the Dirac measures $\delta_z, z \in \mathbb{R}^d$, due to the following simple consideration[19].

onv-deltax (31) $$C_{\delta_x} f(z) = \delta_x(T_z \check{f}) = [T_z \check{f}](x) = \check{f}(x - z) = f(z - x) = [T_x f](z),$$

and thus $C_{\delta_x} = T_x$, in particular $C_{\delta_0} = T_0 = Id$, resp.

convDirac0 (32) $$f = \delta_0 * f \quad \text{and} \quad T_x f = \delta_x * f, \quad f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

Remark 8. In the literature there another, quite non-obvious definition of the convolution of two bounded measures is given. It makes a non-trivial proof of the associativity necessary, which is *not necessary* in the case of our definition. So the classical way of describing the convolution of two measures (for now the order matters) is as follows: Given $\mu_1, \mu_2 \in \boldsymbol{M}_b(\mathbb{R}^d)$ the action of $\mu_1 * \mu_2$ on $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ can be characterized by

conv-meas1 (33) $$\mu_1 * \mu_2(f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y + x) d\mu_2(y) d\mu_1(x).$$

Proof. First recall the definition, i.e. that $\mu_1 * \mu_2$ is the linear functional corresponding to $T := C_{\mu_1} \circ C_{\mu_2} \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ satisfies

$$\mu_1 * \mu_2(f) = T(\check{f})(0) = [(C_{\mu_1} \circ C_{\mu_2})(\check{f})](0) = [C_{\mu_1}(C_{\mu_2}(\check{f})](0) = \mu_1(g)$$

with $g(x) = [C_{\mu_2}(\check{f})]\check{}(x) = C_{\mu_2}(\check{f})(-x) = \mu_2(T_x f)$. Putting everything into the standard notation we have $g(x) = \int_{\mathbb{R}^d} f(y + x) d\mu_2(y)$ which implies the result stated above. □

Remark 9. The different realizations of convolution can be conveniently disguised by using (mostly from now on throughout the rest of the manuscript) the symbol $*$ for convolution, i.e. we write $\mu * f$ instead of $C_\mu(f)$ from now one, or $g * f$ for the convolution of $\boldsymbol{L}^1(\mathbb{R}^d)$-functions using the Lebesgue integral. We have discussed that the reinterpretation of those things in varying contexts is - from a pedantical point of view - a delicate matter, but having discussed this problem in detail for a few cases we leave the rest to the reader if there is interest in the subject. We just want to convince all our readers that e.g. associativity or commutativity of

[19] We start writing $\mu * f$ for $C_\mu(f)$, the convolution of μ with f. This is justified by the fact that in the case of double interpretation, e.g. for $\mu = \mu_g$ for some $g \in \boldsymbol{C}_c(\mathbb{R}^d)$ either as a pointwise defined integral or the action of a measure on test functions one can *easily* check that there is consistency among the different view-points. From now on we will not discuss this issue in all details and and leave the verification of details to the reader.

convolution is always well justified in the context that we are providing, and that in this sense the user of the calculus does not have to worry about possible inconsistencies.

Although the above result, combined with Fubini's theorem indicates that convolution is *commutative* we do not make use of this fact, because we do not want to invoke results from measures theory at this point. In fact, using an approximation argument (using discrete measures) we can get the result. Looking back we can say however that this is also more or less they way how we prove Fubini, using suitable resummation of suitable Riemannian sums.

Remark 10. The fact that the Banach space on which the action of the (then) Banach space $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ on which the Banach convolution algebra $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ has the *extra relation* the this Banach algebra by *duality* implies that the action of the algebra on $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ has and adjoint action on the dual space. Since we would like to call this action also "convolution" we have to check compatibility of the two operations. In fact, what we observe, convolution *within* $\boldsymbol{M}_b(\mathbb{R}^d)$ turns out to be the adjoint action to convolution combined with inversion (note that this makes even in a non-commutative setting the adjoint action covariant (same associativity rule) as in $\boldsymbol{M}_b(G)$).

lip-convCO

$$(\mu, f) \mapsto \check{\mu} * f = (\mu * \check{f})^{\check{}}\ . \tag{34}$$

NOTE: Probably we should use the checkmark symbol (cf. [41]: $f^{\checkmark}$.

In order to verify this other consistency we can argue as follows: By definition we have

$$C_{\mu_1 * \mu_2} h(z) = C_{\mu_1}(C_{\mu_2} h)(z)$$

for all $h \in \boldsymbol{C}_0(\mathbb{R}^d)$ and $z \in \mathbb{R}^d$, hence for $z = 0$, hence by our conventions

$$(\mu_1 * \mu_2)(\check{h}) = \mu_1((\mu_2 * h)^{\check{}}) == \mu_1(\mu_2{}^{\check{}} * \check{h})$$

or by writing $f = \check{h} \in \boldsymbol{C}_0(\mathbb{R}^d)$:

dist-conv

$$(\mu_1 * \mu_2)(f) = \mu_1(\mu_2{}^{\check{}} * f). \tag{35}$$

This form of defining the convolution of to "abstract objects" is very much like the typical definition of the convolution of distributions. It makes sense as long as the convolution of μ_2 with test functions makes sense[20]

Since the algebra $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ is *not* commutative it is not at all clear from this definition why $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_M, *)$ should be a commutative Banach algebra, which is in fact true.

In order to prepare for this statement we have to provide a few more statements.

The compatibility of the (isometric) dilation operators D_ρ with translations, i.e. the rule **[Rd-specific!]**

dil-trans

$$D_\rho T_z = T_{z/\rho} D_\rho \tag{36}$$

makes it possible to define another norm preserving *automorphism* for the Banach algebra $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_M, *)$. [21].

[20]Then the right hand side makes sense and can be used to give the expression $\mu_1 * \mu_2$ a meaning. Note however that this is *dangerous terrain*, because it may happen that such an individually defined convolution between distributions - similar to pointwise products - may turn out to be *non-associative*!

[21]We will see that these two operations on $\boldsymbol{C}_0(\mathbb{R}^d)$ resp. $\boldsymbol{M}(\mathbb{R}^d)$ are adjoint to each other

Strhodef **Definition 21.** The *adjoint* action of the group $\mathbb{R}^+$ on $\boldsymbol{M}(\mathbb{R}^d)$ is defined as the family of adjoint operators on $\boldsymbol{M}(\mathbb{R}^d)$ via:

def-Stroh
$$St_\rho\mu(f) := \mu(D_\rho f), \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d),\ \rho > 0. \tag{37}$$

Check

$$St_\rho\mu(f) = \mu(St_\rho f) \quad \text{and} \quad D_\rho\mu(f) = \mu(St_\rho(f))$$

Being defined as adjoint operators each of the operators St_ρ is not only isometric on $\boldsymbol{M}(\mathbb{R}^d)$, but also w^*-w^*-continuous on $\boldsymbol{M}(\mathbb{R}^d)$.

Definition 22. ALTERNATIVE DESCRIPTION (replaced later >> theorem): Given $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ the uniquely determined measure corresponding to the operator $D_\rho^{-1} \circ C_\mu \circ D_\rho$ will be denoted by $St_\rho\mu$.

We collect the basic facts for this new mapping:

conv-strho
$$St_\rho\mu_1 * St_\rho\mu_2 = St_\rho(\mu_1 * \mu_2) \tag{38}$$

Proof. To be provided later. It shows that the two alternative definitions above are indeed equivalent!! □

We have **[Rd-specific!]**

Strho-isom
$$\|St_\rho\mu\|_{\boldsymbol{M}} = \|\mu\|_{\boldsymbol{M}}. \tag{39}$$

While the (mass-preserving) operator St_ρ is well compatible with convolution, the (value-preserving) operators D_ρ is ideally compatible with pointwise multiplications, since obviously

rho-ptwmul
$$D_\rho(f \cdot g) = D_\rho f \cdot D_\rho g. \tag{40}$$

If one used D_ρ in connection with convolution one has to take into account the following relation:

Drho-conv1
$$St_\rho\mu * D_{1/\rho} f = D_{1/\rho}(\mu * f) \tag{41}$$

Stroh-map **Lemma 15.** *The mapping $\rho \mapsto St_\rho$ is continuous from $\mathbb{R}^+$ into $\mathcal{L}(\boldsymbol{M}(\mathbb{R}^d))$, endowed with the strong operator topology. It is also true that the mapping $(f, \rho) \mapsto C_{St_\rho\mu}(f)$ is continuous from $\boldsymbol{C}_0(\mathbb{R}^d) \times \mathbb{R}^+$ into $\big(\boldsymbol{C}_0(\mathbb{R}^d),\ \|\cdot\|_\infty\big)$. Obviously the operators $(Strho)_{\rho>0}$ form a commutative group of isometric operators on $\boldsymbol{M}(\mathbb{R}^d)$ (but also - since they leave $\boldsymbol{L}^1(\mathbb{R}^d)$ invariant - on $\big(\boldsymbol{L}^1(\mathbb{R}^d),\ \|\cdot\|_1\big)$).*

stro-group
$$St_{\rho_1} \circ \mathrm{St}_{\rho_2} = St_{\rho_1 \cdot \rho_2} = St_{\rho_2} \circ \mathrm{St}_{\rho_1} \tag{42}$$

Proof. Proof to be typed later on. □

rho-deltas
$$St_\rho \delta_x = \delta_{\rho x}, \tag{43}$$

Remark 11. Due to the w^*-density of finite discrete measures in $\boldsymbol{M}(\mathbb{R}^d)$ one can characterize St_ρ as the uniquely determined norm-to-norm continuous *and* $w^* - w^*$-continuous mapping which is isometric on $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ and satisfies formula (43). Strho-deltas

3. Basic Functional Analytic Considerations

A series of lemmata (lemmas) making use of density.

Lemma 16. *Assume a bounded linear mapping between two Banach spaces is given on a dense subset only, then it can be extended in a unique way to a bounded linear operator of equal norm on the full space.*

Proof. It is enough to know a bounded, linear mapping T on a dense subset D of a normed space, in order to observe that for any element v in the domain of T one has a convergent sequence $(d_n)_{n\in\mathbb{N}}$ in D with limit x. Hence $(d_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, so that the boundedness of T implies that $(Td_n)_{n\in\mathbb{N}}$ is a again a Cauchy sequence in the range, hence (by completeness) has a limit. The uniqueness of this extension and the fact that this extension has the same norm, i.e. that

$$|||T||| := \|T\|_{Op} = sup_{\|d\|\leq 1}\|Td\|$$

is also an immediate consequence. □

AIsubspace **Lemma 17. Test of BAI on a dense subspace:**
A bounded family $(e_\alpha)_{\alpha\in I}$ in a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a BAI for $\boldsymbol{A}$ if (only) for some total subset $D \subseteq \boldsymbol{A}$ one has:

$$\|e_\alpha \cdot d - d\|_{\boldsymbol{A}} \to 0 \quad \forall d \in D.$$

Proof. It is easy to derive - using the properties of a net - that one has (uniform) convergence for finite linear combinations of elements from D, and then by a density argument one verifies convergence for all elements. [22] □

doubnetlim **Lemma 18.** *A statement about iterated bounded, strongly convergent nets of operators.*
Assume that two bounded nets of operators between normed spaces, $(T_\alpha)_{\alpha\in I}$ and $(S_\beta)_{\beta\in J}$ are strongly convergent to some limit operators T_0 and S_0 respectively. Then the iterated limit of **any order** *exists and the two limits are equal.*
In fact, the index set $(\alpha, \beta) \in I \times J$ with the natural order [23] is turning $(S_\beta \circ T_\alpha)$ into a strongly convergent net.

Proof. ... Without loss of generality we may assume $T_0 = 0$ and $S_0 = 0$ (otherwise treat $T_\alpha - T_0$ etc.). THE REST OF THE PROOF is LEFT TO THE READER. □

AIoncomp **Lemma 19. One can choose the BAI elements from a dense SUBSPACE .**
Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with a bounded approximate unit, and D some dense subset of $\boldsymbol{A}$. Then there exists also approximate units $(d_\alpha)_{\alpha\in I}$ in D (if D is a dense subspace the new family $(d_\alpha)_{\alpha\in I}$ can even be chosen to be of equal norm).

Proof. One just has to choose d_α close enough to e_α, especially for α large enough (to be expressed properly). By the density of D one can do this. If D is a subspace (not just a subset) one can renormalize the new elements so that they have the same norm in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ as the original elements $(e_\alpha)_{\alpha\in I}$. □

[22] Of course this is more or less a statement about strong operator convergence for a net of bounded operators.

[23] explanation: the natural ordering is given by the fact that $(\alpha, \beta) \succeq (\alpha_0, \beta_0)$ if $\alpha \succeq \alpha_0$ and $\beta \succeq \beta_0$.

Lemma 20. Uniform action of BAIs on compact subsets
By the definition a family $(e_\alpha)_{\alpha\in I}$ acts pointwise like an identity in the limit case, for each "point". However, the action is even uniformly over finite sets, and hence over compact sets, by approximation (we shall use the acronym unifcomp-convergence for this situation in the future).

Proof. That one has uniform convergence over finite subsets is easily verified, using the property of a net. The inductive step is based on the following argument: Assume that one has found α_0 such that

$$\|e_\alpha \cdot a_i - a_i\| \leq \varepsilon \quad \forall \alpha \succ \alpha_0,\, 1 \leq i \leq m.$$

Since $\|e_\alpha \cdot a_{m+1} - a_{m+1}\| \leq \varepsilon$ for all $\alpha \succ \alpha_1$ we just have to choose some index α_2 with $\alpha_2 \succ \alpha_0$ *and* $\alpha_2 \succ \alpha_1$ (which is possible due to the definition of directed sets). Obviously

$$\|e_\alpha \cdot a_i - a_i\| \leq \varepsilon \quad \forall \alpha \succ \alpha_2,\, 1 \leq i \leq m+1.$$

In order to come up with uniform convergence over compact sets, we use again a typical approximation argument. Given any compact set $M \subseteq \boldsymbol{C}_0(\mathbb{R}^d)$ and $\varepsilon > 0$ we have to find some index α_3 such that

$$\|e_\alpha \cdot a - a\| < \varepsilon \quad \forall a \in M.$$

Recalling that $(e_\alpha)_{\alpha\in I}$ is bounded, i.e. $\|e_\alpha\| \leq C$ for some $C \geq 1$ for all $\alpha \in I$, we may choose some finite subset $F \subseteq M$ such that for any $a \in M$ there exists $a_i \in F$ with $\|a_i - a\| \leq \varepsilon/(3C) > 0$ (which is just another positive constant, known once we know C and ε). Hence for any given $a \in M$ one can use one of such elements $a_i \in F$ in order to argue that the triangular inequality implies (adding and subtracting the term $e_\alpha \cdot a_i$).

$$\|e_\alpha \cdot a - a\| \leq \|e_\alpha \cdot (a - a_i)\| + \|e_\alpha \cdot a_i - a_i\| + \|a_i - a\|.$$

If we choose now α_3 such that $\|e_\alpha \cdot a_i - a_i\| \leq \varepsilon/3 \quad \forall \alpha \succ \alpha_3$ and $a_i \in F$ we obtain altogether (more details are left to the reader, ...),

$$\|e_\alpha \cdot a - a\| \leq \varepsilon$$

□

boundedAI1 **Lemma 21.** *The following properties are equivalent.*

- *there is a bounded approximate identity in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$;*
- *there exists $C > 0$ such that for every finite subset $F \in \boldsymbol{A}$ and $\varepsilon > 0$ there exists element $h \in \boldsymbol{A}$ with $\|h\| \leq C$, such that*

$$\|h \cdot a - a\| < \varepsilon \quad \forall a \in F.$$

The argument to turn this family into a bounded net is the obvious one. One just has to set $\alpha := (F, \varepsilon)$, and defining such a pair "stronger than another pair $\alpha_1 := (F_1, \varepsilon_1)$ if $F_1 \supseteq F$ and $\varepsilon_1 \leq \varepsilon$. It is easy to verify that this defines a directed set, and that the choice $e_\alpha = h$ (corresponding to the pair (K, ε) as describe above) turn $(e_\alpha)_{\alpha\in I}$ into a bounded and convergent net, hence constitutes a BAI for $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$.

Note: In words: The existence of a BAI is equivalent to the existence of a bounded net in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ such that the action of "pointwise multiplication" (each element e_α is identified with the left algebra multiplication operator $a \mapsto e_\alpha \cdot a$) is convergent to

the identity operator in the strong operator norm topology (which is just the pointwise convergence of operators).

Note: Sometimes one observes that one has unbounded approximate identities, where the "cost" (i.e. the norm of e_α grows as the required quality of approximation, expressed by the smallness of ε), tends to the ideal limit zero. It makes sense to think of limited costs (given by $C > 0$) for arbitrary good quality of approximation which makes the BAI so useful.

Theorem 7. *(The Cohen-Hewitt factorization theorem, without proof, see [24])*
Let $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with some BAI, then the algebra factorizes, which means that for every $a \in \boldsymbol{A}$ there exists a pair $a', h' \in \boldsymbol{A}$ such that $a = h' \cdot a'$, in short: $\boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{A}$. In fact, one can even choose $\|a - a'\| \leq \varepsilon$ and $\|h'\| \leq C$.

There is a more general result, involving the terminology of **Banach modules**.

BanMod **Definition 23.** A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach module* over a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if one has a bilinear mapping $(a, b) \mapsto a \bullet b$, from $\boldsymbol{A} \times \boldsymbol{B}$ into $\boldsymbol{B}$ with

$$\|a \bullet b\|_{\boldsymbol{B}} \leq \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{B}} \quad \forall\, a \in \boldsymbol{A}, b \in \boldsymbol{B}$$

which behaves like an ordinary multiplication, i.e. is associative, distributive, etc.:

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}.$$

Ex1Banmod **Lemma 22.** *(Ex. to Def. BanMod 23) A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is an (abstract) Banach module over a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$* [24] *if and only if there is a non-expansive (hence continuous) linear algebra homomorphism J from $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ into $\mathcal{L}(\boldsymbol{B})$.*[25]

Proof. For a Banach module the mapping: $a \mapsto J(a) : [b \mapsto a \bullet b]$ defines a linear mapping from $\boldsymbol{A}$ into $\mathcal{L}(\boldsymbol{B})$.

Conversely, one can define a $\boldsymbol{A}$-Banach module structure on $\boldsymbol{B}$ by the definition: $a \bullet b := J(a)(b)$.

Without going into all necessary details let as recall that the associativity law

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}.$$

It is a consequence of the homomorphism property of J: The left hand side $a_1 \bullet (a_2 \bullet b)$ translates into $J(a_1)[J(a_2)b]$, while the right hand side equals $J(a_1 \cdot a_2)(b)$. □

ess-Banmod **Definition 24.** A Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ over some Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is called *essential* if it coincides with the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B} = \{a \bullet b \,|\, a \in \boldsymbol{A}, b \in \boldsymbol{B}\}$.

For a general Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B}$ is denoted by $\boldsymbol{B}_e$ or $\boldsymbol{B}_{\boldsymbol{A}}$ (especially if there are different Banach algebras acting on the same space).

The notion of "essential Banach modules" is of course trivial in case the Banach algebra $\boldsymbol{A}$ has a unit element which is mapped into the identity operator, i.e. if there exists $u \in \boldsymbol{A}$ such that $u \cdot a = a \,\forall a \in \boldsymbol{A}$ and also $u \bullet b = b$, $\forall b \in \boldsymbol{B}$.

[24] $\boldsymbol{A}$ may be commutative or non-commutative, with our without unit.

[25] This viewpoint immediately suggests to discern between general embedding and those having special properties, e.g. the case where J is an injective mapping, i.e. for every $a \in \boldsymbol{A}$ there exists some $b \in \boldsymbol{B}$ such that $a \bullet b \neq 0$. Such modules are called *true!?* modules: in German: treue Moduln.

Lemma 23. *Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with BAIs $(e_\alpha)_{\alpha\in I}$. Then a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is essential if and only if*

essmod-BAI (44) $$\|e_\alpha \bullet b - b\|_{\boldsymbol{B}} \to 0 \quad \forall b \in \boldsymbol{B}$$

In particular, relation (44) holds true for one such BAI if and only if it is true for every BAI in $\boldsymbol{A}$.

dual-alg **Definition 25.** For any Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ over the Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ the dual space$(\boldsymbol{B'}, \|\cdot\|_{\boldsymbol{B'}})$ can be turned naturally into a $\boldsymbol{A}$-Banach module via the action

dual-alg1 (45) $$[a \bullet \sigma](b) := \sigma(a \cdot b), \quad \forall a \in \boldsymbol{A}, b \in \boldsymbol{B}, \sigma \in \boldsymbol{B'}.$$

Lemma 24. *(Ex. to Def. (25) $h\delta_x = h(x)\delta_x$, i.e. multiplication of a Dirac measure is realized as scalar multiplication of this Dirac measure by the point value of the continuous function h at that point.*

For *convenience* we will write $h \cdot \mu$ instead of the abstract symbol $\bullet$ in order to indicate pointwise multiplication between functionals μ and functions h.

Theorem 8. *The Banach space $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is an essential Banach module over $\boldsymbol{C}_0(\mathbb{R}^d)$ with respect to the natural (dual) action of pointwise multiplication.*

Proof. In fact we will show much more: for any BUPU Φ one has

$$\mu = \sum_{i\in I} \phi_i \mu$$

as absolutely convergent sum in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, hence

$$\|\mu - \mu \cdot \sum_{i\in F} \phi_i\|_{\boldsymbol{M}} \to 0 \quad \text{for} \quad F \,\text{finite},\, F \uparrow I.$$

□

Definition 26. It will be convenient to use the symbol ψ_J for the function $\sum_{i\in J} \psi_i$. For the choice $F = F(i) = \{j | \psi_j \cdot \psi_i \neq 0\}$ we write ψ_i^* for $\psi_{F(i)}$.

Then the above claim translates into $\|\mu \cdot \psi_F - \mu\|_{\boldsymbol{M}} \to 0$. The convenient and very useful relation

platrepr (46) $$\psi_i \cdot \psi_i^* = \psi_i \quad \text{for all} \quad i \in I.$$

More or less a reformulation

Lemma 25. *Let $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ and $\varepsilon > 0$ be given. Then there exists $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\|k\|_\infty \leq 1$ such that $\mu(k) \geq 0$ and $\mu(k) \geq \|\mu\|_{\boldsymbol{M}}(1-\varepsilon)$.*

Proof. Using the definition of the functional norm one only has to take into account phase factors and multiply, if necessary, k by $|\mu(k)|/\mu(k)$, which does not change $\|k\|_\infty$[26]. □

ascompsupp **Corollary 2.** *Every $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ is the limit of "compactly supported" measures of the form $(\psi_F \cdot \mu)$, where F is running through the finite subsets of I.*

In the terminology of Banach modules we can restate the last corollary in the form:

[26]See the embedding of $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$ into $\boldsymbol{M}_b$ given below for more details.

Corollary 3. *The dual space to* $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$*, i.e.* $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$*, is an essential Banach module over the Banach algebra* $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ *with respect to the action of pointwise multiplication.*

We will derive therefrom that one can also "integrate" arbitrary elements $h \in \boldsymbol{C}_b(\mathbb{R}^d)$.

Lemma 26. Integration of bounded functions against bounded measures
For any $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ *and* $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ *the net* $\mu(p_K \cdot h) = p_K \cdot \mu(h)$ *is a Cauchy-net in* $\mathbb{C}$*. Therefore it makes sense to define* $\mu(h) = \lim_K \mu(p_K \cdot h)$*. It is clear that in this way* μ *extends in a unique way to a bounded linear functional on* $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$*, and that the norm of this extension equals* $\|\mu\|_{\boldsymbol{M}}$*.*

Remark 12. If the net of bounded measures is (bounded and) tight, then it is an exercise to show, that it is *vaguely convergent* [27] if and only if it is w^*-convergent, resp. if and only if $\mu_\alpha(h) \to \mu_0(h)\ \forall h \in \boldsymbol{C}_b(\mathbb{R}^d)$. It is a bit more work (but still an exercise) to check out that this "pointwise convergence" even takes place *uniformly over compact subsets* of $\boldsymbol{C}_b(\mathbb{R}^d)$.

Remark 13. (MAYBE MISPLACED REMARK)
Recall that the "mass-preserving" stretching/compression operator St_ρ can be extended to $\boldsymbol{M}_b(\mathbb{R}^d)$ by the definition **[Rd-specific!]**

$$\mathrm{St}_\rho\, \mu(f) := \mu(D_\rho f).$$

Check that one has $\mathrm{St}_\rho\, \delta_x = \delta_{\rho x}$, and that $lim_{\rho\to 0}\, \mathrm{St}_\rho\, \mu = \mu(1)\delta_0$. In fact, for each $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ one has: $D_\rho f$ is uniformly bounded with respect the sup-norm for $\rho \to 0$ one has: $f(x) \to f(0)$, uniformly over compact sets. Since we can approximate the measure μ (by localizing it) to a measure with compact support its action is defined on sufficiently large compact sets, where $D_\rho f$ is like the constant function $f(0) = \delta_0(f)$, while the action on $Const \equiv 1$ is denoted by $\mu(1)$.

One can use this fact to find out that it is even possible to extend the convolution operators $f \mapsto C_\mu f$ to all of $\boldsymbol{C}_b(\mathbb{R}^d)$ (still with the equality of operator norm on $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ with the functional norm of μ), and with the property that the operators arising in such a way commute with translations. However, by means of the Hahn-Banach theorem one can construct translation invariant *means* on $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ which in turn allow to construct bounded linear operators on $\boldsymbol{C}_b(\mathbb{R}^d)$ which commute with all the translation operators without being of the form of a convolution by some bounded measure. In fact, those operators are non-zero operators on $\boldsymbol{C}_b(\mathbb{R}^d)$, but they map all of $\boldsymbol{C}_0(\mathbb{R}^d)$ onto the zero function. It is also not much more than a simple exercise to find out (using the characterization of $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ within $\boldsymbol{C}_b(\mathbb{R}^d)$ given early on) to check that any operator on $\boldsymbol{C}_b(\mathbb{R}^d)$ commuting with translations will map $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ into itself (in fact, this argument was used at the beginning of the identification theorem.)

We are now in the position to define the **Fourier transform** of a bounded measure. In the classical literature this is often referred to as the Fourier-Stieltjes transform of a measure, because it can be carried out technically over $\mathbb{R}$ using Riemann-Stieltjes integrals.

[27] This means that it is convergent in the $\sigma(\boldsymbol{M}_b(\mathbb{R}^d), \boldsymbol{C}_c(\mathbb{R}^d))$-topology, resp. $\mu_\alpha(k) \to \mu_0(k) \quad \forall k \in \boldsymbol{C}_c(\mathbb{R}^d)$.

Such a R-St-integral is the difference of two R-St-integrals with respect to bounded, non-decreasing "distribution" functions F. So in such a definition the ordinary Riemanian sum is replaced by an sum of the same form, but instead of the "natural length" of the interval $[a,b]$, which is $|b-a|$, one uses the length in the sense of F which is $F(b)-F(a)$.

Definition 27. A *character* is a continuous function from a topological group into the torus group $\mathbb{U} = \{z \in \mathbb{C} \,|\, |z| = 1\}$. In other words, χ is a character if $\chi(x+y) = \chi(x)\cdot\chi(y)$ for all $x, y \in G$. Moreover, since $|\chi(x)| = 1$ one has $\overline{\chi(x)} = 1/\chi(x)$ for all $x \in G$.

Definition 28. The set of all character is called the *dual group*, because those characters form an Abelian group under pointwise multiplication (!Exercise!). We write $\hat{G}$ for the dual group corresponding to G (the group law written as addition in this case).

Theorem 9. *In the case of $(\mathbb{R}^d, +)$ the dual group consists of the characters χ of the form $\chi_s : t \mapsto exp(2\pi i s\cdot t)$, with $s \in \mathbb{R}^d$. Due to the exponential law the pointwise multiplication of characters turns into addition of the parameters s (describing the "frequency content" of χ_s).*

Definition 29. The Fourier transform of $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ is defined by

eas-FT-def

$$\hat{\mu}(s) = \mu(\chi_{-s}) = \mu(\overline{\chi_s}). \tag{47}$$

Note that $\chi_s \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$ and that $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ can be applied according to ??

emLebLemma **Proposition 1** (Riemann-Lebesgue Lemma)**.** *The Fourier transform is a linear and non-expansive mapping from $\boldsymbol{M}_b(\mathbb{R}^d)$ into $\big(\boldsymbol{C}_{ub}(\mathbb{R}^d), \|\cdot\|_\infty\big)$*[28].

Proof. The uniform continuity results from the essential concentration of bounded measures over compact sets, the "usual rule" $T_s\,\mathcal{F}(\mu) = \mathcal{F}(M_s\mu)$., and the properties of characters.

In fact, given $\varepsilon > 0$ (alternative typing: $\varepsilon > 0$, and $\chi_0 \in \hat{G}$, we can find a compact subset $Q \subseteq G$ such that $\|\mu - \psi_Q\mu\|_{\boldsymbol{M}} < \varepsilon/3$. Hence there is a neighborhood W of the identity in $\hat{G}$ such that for all $\chi \in \chi_0 + W$ one has $|\chi(y) - \chi_0(y)| = |\chi(y)/chi_0 - 1| < \varepsilon/3$ for all $y \in Q$, by the definition of neighborhoods in $\hat{G}$ (compact open topology), and the fact that the quotient χ/χ_0 belongs to W. Expressed differently we have $\|\psi_Q(\chi_0 - \chi)\|_\infty < \varepsilon/3$. Altogether we have

$$|\hat{\mu}(\chi_0) - \hat{\mu}(\chi)| \le |\,\mathcal{F}(\mu - \mu\psi_Q)(\chi_0)| + |\,\mathcal{F}(\mu\psi_Q)(\chi_0 - \chi)| + |\,\mathcal{F}(\mu\psi_Q - \mu)(\chi)|$$

and consequently

$$|\hat{\mu}(\chi_0) - \hat{\mu}(\chi)| \le 2\|\mu - \mu\psi_Q\|_{\boldsymbol{M}} + \varepsilon/3 \le \varepsilon.$$

□

Although little can be said about the connection between w^*-convergence (in $\boldsymbol{M}(\mathbb{R}^d)$) and pointwise convergence "on the Fourier transform side" in general one has the following useful fact:

[28] In the same way as the Fourier transform from $\boldsymbol{L}^1(\mathbb{R}^d)$ into $\boldsymbol{C}_0(\mathbb{R}^d)$ is injective, but not surjective we also have here (for "local reasons") that the mapping is injective but not surjective.

wsttoFOUR1 **Lemma 27.** *Let (μ_α) be a w^*-convergent and tight net in $\boldsymbol{M}_b(\mathbb{R}^d)$, with $\mu_0 = w^*-lim_\alpha \mu_\alpha$. Then we have $\widehat{\mu_\alpha}(s) \to \widehat{\mu_0}(s)$, uniformly over compact subsets of $\widehat{G}$.*

Proof. Pointwise convergence of the Fourier transform is a consequence of the fact that due to the tightness of the whole family the action can be reduced to the case of measures with compact joint support. SHOULD BE MORE EXPLICIT!. □

Corollary 4. *Consider the family $D_\Psi \mu$ with $|\Psi| \leq 1$. Then their Fourier (Stieltjes) transforms is uniformly convergent over compact sets. They are uniformly bounded as well as equi-continuous on $\mathbb{R}^d$.*

$$e^{-2i\pi x\cdot y}$$

The basic relation

deltFdelt (48) $$\delta_y(\widehat{\delta_x}) = e^{-2\pi i x\cdot y} = \delta_x(\widehat{\delta_y}).$$

For any fixed BUPU Φ and Ψ (48) implies the following relation, observing that the convergence of the possible infinite sequence of the resulting discrete measures is absolute (hence in norm, in $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$).

measFmeas3 (49) $$D_\Psi\mu(\widehat{D_\Phi\nu}) = D_\Phi\nu(\widehat{D_\Psi\mu}) \quad \text{for any pair} \quad \mu,\nu \in \boldsymbol{M}_b(\mathbb{R}^d).$$

By linearity and then passing to the limit (using Proposition 2 below) we can derive from the following important *Fundamental Formula for the Fourier Stieltjes Transform*:

measFmeas (50) $$\nu(\hat{\mu}) = \mu(\hat{\nu}) \quad \text{for any pair} \quad \mu,\nu \in \boldsymbol{M}_b(\mathbb{R}^d).$$

For integrable functions this relation turns into what H. Reiter has called the *Fundamental Relation of the Fourier Transform*

L1FourL1 (51) $$\int_{\mathbb{R}^d} f(t)\hat{g}(t)dt = \int_{\mathbb{R}^d} g(z)\hat{f}(z)dz, \quad \text{for all} \quad f,g \in \boldsymbol{L}^1(\mathbb{R}^d).$$

As an immediate consequence of identity (50) we can derive the uniqueness theorem for Fourier Stieltjes transforms. If $\hat{\mu}$ is the zero-function, we can apply $\nu = \mu_g$, for any possible $g \in \boldsymbol{C}_c(\mathbb{R}^d)$ (or $\boldsymbol{L}^1(\mathbb{R}^d)$). Hence $\mu(\hat{g}) = 0$ for every $\hat{g} \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$. Since $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is dense in $\boldsymbol{C}_c(\mathbb{R}^d)$ with respect to the $\|\cdot\|_1$-norm this implies that μ is the zero functional.

Note that an easy why to show that $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is dense in $\boldsymbol{C}_0(\mathbb{R}^d)$ is based on the following sequence of observations:

(1) $\mathcal{F}\boldsymbol{L}^1(\mathbb{R})$ contains the triangular function Δ, which can be viewed as B-spline of order 1 (piecewise linear function) and whose Fourier transform is the (non-negative and even) function $SINC^2$;
(2) the shifted triangular functions $T_n\Delta$ forms a BUPU (bounded in $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d),\ \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$, since translation corresponds to modulation in $\left(\boldsymbol{L}^1(\mathbb{R}^d),\ \|\cdot\|_1\right)$ and is this isometric)
(3) for $d > 1$ one obtains BUPUs by looking at tensor products of $1D$-Δ-functions. Moreover one has $\|f \otimes g\|_1 = \|f\|_1\|g\|_1$ and

tensorFour (52) $$\mathcal{F}(f \otimes g) = \mathcal{F}(f) \otimes \mathcal{F}(g),$$

based on the following convention (using the *tensor product symbol*):

tensordef0 (53) $$f \otimes g(x,y) := f(x) \cdot g(y), \quad x,y \in \mathbb{R}^d,$$

(4) Since $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ is isometrically dilation invariant under $St_\rho, \rho > 0$ we find that $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is isometrically dilation invariant under the family $D_\rho, \rho > 0$, hence one obtains arbitrary fine BUPUs of the form $D_\rho T_n \Delta, n \in \mathbb{Z}^d, \rho > 0$;

(5) for $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ one has convergence of the corresponding spline-type approximations with respect to the $\|\cdot\|_\infty$-norm, i.e. $\|Sp_{\rho\Delta}k - k\|_\infty \to 0$.

The argument is based on the following general result:

nvmeasfcts **Proposition 2.** *Assume that a (bounded) and tight net (μ_α) is w^*-convergent to μ_0 and that (h_β) is a bounded net in $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ which is uniformly convergent to h_0 over compact subsets. Then for every $\varepsilon > 0$ there exist indices α_0 and β_0 such that for every $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$*

convmeas2 (54) $$|\mu_\alpha(h_\beta) - \mu_0(h_0)| \leq \varepsilon.$$

which is expressed equally by the formulation

$$\lim_{\alpha,\beta} \mu_\alpha(h_\beta) = \mu_0(h_0).$$

Proof. Given $\varepsilon > 0$ we recall that $\|h_\beta\|_\infty \leq C_0$ and $\|\mu_\alpha\|_{\boldsymbol{M}} \leq C_1$ for all α. By the tightness of the net (μ_α) we can find some plateau-like function $p \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\|p\|_\infty \leq 1$ such that $\|\mu_\alpha - p \cdot p \cdot \mu_\alpha\|_{\boldsymbol{M}} < \varepsilon {C_0}^{-1}$ for all α. Writing $K := \mathrm{supp}(p)$ we can find some index β_0 such that $|h_\beta(x) - h_0(x)| \leq \varepsilon/C_1$ for all $x \in K$ and all $\beta \geq \beta_0$. Altogether we then have

$$|\mu_\alpha(h_\beta) - \mu_0(h_0)| \leq$$
$$|\mu_\alpha(h_\beta) - p\cdot\mu_\alpha(h_\beta)| + |\mu_\alpha(p\cdot h_\beta) - \mu_\alpha(p\cdot h_0)| + |\mu_\alpha(p\cdot h_0) - \mu_0(p\cdot h_0)| + |p\cdot\mu_0(h_0) - \mu_0(h_0)| \leq$$
$$|[\mu_\alpha - p\cdot\mu_\alpha](h_\beta)| + |\mu_\alpha(p\cdot(h_\beta - h_0))| + |[\mu_\alpha - \mu_0](p\cdot h_0)| + |[p\cdot\mu_0 - \mu_0](h_0)| \leq$$
$$\|\mu_\alpha - p\cdot\mu_\alpha\|_{\boldsymbol{M}}\|h_\beta\|_\infty + \|\mu_\alpha\|_{\boldsymbol{M}}\|p\cdot(h_\beta - h_0)\|_\infty + |[\mu_\alpha - \mu_0](p\cdot h_0)| + \|p\cdot\mu_0 - \mu_0\|_{\boldsymbol{M}}\|h_0\|_\infty \leq$$
$$\varepsilon\cdot C_0 + C_1\cdot\|p\cdot(h_\beta - h_0)\|_\infty + |[\mu_\alpha - \mu_0](p\cdot h_0)| + \varepsilon C_0 \leq C_2\cdot\varepsilon.$$

for some constant $C_2 > 0$, and for all $\alpha \succ \alpha_0, \beta \succ \beta_0$. WELL: WE HAVE TO CHECK THE DETAILS Note that the w^*-convergence of the net (μ_α) implies that $\|\mu_0\|_{\boldsymbol{M}} \leq \liminf_\alpha \|\mu_\alpha\|_{\boldsymbol{M}}$. The first and last part is due to the tightness and boundedness, independent from the , while the is due to uniform convergence of (h_β) over $K = \mathrm{supp}(p)$ and the third term is getting small due to the w^*-convergence of μ_α to μ_0. □

Corollary 5. *Given $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$. Then there exists $\delta > 0$ such that the family $D_\Psi\mu$ is equicontinuous and there exists δ_0 such that*

erer er (55) $$|\widehat{D_\Psi\mu}(s) - \widehat{D_\Psi\mu}(s-h)| \leq \varepsilon \quad \textit{for all} \quad |h| \leq \delta, |\Psi| \leq 1.$$

Using the last lemma we can derive formula (50) from (48) via the following relation: measFmeas deltFdelt

We thus have the following *Fundamental Relation for the Fourier-Stieltjes transform*:

measFmeas2 (56) $$\mu(\hat{\nu}) = \nu(\hat{\mu}) \quad \text{for} \quad \mu,\nu \in \boldsymbol{M}_b(\mathbb{R}^d).$$

then follows using exactly Proposition 2 and Lemma 27. convmeasfcts wsttoFOUR1

convFTmeas **Theorem 10.** *The FT on $\boldsymbol{M}_b(\mathbb{R}^d)$ is injective, and turns convolution into pointwise multiplication, i.e. in fact, it is a homomorphism of Banach algebras. This also implies (once more) that convolution is commutative (because obviously pointwise multiplication is a commutative operation).*

The compatibility with convolution is an easy exercise for discrete measures, and can be transferred to the general case using a weak-star argument. Recall again that w^*-convergence of bounded nets of measures implies pointwise convergence of their Fourier transforms.

There is an alternative way of proving commutativity of convolution. It is easy to see that the convolution of (finite) discrete measures is commutative, and the general case follows from this (by approximation in the strong operator topology).

Material on Banach Modules

The Banach module is called "true" if the mapping J described above is injective.

If one only has a continuous (but not necessarily non-expansive algebra homomorphism J) one can replace the norm on $\boldsymbol{A}$ by another equivalent norm (just some constant multiple of the original one) in order to ensure this (harmless) extra property.

Recall the notions of *weak topology* on any Banach space (such as $\left(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\right)$, and the *$w^*$-topology*) on any dual space, such as $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

Theorem 11. *A sequence (or indeed a bounded net) of functions in $\left(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\right)$ is weakly convergent if and only if it is pointwise convergent (while in contrast norm-convergence means uniform convergence over $\mathbb{R}^d$).*

Proof. Since the Dirac measures are specific linear functionals on $\left(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\right)$ weak convergence of a sequence (f_n) in $\boldsymbol{C}_0(\mathbb{R}^d)$ implies $f_n(x) = \delta_x(f_n) \to \delta_x(f_0) = f_0(x)$ for any $x \in \mathbb{R}^d$. Conversely, the possibility of approximating a general measure in a bounded way by linear combinations of Dirac measures implies that pointwise convergence indeed implies weak convergence. If one goes into the details of the proof the boundedness of the set of approximating measures as well as the boundedness of the sequence (resp. a net (f_α)

□

Remark 14. For *equicontinuous* families one can show that weak (or pointwise) convergence is equivalent to "uniform convergence over compact set". A (?bounded) net (f_α) is weakly convergent if and only if it is UCOCS, etc. dots

Remark 15. How can we characterize w^*-convergence in $\boldsymbol{M}(\mathbb{R}^d)$? (for bounded sets): cf. Bernoulli convergence, vague convergence ($= \sigma(\boldsymbol{C}_c(G), \boldsymbol{R}(G))$) using standard topological constructions in functional analysis), pointwise convergence of the STFT or of the Fourier transform etc...

Alternative description of "multipliers" on $\boldsymbol{C}_0(\mathbb{R}^d)$ resp. translation invariant BIBOS (bounded input bounded output systems, with the property of mapping $\boldsymbol{C}_0(\mathbb{R}^d)$ into itself).

terizeHGCO **Theorem 12.** *Let H be any w^*- total subset of $\boldsymbol{M}(\mathbb{R}^d)$, and assume that a bounded linear operator $T \in \mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ commutes with the action of H, i.e., that the commutators $[C_h, T] \equiv 0$ for all $h \in H$. Then $T \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$.*

Note that in the original definition the set H was just the set of convolution operators by Dirac measures $\delta_x, x \in \mathbb{R}^d$ (or at least from some dense subset).

UPCOMING MATERIAL:

Embedding of test functions into $\boldsymbol{M}(\mathbb{R}^d)$ (over groups this requires the use of the [invariant] Haar measure, which indeed is a linear functional on $\boldsymbol{C}_c(G)$). Compatibility of operators which are now available on both the functions and the measures (resp. functionals). E.g. we can now do an internal convolution of functions (viewed as bounded measures) or an external action (one is acting as a bounded measure, the other is consider as the $\boldsymbol{C}_0(\mathbb{R}^d)$ element on which the action takes place). Associativity of convolution in the most general situation (also of course **commutativity**, etc.).

Further notes:

$\boldsymbol{C}_{ub}(\mathbb{R}^d)$ is a (closed) subspace of the dual of $\boldsymbol{L}^1(\mathbb{R}^d)$. Hence it carries a $\sigma(\boldsymbol{C}_{ub}(\mathbb{R}^d), \boldsymbol{L}^1(\mathbb{R}^d))$ topology which can be shown to be equivalent (at least on bounded sets!?, or more) to the uniform convergence over compact sets (?true).

STATEMENT: Every $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ is a limit (in the sense of uniform limit over compact sets) of a *bounded* sequence of functions from $\boldsymbol{C}_0(\mathbb{R}^d)$ resp. even from $\boldsymbol{C}_c(\mathbb{R}^d)$. In fact, on can take the sequence $p_n \cdot f$, where (p_n) is a BAI for $\boldsymbol{C}_0(\mathbb{R}^d)$ consisting of (increasing) plateau functions).

EXTENSION PRINCIPLE. Let (p_n) be as above, and $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ and $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ be given. Then the sequence $\mu(p_n \cdot f)$ is a Cauchy sequence, hence convergent in $\mathbb{C}$. In fact, the limit is the same for any other BAI in $\boldsymbol{C}_0(\mathbb{R}^d)$. Therefore it makes sense to define $\mu(f) := lim_{n\to\infty} \mu(p_n f)$.

REMARK: this will be important to define the Fourier Stieltjes transforms for bounded measures, i.e. for $\hat{\mu}(s) = \mu(\chi_{-s})$ later on!

Lemma 28. *The convolution operators form a (commutative) Banach algebra of operators. It turns out that the characters can be identified with the joint eigenvectors for this whole class of operators. Indeed, we have $C_\mu(\chi_s) = \hat{\mu}(s)\chi_s$ for any $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ and any character χ_s on $\mathbb{R}^d$.*

Proof. The claim is valid for $\mu = \delta_x$, for any $x \in \mathbb{R}^d$, and hence for finite linear combinations of measures and also immediately for discrete measures (due to a simple approximation argument). However, it is enough to use the fact that one has Bernoulli convergence of $D_\Psi \mu$ to μ (i.e. w^*-convergence by a tight net of discrete, bounded measures with limit μ).

Maybe even more convincing is the following observation. Due to the exponential law one has $\chi_s(-z) = \chi_{-s}(z)$ and $T_z(\chi_{-s}) = \chi(z) \cdot \chi(-s)$ and therefore

ourConvEig (57) $\quad$ "$\mu * \chi_s(z)'' = C_\mu(\chi_s)(z) = \mu(T_z(\chi_{-s})) = \mu(\chi_{-s}) \cdot \chi_s(z) = \hat{\mu}(s)\chi_s(z)$,

i.e. $C_\mu(\chi_s) = \mu(\chi_{-s} \cdot \chi_s \cdot \chi_s$, i.e. □

IT SEEMS THAT WE HAVE TO SHOW UNIQUENESS OF THE FOURIER STIELTJES TRANSFORM!

Although the following result will be a consequence of basic facts about a more general (distributional) Fourier transform let us claim (and verify) the injectivity of the Fourier Stieltjes transform just defined:

FourStinj **Proposition 3** (Uniqueness of Fourier Stieltjes transform)**.** *Let $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ be given, and assume that $\hat{\mu}(\chi) \equiv 0$. Then $\mu = 0$ in $\boldsymbol{M}_b(\mathbb{R}^d)$. Consequently the bounded linear mapping $\mu \to \hat{\mu}$ from $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ into $\big(\boldsymbol{C}_{ub}(\mathbb{R}^d), \|\cdot\|_\infty\big)$ is injective.*

Proof. We use the density of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ first, which is a consequence of the fact that $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is a Banach algebra (due to the convolution theorem) which if of course closed under conjugation ($h = \hat{f}$ ..) and point separating, hence it is dense as a consequence of the Stone-Weierstrass approximation theorem. Being a subspace of $\boldsymbol{M}_b(\mathbb{R}^d)$ we also know that we can approximate any $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ in the w^*-sense by discrete measures, in such a way that the corresponding Fourier Stieltjes transforms (they are trigonometric polynomials) converge to $\hat{g}$ uniformly over compact sets. □

4. Identifying "ordinary functions with functionals"

There is a natural way to identify "ordinary functions" (say $k \in \boldsymbol{C}_c(\mathbb{R}^d)$) with linear functions $\mu \in \boldsymbol{M}(\mathbb{R}^d)$, by the following trick: Given

ttomeasure
$$\mu = \mu_k, \quad \text{resp.} \quad \mu(f) = \int_{\mathbb{R}^d} f(x)k(x)dx \tag{58}$$

This is also possible over general locally compact Abelian groups, but requires the existence of the Haar measure (we will not go into this direction, a good explanation is given in Deitmar's book [9]).

Lemma 29. *The mapping $k \to \mu_k$ described above defines an isometric embedding from $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$ into $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Hence we may identify the closure of $\boldsymbol{M}_{C_c} = \{\mu_k \,|\, k \in \boldsymbol{C}_c(\mathbb{R}^d)\}$ with the completion of the normed space $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$.*

isomL1emb **Proposition 4** (Functions to Measures)**.** *There is a natural,* isometric *embedding of $(\boldsymbol{C}_c(G), \|\cdot\|_1)$ into $\boldsymbol{M}_b(G)$, given by*

$$k \mapsto \mu_k : \mu_k(f) = \int_G f(x)k(x)dx.^{29}$$

Proof. It is obvious that each μ_k is in fact a bounded linear functional on $\boldsymbol{C}_0'(G)$ and that the mapping $k \mapsto \mu_k$ is linear and nonexpansive, since evidently for each $f \in \boldsymbol{C}_0(G), k \in \boldsymbol{C}_c(G)$ one has:

$$|\mu_k(f)| = |\int_G f(x)k(x)dx| \le \int_G |f(x)||k(x)|dx \le \|f\|_\infty \|k\|_1.$$

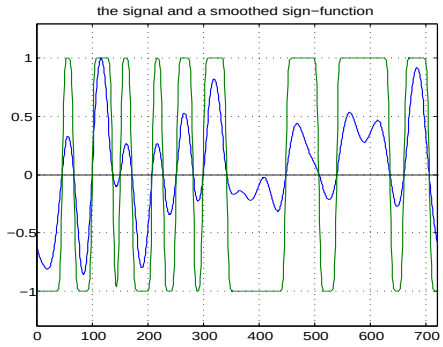
The converse is a bit more involved. In principle one would choose, for the case of a real-valued function k a function $f \in \boldsymbol{C}_0(G)$ which is a minimal (but continuous!) modification of the signum function, which turns (when integrated against k) the negative parts into positive parts, thus turning $\mu_k(f)$ in a good approximation of $\|k\|_1$.

[29]The integration is with respect to the Haar measure on the group G.

To be more formal let us consider $f \in \boldsymbol{C}_c(G)$, let us consider for any $\eta > 0$ the "essential" support $K_\eta := \{z \in G \,|\, |k(z)| \geq \eta\}$. Then K_η is a compact set and we can find continuous function [30] h_η with values in $[0,1]$ such that $h_\eta(z) = 1$ on K_η and with support of h_η within (the interior) of $K_{\eta/2}$. The function $f_\eta(x) := h_\eta(x)|k(x)|/k(x)$ is then well defined (because $k(x) \neq 0$ for any point in the support of h_η, and $\|f_\eta\|_\infty \leq 1$). We observe that

$$\mu_k(f) = \int_G f_\eta(x)k(x)dx = \int_G |k(x)|h_\eta(x)dx.$$

It remains to verify that this tends to $\|k\|_1$ for $\eta \to 0$.



Writing K_0 for $\mathrm{supp}(k)$ this is a direct consequence of the following estimate

$$\int_G |k(x)(1-h_\eta(x))| \leq \int_{K_0} |k(x)|(1-h_\eta(x))dx \leq Vol(K_0)^{31} \cdot \|k(1-h_\eta)\|_\infty \to 0.$$

[30] The existence of h_η is guaranteed by Tietze's theorem, on of the important theorems concerning locally compact, hence completely regular topological spaces. It helps to avoid the potential problem of a phase discontinuity, i.e. problems with the continuity of $k(x)/|k(x)|$ near the zeros of k.

[31] $Vol(K_0)$ stands for the Haar measure of the set K_0, but the $\|\cdot\|_1$ of a plateau-function $p(x)$ with p with $p(x) \cdot k(x) = k(x)$ would do. In fact, the "measure of K_0, i.e. $Vol(K_0) = \mu(K_0)$ can be shown to be equal to the infimum over all those $\|\cdot\|_1$-norms.

□

Notation: $\boldsymbol{M}_{cs}$ = continuout shift.

Li-Def **Definition 30.** We define $\boldsymbol{L}^1(\mathbb{R}^d)$ as the closure of $\boldsymbol{M}_{cs}$ within $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

Following the Riemann-Lebesgue Lemma we derive:

Theorem 13. *Let $f, g \in \boldsymbol{L}^1(\mathbb{R}^d)$ be given. Then*

d-Fourier0 (59)
$$\int_{\mathbb{R}^d} f(t)\hat{g}(t)dt = \int_{\mathbb{R}^d} \hat{f}(x)g(x)dx$$

Proof. The Fourier transforms $\hat{f}$ and $\hat{g}$ are bounded and continuous, hence both integrands are in $\boldsymbol{L}^1(\mathbb{R}^d)$. The relation (59) then follows via Fubini's theorem, since $e^{2\pi itx}f(t)g(x) \in \boldsymbol{L}^1(\mathbb{R}^{2d})$:

fund-Fourier0

(60)
$$\int_{\mathbb{R}^d} f(t)\hat{g}(t)dt = \int_{\mathbb{R}^d} f(t)\left(\int_{\mathbb{R}^d} e^{-2\pi ixt}g(x)dx\right) = \int_{\mathbb{R}^d} g(x)\left(\int_{\mathbb{R}^d} e^{-2\pi ixt}f(t)dt\right)dx = \int_{\mathbb{R}^d} f(x)\hat{g}(x)dx.$$

□

Of course one has to justify this definition, by recalling that the usual definition of $\boldsymbol{L}^1(\mathbb{R}^d)$ based on Lebesgue's integrability criterion provides us with a Banach space (of equivalence classes of measurable functions, identifying two functions if they are equal almost everywhere), which contains $\boldsymbol{C}_c(\mathbb{R}^d)$ as a dense subspace (cf. more or less any book on measure theory for details on this matter: in fact, it is sufficient to approximate - in the $\boldsymbol{L}^1$-norm - indicator functions of parallel-epipeds by continuous functions with compact support, i.e. something like a trapezoidal function sufficiently close to a "box-car"-function in the 1-dimensional case).

It is also of interest to introduce **the concept of a support to measures**, in a way which is compatible with the notion $\mathrm{supp}(k)$ for $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ given above:

as-support **Definition 31.** A point x does *not* belongs to the support of a measure $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ if there exists some $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $k(x) = 1$, but nevertheless $k \cdot \mu \neq 0$. The complement of this set is denoted by $\mathrm{supp}(\mu)$.

Lemma 30. .

- $\mathrm{supp}(\mu)$ *is a closed subset of* $\mathbb{R}^d$,
- *there is consistency with the concept already defined for* $k \in \boldsymbol{C}_c(\mathbb{R}^d)$, *in other words:* $\mathrm{supp}(k)$ *(in the old sense) coincides with* $\mathrm{supp}(\mu_k)$, *just defined.*
- *For a discrete measure the support is given by the closure of the union of all points involved, i.e., for* $\mu = \sum_{k=1}^{\infty} c_k\delta_{t_k}$ *we have*[32] $\mathrm{supp}(\mu) = (\bigcup_k t_k)^-$.
- *The notion of support is compatible with pointwise products: i.e., for any* $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ *on has* $\mathrm{supp}(h\mu) \subseteq \mathrm{supp}(h) \cap \mathrm{supp}(\mu)$ *(as for functions).*
- *consequently every* $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ *can be approximated by compactly supported measures of the form* $p \cdot \mu$, *with* $\mathrm{supp}(p \cdot \mu) \subset \mathrm{supp}(\mu)$.

[32]Assuming of course the canonical representation of μ, with $t_k \neq t_l$, for $k \neq l$ and $c_k \neq 0$.

We have the following equivalent description of the $\mathrm{supp}(\mu)$ (which is actually the usual definition):

as-charact **Lemma 31.** *The following properties are equivalent:*

- $z \in \mathrm{supp}(\mu)$*;*
- *for any* $\varepsilon > 0$ *there exists some* $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ *with* $\mathrm{supp}(h) \subseteq B_\epsilon(z)$ *with* $\mu(h) \neq 0$*.*
- $\mathrm{supp}(\mu)$ *coincides with the intersection of all supports of [plateau-]functions* p *such that* $p\mu = \mu$*.*

The following results indicates the w^*-continuity of the concept of a support.

Lemma 32. *Assume that* $\mu_0 = w^* - lim_\alpha \mu_\alpha$*, then* $\mathrm{supp}(\mu_0) \subseteq \bigcap_\alpha \mathrm{supp}(\mu_\alpha)$

The notion of support is also compatible with convolution: [33]

nv-support **Lemma 33.** *For* $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ *and* $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ *one has*

conv-supp
$$\mathrm{supp}(\mu * f) \subseteq \mathrm{supp}(\mu) + \mathrm{supp}(f). \tag{61}$$

Lemma 34. *Assume that* $\mu_0 = lim_{w^*} \mu_\alpha$*. Then also for any BUPU* Ψ *the family* $D_\Phi \mu_\alpha$ *is* w^**-convergent to* $D_\Phi \mu_0$*. Even more, the family* $D_\Phi \mu_\alpha$ *is uniformly tight and* w^**-convergent to* μ_0 *as* $|\Phi| \to 0$*.*

Finally we claim that the family $D_\Phi(p_K \mu_\alpha)$*, where* p_K *runs through the family of all plateau functions (with* $K \to \mathbb{R}^d$*), satisfies the same relation. Note that the resulting measures are in fact FINITE discrete measures.*[34]

It is an important and not completely trivial claim that a measure supported within the zero-set of a continuous function the action is zero. More precisely:

pinzeroset **Lemma 35.** $\mathrm{supp}(\mu) \subseteq \{x \,|\, h(x) = 0\}$ *implies* $\mu(h) = 0$*.*

Proof. It is enough to verify that any function $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ can be approximated in the sup-norm (not so in other norms, like the $\mathcal{F}\boldsymbol{L}^1$-norm!) by other functions which vanish near the zero-set of h. This can be achieved by multiplying h with a plateau-type function which vanishes on $\{x \,|\, |h(x)| \leq \varepsilon/3\}$ and has a plateau on the set $\{x \,|\, |h(x)| \geq 2\varepsilon/3\}$. etc. ...

Alternatively, one my first show that it is no loss of generality to assume that h has compact support and thus that $h \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$. In this case one can replace h by $Sp_\Phi h$, for sufficiently fine Φ, and then discard all the elements of the form $h(x_j)\phi_j$ with $|h(x_j)| \leq \varepsilon/3$. The effect is practically the same. □

[33] We probably still have to take care of the notion of support for the case that the measure does not have compact support.

[34] Alternatively one could use only finite subfamilies from the partition of unity, or put p_k on the outside, i.e. write $p_k \cdot D_\Phi \mu_\alpha$. The consequences remain the same for all of these variants!

5. Basic properties of $\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$

We have defined $\boldsymbol{L}^1(\mathbb{R}^d)$ as the closure of $\boldsymbol{C}_c(\mathbb{R}^d)$ (identified via $k \mapsto \mu_k$ with a subspace of $\boldsymbol{M}(\mathbb{R}^d)$) in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Since this is a Banach space, it is a Banach space itself, and identical with the (abstract) completion of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\boldsymbol{M}(\mathbb{R}^d)$.

The next theorem gives us some more information about the containment of $\boldsymbol{L}^1(\mathbb{R}^d)$ in $\boldsymbol{M}(\mathbb{R}^d)$:

L1-Basic **Theorem 14.** *(Basic properties of $\boldsymbol{L}^1(\mathbb{R}^d)$)*

- *$\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$ is a closed ideal within $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. It is a Banach algebra with a BAI (so-called Dirac sequences).*
- *$\boldsymbol{L}^1(\mathbb{R}^d)$ can be characterized as the closed subspace of with continuous shift, i.e. a bounded measure μ is of the form $\mu = \mu_g$, resp. $\mu(f) = \int_{\mathbb{R}^d} f(x)g(x)dx$ if and only if $\|T_x\mu - \mu\|_{\boldsymbol{M}} \to 0$ for $x \to 0$.*
- *$\boldsymbol{L}^1(\mathbb{R}^d)$ is w^*-dense in $\boldsymbol{M}(\mathbb{R}^d)$, in fact, for every $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ there exists a tight (hence bounded) sequence (f_k) in $\boldsymbol{L}^1(\mathbb{R}^d)$, with f_k resp. $\mu_{f_k} \to \mu$ in the w^*-topology.*

Proof. The main arguments are the identification of the "internal convolution" within $\boldsymbol{M}(\mathbb{R}^d)$ with the usual convolution formula

conf-CcRd
$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy, \quad \text{for} \quad f, g \in \boldsymbol{C}_c(\mathbb{R}^d). \tag{62}$$

but also the external action of $\boldsymbol{M}(\mathbb{R}^d)$ on the homogeneous Banach space

$$\boldsymbol{M}(\mathbb{R}^d)_e = \{\mu \,|\, \|T_x\mu - \mu\|_{\boldsymbol{M}} \to 0 \text{ for } x \to 0.\}$$

The typical bounded approximate units are of the form $(St_\rho g)_{\rho>0}$, for an arbitrary $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\hat{g}(0) = \int_{\mathbb{R}^d} g(t)dt = 1$.

It is easy to verify that this net is tight and tends to δ_0 in the w^*-topology. In a similar way one can approximate a finite and discrete measure by a linear combination of such Dirac sequences. Since the Dirac measures form a total subset in $\boldsymbol{M}(\mathbb{R}^d)$ with respect to the w^*-topology the w^*-density of $\boldsymbol{L}^1(\mathbb{R}^d)$ in $\boldsymbol{M}(\mathbb{R}^d)$ is established. □

Lesson of May 29th: Bounded measures operate not only on $\boldsymbol{C}_0(\mathbb{R}^d)$ but also on any homogeneous Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. The proof of this fact is essentially based on the idea that it is enough to establish this fact for bounded discrete measures (which is easy) and then show that for any sequence of discretizations of a given measure (where the diameter of the support of the corresponding BUPUs shrinks to zero) generates a sequence μ_k which is uniformly tight and bounded, but also produces a Cauchy sequence in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ of the form $(\mu_k * f)$, for any given $f \in \boldsymbol{B}$. Obviously it makes sense to define the limit (which does not depend on the choice of discretizations via BUPUs) by $\mu * f$ (although it is formally a new operation, and the "star" just introduced should be distinguished for a little while from the "known" star which denotes convolution within $\boldsymbol{M}(\mathbb{R}^d)$): $\mu * f \in \boldsymbol{B}$ and NORMS

Dirac-Conc **Lemma 36.** *A bounded net of functions $(h_\alpha)_{\alpha\in I}$ is a BAI for $\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$ if the following property is satisfied: For every $\varepsilon > 0$ there exists some α_0 such that for $\alpha \succ \alpha_0$*

one has:

Dirac-eps

$$|\int_{B_\varepsilon(0)} h_\alpha(t)dt - 1| \leq \varepsilon \quad \textit{and} \quad \int_{|x|\geq\varepsilon} |h_\alpha(t)|dt \leq \varepsilon. \tag{63}$$

Proof. Argument: Due to the density of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\boldsymbol{L}^1(\mathbb{R}^d)$ one can reduce the discussion to functions $k \in \boldsymbol{C}_c(\mathbb{R}^d)$, i.e., it is enough to show that $h_\alpha * k \mapsto k$ for any $k \in \boldsymbol{C}_c(\mathbb{R}^d)$. The second condition allows to restrict the attention to a net with common compact support K. Consequently one has $h_\alpha * k(x) \neq 0$ only for $x \in K + \mathrm{supp}(k)$. Furthermore we obtain $h_\alpha * k(x) = \int_{\mathbb{R}^d} h_\alpha(y)k(x-y)dy \mapsto$ MORE TO BE DONE TOMORROW!! □

Remark 16. Of course one can also consider $\boldsymbol{M}(\mathbb{R}^d)$ as a Banach module over the Banach algebra $\boldsymbol{L}^1(\mathbb{R}^d)$ (with respect to convolution). Then the $\boldsymbol{L}^1(\mathbb{R}^d)$-essential part of $\boldsymbol{M}(\mathbb{R}^d)$ is equal to $\boldsymbol{L}^1(\mathbb{R}^d)$ itself.

On the other hand we can consider $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ as a Banach module over $\boldsymbol{L}^1(\mathbb{R}^d)$ (again with respect to convolution), and then this is an essential Banach module.

6. Tight subsets

A given $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ is of course "essentially concentrated" on a compact set (and uniformly small outside a sufficiently large compact set, by definition). We also have shown that a functional $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ is having most of its "mass" sitting within a compact set, while its action outside of this compact set is small. Indeed, since for any BUPU Φ we have $\mu = \sum_{iinI} \phi\mu$ as absolutely convergent sum the tails μ minus a large partial sum is small in the $\boldsymbol{M}(\mathbb{R}^d)$-sense.

Next we want to extend this "concentration over compact sets" concept to general bounded subsets of $\boldsymbol{M}(\mathbb{R}^d)$ (and later other functional spaces):

ht-measdef

Definition 32. A bounded subset $H \subset \boldsymbol{M}(\mathbb{R}^d)$ is called (uniformly) tight if for every $\varepsilon > 0$ there exists $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|\mu - k \cdot \mu\|_{\boldsymbol{M}} < \varepsilon$ for al $\mu \in H$.

In a similar way we define tightness in $\boldsymbol{C}_0(\mathbb{R}^d)$:

Definition 33. A bounded subset $H \subset \boldsymbol{C}_0(\mathbb{R}^d)$ is called (uniformly) tight if for every $\varepsilon > 0$ there exists $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|h - k \cdot h\|_\infty < \varepsilon$ for al $h \in H$.

Note: For a general $\boldsymbol{C}_0(\mathbb{R}^d)$ module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ one can define tightness as follows:

Definition 34. A bounded subset $H \subset (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called (uniformly) tight if for every $\varepsilon > 0$ there exists $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|h - k \cdot h\|_\infty < \varepsilon$ for al $h \in H$.

The concept of *tightness* plays a big role in the characterization of *relatively compact subsets* (hence compact operators)

Theorem 15. *Assume that W is a tight set within $\boldsymbol{M}(\mathbb{R}^d)$ and that H is a tight subset within $\boldsymbol{C}_0(\mathbb{R}^d)$. Then $W * H = \{\mu * h \,|\, \mu \in W, h \in H\}$ is a tight subset in $\boldsymbol{C}_0(\mathbb{R}^d)$.*

cf. the "compactness paper" p.307 (bottom):
`http://univie.ac.at/nuhag-php/bibtex/open_files/fe84_compdist.pdf`

Indeed, for any plateau-function τ which satisfies $\tau(x) \equiv 1$ on $\mathrm{supp}(k^1) + \mathrm{supp}(k^2)$, hence the following estimate holds:

$$(1-\tau)(f^1 * f^2) = (1-\tau)(f^1 * f^2 - f^1k^1 * f^2k^2)$$

$$(1-\tau)(\mu * f) = (1-\tau)(\mu * f - \mu k^1 * fk^2)$$

Applying norms to both sides and using the triangle inequality we obtain the following estimate in the sup-norm:

$$\|(1-\tau)(\mu * f)\| = \|(1-\tau)(\mu * f - \mu k^1 * fk^2)\| \leq$$

$$\leq \|(1-\tau)\|\|(1-k^1)\mu * f\| + \|k^1\mu * (1-k^2)f\| \leq$$

$$\leq \|(1-\tau)\|\|(1-k^1)\mu\|_{\boldsymbol{M}}\|f\| + \|k^1\|\|\mu\|_{\boldsymbol{M}}\|(1-k^2)f\|.$$

7. The Fourier transform for $\boldsymbol{L}^1(\mathbb{R}^d)$

The Fourier transform maps $\boldsymbol{M}(\mathbb{R}^d)$ into $\boldsymbol{C}_{ub}(\mathbb{R}^d)$. It will be seen as a non-expansive Banach algebra homomorphism from the closed ideal $\boldsymbol{L}^1(\mathbb{R}^d)$ into the closed ideal $\boldsymbol{C}_0(\mathbb{R}^d)$ of $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ (this result is usually known as Riemann-Lebesgue Lemma).

The range of the Fourier transform is a dense subalgebra (closed under complex conjugation), due to the "locally compact version" of the Stone-Weierstrass theorem.

Recall the standard version of the Stone-Weierstrass theorem:

ierstr-thm **Theorem 16.** *Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra within $\boldsymbol{C}(X)$, where X is some compact topological space. Then $\boldsymbol{A}$ is dense with respect to the uniform norm if $\boldsymbol{A}$ contains the constant functions, is closed under conjugation, and separates points, i.e., , if for any pair of points $x_1, x_2 \in X$, with $x_1 \neq x_2$ there exists some $f \in \boldsymbol{A}$ such that $f(x_1) \neq f(x_2)$.*

Since $\mathcal{F}\boldsymbol{L}^1$ does not have a unit (for pointwise multiplication), due to the fact that $\boldsymbol{L}^1(\mathbb{R}^d)$ does not contain a unit (the unit with respect to convolution is the Dirac measure δ_0, which do *not* cannot be approximated in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ from within $\boldsymbol{C}_c(\mathbb{R}^d)$), one has to modify the above result to the locally compact case, by "adding" the constant functions, and replacing $\mathbb{R}^d$ by its Alexandroff (one-point) compactification X of $\mathbb{R}^d$. Indeed, $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ can be identified with a closed subalgebra of all continuous functions vanishing "at infinity". In fact, if $C_0 + h$ in $\boldsymbol{C}(X)$ is approximated by a sequence of the form $C_n + f_n$, then $|C_n - C_0| \to 0$ for $n \to \infty$, so $\|f_n - h\|_\infty$ for $n \to \infty$ (details left to the reader).

L1-algprop **Lemma 37.** *$\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ is a Banach algebra with respect to the standard norm $\|\hat{f}\|_{\mathcal{F}\boldsymbol{L}^1} := \|f\|_1$ for $f \in \boldsymbol{L}^1(\mathbb{R}^d)$, which is closed under translation, modulations and dilations (in fact, with continuous dependence on the shift resp. dilation parameters). It is also closed under complex conjugation.*

Proof. All these properties follow from the algebraic properties of the Fourier transform (intertwining of modulation and translation operators), and interwining of St_ρ and D_ρ operators, as well as $f \mapsto f^*$ with $f^*(x) = conj(f(-x))$, which intertwines with ordinary conjugation of $\hat{f}$. [more concrete formulas?] □

For $\mathbb{R}^d$ one can of course use alternative arguments, e.g. by observing that certain functions, such as the Schwartz functions, belong to the Fourier algebra, so there is enough richness in the function space $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ in order to show the density of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, but the above argument applies the general LCA groups.

FliR-dens1 **Lemma 38.** *For every $k \in \boldsymbol{C}_c(\mathbb{R})$ and Ψ the family of B-splines of any fixed order $s \in \mathbb{N}$ (hence a BUPU of size $s/2$). Then for every $\delta > 0$ the spline-approximation $Sp_{\delta\Psi}k$ is a finite sum of shifted B-spline functions, hence a function having $(s-1)$ continuous derivatives. Also for $s \geq 2$ the B-splines are continuous and belong to $\mathcal{F}\boldsymbol{L}^1(\mathbb{R})$, because their Fourier transforms are of the form $SINC^s$. In particular, $\boldsymbol{C}^m$ as well as $\mathcal{F}\boldsymbol{L}^1$ are dense subspaces of $\boldsymbol{L}^1(\mathbb{R})$ (and other $\boldsymbol{L}^p$-spaces, for $p < \infty$).*

On of the central statements concerning the Fourier transform is *Plancherel's theorem*, stating that the Fourier transform can be considered as a unitary linear automorphism of the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ onto itself. This is in complete analogy to the statement that the for the case of $\mathbb{C}^n$ the Discrete Fourier transform (often realized in the form of the FFT) is a change of base from the orthonormal basis of unit vectors to the orthogonal system of pure frequencies. Since the vectors representing the pure frequencies (which are exactly the joint eigenvalues to all the translation operators) are of absolute value one, they are all of norm $\sqrt{n}$ the inverse FFT is essentially the conjugate (transpose) of the Fourier matrix, with a compensating factor of the form $1/n$. The advantage of our normalization in the continuous case (with the factor 2π as part of the exponent) has the advantage that the inverse Fourier transform will come in the form

nv-Fourdef (64) $$h(t) = \int_{\mathbb{R}^d} \hat{f}(s) e^{2\pi i t\cdot s} dt$$

Since the integral definition of the FT of its inverse do not apply to general functions $\hat{f} \in \mathcal{F}\boldsymbol{L}^1$, part of the discussion of the **Fourier-Plancherel Theorem** is concerned with technical questions around problems of this kind (how to overcome lack of integrability, e.g., by applying so-called *summability* methods, which are a generalization of the idea of an infinite integral, taken as limit of finite integrals).

Lemma 39. *It is enough to verify that for some dense subspace $\boldsymbol{B}$ of $\boldsymbol{L}^2(\mathbb{R}^d)$ within $\boldsymbol{L}^1(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{C}_0(\mathbb{R}^d)$ one can find that the mapping $f \mapsto \hat{f}$ is well defined and isometric, and with dense range, in order to be able to extend the "classical" Fourier transform and its inverse (given by the integral) to an isometry from $\boldsymbol{L}^2(\mathbb{R}^d)$ onto itself.*

Proof. Since we assume that $\boldsymbol{B} \subseteq \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ one can claim that the direct and the inverse Fourier transform given via (absolutely convergent) Riemannian integrals is valid. For the rest one only has to show that for an arbitrary $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ and any sequence f_n, with $f_n \in \boldsymbol{B}$ with $\|f - f_n\|_2 \to 0$ for $n \to \infty$ one finds that $\|\hat{f} - \hat{f}_n\|_2 = \|f - f_n\|_2 \to 0$, so by the completeness of $\boldsymbol{L}^2(\mathbb{R}^d)$ the Cauchy sequence $\hat{f}_n$ must have a limit, which may be denoted (by a so-called abuse of language) as the Fourier

transform $\hat{f}$ of f. The extended (still isometric) mapping has dense range according to our assumptions, and therefore the same argument can be applied to the inverse Fourier transform in order to realize that the extended mapping (often called the *Fourier-Plancherel* or just *Plancherel* transform defines an isometric automorphism of $\boldsymbol{L}^2(\mathbb{R}^d)$. Due to the polarization identity

$$\langle f, g\rangle = \sum_{k=0}^{3} i^k \|f + i^k g\|^2.$$

such a mapping also preserves scalar products in general. □

For the proof of Plancherel's theorem one may use $\boldsymbol{B} = \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ or the linear span of all the time-frequency shifted and dilated version of a Gauss function (we do not require any norm on $\boldsymbol{B}$). Ideally one can or should use a space of functions which is invariant under the Fourier transform. Details have been given in the FA (= functional analysis) course WS0506 by HGFei.

Of course the extended Fourier transform is still compatible with convolution resp. pointwise multiplication. In other words, convolution on one side of the FT goes into pointwise product on the other side (and vice versa). As a consequence one obtains a characterization of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$: A function belongs to $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ if and only if it can be written as a convolution product of two functions in $\boldsymbol{L}^2(\mathbb{R}^d)$, i.e. $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ if and only if there exist two functions $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$ such that $h = f * g$. The direct direction (i.e. convolution products are in $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$) is easy, because their Fourier transforms give a function which is a pointwise product of two $\boldsymbol{L}^2$-functions, and hence according to the Cauchy-Schwartz inequality $\hat{h} = \hat{f} \cdot \hat{g} \in \boldsymbol{L}^2 \cdot \boldsymbol{L}^2 \subseteq \boldsymbol{L}^1$, or equivalently $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.

The easiest argument for the converse is again on the Fourier transform side. Write $\hat{h} \in \boldsymbol{L}^1(\mathbb{R}^d)$ as a pointwise product of two $\boldsymbol{L}^2$-functions. If $\hat{h}$ was non-negative there is a natural solution to this problem, just take $\sqrt{\hat{h}}$. If $\hat{h}$ is a complex-valued function, one can apply this trick only to $|\hat{h}|$, and can assign the phase factor to one of the two non-negative square roots (details are left to the reader).

May 2010: We are deriving (a weak form) of the Fourier inversion theorem from Proposition 50.

meas↑meas

er_invers2

Theorem 17. *Assume $f \in \boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d)$ with (the additional assumption that) $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$* [35] *then for every $t \in \mathbb{R}^d$ one has:*

$$f(t) = \int_{\mathbb{R}^d} e^{2i\pi st} \hat{f}(s) ds.$$

Proof. Recall that we have the consistency of the Fourier transform for $\boldsymbol{L}^1(\mathbb{R}^d)$-functions f with the corresponding measure μ_f, since

$$\hat{f}(s) = \int_{\mathbb{R}^d} f(t)\overline{\chi_s(t)} dt$$

[35] Because $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{C}_0(\mathbb{R}^d)$ we could express this a bit more symmetric by assuming that $f \in \boldsymbol{L}^1 \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$: Note that this is a Fourier invariant Banach space, and even a Banach algebra, both with respect to pointwise multiplication and convolution!

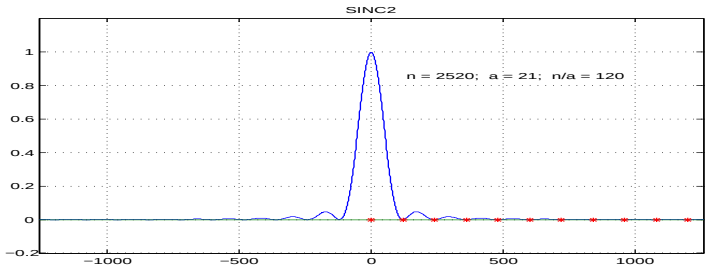
In order to retrieve the inversion formula we just have to use (50) with the following choices:

□

FTmeasinj **Lemma 40.** *$\hat{\mu} = 0$ implies $\mu = 0$.*

Proof. If $\hat{\mu} = 0$, we find from (50) that $\mu(\hat{\mu}) = 0$ for every $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$, in particular $\mu(\hat{f}) = 0$ for all $f \in \boldsymbol{L}^1(\mathbb{R}^d)$. Since $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is a dense subspace of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ this implies that $\mu = 0$. □

Arguments why $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is dense in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$: either using Stone-Weierstrass, or by approximation of any $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ by piecewise linear functions, and the observation that these functions are sums of triangular functions, which in turn belong to the *Fourier algebra* because they correspond to convolution squares of rectangular functions and hence their Fourier transforms are (up to dilation) just squared SINC-functions, i.e. of the form $\hat{\Delta}(s) = [sin(\pi s)/(\pi s)]^2$. **[Rd-specific!]**



8. Wiener's algebra $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$

Because it can be defined without the existence of a Haar measure the following space plays an important role within Harmonic Analysis. We define $W(G)$ as follows;

Then we define $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ as follows:

WCdefphi **Definition 35.** Let φ be any non-zero, non-negative function on $\mathbb{R}^d$.

WCdefphi1 (65) $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1) = \{\, f \in \boldsymbol{C}_0(\mathbb{R}^d) \,|\, \exists (c_k)_{k\in\mathbb{N}} \in \boldsymbol{\ell}^1, (x_k)_{k\in\mathbb{N}} \text{ in } \mathbb{R}^d, |f(x)| \le \sum_{k\in\mathbb{N}} c_k \varphi(x - x_k) \,\}$

We define

$$\|f \,|\, \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)\| := \inf \{\|\mathbf{c}\|_{\boldsymbol{\ell}^1} = \sum_{k\in\mathbb{N}} |c_k| \,\}$$

where the infimum is taken over all "admissible dominations" of f as in (65). WCdefphi1

It is obvious that $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ is continuously embedded into $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ [36] since $\|f\|_\infty \le (\sum_{k\in\mathbb{N}} |c_k|)\|\varphi\|_\infty$. By a similar argument we have a continuous embedding of $\big(\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d),\, \|\cdot\|_{\boldsymbol{W}}\big)$ into $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$ or in any other $\boldsymbol{L}^p$-space.

A convenient characterization of Wiener's algebra over $\mathbb{R}^d$ can be given using (arbitrary, non-negative, regular) BUPUs:

ener-BUPU2 **Lemma 41.** *Assume that Ψ is a regular BUPU. Then a continuous function belongs to $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ if and only if the decomposition $f = \sum_{\lambda\in\Lambda} \psi_\lambda \cdot f$ is absolutely convergent, i.e. if and only if $\|f\|_{\boldsymbol{W},\Psi} = \sum_{\lambda\in\Lambda} \|\psi_\lambda \cdot f\|_\infty < \infty$. Moreover, every such norm $\|f\|_{\boldsymbol{W},\Psi}$ is equivalent to the one given above (in the definition).*

NEW (May 2010)

iener-Conv **Lemma 42.** *$\big(\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d),\, \|\cdot\|_{\boldsymbol{W}}\big)$ is a Banach algebra with respect to convolution. In fact it is a Banach ideal (and a so-called Segal algebra) within $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$.*

Proof. Since every element in $\big(\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d),\, \|\cdot\|_{\boldsymbol{W}}\big)$ is an absolutely convergent sum of elements which are (up to translation) all bounded and have common compact support (within $\mathrm{supp}(\psi)$) is is enough split a convolution product of $g \in \boldsymbol{L}^1(\mathbb{R}^d) \subseteq \boldsymbol{M}_b(\mathbb{R}^d)$ with $f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ into little blocks and estimate

as-WRdconv (66) $$\|\mu\psi_k * f\psi_n\|_\infty \le \|\mu\psi_k\|_{\boldsymbol{\mathcal{M}}} \cdot \|f\psi_n\|_\infty$$

and observe, that each such convolution product is split into a finite (controlled number of terms). PLOT for demonstration. □

Similar statements can be made for other function spaces, in particular for the pointwise algebra $(\boldsymbol{A},\, \|\cdot\|_{\boldsymbol{A}}) = \big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$.

This is a special case of a Segal algebra (see next section).

For us the case $(\boldsymbol{A},\, \|\cdot\|_{\boldsymbol{A}}) = \big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$ will be of upmost relevance. It is called $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ (the zero in the subscript stands for the *minimality* of this space), also known in the literature meanwhile ([36]) as *Feichtinger's algebra*, introduced around 1979 (see [14] for the official paper on the subject).

Within the family of Wiener amalgam spaces (defined via BUPUs) we can give the following definition:

SO-WFLili **Definition 36.**

$$\boldsymbol{S}_0(\mathbb{R}^d) := \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^1)(\mathbb{R}^d).$$

One of the first and most important properties is the invariance of the space under the Fourier transform!

[36] by the same argument $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ is also contained in many other Banach spaces of functions with the property that translations are isometric and that the space is solid.

SO-FOURinv **Theorem 18.**

$$\mathcal{F}[\boldsymbol{S}_0(\mathbb{R}^d)] = \boldsymbol{S}_0(\mathbb{R}^d),$$

with equivalence of norms[37].

Moreover, both the time-shift operators and the modulation operators act uniformly bounded on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$*, and furthermore* $\|T_x f - f\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \to 0$ *for* $x \to 0$ *as well as* $\|M_s f - f\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \to 0$ *for* $s \to 0$*, for each* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$*.*

As a consequence one has $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d)$ *and* $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d)$

Proof. The uniform boundedness of both time and frequency shift operators can be considered as an exercise. Since obviously the compactly supported (partial sums) elements are dense in $\boldsymbol{S}_0(\mathbb{R}^d)$. But over compact sets the $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$-norm and the $\boldsymbol{S}_0(\mathbb{R}^d)$-norms are equivalent, and hence to continuous dependencies claimed above are valid.

As a further consequence by (vector-valued) integration we get $\boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d)$, and hence in particular $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)$, with $\|g * f\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \leq \|g\|_1 \|f\|_{\boldsymbol{S}_0}$, for all $g \in \boldsymbol{L}^1, f \in \boldsymbol{S}_0$.

Since by the definition the decomposition $f = \sum_{\lambda\in\Lambda} \psi_\lambda \cdot f$ is an absolutely convergent series in $\big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$ it follows that $\hat{f} = \sum_{\lambda\in\Lambda} \widehat{\psi_\lambda \cdot f}$ is absolutely convergent in $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$. But we want more. The series should also be convergent in the (much smaller) Banach space $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^1)(\mathbb{R}^d)$. In order to do this we take *any* function in $\boldsymbol{S}_0(\mathbb{R}^d)$ (e.g. $SINC^2$) such that $\hat{g}$ belongs to $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ and has compact support (e.g. the Vallee DePoussin kernel), and such that $\psi \cdot \hat{g} = \psi$, hence

$$T_\lambda(\psi \cdot \hat{g}) = T_\lambda\psi \cdot T_\lambda\hat{g} = T_\lambda\psi \cdot \widehat{M_\lambda g}$$

and consequently we can derive

$$\|\widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{S}_0(\mathbb{R}^d)} = \|T_\lambda\hat{g} \cdot \widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \leq \|M_\lambda g\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \|\widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{L}^1(\mathbb{R}^d)} = \|g\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \|\widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{L}^1(\mathbb{R}^d)}.$$

and consequently

$$\|\hat{f}\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \leq \sum_{\lambda\in\Lambda} \|\widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \leq \|g\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \sum_{\lambda\in\Lambda} \|\widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{L}^1(\mathbb{R}^d)} \leq \|g\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \cdot \|f\|_{\boldsymbol{S}_0(\mathbb{R}^d)}.$$

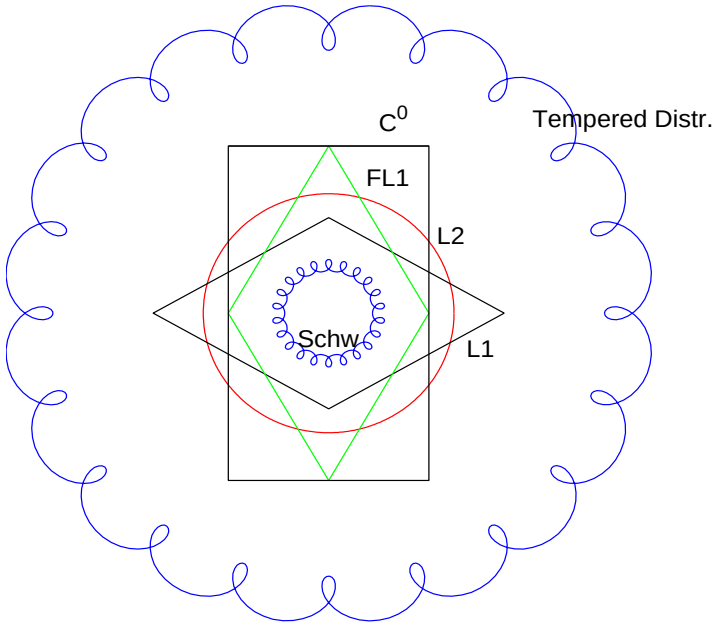
□

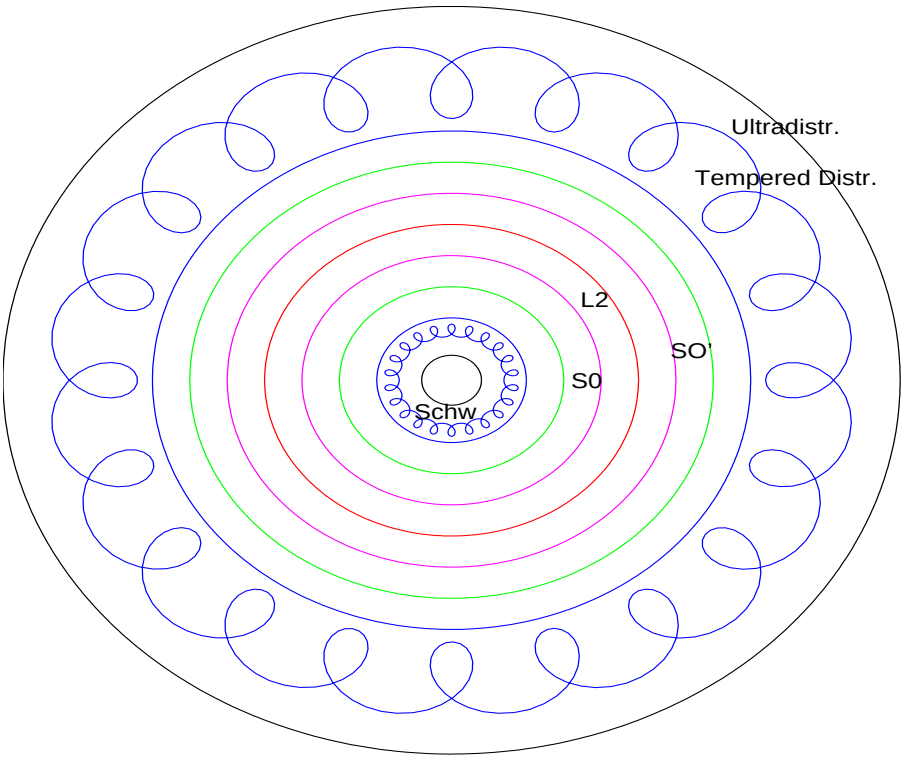
9. The Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ and Banach Gelfand triples

There are different ways of defining $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^1)$ should be renamed `WFLili` `instead of WFlili.`

[37]Later on we will see that with a specific norm one can make the Fourier transform an isometric automorphism on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$.

We have now a Banach space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ which is Fourier invariant, contained (continuously embedded, and densely) in $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ ... Together with the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ we will have a convenient setting for a theory of generalized Fourier transforms.




Ultradistr.
Tempered Distr.
L2
SO'
S0
Schw

10. The inverse STFT: Material thanks to Severin Bannert!

Definition 37. The *Short Time Fourier Transform (STFT)* of a function $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ with respect to a window $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is defined as

$$\begin{aligned} V_g f(x,\omega) &= \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}dt \\ &= \int_{\mathbb{R}^d} f(t)\overline{M_\omega T_x g(t)}dt \\ &= \langle f, M_\omega T_x g\rangle \end{aligned} \tag{67}$$

eq:stft

for $(x,\omega) \in \mathbb{R}^{2d}$.

Theorem 19. *(Orthogonality relations of the STFT) Let $f_1, f_2, g_1, g_2 \in \boldsymbol{L}^2(\mathbb{R}^d)$, then*

$$\langle V_{g_1} f_1, V_{g_2} f_2\rangle = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}. \tag{68}$$

h-rel-stft

Furthermore $V_{g_k} f_k \in \boldsymbol{L}^2(\mathbb{R}^{2d})$ for $k = 1, 2$.

Proof. See [22], p. 42 □

The orthogonality relations immediatly lead to the following corollary.

Corollary 6. *(Moyals Formula) Let $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$, then*

$$\|V_g f\|_{\boldsymbol{L}^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} = \|g\|_{\boldsymbol{L}^2(\mathbb{R}^d)}\|f\|_{\boldsymbol{L}^2(\mathbb{R}^d)} \tag{69}$$

eq:moyals

Remark 17. If in particular $\|g\|_{\boldsymbol{L}^2}\| = 1$ (for example if g is the normed gaussian), then (69) says that the STFT is an isometry, $V_g : \boldsymbol{L}^2(\mathbb{R}^d) \to \boldsymbol{L}^2(\mathbb{R}^{2d})$, and thus injective, i.e each $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is uniquely determined by its STFT. Next we will find a way to reconstruct f from its STFT.

With the previous results we have the right tools at hand to formulate and proof the inversion formula of the STFT.

Theorem 20. *(The Inversion formula of the STFT) Let $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ with $\|g\|_{\boldsymbol{L}^2} = 1$, then for $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ we have*

$$f = \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} V_g f(x,\omega) M_\omega T_x g \, d\omega dx \tag{70}$$

q:inv-stft

Proof. Since $\|g\| = 1$, (68) implies that

eq:orth-rel-stft

$$\langle V_g f_1, V_g f_2\rangle = \langle f_1, f_2\rangle \qquad \forall f_1, f_2 \in \boldsymbol{L}^2(\mathbb{R}^d),$$

which leads to

$$\langle V_g^* V_g f_1, f_2\rangle = \langle f_1, f_2\rangle, \tag{71}$$

Vg*Vgf1-f2

which in turn implies $V_g^* V_g = \mathrm{id}$, where V_g^* denotes the adjoint operator of V_g. What we have to show now is that

$$V_g^* F = \int_{\mathbb{R}^d\times\mathbb{R}^d} F(x,\omega) M_\omega T_x g \, dx d\omega, \tag{72}$$

F-inv-stft

for $F \in \boldsymbol{L}^2(\mathbb{R}^{2d})$. This is done by the following computations:

$$\begin{aligned}\langle V_g f, F\rangle &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_g f(x,\omega)\overline{F(x,\omega)}\,dxd\omega \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(t)\overline{T_x g(t)}e^{-2\pi it\omega}dt\right)dxd\omega \\ &= \int_{\mathbb{R}^d} f(t)\overline{\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x,\omega)M_\omega T_x g(t)\,dxd\omega\right)}dt \\ &= \langle f, V_g^* F\rangle.\end{aligned}$$

Hence, by setting $F = V_g f$ and with the help of (71) the proof is complete. □

Remark 18. We can omit the assumption that $\|g\|_{\boldsymbol{L}^2} = 1$ in the previous theorem. The inversion formula then reads

$$f = \frac{1}{\|g\|_2^2}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} V_g f(x,\omega)M_\omega T_x g\,dxd\omega.$$

About the actual convergence of the integral (describing the adjoint linear mapping, applied to F) one has to say: it is only defined in the "weak sense" (i.e. it is understood in a kind of symbolic sense in general). For $F \in \boldsymbol{L}^1(\mathbb{R}^{2d})$, specifically for $F = V_g(f)$ for some $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ the convergence in the spirit of a Riemannian sum is taking place in the $\boldsymbol{S}_0$-sense, hence also uniformly, in $\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$ as well as pointwise (cf. [43]).

$$f(t) = \frac{1}{\|g\|_2^2}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} V_g f(x,\omega)M_\omega T_x g(t)\,dxd\omega,$$

where in fact it would be enough to use an absolutely convergent (improper) Riemannian integral with respect to $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. For *good windows* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ one even has norm convergence (but to my knowledge not necessarily pointwise convergence) for $f \in \boldsymbol{L}^2(\mathbb{R}^d)$.

Material for course October 2006

Given a Banach module $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$ over a Banach algebra $(\boldsymbol{A},\, \|\cdot\|_{\boldsymbol{A}})$ with bounded approximate units we define the essential part $\boldsymbol{B}_{\boldsymbol{A}}$ and the relative completion of A, which is given as $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A},\boldsymbol{B})$. Note that this is again in a natural way a Banach module (with respect to the operator norm) over the original Banach algebra $(\boldsymbol{A},\, \|\cdot\|_{\boldsymbol{A}})$, and that $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$ can be mapped into this Banach module in a natural way as a closed subspace (at least of $\boldsymbol{B} = \boldsymbol{B}_{\boldsymbol{A}}$ and if $(\boldsymbol{A},\, \|\cdot\|_{\boldsymbol{A}})$ has bd. approx. units).

ARGUMENT: One has to identify each element $b \in \boldsymbol{B}$ with the operator $T_b \in \mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B})$ obtained by something like the "right regular representation", i.e., the operator $T_b : a \mapsto a \bullet b$. It is always true that $\|T_b\|_{op} \le \|b\|_{\boldsymbol{B}}$, and by applying T_b to the elements of some bounded approximate unit in $\boldsymbol{A}$ one finds the converse estimate, i.e., $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$ can be identified with a closed subspace of all bounded linear operators from $\boldsymbol{A}$ to $\boldsymbol{B}$.

Note that one should not forget that one has to impose the "natural" $\boldsymbol{A}$-module structure on $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}_1, \boldsymbol{B}_2)$, before making the claim that the $\boldsymbol{A}$-module $\boldsymbol{B}$ can be embedded via an $\boldsymbol{A}$-module homomorphism (embedding) into the larger $\boldsymbol{A}$-module $\boldsymbol{B}^{\boldsymbol{A}}$.

Exercise: For the case of the pointwise algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}}) = \big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ one finds that $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{A}) = \big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ in a "natural way. Note that the "identity operator always belongs to $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{A})$ (obviously it commutes with any other operator), and therefore the "enlargement" from $\boldsymbol{A}$ to $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A})$ also implies the adjunction of a unit element to the Banach algebra $\boldsymbol{A}$, but typically much more than this. So in a way the (in this case isometric) embedding of $\boldsymbol{C}_0(\mathbb{R}^d)$ into $\boldsymbol{C}_b(\mathbb{R}^d)$ (both with the sup-norm) can be seen as an embedding of $\boldsymbol{A} = \boldsymbol{C}_0(\mathbb{R}^d)$ into the maximal algebra "with the same norm" (and a unit).

Plancherel's theorem can be used (we skip those details) to identify $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^2, \boldsymbol{L}^2)$ (or equivalently the set of all bounded linear operators from $\boldsymbol{L}^2(\mathbb{R}^d)$ into $\boldsymbol{L}^2(\mathbb{R}^d)$ which commute with translation) with $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{L}^2, \boldsymbol{L}^2)$ (for $\boldsymbol{A} = \mathcal{F}\boldsymbol{L}^1$, i.e., the operators from $\boldsymbol{L}^2(\mathbb{R}^d)$ into $\boldsymbol{L}^2(\mathbb{R}^d)$ which commute with pointwise multiplications with elements from $\boldsymbol{A} = \mathcal{F}\boldsymbol{L}^1$, or equivalently, just with the multiplication with pure frequencies resp. characters on $\mathbb{R}^d$, which are the functions $x \mapsto e^{2\pi i s\cdot x}$. These are again pointwise multiplication operators, and it is not hard to find out that a pointwise multiplier h of $\boldsymbol{L}^2(\mathbb{R}^d)$ is has to be a measurable function which is essentially bounded, i.e., $h \in \boldsymbol{L}^\infty(\mathbb{R}^d)$.

Let us just sketch the basic idea behind this fact:

Lemma 43. *Assume that $\boldsymbol{B}$ is a Banach module with respect to a pointwise Banach algebra $\boldsymbol{A}$ (and assume that $\boldsymbol{A}_c(\mathbb{R}^d) = \boldsymbol{C}_c(\mathbb{R}^d) \cap \boldsymbol{A}$ is dense in $\boldsymbol{A}$), and assume that $\boldsymbol{A} \cap \boldsymbol{B}$ contains arbitrary large plateau-functions, i.e., with the property, that for each compact set $K \subseteq \mathbb{R}^d$ there exists some $q \in \boldsymbol{B} \cap \boldsymbol{A}$ such that $q(x) \equiv 1$ on K. Then the elements in $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}, \boldsymbol{B})$ are pointwise multipliers with suitable functions h which belong locally to $\boldsymbol{A}$.*

Proof. TO BE GIVEN LATER ON! □

twmultalg1

Lemma 44. *Assume that an operator S on some function space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ satisfies the property $S(h \cdot f) = h \cdot S(f)$ for a sufficiently rich family of pointwise multipliers h on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Then $S(f) = h_0 \cdot f$ for some function h_0.*

Proof. Note that in the case $\mathscr{G} = \mathbf{Z}_N$ the claim is simply: Assume a linear mapping from $\mathbb{C}^N$ into itself, represented by a matrix, commutes with pointwise multiplication with unit vectors then it must be a diagonal matrix. In fact, such linear mappings do not increase the support of a function. In fact, assume that some coordinates of f are zero, then f does not change of it is multiplied with the sequence h taking the value one on those specific coordinates. On the other hand after pulling out of the argument one finds that also $h \cdot S(f) = S(h\dot{f}) = S(f)$ must be zero at the same coordinated, which easily implies that $N \times N$ matrix representing the linear mapping $f \mapsto S(f)$ must be a diagonal matrix, resp. S must be a multiplication operator (with $h_0 = \mathrm{diag}(S)$).

Let us now do the continuous version of the argument. For simplicity we assume that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ allows pointwise multiplications by the elements of some nice, regular Banach

algebra (such as $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$), and that we can make use of BUPUs, i.e. we have a bounded family (of pointwise multipliers on $(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}})$) such that $\mathbf{1} = \sum_{\lambda\in\Lambda} \psi_\lambda$. Using (as usual now) we have $\psi = \psi\cdot\psi^*$, hence also $f = \sum_{\lambda\in\Lambda} \psi_\lambda \cdot \psi_\lambda^* f$, hence (assuming now that the series is norm convergent in $\boldsymbol{L}^2$ or $\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$), for example, and setting $h_0 := \sum_{\lambda\in\Lambda} S(\psi_\lambda)$. Note that the sequence is pointwise well convergent, since $\psi_\lambda^*\cdot\psi_\lambda = \psi_\lambda$ implies the $\mathrm{supp}(S\psi_\lambda) \subseteq \mathrm{supp}(\psi_\lambda^*)$, but only finitely many of them overlap! On the other hand one has

intw-mult1 (73) $$Sf = S\left(\sum_{\lambda\in\Lambda} \psi_\lambda \cdot f\right) = \sum_{\lambda\in\Lambda} S(f\cdot\psi_\lambda) = f\cdot\sum_{\lambda\in\Lambda} S(\psi).$$

□

For us an important Banach algebra is $\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$, endowed with convolution as multiplication (which is commutative, due to the commutativity of addition in $\mathbb{R}^d$). It does not have any units, but we have shown earlier (?) that $\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$ has bounded approximate units. Typically such a family is obtained by taking any sequence f_n (or net) of functions, e.g. in $\boldsymbol{C}_c(\mathbb{R}^d)$, with $\int_{\mathbb{R}^d} f_n(x)dx = 1$ and "shrinking support". This can be obtained by choosing the shape of of f_n arbitrarily, but assuming that $f_n(x) = 0$ for $|x| \geq \delta_n$ for some null-sequence $\delta_n \to 0$ for $n\to\infty$. Alternatively one compresses a given function $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ or even in $\boldsymbol{L}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} f_n(x)dx = 1$, and chooses $f_n = St_{\rho_n} f$, for some sequence $\rho_n \to 0$ for $n\to\infty$. The choice $f(x) = e^{-\pi x^2}$ is a popular choice (which also shows that it is not important for f to be compactly supported).

We will see shortly that $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1) = \mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1, \boldsymbol{L}^1)$ can be identified with the space $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ of all bounded measures (resp. with $\left(\boldsymbol{C}_0{}'(\mathbb{R}^d),\, \|\cdot\|_{\boldsymbol{M}}\right)$). This result is called "Wendel's theorem" ([29, 44]). It has of course two parts: First of all one has to show that convolution operators induced by elements from $\boldsymbol{M}_b(\mathbb{R}^d)$ leave the closed subspace $\boldsymbol{L}^1(\mathbb{R}^d)$ invariant. In the second part one has to verify that any abstract (bounded and linear) operator on $\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$ commuting with all the translation must be such a convolution operator.

endel-thm1 **Theorem 21.** *The space of $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1, \boldsymbol{L}^1)$ all bounded linear operators on $\boldsymbol{L}^1(G)$ which commute with translations (or equivalently: with convolutions) is naturally and isometrically identified with $(\boldsymbol{M}_b(\mathscr{G}), \|\cdot\|_{\boldsymbol{M}_b})$.*

Proof. For the first part we have to verify that $\boldsymbol{L}^1(\mathbb{R}^d)$ is a closed ideal of $\boldsymbol{M}_b(\mathbb{R}^d)$, i.e., that $\boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{L}^1(\mathbb{R}^d) \subseteq \boldsymbol{L}^1(\mathbb{R}^d)$. One way to do that is to check that $\left(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\right)$ is a *homogeneous Banach space*, i.e., to show that the group $\mathbb{R}^d$ acts in a continuous and isometric way on $\boldsymbol{L}^1(\mathbb{R}^d)$. This means that we have to verify the following conditions: $\|T_x f\|_1 = \|f\|_1$ for all $x\in\mathbb{R}^d$ and all $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ (hence all $f\in\boldsymbol{L}^1(\mathbb{R}^d)$) and also

ntLi-shift (74) $$lim_{x\to 0}\|T_x f - f\|_1 = 0 \quad \text{for} \quad x\to 0,$$

for any $f\in\boldsymbol{C}_c(\mathbb{R}^d)$, hence (by approximation for any $f\in\boldsymbol{L}^1(\mathbb{R}^d)$).

That $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ (viewed as a Banach algebra with respect to convolution) is acting boundedly on *any homogeneous Banach space* will be discussed separately.

Alternatively one can even describe $\boldsymbol{L}^1(\mathbb{R}^d)$ as the subset of all bounded measures which have continuous translation, in other words, one can show (see a classical paper by Plessner, 1929) that $\|T_x\mu - \mu\|_{\boldsymbol{M}_b} \to 0$ for $x \to 0$ implies that μ is an "absolutely continuous" measure, i.e., belongs to $\boldsymbol{L}^1(\mathbb{R}^d)$.

The argument for this result typically relies on a compactness argument (w^*- compactness of the unit ball in the dual Banach space $\boldsymbol{M}_b(\mathbb{R}^d) = \big(\boldsymbol{C}_0{}'(\mathbb{R}^d),\, \|\cdot\|_{\boldsymbol{M}}\big)$). One applies the given operator $T \in \mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1)$ to any *Dirac- sequence* f_n which forms a bounded approximate unit in $\boldsymbol{L}^1(\mathbb{R}^d)$. By the boundedness of (f_n) and the operator T the image sequence $T(f_n)$ is also bounded in $\boldsymbol{L}^1(\mathbb{R}^d)$, hence in the larger (dual) space $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$. By the w^*-compactness of bounded balls in this space we obtain a w^*-convergent subnet, with some limit $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$. It remains to show that this limit is inducing the operator, i.e., one has to verify that $T = C_\mu$. □

Corollary 7. *In the situation described in Wendel's theorem we have: Every multiplier of* $\big(\boldsymbol{L}^1(G),\, \|\cdot\|_1\big)$ *has the property that the restriction to* $\boldsymbol{L}^1 \cap \boldsymbol{C}_0(G)$ *(endowed with the natural norm* $\|f\|_S := \|f\|_1 + \|f\|_\infty$*) is also a multiplier on that space, which in turn is dense both* $\big(\boldsymbol{L}^1(G),\, \|\cdot\|_1\big)$ *as well as* $\big(\boldsymbol{C}_0(G),\, \|\cdot\|_\infty\big)$*. The same is true for multipliers on* $\big(\boldsymbol{C}_0(G),\, \|\cdot\|_\infty\big)$*.*

Also the *converse* is true, at least for $G = \mathbb{R}^d$: every multiplier on the Segal algebra $\boldsymbol{L}^1 \cap \boldsymbol{C}_0(G)$. A proof should be found in Larsen's book [29], Corollary 3.5.2.

Material of Nov. 9-th (! given below) is partially covered by the paper "Banach spaces of distributions having two module structures J. Funct. Anal. (1983)" ([6]). The main result of this paper is a "chemical diagram" that can be attached to each of the spaces in >>> standard situation:

Another Segal algebra is $\boldsymbol{L}^1 \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ with the natural norm $\|f\|_S := \|f\|_1 + \|\hat{f}\|_1$.

Some comments on the classical Riemann-Stieltjes integral (in German)

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http://de.wikipedia.org/wiki/Stieltjes-Integral
http://de.wikipedia.org/wiki/Beschr%C3%A4nkte_Variation
http://de.wikipedia.org/wiki/Absolut_stetig
http://de.wikipedia.org/wiki/Satz_von_Radon-Nikodym
http://de.wikipedia.org/wiki/Lebesgue-Integral
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Some of the material in this course has been already given in courses in Heidelberg (1980), Maryland (1989/90) or at the university of Vienna in the last 20 years.

The material concerning the Segal algebra $\boldsymbol{S}_0(G)$ is going back to various original publications by the author, see for example [14], where this particular Segal algebra has been introduced and where it is shown that it is the *minimal* TF-isometric homogeneous Banach space (and many other properties). The role of the dual space has been described already in [13] (both papers downloadable from the NuHAG site). The *double module view-point* is described in much detail in [6] (Banach spaces of distributions having two module structures, J. Funct. Anal.). A detailed account of notions of *compactness* (and also a clean description of tightness, etc.) is given in [15]. The first atomic characterization of modulation spaces ($\boldsymbol{S}_0(\mathbb{R}^d)$ is among them) has been given at a conference in Edmonton in 1986 (published then in [16]).

There are many places where especially the role of the Segal algebra $\boldsymbol{S}_0(G)$ for the discussion of basic questions in Gabor Analysis has been described. The very first systematic discussion as been probably given in the Chapter by Feichtinger and Zimmermann in the first Gabor book of 1998 ([19]). Another relevant paper is the one by Feichtinger and Kaiblinger ([17]) where it is shown, that (in the $\boldsymbol{S}_0(\mathbb{R}^d)$ context) the dual window is depending continuously on the lattice constants in the case of Gabor frames resp. Gabor Riesz bases.

Preview: In order read about Gabor multipliers the best source is probably the survey article in the second (blue) Gabor book, "Advances in Gabor Analysis", by Feichtinger and Nowak ([18]).

General references are: Hans Reiter's book on Harmonic Analysis (including a very fine and compact introduction to Integration Theory over Locally Compact Groups, but without the proof of the existence of the Haar measure) [33]. An updated version (edited by his former PhD student Ian Stegeman is [36]). The book describes (see also [34]) the concepts of Segal algebras (such as $S_0(G)$), and Beurling algebras $\boldsymbol{L}^1_w(G)$ (with respect to multiplicative weights). Both books are available in the NuHAG library.

A nice introduction into "abstract harmonic analysis" is that of Deitmar ([9]), and of course always Katznelson (starting with classical Fourier series, but also talking about the Gelfand theory for commutative C^*-algebras) is [28]. It also contains the first "propagation" of the concept of *homogeneous Banach spaces*. A similar unifying viewpoint is taken in the book of Butzer and Nessel ([8]) and of course several of the books of Hans Triebel (see BIBTEX collection and book-list). These two mathematicians can also be seen as pioneers of interpolation theory and the so-called theory of function spaces. (Abstract) Homogeneous Banach spaces are also treated in the Lecture Notes by H. S.

Shapiro ([38]), having approximation theoretic questions in mind (he makes the associativity of the action of bounded measures an axiom, obviously because he could proof it only in concrete cases).

Solid Banach spaces of function (under the name of *Banach function spaces*) appear in the work of Zaanen: [45] (or [46]), both books should be available in the NuHAG library (Alserbachstrasse 23, Room 8)

A very good source to learn about Besov spaces is Jaak Peetre's book entitled "New Thoughts on Besov Spaces" ([31]).

Banach modules: Rieffel's work [37] [some tex-nical problem with BIBTEX]

Generally interesting references about "mathematics and signal processing": Richard Holmes: [26]

Protocol for the course: Nov 9th, 2006 HGFei

TOPIC: **Standard Spaces**

What are standard spaces?? Banach spaces of functions or distributions which allow sufficiently many regularization operators, e.g. localization (by pointwise multiplication) and regularization (by convolution).

Definition 38. A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *(restricted) standard space* if

(1) $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (continuous embeddings);
(2) $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$, with $\|h \cdot f\|_{\boldsymbol{B}} \leq \|h\|_{\mathcal{F}\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}$;
(3) $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{B} \subseteq \boldsymbol{B}$ with $\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}$; for $g \in \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B}$;

It is clear that almost all the spaces used "normally" in Fourier analysis are such "standard spaces". It is sufficient that a space (of locally integrable functions or Radon measures) is isometrically invariant under the time-frequency shifts $\pi(\lambda) = M_\omega T_t$ for $\lambda = (t, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and that e.g. the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is contained in $\boldsymbol{B}$ as a dense subspace, to ensure that the above conditions are satisfied. Let us formulate this claim as a lemma:

ndardonRd1 **Lemma 45.** *? DUPLICATE!? Assume that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a Banach space of locally integrable functions on $\mathbb{R}^d$ such that $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is contained in $\boldsymbol{B}$ as a dense subspace and that $\|M_\omega T_t f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ for all $\lambda = (t, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then it is a standard space.*

Which kind of objects do we want: Banach spaces of continuous functions? Banach spaces of locally (Lebesgue-) integrable functions? Banach spaces of Radon measures, or (tempered?) distributions? Should we allow even ultra-distributions? Wishes: We would like to be able to do functional analysis, so with each spaces we would like to have the dual space in the same family (as long as it can be viewed as a Banach space of distributions, hence only if it can be completely characterized by the sum of the local actions).

With each space the Fourier transform of the space should be in the same family, etc. etc.

Formal suggestion in this direction is given by the following definition, which describes at "reasonable" generality a family of Banach spaces which is *not* restricted to Banach

spaces of functions, because such a family will typically *not* be closed under duality (an exception is the family of $\boldsymbol{L}^p$-classes, but already the dual space of $\boldsymbol{C}_0(\mathbb{R}^d)$ contains discrete measures which are not represented by (integrals against measurable) functions.

Definition 39. A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *(restricted) standard space* if

(1) $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (continuous embeddings);
(2) $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$, with $\|h \cdot f\|_{\boldsymbol{B}} \leq \|h\|_{\mathcal{F}\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}$;
(3) $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{B} \subseteq \boldsymbol{B}$ with $\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}$; for $g \in \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B}$;

Remark 19. The main idea behind this specific definition (also the reason why it is called for a while the "restricted standard situation" is the fact that the fact that $\boldsymbol{S}_0(\mathbb{R}^d)$ and its dual $\boldsymbol{S}_0'(\mathbb{R}^d)$, or that the whole Banach algebra $\boldsymbol{L}^1(\mathbb{R}^d)$ is acting on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ via convolution, can be seen as a matter of convenience. In this way we avoid a number of technical conditions involving weights and still have a fairly large collection of examples available. We will be able to demonstrate the roles of pointwise multiplication and convolutive action in the present context, and it will be easy for the reader to generalize the observations to more general situations.

A typical alternative view in the context of $\mathcal{G} = \mathbb{R}^d$ is the following setting:

Definition 40 (convenient description)**.** .
A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *tempered standard space* on $\mathbb{R}^d$ if

(1) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ (continuous embeddings);
(2) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$
(3) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) * \boldsymbol{B} \subseteq \boldsymbol{B}$

Aside from the fact that one needs some functional analytic argument in order to establish the equivalence between this "convenient" and another more technical one (which is however what one needs in order to make those concepts useful). It will be convenient for this purpose to make use of polynomial (submultiplicative) weights w_s, given by $w_x : x \mapsto (1 + |z|^2)^{s/2}$:

Definition 41 (technical definition)**.** A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *tempered standard space* on $\mathbb{R}^d$ if

(1) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ (continuous embeddings);
(2) There $s \geq 0$ such that $\boldsymbol{L}^1_{w_s}(\mathbb{R}^d)$ acts on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ by convolution and $\|g * f\|_{\boldsymbol{B}} \leq C_s \|g\|_{1,w_s} \|f\|_{\boldsymbol{B}}$; for $g \in \boldsymbol{L}^1_{w_s}, f \in \boldsymbol{B}$, for some $C_s > 0$;
(3) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$ and there exists some constant $\|h \cdot f\|_{\boldsymbol{B}} \leq \|\widehat{h}\|_{1,w_r} \|f\|_{\boldsymbol{B}}$;

Remark 20. The typical examples of *reduced standard spaces* arise from Banach spaces of say tempered distributions which are isometrically invariant under TF-shifts, i.e., which satisfy

$$\|\pi(t,\omega) f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}} \quad \forall f \in \boldsymbol{B}.$$

and which contain $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ or $\boldsymbol{S}_0(\mathbb{R}^d)$ as a dense subspace. In fact, in such a case on can argue that the isometric invariance of the space implies that the continuity of the mapping (t, ω) into $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ resp. $\boldsymbol{S}_0(\mathbb{R}^d)$ implies that one can extend the strong continuity to all of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, in other words, one obtains a so-called *time-frequency homogeneous Banach*

space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ in this case, and the mapping (t,ω) to $\pi(t,\omega)(f)$ is continuous for *every* $f \in \boldsymbol{B}$.

One can also discuss from a technical side the need of assuming that the embedding from $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ into $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ should be a continuous one with respect to the occurring natural topologies. In fact, it should be enough, for example, to verify that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ itself is continuously embedded into the space of all locally integrable functions and that for each norm convergent sequence (f_n) in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ there exists a subsequence (f_{n_k}) which is pointwise convergent almost everywhere. etc. ...

Standard spaces are also described within the TEXBLOCK system (part of the NuHAG DB system):

`http://www.univie.ac.at/nuhag-php/tex-blocks/search.php?id=19`

Starting from the observation that for any of the module actions, arising from the pointwise algebra $\boldsymbol{A} = \mathcal{F}\boldsymbol{L}^1$ and the convolutional Banach module structure over $\boldsymbol{L}^1$ one can build two types of completions and also two typos of "essential parts". We will write $\boldsymbol{B}^{\boldsymbol{A}}$ for the $\boldsymbol{A}-$completion of $\boldsymbol{B}$, and $\boldsymbol{B}_{\boldsymbol{A}}$ for the essential part with respect to the pointwise module action. It is easy to verify that an element is in $\boldsymbol{B}_{\boldsymbol{A}}$ if and only if it can be approximated by elements with compact support, or if and only if any bounded sequence of plateau-like functions (forming a bounded approximate unit in $\mathcal{F}\boldsymbol{L}^1$) acts as approximation to the identity operator on the given element.

Analogously we define the completion and the essential part with respect to the Banach algebra $\boldsymbol{L}^1$. Since the action of this algebra usually comes from the group action (by translation), we will use the symbols $\boldsymbol{B}^{\mathcal{G}}$ and $\boldsymbol{B}_{\mathcal{G}}$.

Combining those four operations in a serial way we can come up with a large number of new spaces, derived from any of those spaces. Since the operations of completion and essential part with respect to the *same* algebra action are canceling each other (similar to the operation of taking a closure resp. the interior of a "nice set") we can concentrate in our discussion on "mixed series", such as: $\boldsymbol{B}_{\mathcal{G}}{}^{\boldsymbol{A}}{}_{\mathcal{G}}$, or even longer chains of operations of a similar kind.

The result that has been derived in [6] can be summarized in the following way: JUST the LAST OCCURRENCE of each algebra operation counts, i.e., the last occurrence of the symbol $\mathcal{G}$ and the last occurrence if $\boldsymbol{A}$. So we have $\boldsymbol{B}_{\mathcal{G}}{}^{\boldsymbol{A}}{}_{\mathcal{G}} = \boldsymbol{B}^{\boldsymbol{A}}{}_{\mathcal{G}}$ or $\boldsymbol{B}_{\mathcal{G}}{}^{\boldsymbol{A}}{}_{\mathcal{G}\boldsymbol{A}} = \boldsymbol{B}_{\mathcal{G}\boldsymbol{A}}$.

The most important spaces in this family are the minimal space, which turns out to be the "double essential part", where the order of algebra operations does *not* play a role anymore. It coincides with the closure of the test functions in the standard spaces. So we have

oubesspart

$$\boldsymbol{B}_{\boldsymbol{A}\mathcal{G}} = \overline{\boldsymbol{S}_0}^{\boldsymbol{B}} = \boldsymbol{B}_{\mathcal{G}\boldsymbol{A}} \tag{75}$$

On the other hand here is a (single) double completion, which coincides with the w^*-relative completion of $\boldsymbol{B}$ within $\boldsymbol{S}_0'$ (details to be presented at another time).

doublcompl

$$\boldsymbol{B}^{\boldsymbol{A}\mathcal{G}} = \widetilde{\boldsymbol{B}} = \boldsymbol{B}^{\mathcal{G}\boldsymbol{A}} \tag{76}$$

where we define the vague or w^*-relative completion of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ in $\boldsymbol{S}_0'$ as follows:

el-complet

$$\widetilde{\boldsymbol{B}} = \{\sigma \in \boldsymbol{S}_0' \,|\, \sigma = w^* - lim_\alpha f_\alpha, \sup_\alpha \|f_\alpha\|_{\boldsymbol{B}} < \infty\} \tag{77}$$

The infimum over all the bounds $\sup_\alpha \|f_\alpha\|_{\boldsymbol{B}}$ makes $\widetilde{\boldsymbol{B}}$ into a standard space, which contains $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ as a closed subspace.

Example:
Starting from $\boldsymbol{C}_0(\mathbb{R}^d)$ one can find that is relative completion is just $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$.

A WIKIPEDIA contribution:

http://en.wikipedia.org/wiki/Harmonic_analysis
http://en.wikipedia.org/wiki/Tempered_distribution
http://en.wikipedia.org/wiki/Fourier_analysis
http://en.wikipedia.org/wiki/Discrete-time_Fourier_transform
http://en.wikipedia.org/wiki/Colombeau_algebra

Next we will show that the convolution action of bounded discrete measures on a *homogenous Banach space* can be extended to all of the measures in order to generate an action of $(\boldsymbol{M}_b(\mathscr{G}), \|\cdot\|_{\boldsymbol{M}_b})$ on such a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.

eas-homBR0 **Theorem 22** (Bounded Measures act on Homogeneous Banach Spaces)**.** . *Assume that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a homogeneous Banach-space with respect to some "abstract" group action ρ, i.e. we assume that $x \to \rho(x)f$ is continuous from $\mathscr{G}$ into $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, isometric in the sense that $\|\rho(x)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ for all $x \in \mathscr{G}$ and $f \in \boldsymbol{B}$, and that $\rho(x_1x_2) = \rho(x_1)\rho(x_2)$. Of course we can define $\mu\bullet_\rho f$ for any discrete measure, hence for the family $D_\Psi\mu$ as the (unique) limit of this Cauchy net in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Then one has*

meas-onB1 (78) $$\|\mu\bullet_\rho f\|_{\boldsymbol{B}} \leq \|\mu\|_{\boldsymbol{M}}\|f\|_{\boldsymbol{B}}, \quad \text{for all } f \in \boldsymbol{B}.$$

In fact, $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ becomes a Banach module over $(\boldsymbol{M}_b(\mathscr{G}), \|\cdot\|_{\boldsymbol{M}_b})$ in this way.

Proof. We are going first to define the action of $\mu \in \boldsymbol{M}_b(\mathscr{G})$ on an individual element $f \in \boldsymbol{B}$, by verifying that the net

$$D_\Psi(\mu)\bullet_\rho f := \sum_{i\in I} \mu(\psi_i)\rho(x_i)f$$

is convergent, as $|\Psi| \to 0$.

The idea is to consider the action of $D_\Psi\mu$ on f as Riemann-type sums for the integral $\int_{\mathscr{G}} \rho(x)f d\mu(x)$. Therefore it is natural to check that the action of bounded discrete measures is OK (this is an easy consequence of the assumptions) and then to compare two such expressions, namely $D_\Psi(\mu)\bullet_\rho f$ and $D_\Phi(\mu)\bullet_\rho f$ by making use of their joint refinement, constituted by the (double indexed family) $(\psi_i\phi_j)$.

Let us first estimate the norm of $D_\Psi(\mu)\bullet_\rho f$. Using the isometry of the action of ρ on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ one has, independently from Ψ:

rmeasconv1 (79) $$\|D_\Psi(\mu)\bullet_\rho f\|_{\boldsymbol{B}} \leq \sum_{i\in I} |\mu(\psi_i)|\|\rho(x_i)f\|_{\boldsymbol{B}} \leq \|f\|_{\boldsymbol{B}} \sum_{i\in I} |\mu(\psi_i)| \leq \|\mu\|_{\boldsymbol{M}}\|f\|_{\boldsymbol{B}}.$$

Assume next that there are two families $\Psi = (\psi_i)_{i\in I}$ and $\Phi = (\phi_j)_{j\in J}$ are given, with central points $(x_i)_{i\in I}$ and $(y_j)_{j\in}$. Then we can define the *joint refinement* $\Psi - \Phi$ as the family $(\psi_i\phi_j)_{(i,j)\in I\diamond J}$, where we can agree to call $I \diamond J$ the family of all index pairs such that $\psi_i \cdot \phi_j \neq 0$ (because all the other products are trivial and should be neglected). In fact, if both Ψ and Φ are sufficiently "fine" BUPUs one has: [38]

ann-estim1 (80) $$\|D_\Psi\mu\bullet_\rho f - D_\Phi(\mu)\bullet_\rho f\|_{\boldsymbol{B}} = \sum_{(i,j)\in I\diamond J} \|\rho(x_i)f - \rho(y_j)f\|_{\boldsymbol{B}}|\mu(\psi_i\phi_j)| \leq$$

$$\sup_{(i,j)\in I\diamond J} \|\rho(x_i)[f - \rho(y_j - x_i)f]\|_{\boldsymbol{B}} \sum_{(i,j)\in I\diamond J} \|(\psi_i\phi_j)\mu\|_{\boldsymbol{M}} \leq \varepsilon\|\mu\|_{\boldsymbol{M}},$$

if only Ψ resp. Φ are fine enough. Due to the completeness of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ one finds that there is a uniquely determined limit, which we will call $\mu\bullet_\rho f$. It is then obvious that

stf-estim1 (81) $$\|\mu\bullet_\rho f\|_{\boldsymbol{B}} = \lim_{|\Psi|\to 0} \|D_\Psi\mu\bullet_\rho f\|_{\boldsymbol{B}} \leq \limsup \|D_\Psi\mu\|_{\boldsymbol{M}}\|f\|_{\boldsymbol{B}} = \|\mu\|_{\boldsymbol{M}}\|f\|_{\boldsymbol{B}}.$$

[38] Using that $\psi_i = \sum_{j\in j} \psi_i\phi_j$, hence $\sum_{(i,j)\in I\diamond J} \psi_i\phi_j \equiv 1$ and $\sum_{(i,j)\in I\diamond J} \|(\psi_i\phi_j)\mu\|_{\boldsymbol{M}} = \|\mu\|_{\boldsymbol{M}}$.

Of course it remains to show that the so defined action is associative, i.e. that

assocat (82) $$(\mu * \mu_2)\bullet_\rho f = (\mu_1)\bullet_\rho(\mu_2\bullet_\rho f), \quad \mu_1, \mu_2 \in \boldsymbol{M}_b(\mathscr{G}), f \in \boldsymbol{B},$$

but his is clear because the associativity is valid for the discrete measures $D_\Psi\mu$ and $D_\Phi(\mu)$.[39] □

Remark 21. In the derivation above we have used the isometric property and the fact that $\rho(x_1x_2) = \rho(x_1) \circ \rho(x_2)$. It would have been no problem if this identity was only true "up to some constant of absolute value one", i.e. if one has a *projective representation* of $\mathscr{G}$ only, such as the mapping $\lambda = (t, \omega) \mapsto \pi(\lambda) = M_\omega T_t$ from $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ into the unitary operators on the Hilbert space $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$, which is one of the key players in time-frequency analysis.

For the next step we need a simple observation from abstract Hilbert space theory.

Lemma 46. *Assume that a (complex) linear mapping between two Hilbert space over the complex numbers, $\mathscr{H}_1 \to \mathscr{H}_2$ is isometric, i.e. satisfies*

som-embed1 (83) $$\|T(h)\|_{\mathscr{H}_2} = \|h\|_{\mathscr{H}_1} \quad \forall h_1 \in \mathscr{H}_1.$$

Then the adjoint mapping $T' : \mathscr{H}_2 \to \mathscr{H}_1$ is the inverse on the range, i.e. one has $T'(Tf) = f \quad \forall h_1 \in \mathscr{H}_1$.

Proof. The claim follows from the fact that an isometric embedding also preserves scalar products, as a consequence of the polarization identity

arization (84) $$\langle f, g\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k\langle f + {}^k g, f + i^k g\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k\|f + i^k g\|^2$$

Hence

som-adjinv (85) $$\langle T'(Tf), g\rangle = \langle Tf, Tg\rangle = \langle f, g\rangle \quad \forall f, g \in \mathscr{H}_1.$$

Since this is true for every $g \in \mathscr{H}_1$ the required claim is valid. Usually one says that $T'(h_2)$ is defined in the weak sense for $h_2 \in \mathscr{H}_2$, through the identity

weak-Tadj (86) $$\langle T'h_2, h_1\rangle_{\mathscr{H}_1} = \langle h_2, T(h_1)\rangle_{\mathscr{H}_2}, \quad h_1 \in \mathscr{H}_1, h_2 \in \mathscr{H}_2.$$

□

Application: $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ is defined via its STFT: $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ if and only if $V_{g_0}f \in \boldsymbol{L}^1(\mathbb{R}^{2d})$, where g_0 is the Gauss-function (or any other nonzero Schwartz-function). Since $f \mapsto V_g f$ is isometric (assuming that $\|g_0\|_2 = 1$) we have according to the above lemma the (weak) reconstruction formula

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_{g_0}f(\lambda)\pi(\lambda)g_0 \; d\lambda,$$

[39] Note that H.S.Shapiro (cf. [38]) is making this associativity an extra axiom, apparently because he could not proof it directly for technical reasons, because he defines the action of the bounded measures on an "abstract homogeneous Banach space". H.C. Wang exhibits in [42] an example of what he calls a *semi-homogeneous Banach space* (without strong continuity of the action of $\mathscr{G}$ on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which does not allow the extension to all of the bounded measures. Indeed, it is a Banach space of measurable and bounded functions on $\mathbb{R}$ which is non-trivial, but which does not contain any non-zero *continuous* function!

but if $V_{g_0} f \in \boldsymbol{L}^1(\mathbb{R}^{2d}) \subset \boldsymbol{M}(\mathbb{R}^{2d})$ then we have

$$f = V_{g_0} f \bullet_\pi g_0$$

in the spirit of the above abstract statement (for $\rho = \pi$). It follows that one has for every TF-homogeneous Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. for every Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ such that $\|\pi(\lambda)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ and $\|\pi(\lambda)f - f\|_{\boldsymbol{B}} \to 0$ for $\lambda \to 0$:

o-minimal1 (87) $$\|f\|_{\boldsymbol{B}} = \|V_{g_0} f \bullet_\pi g_0\|_{\boldsymbol{B}} \leq \|V_{g_0} f\|_{\boldsymbol{L}^1(\mathbb{R}^{2d})} \|g_0\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{S}_0} \|g_0\|_{\boldsymbol{B}}.$$

Proof. The proof relies on the fact that for any net of convergent nets Ψ_β (all sufficiently "fine") the overall family ($D_{\Psi_\beta}\mu_\alpha$ is still bounded and uniformly tight! [40] Moreover it is clear that for each fixed β the net $D_{\Psi_\beta}\mu_\alpha$ is w^*-convergent to $D_{\Psi_\beta}\mu_0$. Given the tightness of the family only a finite number of indices of the family $(\psi_i^\beta)_{i\in I}$ is relevant for the convergence, hence $\mu_\alpha(\psi_i^\beta) \to \mu_0(\psi_i^\beta)$ implies that

$$D_{\Psi_\beta}\mu_\alpha \bullet_\rho f \to D_{\Psi_\beta}\mu_0 \bullet_\rho f.$$

FURTHER DETAILS HAVE TO BE CHECKED IN A CLEAN FORM LATER ON!

Psi-tight1 **Lemma 47.** *Let Ψ run through the family of all BUPUs of a given size, e.g. with $supp(\psi_i) \subseteq x_i + Q$, for some compact set Q. Then for any $\mu \in \boldsymbol{M}_b(G)$ on has: $(D_\Psi \mu)_\Psi$ is a tight set.*

Proof. We can start from any fixed BUPU Φ, and recall that $\sum_j \|\phi_j \cdot \mu\|_{\boldsymbol{M}} \leq \|\mu\|_{\boldsymbol{M}}$.

We can use this to approximate μ by a partial sum, i.e. by choosing for $\varepsilon > 0$ a finite set $F \subset J$ can be found such that

$$\|\sum_{j\in F} \phi_j \cdot \mu - \mu\|_{\boldsymbol{M}} = \sum_{j\notin F} \|\phi_j \mu\|_{\boldsymbol{M}} < \varepsilon.$$

or equivalently, writing $\Phi_F := \sum_{j\in F} \phi_j$:

$$\|\Phi_F \cdot \mu - \mu\|_{\boldsymbol{M}} < \varepsilon.$$

Let now $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ by any plateau-function such that $h(x) \equiv 1$ on some neighborhood of $\mathrm{supp}(\Phi_F)$. More precisely, given the uniform size of the BUPUs Ψ to be considered, we can have $h \cdot \psi = \psi$ for all indices $i \in I$ such that $p \cdot \psi \neq 0$.

Then for any BUPU $\Psi = (\psi_i)_{i\in I}$ of size Q $\|h(D_\Psi \mu) - D_\Psi \mu\|_{\boldsymbol{M}}$ can be controlled in a uniform way[41].

ARGUMENTS: For every $\phi \in \boldsymbol{C}_c(\mathbb{R}^d)$ or even $\boldsymbol{C}_b(\mathbb{R}^d)$ one has:

s-estimat1 (88) $$|\mu(\phi)| \leq \|\phi \cdot \mu\|_{\boldsymbol{M}}$$

because in case $\phi \in \boldsymbol{C}_c(\mathbb{R}^d)$ one can choose any other function $\phi^* \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|\phi^*\|_\infty = 1$ and $\phi^* \cdot \phi = \phi$, hence one has

abs-estim2 (89) $$|\mu(\phi)| = |\mu(\phi^*\phi)| \leq |[\phi \cdot \mu](\phi^*)| \leq \|\phi \cdot \mu\|_{\boldsymbol{M}},$$

thus completing the argument. The general case follows from this using the standard reduction to compactly supported measures.

[40] It is a good exercise to check the technical details yourself!

[41] Despite the fact that $p \cdot D_\Psi \mu$ is not equal to $D_\Psi(p\mu)$

Given F we can find some $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $h(x) \in [0,1]$, hence with $\|h\|_\infty = 1$, such that $h(x) \equiv 1$ on $\bigcup_{j\in F} \mathrm{supp}(\phi_j)$. (due to the limited size of the support this is a compact set which can be choosen independently from the concrete choice of Ψ). Consequently

psi-estim3 (90)
$$\|h \cdot D_\Psi \mu - D_\Psi \mu\|_{\boldsymbol{M}} \leq \| \sum_{\{i \,|\, \psi_i \cdot p = 0\}} \mu(\psi_i)\delta_{x_i}\|_{\boldsymbol{M}} = \sum_{\{i \,|\, \psi_i \cdot p = 0\}} |[\mu - p\mu](\psi_i)| \leq \|\mu - p\mu\|_{\boldsymbol{M}} \leq \varepsilon.$$

□

Requiring the argument that one has for any $\mu \in \boldsymbol{M}_b(\mathscr{G})$:

$$\|\mu \bullet_\rho f - D_\Psi \mu \bullet_\rho f\|_{\boldsymbol{B}} \leq \varepsilon \|\mu\|_{\boldsymbol{M}}$$

depending only on the element $f \in \boldsymbol{B}$ and the level of "refinement" of Ψ but NOT on the individual choice of μ. □

Next we want to show that there is an important form of continuity from in this action from $\boldsymbol{M}(\mathscr{G}) \times \boldsymbol{B} \to \boldsymbol{B}$, with respect to the w^*-topology on $\boldsymbol{M}_b(\mathscr{G})$.

st-to-norm **Theorem 23.** *Assume that $(\mu_\alpha)_{\alpha\in I}$ is a bounded and tight net of bounded measures, which is w^*-convergent to some limit measure $\mu_0 \in \boldsymbol{M}(\mathbb{R}^d)$. Then for every $f \in \boldsymbol{B}$:*

t-to-norm1 (91)
$$\|\mu_\alpha \bullet_\rho f - \mu_0 \bullet_\rho f\|_{\boldsymbol{B}} \to 0 \quad \textit{for} \quad \alpha \to \infty.$$

Proof. One can verify this relation by observing that the family $D_\Psi \mu_\alpha$ running through any net of BUPUs with $|\Psi| \leq 1$ and $\alpha \in I$ is a tight family, hence up to some $\varepsilon > 0$ one can replace the given net (μ_α) by a family of measures with joint compact support. Hence all the BUPUs will consist of finitely many discrete measures, and in particular $\|D_\Psi \mu_\alpha \bullet_\rho f - D_\Psi \mu_0 \bullet_\rho f\|_{\boldsymbol{B}} \to 0$ for any $f \in \boldsymbol{B}$, for fixed Ψ. □

Corollary 8. *For every homogeneous Banach space $\left(\boldsymbol{B} \,\|\cdot\|_{\boldsymbol{B}}\right)$ one has: For every $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ and every $f \in \boldsymbol{B}$: $\mu * f$ is the limit finite linear combinations translates of f. In particular one has: Given μ and f and $\varepsilon > 0$ there exists a finite sequence $(x_i)_{i\in F}$ and a finite sequence of complex coefficients $(c_i)_{i\in F}$ such that*

pproxtrans (92)
$$\|\mu * f - \sum_{i\in F} c_i T_{x_i} f\|_{\boldsymbol{B}} < \varepsilon.$$

Proof. We just have to remember that the discrete bounded measures $D_\Psi \mu$ resp. their partial sums form a tight family of measures, which is w^*-convergent towards μ. Since each of the approximating measures $D_\Psi \mu$ is of the form $\sum_{i\in I} \mu(\psi_i)\delta_{x_i}$ and can be approximated (even in the norm of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$) by finite sums we just have to put $c_i = \mu(\psi_i)$ and observe that

$$\left(\sum_{i\in F} c_i \delta_{x_i}\right) * f = \sum_{i\in F} c_i T_{x_i} f.$$

□

Remark 22. A more precise way of expressing what is going on is the use of suitable indexing, allowing to express that it is enough that "sufficiently many" elements from a "sufficiently fine" BUPU Ψ have to be used. Let us choose as index set pairs of the form (K, δ), with $\delta > 0$ and K a compact subset of $\mathbb{R}^d$. They have a partial order, with

$(K_1, \delta_1) \succeq (K_2, \delta_2)$ if $K_2 \supseteq K_1$ and $\delta_2 \leq \delta_1$. Then we say that $\sum_{i\in F} c_i \delta_{x_i}$ has the index (K, δ) if Ψ is a δ-BUPU, and $F \supseteq \{i \in I \,|\, x_i \in K\}$. One could also talk of a local BUPU, by assuming (slightly differently, but technically equivalent) that $\sum_{i\in F} \psi(x) \equiv 1$ on K.

In this setting one can say: For every $\mu, f, \varepsilon > 0$ there exists a pair (K_o, δ_0) such that for all local BUPUs which satisfy at least $\sum_{i\in F} \psi(x) \equiv 1$ on K and are at least δ_0-fine, one has

vtransapp2

$$\|\mu * f - \sum_{i\in F} \mu(\psi_i) T_{x_i} f\|_{\boldsymbol{B}} < \varepsilon. \tag{93}$$

The point of the last remark is that the user may even choose the points by himself, and then suitable coefficients can be found (by making use of an BUPU adapted to the given set), cf. [**?**]

11. Sobolev spaces, derivatives in $\boldsymbol{L}^2(\mathbb{R}^d)$

xxx TEST $\mathcal{H}_s, \mathcal{H}_s(\mathbb{R}^d)$

Sobolev spaces and the relation with the Fourier transform: Relate the classical concept of derivatives or multiple derivatives (as done originally by Sobolev) to the properties of their Fourier transform.

ffer-Four1 **Lemma 48.** *Assume that $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ has continuous partial derivatives. Then their Fourier transforms coincide with $\hat{f}$, multiplied with essentially the coordinate functions.*

Proof. We can restrict the attention to the case $d = 1$. Considering the fact that $f' \in \boldsymbol{C}_c(\mathbb{R}^d)$ by assumption we can guarantee that the difference quotients $\frac{1}{h}(T_h f - f)(x)$ converge *uniformly* to f', hence in the $\boldsymbol{L}^1$-norm.

Consequently their Fourier transforms are uniformly convergent to the Fourier transforms of f', using the mean-value theorem.

diff-four2 (94) $$f(x+h) - f(x) = f'(\xi) \cdot h, \quad \text{for some} \quad \xi \in (x, x+h).$$

Since translation by h goes into multiplication with characters the rule follows by going to the limit

f-fourier1 (95) $$\frac{1}{h}(e^{2\pi i s h} - 1) \to 2\pi i s \cdot e^{2\pi i s t},$$

□

Definition 42. A strictly positive and (without loss of generality) continuous function w is called *submultiplicative* (resp. a *Beurling weight*) if it satisfies

Beurling (96) $$w(x+y) \leq w(x) \cdot w(y), x, y \in G.$$

Examples: $w_s(x) = (1 + |x|)^s$ for $s \geq 0$. It is trivial (Ex!) to prove it for $s = 1$ and then by taking the $s - th$ power it is still valid! The corresponding weighted $\boldsymbol{L}^2$ on $\mathbb{R}^d$ is denoted by $\boldsymbol{L}^2_{w_s}(\mathbb{R}^d)$.

It is easy to verify that this implies a pointwise estimate (using $w(x) \leq w(x-y) \cdot w(y)$ in the convolution integral):

onv-weight (97) $$|(f * g) \cdot w| \leq |f| w * |g| w$$

In fact, we have for every fixed $x \in G$:

convwest1 (98) $$f * g(x) w(x) \leq \int_G |g(x-y)||f(y)| w(x) dy \leq \int_G |g(x-y)| w(x-y) |f(y)| w(y) dy.$$

As an immediate consequence the corresponding weighted $\boldsymbol{L}^1$-space $\boldsymbol{L}^1_w$ is a Banach algebra with respect to convolution, a so-called *Beurling algebra* (cf. [36]).

Another important class are the so-called WSA weights:

WSA-wgt **Definition 43.** A strictly positive and continuous weigth w is called WSA (*weakly sub-additive*) if there is some constant $C > 0$ such that for all x, y:

WSA-def (99) $$w(x+y) \leq C(w(x) + w(y)).$$

It implies in a completely similar manner another useful pointwise estimate:

conv-WSA (100) $$|(f * g) \cdot w| \leq C(|f|w * |g| + |f| * |g|w)$$

recall that $\boldsymbol{L}^2_w := \{f \,|\, fw \in \boldsymbol{L}^2\}$, with the natural norm $\|f\|_{2,w} := \|fw\|_2$.
Using equation 11 and the fact that $\boldsymbol{L}^1 * \boldsymbol{L}^2 \subseteq \boldsymbol{L}^2$ (with corresponding norm inequalites)

$$\|g * f\|_2 = \|\hat{g} \cdot \hat{f}\|_2 \leq \|\hat{g}\|_\infty \cdot \|\hat{f}\|_2.$$

Theorem 24. *Let w be a WSA weight. $\boldsymbol{L}^1 \cap \boldsymbol{L}^2_w$ is a Banach algebra with respect to convolution. In particular, $\boldsymbol{L}^2_w$ is a Banach algebra with respect to convolution if $1/w \in \boldsymbol{L}^2$.*

Proof. The $\boldsymbol{L}^1$-norm is no problem, so we only have to find for $f, g \in \boldsymbol{L}^1 \cap \boldsymbol{L}^2_w$ an estimate conv-WSA for $\|(f * g)w\|_2$, which can be obtained from the estimate ().

□

Argument: then $\boldsymbol{L}^2_w = \boldsymbol{L}^1 \cap \boldsymbol{L}^2_w$ with equivalence of the natural norms associated with these spaces!

Since the Fourier transform maps $\boldsymbol{L}^1$ into $\boldsymbol{C}_0$ (according to the Riemann Lebesgue Lemma) it follows that under the same conditions a Sobolev space is embedded into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. Let us write $\mathcal{H}_s(\mathbb{R}^d)$ for $\mathcal{F}^{-1}\left(\boldsymbol{L}^2_w(\mathbb{R}^d), \|\cdot\|_{2,w}\right)$, with norm
$\|f\|_{\mathcal{H}_s} := \|\hat{f} \cdot w\|_2$.

For integer values of $s \geq 0$ one can show that this space coincides with the space of all $\boldsymbol{L}^2-$ elements which have (in the sense of distributions!) derivatives of order up to s, i.e. the (within the theory of distributions well defined objects f, f', f" etc. up to order s are regular distributions which can be represented by $\boldsymbol{L}^2$-functions, or equivalently: smoothing the distribution f one has uniform control of those derivatives in the classical sense independent of the order of regularization, e.g. by convolution with a very narrow Gauss function);

bolev-emb0 **Corollary 9.** *For $s > d/2$ the Sobolev space $\mathcal{H}_s$ is continuously embedded into $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$, hence also into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$.*

Proof. Observing that $1/w_s \in \boldsymbol{L}^2(\mathbb{R}^d)$ if (and only if) $w > d/2$ on finds that $\boldsymbol{L}^2_w(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^1(\mathbb{R}^d)$ via Cauchy-Schwartz, writing $h \in \boldsymbol{L}^2_w(\mathbb{R}^d)$ as a product of hw with $1/w$:

LtwinLi (101) $$\|h\|_1 = \|hw \cdot 1/w\|_1 \leq \|hw\|_w \cdot \|1/w\|_2.$$

Applying the Fourier transform on both sides we obtain that $\mathcal{H}_s(\mathbb{R}^d) \hookrightarrow \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$, with the natural norm estimates. □

Note that the above argument shows that $\mathcal{H}_s(\mathbb{R}^d)$ is not only continuously embedded into $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$ (according to Plancherel, since $\boldsymbol{L}^2_w(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^2(\mathbb{R}^d)$), but also in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, hence into $\boldsymbol{L}^2 \cap \boldsymbol{C}_0(\mathbb{R}^d)$ with the natural norm (sum of the two norms). Using the so-called theory of Wiener amalgam spaces on can show that $\mathcal{H}_s(\mathbb{R}^d) \hookrightarrow \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^2)(\mathbb{R}^d)$, by the argument that $\boldsymbol{L}^2_w(\mathbb{R}^d) = \boldsymbol{W}(\boldsymbol{L}^2, \boldsymbol{\ell}^2_w) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \boldsymbol{\ell}^2_w) \hookrightarrow \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \boldsymbol{\ell}^1)$ (using again Cauchy Schwartz in the last inclusion), which by a variant of the Hausdorff-Young inequality for Wiener amalgam spaces gives

$$\mathcal{H}_s(\mathbb{R}^d) = \mathcal{F}^{-1}\,\boldsymbol{L}^2_w \hookrightarrow \mathcal{F}^{-1}\,\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \boldsymbol{\ell}^1) \hookrightarrow \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^2) \hookrightarrow \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^2)(\mathbb{R}^d),$$

which is a space strictly contained in $\boldsymbol{L}^2 \cap \boldsymbol{C}_0(\mathbb{R}^d)$.

Note that $\mathcal{H}_s(\mathbb{R}^d)$ is not only a Banach space with respect to the natural norm $\|f\|_{\mathcal{H}_s} := \|\hat{f}w\|_2$, but even a Hilbert spaces, because the norm obviously comes from the scalar product

S-scalprod

$$\langle f, g\rangle_{\mathcal{H}_s} := \langle \hat{f}w, \hat{g}w\rangle_{\boldsymbol{L}^2} = \int_{\mathbb{R}^d} \hat{f}(s)\,\overline{\hat{g}(s)}\; w^2(s)ds. \tag{102}$$

Hs-RKHS **Corollary 10.** *$\mathcal{H}_s(\mathbb{R}^d)$ is a reproducing kernel Hilbert space, i.e. for each $t \in \mathbb{R}^d$ the Dirac measure $\delta_x : f \mapsto f(x)$ is a continuous linear functional on the Hilbert space $\mathcal{H}_s(\mathbb{R}^d)$. The kernel $K(x, y)$, consisting of functions $K(\cdot, y) = k_x(y)$ in $\mathcal{H}_s(\mathbb{R}^d)$ with*

Hs-RKHSphi

$$f(x) = \langle f, k_x\rangle_{\mathcal{H}_s} \quad \textit{for all} \quad x \in \mathbb{R}^d, f \in \mathcal{H}_s \tag{103}$$

is obtain as collections of shifts $T_x\varphi$, with $\varphi = \mathcal{F}^{-1}(1/w^2)$.

Proof. Since $\mathcal{H}_s(\mathbb{R}^d)$ is continuously embedded into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ it is clear that the family of point measures acts even *uniformly bounded* on $\mathcal{H}_s(\mathbb{R}^d)$. Hence we only have to prove the explicit representation of δ_0 on $\mathcal{H}_s(\mathbb{R}^d)$ and then (using the definitions) the covariance of the situation: shifting the point evaluation (δ_0 to δ_x) corresponds to shift the generator representing φ. We get the representation using the Fourier inversion formula (it is easy to recognize that we do not need that f itself is in $\boldsymbol{L}^1(\mathbb{R}^d)$, the inverse Fourier transform a priori defined as an $\boldsymbol{L}^2$-FT has the usual form as integral if $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$).

-deltaOrep

$$f(0) = \int_{\mathbb{R}^d} \hat{f}(s)ds = \int_{\mathbb{R}^d} \hat{f}(s)w(s) \cdot \frac{1}{w^2}(s)w(s)ds =: \int_{\mathbb{R}^d} \hat{f}(s)\,\overline{\hat{\varphi}(s)}\; w^2(s)ds, \tag{104}$$

if we put $\varphi = \mathcal{F}^{-1}(1/w^2)$.

Now we can apply the shift invariance of the scalar product (excercise):

-scalshift

$$\langle T_x f, T_x h\rangle_{\mathcal{H}_s} = \langle f, h\rangle_{\mathcal{H}_s} \quad \text{for all} \quad f, h \in \mathcal{H}_s(\mathbb{R}^d), \tag{105}$$

because we know that translation goes to modulation on the Fourier transform side, but having the same modulations within a scalar products means (due to the fact that one is taking a conjugation in the scalar product) that it is unchanged. Technically speaking one could argue that translation goes into modulation, but for *all weights* modulations are unitary on the corresponding weighted $\boldsymbol{L}^2$-spaces.

□

Another interesting embedding reads as follows:

b-decay-SO **Theorem 25.** *For $s > d$ the intersection of $\mathcal{H}_s(\mathbb{R}^d)$ with $\boldsymbol{L}^2_{w_s}(\mathbb{R}^d)$ with its natural norm (sum of the two norms) is continuously (and densely) embedded into $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$.*

Remark 23. Cf. the work of K. Gröchenig: there are also non-symmetric conditions and even conditions with weighted $\boldsymbol{L}^p-$norms on the time side and other weighted $\boldsymbol{L}^q$-conditions on the Fourier transform side which can be used to show that function with a certain amount of both time- and frequency concentration (as expressed by these conditions) are necessarily lying within $\boldsymbol{S}_0(\mathbb{R}^d)$. Of course one can easily adapt those conditions to conditions which are sufficient conditions for other modulation spaces, especially to be within Shubin classes $\boldsymbol{Q}(\mathbb{R}^d)$, or modulation spaces $\boldsymbol{M}^1_w(\mathbb{R}^d)$.

details are in [11], and about WSA functions in the work of Brandenburg [5], online at
http://univie.ac.at/nuhag-php/bibtex/open_files/1407_9550001.pdf

For other purposes we mention also the concept of *moderate weights*
COMES TWICE!

A weight function v on $\mathbb{R}^{2d}$ is called *submultiplicative*, if

ubmultdef1 (106) $$v(z_1 + z_2) \leq v(z_1)v(z_2),$$

for all $z_1 = (x_1, \xi_1), z_2 = (x_2, \xi_2) \in \mathbb{R}^{2d}$.

A weight function w on $\mathbb{R}^{2d}$ is v-moderate if

(107) $$w(z_1 + z_2) \leq cv(z_1)w(z_2)(*)$$

for all $z_1 = (x_1, \xi_1), z_2 = (x_2, \xi_2) \in \mathbb{R}^{2d}$.

Two weights w_1 and w_2 are equivalent, written $w_1 \asymp w_2$, if

(108) $$C^{-1}w_1(z) \leq w_2(z) \leq Cw_1(z)$$

for all $z = (x, \xi) \in \mathbb{R}^{2d}$ and some positive constant C.

For references see [11, 12, 23].

12. Some pointwise estimates

Pointwise estimates: Convolution preserves monotonicity We have to define $|\mu|$ for a given measure, and have to show that $\| \, |\mu| \, \|_{\boldsymbol{M}} = \|\mu\|_{\boldsymbol{M}}$ for each $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$.

point-conv (109) $$|\mu * f| \leq |\mu| * |f|$$

oint-conv1 (110) $$|f| \leq |g| \Rightarrow |\mu| * |f| \leq |\mu| * |g|$$

ost-estim1 (111) $$[(D_\Psi f - f) * g](x) \; \leq \; [|f| * \mathrm{osc}_\delta(g)](x), \; \forall x \in \mathbb{R}^d.$$

or in short, a pointwise estimate of the form

st-estim1b (112) $$[(D_\Psi f - f) * g] \; \leq \; [|f| * \mathrm{osc}_\delta(g)].$$

MAIN ESTIMATE

in-estim11 (113) $$|D_\Psi \mu * f - \mu * f| \leq |\mu| * |\,\mathrm{Sp}_\Psi \, f - f| \leq |\mu| * \mathrm{osc}_\delta \, f$$

if $\mathrm{diam}(\psi) \leq \delta$.
It relies on a couple of "simple" estimates, such as

$$\mathrm{osc}_\delta \, \check{f} = (\mathrm{osc}_\delta \, f)\check{}$$

and

$$\mathrm{osc}_\delta(T_x f) = T_x(\mathrm{osc}_\delta \, f)$$

Obviously

$$|\,\mathrm{Sp}_\Psi \, f(x) - f(x)| \; \leq \; \mathrm{osc}_\delta \, f(x), \; \forall x \in \mathbb{R}^d.$$

Moreover the fact that the discretization operator $D_\Psi : \boldsymbol{M}_b(\mathbb{R}^d) \mapsto \boldsymbol{M}_b(\mathbb{R}^d)$ is the adjoint of the spline operator $\mathrm{Sp}_\Psi : \boldsymbol{C}_0(\mathbb{R}^d) \mapsto \boldsymbol{C}_0(\mathbb{R}^d)$, implies also that we have:

conv-Dpsi (114) $$D_\Psi \mu * f = \mu * \mathrm{Sp}_\Psi \, f$$

Lemma 49. *If $f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)$ then also $\mathrm{osc}_\delta \, f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)$.*

Lemma 50. *A function $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ belongs to $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)$ if and only if $f \in \boldsymbol{L}^p(\mathbb{R}^d)$ and $\mathrm{osc}_\delta \, f \in \boldsymbol{L}^p(\mathbb{R}^d)$.*

13. Discretization and the Fourier transform

TEST: We shall define here $\sqcup\!\sqcup_a := \sum_k \delta_{ak}$ and $\sqcup\!\sqcup^a = \frac{1}{a}\sum_n \delta_{\frac{n}{a}}$. Then $\mathcal{F}\sqcup\!\sqcup_a = \sqcup\!\sqcup^a$. In fact, one has for $a = 1$ according to Poisson's formula $\mathcal{F}\sqcup\!\sqcup_1 = \sqcup\!\sqcup^1$, and the general formula follows from this by a standard dilation argument: Mass preserving compression St_ρ is converted into "value-preserving" dilation D_ρ on the Fourier transform side, and $D_\rho \sqcup\!\sqcup^1 = \sqcup\!\sqcup^{1/\rho}$.

Let us put a few observations of importance at the beginning of this section:

- The periodic and discrete (unbounded) measures are exactly those which arise as periodic repetitions of a fixed finite sequence of the form $\sum_{k=0}^{N-1} a_k\delta_k$.
- The Fourier transform of such a sequence can be calculated directly using the FFT
- for any (sufficiently nice) function f (e.g. $f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$) one has for $b = 1/a$:

frepper1 (115) $$\mathcal{F}\big[\sqcup\!\sqcup_{aN} * (\sqcup\!\sqcup^a \cdot f)\big] = \sqcup\!\sqcup^{Nb} \cdot (\sqcup\!\sqcup_b * \hat{f}) = \sqcup\!\sqcup_b * (\sqcup\!\sqcup^{Nb} \cdot \hat{f})$$

The last step in the proof of formula frepper1 115 is easily verified directly: sampling and periodization commute if (and only if) the periodization constant (bN in our case) is a multiple of the sampling period (in our case b).

The question of approximately obtaining the continuous Fourier transform $\hat{f}$ of a "nice function" f from the FFT of it's sampled version can be derived from this fact.

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

Given $h > 0$ and some prescribed function ψ on $\mathbb{R}^d$, such as a cubic B-spline, the quasi-interpolation $Q_h f = Q_h^\psi f$ of a continuous function f on $\mathbb{R}^d$ is defined by

eqdefQh (116) $$Q_h f(x) = \sum_{k\in\mathbb{Z}} f(hk)\psi(x/h - k), \qquad x \in \mathbb{R}^d.$$

For suitable ψ, this formula describes an approximation to f from its samples on the fine grid $h\mathbb{Z}^d \subset \mathbb{R}^d$.

theoSO **Theorem 26.** *Assume that $\psi \in \boldsymbol{S}_0(\mathbb{R}^d)$ satisfies $\sum_{k\in\mathbb{Z}^d} \psi(x-k) \equiv 1$, i.e., that the family $(T_k\psi)_{k\in\mathbb{Z}^d}$ forms a partition of unity. Then for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ we have $\|Q_h f - f\|_{\boldsymbol{S}_0} \to 0$ as $h \to 0$.*

Note, that under the same restrictions on ψ one also has convergence of the quasi-interpolation scheme in the Fourier algebra $\mathcal{F}\boldsymbol{L}^1$ for all $f \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$. Consequently one has $Q_h^*\sigma \to \sigma$ in the weak*-sense for each $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$. But $Q_h^*\sigma = \sum_k \sigma(\psi_k)\delta_{hk}$. Hence the discrete measures are w^*-dense in $\boldsymbol{S}_0'(\mathbb{R}^d)$.

Comment on the consistency of distributional Fourier transform with the classical one (defined on $\boldsymbol{L}^1(\mathbb{R}^d)$, using the Lebesgue integration formula).

For any Fourier invariant space of test functions (such as $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ or the Schwartz space of *rapidly decreasing functions* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$) one defines a *generalized Fourier transform* by the formula

Definition 44.

Four-def2

$$\mathcal{F}\sigma = \hat{\sigma} : (\hat{\sigma})(f) := \sigma(\hat{f}) \tag{117}$$

The above formula can then be interpreted as a consistency relation. Write σ_g and $\sigma_{\hat{g}}$ for the distributions generated by the functions $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ and $\hat{g} \in \boldsymbol{C}_0(\mathbb{R}^d)$ respectively [42] then the formula (59) tell us that

Fund-Fourier0

$$\widehat{\sigma_g} = \sigma_{\hat{g}}.$$

see e.g. [20], Lesson 17, page 156.

[42] so the space of test functions must be contained in $\boldsymbol{L}^1(\mathbb{R}^d)$ to ensure that $\boldsymbol{C}_0(\mathbb{R}^d)$-functions define continuous linear functionals.

14. Quasi-Interpolation

The piecewise linear interpolation operator for data available on the lattice of integers $\mathbb{Z}$, say $(c_k)_{k\in\mathbb{Z}}$, can be described as a sum of shifted triangular functions $\Delta(0) = 1, \Delta(k) = 0$ for $k \notin \mathbb{Z}$. Hence it can be written as a convolution product of the form

$$\big(\sum_{k\in\mathbb{Z}} c_k\delta_k\big) * \Delta.$$

It is easy to show that the resulting sum (the interpolant) belongs to $\boldsymbol{L}^p(\mathbb{R})$ if the sequence $\mathbf{c}$ is from $\boldsymbol{\ell}^p(\mathbb{Z})$. But this is true for much more general functions than then triangular function. It suffices to have $\varphi \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R})$ in order to find out that $\sum_{k\in\mathbb{Z}} T_k\varphi$ belongs to $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R})$ for $\mathbf{c} \in \boldsymbol{\ell}^p(\mathbb{Z})$. In fact, this assumption implies $\sum_k c_k\delta_k \in \boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^p)$ and hence the convolution relations for Wiener amalgam spaces imply:

$$f = \sum_{k\in\mathbb{Z}} T_k\varphi = \big(\sum_{k\in\mathbb{Z}} c_k\delta_k\big) * \varphi \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}).$$

As a consequence f is a continuous function and can be sampled, e.g., over the integers, but in most cases $f(k)$ will be perhaps close to, but different from the original sequence $(c_k)_{k\in\mathbb{Z}}$, hence the name *quasi-interpolation.* [43]

The so-called quasi-interpolation operators make sense for functions from $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d)$, to choose the appropriate generality from now on. For those functions one can guarantee that for some $C > 0$ and all $p \in [1, \infty]$ one has:

$$\|(f(k))_{k\in\mathbb{Z}^d}\|_p \le C\|f \,|\, \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)\| \quad \forall f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d).$$

The same is true for any other lattice $\Lambda \lhd \mathbb{R}^d$, with

$$\|(f(\lambda))_{\lambda\in\Lambda}\|_p \le C_\Lambda\|f \,|\, \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)\| \quad \forall f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d).$$

Hence the operator

$$f \mapsto \sum_{\lambda\in\Lambda} f(\lambda)T_\lambda\varphi$$

is a well defined operator on $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d)$ (even uniformly bounded with respect to the range $p \in [1, \infty]$. We will call such an operator the *quasi-interpolation operator* with respect to the pair (Λ, φ).

Among the quasi-interpolation operators those which arise from BUPUs, i.e., from functions $\varphi \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ satisfying

$$\sum_{\lambda\in\Lambda} T_\lambda\varphi(x) \equiv 1$$

are the most important ones. We are going to show that quasi-interpolation operators with respect to "fine lattices" Λ are good approximation operators.

The interesting phenomenon is the behaviour of piecewise linear interpolation over lattices of the form $\alpha\mathbb{Z}^d$, for $\alpha \to 0$.

[43] Note that SINC is not covered by this example, although for $p \in (1, \infty)$ it shares more or less all the properties described above.

Let us recall that $(T_k\varphi)_{k\in\Lambda}$ is a BUPU for some $\varphi \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ if and only if $\hat{\varphi}$ is a Lagrange interpolator over the orthogonal lattice $\Lambda^\perp = \{\chi \,|\, \langle \chi, \lambda\rangle \equiv 1 \quad \forall \lambda \in \Lambda\}$, i.e., that

Lagr-intp1 (118) $$\widehat{\varphi}(\lambda') = \delta_{0,\lambda'} \quad \forall \lambda' \in \Lambda^\perp.$$

Proof. We can reinterpret the BUPU condition as $\sqcup\!\sqcup_H * \varphi \equiv 1$, which turns into

$$\mathcal{F}(\sqcup\!\sqcup_H) \cdot \hat{\varphi} = \mathcal{F}(1) = \delta_0.$$

Since $\mathcal{F}(\sqcup\!\sqcup_H) = C_H \sqcup\!\sqcup_{H^\perp}$ this condition reduces to (using $f \cdot \delta_x = f(x)\delta_x$):

$$C_H \sqcup\!\sqcup_{H^\perp} \cdot \hat{\varphi} = \sum_{h' \in H^\perp} \hat{\varphi}(h')\delta_{h'} = \delta_0,$$

which in turn is true if and only if $\hat{\varphi}(h') = 0$ for $h' \neq 0$ for all $h' \in H^\perp$.

□

Remark 24. The condition described above is invariant with respect to pointwise powers on the Fourier transform side, i.e., $\hat{\varphi}$ satisfies (118) then the same is true for $\hat{\varphi}^2 = \widehat{\varphi * \varphi}$.

The quasi-interpolation operator $Q_{\Lambda,\varphi}$ can thus be described as the mapping

quasint (119) $$f \mapsto (\sqcup\!\sqcup_H \cdot f) * \varphi)$$

Note that this operators is bounded on $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d)$ because $\sqcup\!\sqcup_H \cdot f \in \boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^\infty)(\mathbb{R}^d) \cdot \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d) \subseteq \boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^p)(\mathbb{R}^d)$, hence

$$(f \cdot \sqcup\!\sqcup_H) * \varphi \subseteq \boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^p) * \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1) \subseteq \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d).$$

It is of interest to check the behaviour of quasi-interpolation for the lattices $h\mathbb{Z}^d$, with $h \to 0$:
Given $h > 0$ and some prescribed function ψ on $\mathbb{R}^d$, such as a B-spline, the quasi-interpolation $Q_h f = Q_h^\psi f$ of a continuous function f on $\mathbb{R}^d$ is defined by

eqdefQh (120) $$Q_h f(x) = \sum_{k\in\mathbb{Z}} f(hk)\psi(x/h - k), \qquad x \in \mathbb{R}^d.$$

For suitable ψ, this formula describes an approximation to f from its samples on the fine grid $h\mathbb{Z}^d \subset \mathbb{R}^d$.

theoSO **Theorem 27.** *Assume that $\psi \in \boldsymbol{S}_0(\mathbb{R}^d)$ satisfies $\sum_{k\in\mathbb{Z}^d} \psi(x-k) \equiv 1$, i.e., that the family $(T_k\psi)_{k\in\mathbb{Z}^d}$ forms a partition of unity. Then for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ we have $\|Q_h f - f\|_{\boldsymbol{S}_0} \to 0$ as $h \to 0$.*

Note, that under the same restrictions on ψ one also has convergence of the quasi-interpolation scheme in the Fourier algebra $\mathcal{F}\boldsymbol{L}^1$ for all $f \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.

15. Pseudo-measures and other auxiliary terms

seudomeas1 **Definition 45.** The space $\mathcal{F}\boldsymbol{L}^\infty$ is defined as the (inverse) image under the (generalized) Fourier transform of $\boldsymbol{L}^\infty(\mathbb{R}^d) = [\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)]'$.

An alternative definition of pseudo-measures (used e.g. in the work of G. Gaudry or R. Larsen) is based on the following characterization:

urLinfdual **Lemma 51.** *There is a natural identification of* $\mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$ *with the* $[(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})]'$, *via* $\sigma(\hat{h}) = \sigma_h(\widehat{\sigma}) = \int_{\mathbb{R}^d} \widehat{\sigma}(s)h(s)ds$.

Even in the context of non-Abelian Groups $\mathscr{G}$ this makes sense, if one uses Eymard's Fourier algebra $\boldsymbol{A}(\mathscr{G})$ ([10]).

16. Advantages of a Distributional Fourier Transform

Whereas most books in the field of Fourier analysis describe the Fourier transform at various levels, typically starting from the classical case of the Fourier transform for periodic functions [9, 27]. Sometimes the need of a generalized Fourier transform is motivated by the fact, that certain objects (like the "pure frequencies") do not have a Fourier transform in the usual sense, because first of all the classical Fourier transform is bound to diverge, while on the other hand the Fourier transform (which is in the generalized calculus a Dirac measure) is not an ordinary function, but has to be a kind of generalized function (in fact a bounded measure in that case), cf. [2].
In this section we want to emphasize (by demonstrating the situation, valid even for locally compact groups in full generality, through the example $\mathcal{G} = \mathbb{R}^d$).
Not only does the distributional Fourier transform (and it suffices to know the $\boldsymbol{S}_0'$-theory for this purpose) allow to define the Fourier transform for decaying objects (like functions in any of the $\boldsymbol{L}^p$-spaces), but also for periodic objects (such as periodic functions belonging locally to $\boldsymbol{L}^p$), even with different periods (which brings us already close to the discussion of almost periodic functions).
As a central topic let us therefore discuss the Fourier transform of periodic functions (or measures, or distributions) as the "infinite limit" of it's periodic repetitions. First of let us recall that it is easy to find for each lattice $\Lambda = \boldsymbol{A} * \mathbb{Z}^d$, for some non-singular $d \times d$-matrix $\boldsymbol{A}$ a fundamental domain (equal to $Q = \boldsymbol{A} * [0,1)^d$) and also bounded partitions of unity of the form $(\varphi_\lambda)_{\lambda\in\Lambda} = (T_\lambda\varphi)_{\lambda\in\Lambda}$, with $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$ or even in $\mathcal{S}(\mathbb{R}^d)$. For the next lemma we need $\varphi \in \boldsymbol{S}_0(\mathbb{R}^d)$, or even better, in $\mathcal{S}(\mathbb{R}^d)$.

Lemma 52. *A function* f *(or distribution in* $\boldsymbol{S}_0'(\mathbb{R}^d)$*)* [44] *is periodic with respect to* $\Lambda \lhd \mathbb{R}^d$ *if and only if it is of the form*

$$f = \sum_{\lambda\in\Lambda} T_\lambda f^\circ, \tag{121}$$

for some compactly supported pseudo-measure $f^\circ \in \mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$.

[44] A distribution from $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ which is $\Lambda-$periodic for any co-compact lattice Λ in $\mathbb{R}^d$ does in fact belong automatically to $\boldsymbol{S}_0'(\mathbb{R}^d)$!

Proof. If f is Λ-periodic, i.e., if $T_\lambda f = f$ for all $\lambda \in \Lambda$, then we can choose $f^\circ = f\varphi$, for a function φ with compact support, generating a Λ-BUPU as described above, because

$$f = \sum_{\lambda\in\Lambda} T_\lambda(T_{-\lambda}(f\varphi_\lambda)) = \sum_{\lambda\in\Lambda} T_\lambda f^\lambda = \sum_{\lambda\in\Lambda} T_\lambda f^\circ.$$

because $f^\lambda = T_{-\lambda}(f\cdot\varphi_\lambda) = T_{-\lambda}f \cdot T_{-\lambda}\varphi_\lambda = f\cdot\varphi =: f^\circ$ by the periodicity of f.
Conversely, let f° a compactly supported pseudo-measure or even in $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$. Since $\boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^\infty) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \boldsymbol{\ell}^1) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^\infty) = \boldsymbol{S}_0'$ the partial sums of the periodization are both uniformly in $\boldsymbol{S}_0'$ as well as w^*-convergent, as can been seen from the interpretation

$$\lim_{F\to\Lambda} \sum_{\lambda\in F} T_\lambda f^\circ = \lim_{F\to\Lambda} \left((\sum_{\lambda\in F} \delta_\lambda) * f^\circ \right).$$

In fact, one can reduce the general case of w^*-convergence to the case where (first of all) f° as compact support, but in testing that the action on arbitrary test functions $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ those in $(\boldsymbol{S}_0)_c = \boldsymbol{A}_c(\mathbb{R}^d)$ are sufficient.
PROBABLY SOME MORE DETAILS TO BE GIVEN □

We remark that a compactly supported pseudo-measure has the property that its Fourier transform is indeed a bounded and continuous function. Indeed, we can find some $\varphi \in \boldsymbol{S}_0(\mathbb{R}^d)$ such that $f^\circ\cdot\varphi$, hence $\mathcal{F}(f^\circ\cdot\varphi) = \widehat{f^\circ} * \hat\varphi \in \boldsymbol{L}^\infty * \boldsymbol{S}_0 \subseteq \boldsymbol{C}_b(\mathbb{R}^d)$. One can also show that the Fourier coefficients of the periodic version are just the values $\widehat{f^\circ}$ over the orthogonal lattice $\Lambda^\perp$. Let us describe this in detail:

Using now the fact that the distributional FT of $\sqcup\!\sqcup_\Lambda$ coincides with a multiple of $\sqcup\!\sqcup_{\Lambda^\perp}$, i.e., $\mathcal{F}\sqcup\!\sqcup_\Lambda = C_\Lambda \sqcup\!\sqcup_{\Lambda^\perp}$ in the $\boldsymbol{S}_0'$-sense, we find that for any periodic function or distribution f we have

FT (122) $$\mathcal{F}f = \mathcal{F}(\sqcup\!\sqcup_\Lambda * f^\circ) = \mathcal{F}(\sqcup\!\sqcup_\Lambda)\cdot \hat{f^\circ} = C_\Lambda \sqcup\!\sqcup_{\Lambda^\perp} \cdot \hat{f^\circ}.$$

Consequently we have $\mathrm{supp}(\hat f) \subseteq \mathrm{supp}(\sqcup\!\sqcup_{\Lambda^\perp}) = \Lambda^\perp$. In fact, we can give a more explicit description of $\hat f$: by carrying out the pointwise multiplication $\sqcup\!\sqcup_{\Lambda^\perp}\cdot \hat{f^\circ}$ we find that

$$\hat f = \sum_{\lambda^\perp\in\Lambda^\perp} \hat{f^\circ}(\lambda^\perp)\, \delta_{\lambda^\perp}.$$

But for $f^\circ \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$ (by the Hausdorff-Young principle for generalized amalgams) we know that $\hat{f^\circ} \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^\infty) \subset \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^\infty) \subset \boldsymbol{C}_b(\mathbb{R}^d)$. Hence we can even claim that $\hat f$ is a sum of Dirac measures located at the points of $\Lambda^\perp$, with a bounded sequence of coefficients in $\boldsymbol{\ell}^\infty(\Lambda)$.
In fact (this has to be shown separately), one can be shown that this is a complete *characterization* all the tempered distributions $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ with $\mathrm{supp}(\sigma) \subseteq \Lambda^\perp$.

Lemma 53. *(Characterization of distributions supported on discrete subgroups) A distribution $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ satisfies* $\mathrm{supp}(\sigma) \subseteq \Lambda$ *if and only if it is of the form*

$$\sigma = \sum_{\lambda\in\Lambda} c_\lambda \delta_\lambda$$

for some sequence $\mathbf{c} = (c_\lambda)_{\lambda\in\Lambda} \in \boldsymbol{\ell}^\infty(\Lambda)$.

Summarizing we have a mapping from the periodic elements (in $\boldsymbol{S}_0'(\mathbb{R}^d)$) into $\boldsymbol{\ell}^\infty(\Lambda)$, of the form $f \to (\hat{f^\circ}(\lambda))_{\lambda\in\Lambda}$ (which is of course independent of the choice of f°). Assume we have a regular distribution, "coming from" some $f \in \boldsymbol{L}^1(\mathbb{U})$. Let us discuss (for simplicity) the situation for $d = 1$ and $\Lambda = \mathbb{Z}$. Then we have $Q = [0, 1)$ and we can choose $f^\circ = f \cdot \mathbf{1}_Q \in \boldsymbol{L}^1(\mathbb{R})$. Since $\mathbb{Z}^\perp = \mathbb{Z}$ the Fourier transform of a periodic (local) $\boldsymbol{L}^1$-function are a sequence on $\mathbb{Z}$ again: $\hat{f^\circ}(n) = \int_0^1 f(t)e^{-2\pi it}dt$ which coincides with the usual ("classical") definition of Fourier coefficients for locally integrable, $\mathbb{Z}$-periodic functions.
Note that in our context local square (or generally p-) integrability corresponds to additional properties on f° which in such a case will belong locally to $\boldsymbol{L}^2(\mathbb{R})$ (resp. $\boldsymbol{L}^p(\mathbb{R})$), resp. to $\boldsymbol{L}^2(\mathbb{U})$ or $\boldsymbol{L}^p(\mathbb{U})$. In our terminology $f^\circ \in \boldsymbol{W}(\boldsymbol{L}^2, \boldsymbol{\ell}^1)(\mathbb{R})$ resp. $W(\boldsymbol{L}^p, \ell^1)(\mathbb{R})$, and again by the Hausdorff-Young principle we obtain that $\hat{f^\circ} \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^\infty)(\mathbb{R})$, or the Fourier coefficients of f resp. the values of $(\hat{f^\circ}(n))_{n\in\mathbb{Z}}$ belong to $\boldsymbol{\ell}^{p'}(\mathbb{Z})$.

A topic of interest in connection with standard spaces is the following one, which is based on the fact that for any (restricted) standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ we have the following chain of continuous inclusion:

WB-incl0 (123) $$\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^1) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^\infty)$$

and the fact that we have

WB-incl2 (124) $$\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^p) \hookrightarrow \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^q) \quad \text{if} \quad p \le q.$$

w-uppindex **Definition 46.** Given a (restricted) standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ we define the *lower* resp. *upper index* as follows:

low-index (125) $$\text{lowind}(\boldsymbol{B}) := \sup\{\, p \,|\, \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^p) \hookrightarrow \boldsymbol{B}\}$$

and

upp-index (126) $$\text{uppind}(\boldsymbol{B}) := \inf\{\, q \,|\, \boldsymbol{B} \hookrightarrow \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^q)\}$$

In most cases the supremum resp. infimum will *not be attained*. However, we will have in any case

lowlequpp (127) $$\text{lowind}_{\boldsymbol{B}} \le \text{uppind}_{\boldsymbol{B}}$$

The following condition is in general slightly stronger than the case of equality of indices:

Definition 47. A Banach space is called to be of *global type* p if one has

deftypep (128) $$\boldsymbol{B} = \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^p).$$

Aside from the trivial facts that $\boldsymbol{L}^p(\mathbb{R}^d)$ is of course of type p for any p, one can check that $\boldsymbol{M}_b(\mathbb{R}^d)$ is of type 1, while the usual $\boldsymbol{L}^2$-Sobolev spaces are of type 2. They have therefore been called $\boldsymbol{\ell}^2$-puzzles by P. Tchamitchian in [39, 40] (?)

One of the interesting and non-trivial facts (no proof is given here) is that one has (the most interesting perhaps being the case $p = 1$), see [21]:

Lemma 54. *For* $1 \leq p \leq 2$ $\text{lowind}(\mathcal{F}\boldsymbol{L}^p) = p = \text{lowind}(\mathcal{F}\boldsymbol{L}^{p'})$ *while* $\text{uppind}(\mathcal{F}\boldsymbol{L}^p) = p' = \text{uppind}(\mathcal{F}\boldsymbol{L}^{p'})$. *Hence, except for the case* $p = 2$ *the space* $\mathcal{F}\boldsymbol{L}^p$ *is never of a particular type.*

Another interesting family of spaces is the set of all "multipliers" on $\boldsymbol{L}^p$, for say $1 \leq p < \infty$, which we denote by $H_{\boldsymbol{L}^1}(\boldsymbol{L}^p, \boldsymbol{L}^p)$, which is indeed another (restricted) standard space. It is well known that it coincides (by duality) with $H_{\boldsymbol{L}^1}(\boldsymbol{L}^{p'}, \boldsymbol{L}^{p'})$, that $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1) = \boldsymbol{M}_b(\mathbb{R}^d)$ and $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^2, \boldsymbol{L}^2) = \mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$. Hence one may *conjecture* that lowind and uppind of these spaces equals p and p', but to my knowledge nothing is known about it for $1 < p < 2$.
Another interesting question could be the upper and lower index for modulation spaces (which can be shown to be of local type $\mathcal{F}\boldsymbol{L}^q$). Since the space $\boldsymbol{M}^p(\mathbb{R}^d) = \boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$ for $p = q$ are Fourier invariant an coincide with $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \boldsymbol{\ell}^p)$ they are clearly of *type* p.

17. Ideas on BUPUS, Wiener Amalgam and Spline-Type Spaces

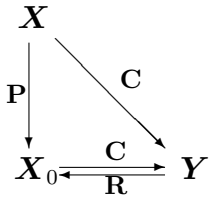
BUPUs are a universal tool for many questions in the theory of function spaces, they are quite useful in order to develop concepts for (conceptual) harmonic analysis, and they are crucial for the definition of Wiener amalgam spaces.
Here is a short list of properties of these families that make them so important:

- They can be defined over arbitrary locally compact groups $\mathscr{G}$. In fact, there use is implicit in the construction of the Haar measure following Cartan resp. A. Weil;
- By means of BUPUs it is easy to show that the discrete measures are w^*-dense in the space of bounded measures $(\boldsymbol{M}_b(\mathscr{G}), \|\cdot\|_{\boldsymbol{M}_b})$.
- Obviously they are extremely useful in defining Wiener amalgam spaces (the discrete description is much more general, at least in order to introduce the spaces, than the "continuous" description);
- Spline-Type are quite important as well; they are obtained as "closed linear span" of a set of function and their translates, within some larger function space, say $\left(\boldsymbol{L}^p(\mathbb{R}^d),\, \|\cdot\|_p\right)$. If we find a *Riesz projection basis* for such a space than typically the discrete $\boldsymbol{\ell}^p$-norm on the coefficients, and the $\boldsymbol{L}^p(\mathbb{R}^d)$ or also the $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d)$-norm are equivalent on the corresponding spline-type space.

For most purposes any result that is valid for regular BUPUs on $\mathbb{R}^d$ can be easily transferred to general statements about arbitrary BUPUs over locally compact, or at least locally compact Abelian groups.

18. Riesz Bases and Banach Frames

In the terminology introduced in XX this means that $\mathbf{R}$ is injective, but not surjective, and $\mathbf{C}$ is a left inverse of $\mathbf{R}$. Thus we have the following commutative diagram.



19. Historical Notes

This is of course a very subjective area. From the point of view taken in these notes functional analysis can be used (practically speaking: jus the theory of Banach and Hilbert spaces is used, together with the main principles of the theory of Banach spaces, operators, completions and the like, such as the *wst*-compactness of the closed unit ball in a dual Banach space $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$).

20. Open Questions, things to do

TEST: $\boldsymbol{B}^*, (\boldsymbol{B}^*, \|\cdot\|_{\boldsymbol{B}^*})$

Show that $\boldsymbol{C}_{ub}(G))$ can be characterized as the subset of $\big(\boldsymbol{L}^\infty(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ with continuous shift, i.e. $\sigma \in \boldsymbol{L}^\prime_1$ belongs to $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ if and only if

ubinLinfRd (129) $$\|\sigma - T_x\sigma\|_{\boldsymbol{L}^\prime_1} \to 0 \quad \text{for} \quad |x| \to 0,$$

viewing $\boldsymbol{L}^\infty$ as a definition as the dual space of $\big(\boldsymbol{L}^1,\, \|\cdot\|_1\big)$.
The argument is essentially of the following nature. First we verify that $\boldsymbol{M} * \boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{L}^1(\mathbb{R}^d)$ (either because $\boldsymbol{L}^1(\mathbb{R}^d)$ is the set of bounded measures with continuous shifts, which is preserved under convolution given as the dual action from the action on $\boldsymbol{C}_0(\mathbb{R}^d)$, lifted to the measures, or just by using the fact that $\boldsymbol{L}^1(\mathbb{R}^d)$ is a homogeneous Banach space, hence $\boldsymbol{M}(\mathbb{R}^d) \bullet \boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{L}^1(\mathbb{R}^d)$, where the abstract action is just convolution. This action is again by a second adjointness relation lifted to $\big(\boldsymbol{L}^1,\, \|\cdot\|_1\big)$ (which is just the natural continuation of the convolution of ordinary functions to the *dual space* of $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$). The elements satisfying CubinLinfRd 129 can be approximated in the norm by elements of the form $g * h \in \boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{L}^\infty(\mathbb{R}^d) \subset \boldsymbol{C}_{ub}(\mathbb{R}^d)$. Since this is a closed subspace of $\boldsymbol{L}^\infty(\mathbb{R}^d)$ (cf. lemma below) the limit, i.e. h must belong (more precisely: must be represented) by a function in $\boldsymbol{C}_{ub}(\mathbb{R}^d)$.
We need the following basic observation:

Lemma 55. *The embedding $h \to \sigma_h$: given by*

CbtoLinf (130) $$\sigma_h(f) = \int_{\mathbb{R}^d} f(x)h(x)dx$$

from $\big(\boldsymbol{C}_b(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ into $\boldsymbol{L}^\prime_1$ (with its natural norm) is isometric, i.e.,

CbLinfisom (131) $$\|h\|_\infty = \sup_{\|g\|_1 \le 1} |\int_{\mathbb{R}^d} g(x)h(x)dx.$$

f_abs_cont **Definition 48** (absolutely continuous, [25], 18.10)**.** Let f be a complex-valued function defined on a subinterval J of $\mathbb{R}$. Suppose that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$(i) \sum_{k=1}^{n} |f(d_k) - f(c_k)| < \epsilon$$

for every finite, pairwise disjoint, family $\{]c_k, d_k[\}_{k=1}^n$ of open subintervals of J for which

$$(ii) \sum_{k=1}^{n} (d_k - c_k) < \delta.$$

Then f is said to be absolutely continuous on J.

_variation **Theorem 28** ([25], 18.12)**.** *Any complex-valued absolutely continuous function f defined on $[a, b]$ has finite variation on $[a, b]$*

meas_deriv **Theorem 29** ([25], 18.14)**.** *If f is a real-valued, nondecreasing function on $[a, b]$, then f' is Lebesgue measurable and*

$$\int_a^b f'(x)dx \le f(b) - f(a).$$

If g is a complex-valued function of finite variation on $[a, b]$, then $g' \in \boldsymbol{L}^1([a, b])$.

derivative **Theorem 30** ([25], 18.15)**.** *Let f be an absolutely continous complex-valued function on $[a,b]$ and suppose that $f'(x)=0$ almost everywhere in $]a,b[$. Then f is a constant.*

l_calculus **Theorem 31** (Fundamental theorem of the integral calculus for Lebesgue integrals, [25], 18.16)**.** *Let f be a complex-valued, absolutely continous function on $[a,b]$. Then $f' \in \boldsymbol{L}^1([a,b])$ and*

al_theorem
$$f(x) = f(a) + \int_a^x f'(t)dt \tag{132}$$

for every $x \in [a,b]$.

f_integral **Theorem 32** ([25], 18.17)**.** *A function f on $[a,b]$ has the form*

$$f(x) = f(a) + \int_a^x \varphi(t)dt$$

for some $\varphi \in \boldsymbol{L}^1([a,b])$ if and only if f is absolutely continous on $[a,b]$. In this case we have $\varphi(x) = f'(x)$ almost everywhere on $]a,b[$.

comments on absolutely continuous functions, classically

ntegral_v1 **Theorem 33** ([25], 18.18)**.** *A function f on $\mathbb{R}$ has the form*

$$f(x) = \int_{-\infty}^x \varphi(t)dt$$

for some $\varphi \in \boldsymbol{L}^1(\mathbb{R})$ if and only if f is absolutely continous on $[-A,A]$ for all $A>0$, $V_{-\infty}^{\infty} f$ is finite, and $\lim_{x\to-\infty} f(x) = 0$.

m_part_int **Theorem 34** ([25], 18.19)**.** *Let f, g be functions in $\boldsymbol{L}^1([a,b])$, let*

$$F(x) = \alpha + \int_a^x f(t)dt,$$

and let

$$G(x) = \beta + \int_a^x g(t)dt.$$

Then

$$\int_a^b G(t)f(t)dt + \int_a^b g(t)F(t)dt = F(b)G(b) - F(a)G(a).$$

r_part_int **Corollary 11** ([25], 18.20)**.** *Let f and g be absolutely continuous functions on $[a,b]$. Then*

$$\int_a^b f(t)g'(t)dt + \int_a^b f'(t)g(t)dt = f(b)g(b) - f(a)g(a)$$

splin-abs1 (133) $$||\operatorname{Sp}_\Psi(f)| - |f||(x) \leq |(\operatorname{Sp}_\Psi(f) - f)(x)| \to 0 \quad \text{for} \quad |\Psi| \to 0.$$

It is of course easy to find some function $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|h\|_\infty = 1$ and $h(x_i) = |k(x_i)|/k(x_i)$ at all the points where (x_i) is the family of centers of the BUPU Φ (and the condition is of course only applied when $k(x_i) \neq 0$. Then $\operatorname{Sp}_\Psi(|k|) = \operatorname{Sp}_\Psi(hk)$. Alternative: LOOK at the function (to be embedded into $\boldsymbol{C}_0'(\mathbb{R}^d)$) by considering the set where it is strictly positive and strictly negative, with some margin.
ANOTHER strategy (comparable with this one) is to show that $|k|$ can be norm-approximated (uniformly, with fixed compact support) by functions of the form $h \cdot k$, where h is a continuous version of the $sign(k(x))$-function. (can we do it using find BUPUS?, or does one need Tietze-Urysohn?).

LEFT OVER MATERIAL

antorovich **Theorem 35.** [Kantorowich Lemma] *Let T_α be a strongly convergent and bounded sequence of invertible operators between Banach spaces, with limit T_0, which is assumed to be invertible itself. Then the inverse operators are strongly convergent as well if the inverse operators are uniformly bounded. If we consider only sequences this is a criterion, because then the strong convergence of $T_n^{-1}(y) \to T_0^{-1}(y)$ for every y implies uniform boundedness of the sequence T_n^{-1}.*

QUESTION: For tight nets of bounded measures w^*-convergence implies pointwise and uniform over compact convergence of their FTs. But also the converse is true!!!! (Exercise). [for non-tight families this is not true, just think of the case $\delta_n, n \to \infty$!

An elementary proof showing that the Gauss function $g_0(t) = e^{-\pi|t|^2}$ is mapped itself by the Fourier transform has been given by Georg Zimmermann, see

http://www.univie.ac.at/NuHAG/FEICOURS/ws0607/efoft.pdf

APPENDIX

Definition 49. A net $\{f_\alpha\}_{\alpha\in A}$ in a Banach space $\mathscr{X}$ is said to be a Cauchy net if for every $\epsilon > 0$, there is a α_0 in A such that $\alpha_1, \alpha_2 \geq \alpha_0$ implies $||f_{\alpha_1} - f_{\alpha_2}|| < \epsilon$.

Proposition 5. *In a Banach space each Cauchy net is convergent.*

Proof. Let $\{f_\alpha\}_{\alpha\in A}$ be a Cauchy net in the Banach space $\mathscr{X}$. Choose α_1 such that $\alpha \geq \alpha_1$ implies $||f_\alpha - f_{\alpha_1}|| < 1$. Having chosen $\{\alpha_k\}_{k=1}^n$ in A, choose $\alpha_{n+1} \geq \alpha_n$ such that $\alpha \geq \alpha_{n+1}$ implies

$$||f_\alpha - f_{\alpha_{n+1}}|| < \frac{1}{n+1}.$$

The sequence $\{f_{\alpha_n}\}_{n=1}^\infty$ is clearly Cauchy and, since $\mathscr{H}$ is complete, there exists f in $\mathscr{H}$ such that $\lim_{n\to\infty} f_{\alpha_n} = f$.
It remains to prove that $\lim_{\alpha\in A} f_\alpha = f$. Given $\epsilon > 0$, choose n such that $\frac{1}{n} < \frac{\epsilon}{2}$ and $\|f_{\alpha_n} - f\| < \frac{\epsilon}{2}$. Then for $\alpha \geq \alpha_n$ we have

$$||f_\alpha - f|| \leq ||f_\alpha - f_{\alpha_n}|| + ||f_{\alpha_n} - f|| < \frac{1}{n} + \frac{\epsilon}{2} < \epsilon.$$

□

f_conv_sum **Definition 50.** Let $\{f_\alpha\}_{\alpha\in A}$ be a set of vectors in the Banach space $\mathscr{X}$. Let $\mathscr{F} = \{F \subset A : F \text{ finite}\}$. If we define $F_1 \leq F_2$ for $F_1 \subset F_2$, then $\mathscr{F}$ is a directed set. For each F in $\mathscr{F}$, let $g_F = \sum_{\alpha\in F} f_\alpha$. If the net $\{g_F\}_{F\in\mathscr{F}}$ converges to some g in $\mathscr{H}$, then the sum $\sum_{\alpha\in A} f_\alpha$ is said to converge and we write $g = \sum_{\alpha\in A} f_\alpha$.

Proposition 6. *If $\{f_\alpha\}_{\alpha\in A}$ is a set of vectors in the Banach space $\mathscr{X}$ such that $\sum_{\alpha\in A} ||f_\alpha||$ converges in the real line $\mathbb{R}$, then $\sum_{\alpha\in A} f_\alpha$ converges in $\mathscr{X}$.*

Proof. It suffices to show, in the notation of Definition def_conv_sum 50, that the net $\{g_F\}_{F\in\mathscr{F}}$ is Cauchy. Since $\sum_{\alpha\in A} ||f_\alpha||$ converges, for $\epsilon > 0$, there exists F_0 in $\mathscr{F}$ such that $F \geq F_0$ implies

$$\sum_{\alpha\in F} ||f_\alpha|| - \sum_{\alpha\in F_0} ||f_\alpha|| < \epsilon.$$

Thus for $F_1, F_2 \geq F_0$ we have

$$\begin{aligned}
||g_{F_1} - g_{F_2}|| &= \left\| \sum_{\alpha\in F_1} f_\alpha - \sum_{\alpha\in F_2} f_\alpha \right\| \\
&= \left\| \sum_{\alpha\in F_1\setminus F_2} f_\alpha - \sum_{\alpha\in F_2\setminus F_1} f_\alpha \right\| \\
&\leq \sum_{\alpha\in F_1\setminus F_2} ||f_\alpha|| + \sum_{\alpha\in F_2\setminus F_1} ||f_\alpha|| \\
&\leq \sum_{\alpha\in F_1\cup F_2} ||f_\alpha|| - \sum_{\alpha\in F_0} ||f_\alpha|| < \epsilon.
\end{aligned}$$

Therefore, $\{g_F\}_{F\in\mathscr{F}}$ is Cauchy and $\sum_{\alpha\in A} f_\alpha$ converges by definition. □

Corollary 12. *A normed linear space $\mathscr{X}$ is a Banach space if and only if for every sequence $\{f_n\}_{n=1}^\infty$ of vectors in $\mathscr{X}$ the condition $\sum_{n=1}^\infty ||f_n|| < \infty$ implies the convergence of $\sum_{n=1}^\infty f_n$.*

Proof. If $\mathscr{X}$ is a Banach space, then the conclusion follows from the preceding proposition. Therefore, assume that $\{g_n\}_{n=1}^\infty$ is a Cauhcy sequence in a normed linear space $\mathscr{X}$ in which the series hypothesis is valid. Then we may choose a subsequence $\{g_{n_k}\}_{k=1}^\infty$ such that $\sum_{k=1}^\infty ||g_{n_{k+1}} - g_{n_k}|| < \infty$ as follows: Choose n_1 such that for $i, j \geq n_1$ we have $||g_i - g_j|| < 1$; having chosen $\{n_k\}_{k=1}^N$ choose $n_{N+1} > n_N$ such that $i, j > n_{N+1}$ implies $||g_i - g_j|| < 2^{-N}$. If we set $f_k = g_{n_k} - g_{n_{k-1}}$ for $k > 1$ and $f_1 = g_{n_1}$, then $\sum_{k=1}^\infty ||f_k|| < \infty$, and the hypothesis implies that the series $\sum_{k=1}^\infty f_k$ converges. It follows from the definition of convergence that the sequence $\{g_{n_k}\}_{k=1}^\infty$ converges in $\mathscr{X}$ and hence so also does $\{g_n\}_{n=1}^\infty$. Thus $\mathscr{X}$ is complete and hence a Banach space. □

21. Fourier Analysis over finite groups approximating LCA groups

On the on hand there is the following *structure theorem* for LCA (*locally compact Abelian*) groups $\mathscr{G}$. Probably details can be found in Hewitt-Ross ([24])

structure1 **Theorem 36.** *Every LCA group can be approximated by elementary LCA groups, which are of the form $\mathbb{Z}^k \times \mathbb{R}^d \times \cdots$. More precisely*
For every LCA group $\mathscr{G}$ there exist arbitrary small compact subgroups $K \lhd \mathscr{G}$ and open subgroups $H \lhd \mathscr{G}$ such that H/K is an elementary group.

This can be used to define the *Schwartz-Bruhat* space (the natural generalization of $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ to LCA groups, see [7]), a much more convenient characterization is given by M.S. Osborne in [30].
It is heavily used in [35]. D. Poguntke has shown that $\boldsymbol{\mathcal{S}}(\mathscr{G}) \subset \boldsymbol{S}_0(\mathscr{G})$ for general LCA groups.

22. Wiener's algebra $\boldsymbol{W}(\mathscr{G})$ over LC groups

Already the construction of the Haar measure (cf. Cartier's proof the existence of an invariant Haar measure, cf. [41], or [9]? uses this space implicitly).

References

amfupa00 [1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Clarendon Press, 2000.

be96 [2] J. J. Benedetto. *Harmonic Analysis and Applications.* Stud. Adv. Math. CRC Press, Boca Raton, FL, 1996.

becz09 [3] J. J. Benedetto and W. Czaja. *Integration and Modern Analysis.* Birkhäuser, 2009.

bo55-1 [4] S. Bochner. *Harmonic Analysis and the Theory of Probability.* University of California Press, 1955.

br75 [5] L. H. Brandenburg. On identifying the maximal ideals in Banach algebras. *J. Math. Anal. Appl.*, 50:489–510, 1975.

brfe83 [6] W. Braun and H. G. Feichtinger. Banach spaces of distributions having two module structures. *J. Funct. Anal.*, 51:174–212, 1983.

br61 [7] F. Bruhat. Distributions sur un groupe localement compact et applications à l etude des représentations des groupes p-adiques. *Bull. Soc. Math. France*, 89:43–75, 1961.

bune71 [8] P. L. Butzer and R. J. Nessel. *Fourier Analysis and Approximation. Vol. 1: One-dimensional Theory.* Birkhäuser, Stuttgart, 1971.

de02 [9] A. Deitmar. *A First Course in Harmonic Analysis.* Universitext. Springer, New York, NY, 2002.

ey64 [10] P. Eymard. L'algèbre de Fourier d'un groupe localement compact. *Bull. Soc. Math. France*, 92:181–236, 1964.

fe79 [11] H. G. Feichtinger. Gewichtsfunktionen auf lokalkompakten Gruppen. *Sitzungsber.d.österr. Akad.Wiss.*, 188:451–471, 1979.

fe79-3 [12] H. G. Feichtinger. Konvolutoren von $L^1(G)$ nach Lipschitz-Räumen. *Anz. sterreich. Akad. Wiss. Math.-Naturwiss. Kl.*, 6:148–153, 1979.

fe80 [13] H. G. Feichtinger. Un espace de Banach de distributions tempérées sur les groupes localement compacts abéliens. *C. R. Acad. Sci. Paris S'er. A-B*, 290(17):791–794, 1980.

fe81-3 [14] H. G. Feichtinger. On a new Segal algebra. *Monatsh. Math.*, 92:269–289, 1981.

fe84 [15] H. G. Feichtinger. Compactness in translation invariant Banach spaces of distributions and compact multipliers. *J. Math. Anal. Appl.*, 102:289–327, 1984.

fe89-1 [16] H. G. Feichtinger. Atomic characterizations of modulation spaces through Gabor-type representations. In *Proc. Conf. Constructive Function Theory*, volume 19 of *Rocky Mountain J. Math.*, pages 113–126, 1989.

feka04 [17] H. G. Feichtinger and N. Kaiblinger. Varying the time-frequency lattice of Gabor frames. *Trans. Amer. Math. Soc.*, 356(5):2001–2023, 2004.

feno03 [18] H. G. Feichtinger and K. Nowak. A first survey of Gabor multipliers. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Appl. Numer. Harmon. Anal., pages 99–128. Birkhäuser, 2003.

fezi98 [19] H. G. Feichtinger and G. Zimmermann. A Banach space of test functions for Gabor analysis. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170, Boston, MA, 1998. Birkhäuser Boston.

gawi99 [20] C. Gasquet and P. Witomski. *Fourier Analysis and Applications Filtering, Numerical Computation, Wavelets Transl From the French by R Ryan.* Springer, 1999.

gr92-2 [21] P. Gröbner. *Banachräume glatter Funktionen und Zerlegungsmethoden.* PhD thesis, University of Vienna, 1992.

gr01 [22] K. Gröchenig. *Foundations of Time-Frequency Analysis.* Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2001.

gr07 [23] K. Gröchenig. Weight functions in time-frequency analysis. In L. Rodino and et al., editors, *Pseudodifferential Operators: Partial Differential Equations and Time-Frequency Analysis*, volume 52 of *Fields Inst. Commun.*, pages 343–366. Amer. Math. Soc., Providence, RI, 2007.

hero70 [24] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis. Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups.* Springer, Berlin, Heidelberg, New York, 1970.

hest69 [25] E. Hewitt and K. Stromberg. *Real and Abstract Analysis. A Modern Treatment of the Theory of Functions of a Real Variable. 2nd Printing corrected.* Springer, Berlin, Heidelberg, New York, 1969.

ho79-4 [26] R. B. Holmes. Mathematical foundations of signal processing. *SIAM Rev.*, 21(3):361–388, jul 1979.

ka76 [27] Y. Katznelson. *An Introduction to Harmonic Analysis. 2nd corr. ed.* Dover Publications Inc, New York, 1976.

ka04-1 [28] Y. Katznelson. *An Introduction to Harmonic Analysis. 3rd Corr. ed.* Cambridge University Press, 2004.

la71-2 [29] R. Larsen. *An Introduction to the Theory of Multipliers.* Springer, 1971.

os75 [30] M. S. Osborne. On the Schwartz-Bruhat space and the Paley-Wiener theorem for locally compact Abelian groups. *J. Funct. Anal.*, 19:40–49, 1975.

pe76 [31] J. Peetre. *New thoughts on Besov spaces.* Duke University Mathematics Series, No. 1. Mathematics Department, Duke University, 1976.

pl29 [32] A. Plessner. Eine Kennzeichnung der totalstetigen Funktionen. *J. Reine Angew. Math.*, 160:26–32, 1929.

re68 [33] H. Reiter. *Classical Harmonic Analysis and Locally Compact Groups.* Clarendon Press, Oxford, 1968.

re71 [34] H. Reiter. *L^1-algebras and Segal Algebras.* Springer, Berlin, Heidelberg, New York, 1971.

re78 [35] H. Reiter. Über den Satz von Weil–Cartier. *Monatsh. Math.*, 86:13–62, 1978.

rest00 [36] H. Reiter and J. D. Stegeman. *Classical Harmonic Analysis and Locally Compact Groups. 2nd ed.* Clarendon Press, Oxford, 2000.

ri69-1 [37] M. A. Rieffel. Multipliers and tensor products of L^p-spaces of locally compact groups. *Studia Math.*, 33:71–82, 1969.

sh71 [38] H. S. Shapiro. *Topics in Approximation Theory*, volume 187 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1971.

tc84 [39] P. Tchamitchian. Généralisation des algèbres de Beurling. *Ann. Inst. Fourier (Grenoble)*, 34(4):151–168, 1984.

tc87 [40] P. Tchamitchian. tude dans un cadre hilbertien des algèbres de Beurling munies d'un poids radial à croissance rapide. (Study in a Hilbertian framework of Beurling algebras equipped with a radial weight having rapid growth). *Ark. Mat.*, 25:295–312, 1987.

vo99 [41] J. von Neumann. *Invariant Measures.* Providence, RI: American Mathematical Society (AMS). xiv, 1999.

wa77 [42] H.-C. Wang. *Homogeneous Banach Algebras.* Marcel Dekker, New York, Basel, 1977.

we07-1 [43] F. Weisz. Inversion of the short-time Fourier transform using Riemannian sums. *J. Fourier Anal. Appl.*, 13(3):357–368, 2007.

we52 [44] J. G. Wendel. Left centralizers and isomorphisms of group algebras. *Pacific J. Math.*, 2:251–261, 1952.

za53 [45] A. C. Zaanen. *Linear Analysis Measure and Integral, Banach and Hilbert Space, Linear Integral Equations*, volume 2 of *Bibliotheca mathematica; a series of monographs on pure and applied mathematics.* North-Holland Publishing Company, Amsterdam, 1953.

za67 [46] A. C. Zaanen. *Integration.* North–Holland, Amsterdam, Completely revises edition of: An introduction to the theory of integration. edition, 1967.

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