Hans G. Feichtinger: Harmonic Analysis based on Functional Analysis

http://www.univie.ac.at/nuhag-php/login/skripten/data/AngAnal15Skript.pdf
and for general functional analysis:
http://www.univie.ac.at/NuHAG/FEICOURS/ws1314/FAws1314Fei.pdf
VERSION of May 2017, hgfei

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We keep updating these notes, following feedback of readers

Typos and inconsistences to be discussed: May&June 2017

The use of the symbol $\boldsymbol{C}_c(G)$, $\boldsymbol{C}_c(\mathbb{R}^d)$, to be checked

SEE SECTIONS MARKED BY "SC"

ARE ALL THE "LATER ON" PROMISES FULFILLED EVENTUALLY!?

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A FUNCTIONAL ANALYTIC APPROACH TO APPLIED ANALYSIS: NUHAG: VERSION OF SEPTEMBER 2017

HANS G. FEICHTINGER

ABSTRACT. **Linear algebra** is dealing with *finite dimensional vector spaces and linear mappings*. In order to describe them one uses bases and *matrices* and thus connection to systems of linear equations is given. Solving such a system is more or less the same as inverting the corresponding linear mapping.

Analysis talks a lot about limits, series, continuity, differentiability, and so on. Functional analysis is dealing with non-finite dimensional vector spaces, but in order to express the possibility of infinite series representation one has to explain how limits are to be understood. Instead of using a specific set of linear functionals (the dual basis to the given basis) one often takes all possible linear functionals on a given normed space. And in order to be in a situation more similar to $\mathbb{R}$ (complete!) and not like $\mathbb{Q}$ (rationals) one often assumes that one is working with *Banach spaces*. Occasionally *Hilbert spaces* are used, which allow to define concepts of *orthogonality*, *unitary* mappings etc.. In order to do this properly one has to make use of basic concepts from **functional analysis**. The corresponding concepts are however also of relevance in **numerical analysis**, where one is usually approximating a continuous problem by a corresponding finite dimensional version of the problem (e.g. obtained by *discretization*).

We do not assume that participants of this course have already taken a course in functional or numerical analysis, but we will try to connect the course material with these fields. **The course will emphasize the use of various Banach spaces of functions, their properties and their role for the description of real world problems.**

Note: Usually the role of the Lebesgue spaces $\boldsymbol{L}^p$, $1 \leq p \leq \infty$ is overemphasized. Obviously the cases $p = 1, 2, \infty$ are very important, also for applications, but we will see that many other spaces are at least equally useful.

2000 *Mathematics Subject Classification.* Primary ; Secondary .

Key words and phrases. Fourier transform, convolution, linear time-invariant systems, bounded measures, Banach modules, approximate units, Banach algebras, essential Banach modules, discrete measures, Plancherel, Fourier Stieltjes transform, compactness, tight, w^*-convergence, strong operator topology, approximate units, support.

1. MATERIAL TO BE USED/REMOVED LATER

Adjustments to be done (list of 15.05.2017, hgfei)

- uniform symbols for G resp. $\mathscr{G}, \widehat{\mathscr{G}}$;
- uniform notation for FLIP operator $\check{f}$ versus $f^{\checkmark}$.

We will write $\mathrm{Sp}_\Psi(f)$ or $\mathrm{Sp}_\Psi f$ etc.; equally $T_x f$ of $T_x(f)$...

1.1. **Prerequisites.** Normed spaces, Banach spaces, Hilbert spaces dual space, linear functionals, norm of linear functionals, bounded linear operators, operator norm (see [future] appendix).

1.2. **Operators and conventions.** Many of the operators defined below (such as translation, dilation, Fourier transform or convolution operators) should be considered as quite flexible, in the sense that the general setup of the presentation is made *in such a way that possible ambiguities* will not come to bear when the user makes his interpretation.

Of course this requires some care (e.g. in the definition of the "existence" of convolutions or pointwise products)

1.3. **The old abstract.** We present a functional analytic approach to harmonic analysis, avoiding heavy measure theoretic tools. The results are non-trivial even for the real line $\mathbb{R}$, but are formulated for the $d-$dimensional Euclidean space $\mathbb{R}^d$, viewed as a prototype for a locally compact Abelian (:LCA) group $\mathscr{G}$ (with the usual addition of vectors and the topology provided by the Euclidian metric).[1] We start with the description of $\boldsymbol{M}_b(\mathscr{G})$, the space of bounded linear measures, as the dual space of $\boldsymbol{C}_0$, which is naturally endowed with a convolution structure.

Convolution will first be defined for measures (via the identification of impulse responses with the corresponding translation invariant linear systems mapping $\boldsymbol{C}_0(\mathbb{R}^d)$ into $\boldsymbol{C}_0(\mathbb{R}^d)$) and only later specialized to "ordinary functions". The Fourier transform is then introduced as the action of such systems on their joint eigenvectors, namely the pure frequencies.

[1] Since any point has a basis of the neighborhood, consisting of the compact closed balls $B_r(x)$ of radius r around the point x this is a locally compact group.

sec:intro

2. Introduction

We start with a few symbols. First note that we will permanently make use of the fact that $\mathbb{R}^d$ is a locally compact Abelian group with respect to addition (dilation will come in only as a convenient but not crucial side aspect Such $\mathbb{R}^d$-specific parts are marked by the symbol $[\mathbf{Rd-specific!}]$!) This means that it is clear when a set is open or closed, and that every point has a basis of neighborhoods consisting of compact balls, the closure of the open balls $B_\gamma(x)$ of radius γ around each $x \in \mathbb{R}^d$. Moreover, clearly addition is continuous, i.e. $a_n \to a_0$ and $b_n \to b_0$ for $n \to \infty$ implies $a_n + b_n \to a_0 + b_0$ for $n \to infty$.

First we define the most simple algebra (pointwise, later on with respect to convolution) of continuous *"test functions"*. [2] Because of the *local compactness* of $\mathbb{R}^d$ the following space of *continuous functions with compact support* is a non-trivial (i.e. *not just* the zero-space) linear space of complex-valued functions on $\mathscr{G}$:

CcRddef1 **Definition 1.**

$$\boldsymbol{C}_c(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C}, \text{ continuous and with compact support}\} \quad .$$

Here we make use of the standard definition of the support [3] of a function:

defsupp1 **Definition 2.** The *support* of a continuous (!) function, in symbols $\mathrm{supp}(f)$, is defined as the closure of the set of "relevant points": [4]

$$\mathrm{supp}(f) := \{x \mid f(x) \neq 0\}^{-}.$$

Lemma 1. *(Ex to Def. defsupp1 2) A continuous, complex-valued function on $\mathbb{R}^d$ is in $\boldsymbol{C}_c(\mathbb{R}^d)$ if and only if there exists $R = R(f) > 0$ such that $f(x) = 0$ for all x with $|x| \geq R$.*

A list of symbols: Operators, names of operators, and their definitions:

Definition 3 (**Operators defined on functions).**

(1) translation by x is the operator T_x given by

translatdef
$$T_x f(z) = [T_x f](z) := f(z - x) \quad x, z \in \mathbb{R}^d; \tag{1}$$

(the graph is preserved but moved by the vector x to another position);

(2) involution $f \mapsto f^{\checkmark}$ with

checkdef1
$$f^{\checkmark}(z) := f(-z) \tag{2}$$

or using other symbols $f^{\checkmark}(z) = f(-z)$.

(3) modulation M_s: Multiplication with the character[5] $x \mapsto exp(2\pi i x \cdot x)$, i.e.

modopdef1
$$[M_s f](z) := e^{2\pi i s \cdot z} f(z) = \chi_s(t) f(z), \quad x, s \in \mathbb{R}^d; \tag{3}$$

(4) **[Rd-specific!]**dilation D_ρ (value preserving) and St_ρ (mass preserving); cf. below

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[2] They can be defined on any topological group, and the space is interesting and non-trivial for any locally compact group. So much of the material given below extends without difficulty to the setting of locally compact, or at least locally compact Abelian group, except for the statements which involve dilations

[3] Why one takes the closure in the above definition will become more clear later on, when the support of a generalized function or distribution will be defined. Otherwise the argument $f \cdot g = 0$ if they do not have any common non-zero argument would be slightly stronger than the claim $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset \Rightarrow f \cdot g = 0$.

[4] The superscript bar stands for "closure" of a set. Hence $\mathrm{supp}(f)$ is by definition a closed set.

[5] Here we write $x \cdot x$ for the standard scalar product in $\mathbb{R}^d$.

(5) Fourier transform $\mathcal{F}, \mathcal{F}^{-1}$ (to be discussed only later): will be given for $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ by the integral (with the *normalization* 2π in the exponent!)

Four-intdef (4)
$$\mathcal{F}\colon f \mapsto \hat{f}: \quad \hat{f}(s) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i s \cdot t} dt$$

HINTS on Brackets, Bindings, Conventions

Remark 1. During these notes we will make a lot of use of operators (such as the ones just defined), but also of all kinds of multiplications, e.g. pointwise multiplications of functions or between functions and measures, or convolution products (denoted by $*$). When such symbols come together we consider the operators to be completely bound/attached to the function symbol. so clearly $T_x f \cdot g$ is to be interpreted as $(T_x f) \cdot g$,which is different from the expression (which would then be spelled out as such) $T_x(f \cdot g)$. There is no fixed convention when two multiplications come together, hence one has to put some kind of brackets to indicate whether one wants to interpret something (!!! please do not use a symbol like $h * g \cdot f$) as $(h * g) \cdot f$ or as $h * (g \cdot f)$.

It may also be worthwhile to note that $T_x : f \mapsto T_x f$ is a mapping between functions, and is *not at all* something which acts on a value. So an expression like $T_x[f(x)]$ would be considered as "non-sense" in our context. At the beginning this fine distinction my look/sound a bit strange, but there is a lot of practical experience behind these conventions, which have proven to be very convenient and helpful when applied strictly and consistently!

Many of the estimates which we will will obtain are realized by first establishing a pointwise estimate for two functions, e.g. $|f(x)| \leq |g(x)|$, $\forall x \in \mathscr{G}$ (for example). In such a case we will simply write $|f| \leq |g|$ (assuming that the domain is clear from the context), which of course implies e.g. $\|f\|_p \leq \|g\|_p$ for any $p \in [1, \infty]$.

We also want to point out that there is no unique convention within functional analysis, how to denote the dual space. Two conventions are quite standard for the description of the dual space, either the symbol $\boldsymbol{B}'$ (and consequently $\boldsymbol{B}$" for the bi-dual space), or $\boldsymbol{B}^*$ (and then $\boldsymbol{B}^{**}$).

Remark 2. More like a FOREWORD/PREFACE to these lecture notes:

The first three operator families defined above are isometric with respect to any of the $\boldsymbol{L}^p$-norms, $1 \leq p \leq \infty$, but we try to introduce these Banach spaces of functions only much later in our notes. In fact, the attempt to *avoid the use of the classical Lebesgue spaces* to a large extent, to rather view them completions of $\boldsymbol{C}_c(G)$ than as equivalence classes of measurable functions and thus emphasize the importance of functional analytic aspects, also conveying the relevance of the concept of *generalized functions*, also called *distributions* early on *without basing it on the classical concepts* but rather through a direct (more integration theoretic than measure theoretic) approach. This feature of our presentation should make it quite unique compared to most of the approaches to Fourier Analysis resp. (Abstract) Harmonic Analysis in the literature, and we hope that both the experts and newcomers to the field may enjoy the path taken.

Despite a lot of comments by many students, coworkers and colleagues the presentation still needs a lot of polishing and hints to short-comings, inconsistencies, repetitions or inaccuracies (solely due to the author) are *very welcome*!

The real stuff begins on the next page, finally! hgfei

Cbdef **Definition 4 (Banach spaces of continuous functions on $\mathbb{R}^d$).**

$$\boldsymbol{C}_b(\mathbb{R}^d) := \{ f : \mathbb{R}^d \mapsto \mathbb{C}, \text{ continuous and bounded, } \quad \text{with norm } \|f\|_\infty = sup_{x\in\mathbb{R}^d} |f(x)| \ \}$$

The spaces $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ and $\boldsymbol{C}_0(\mathbb{R}^d)$ are defined as the subspaces of $\boldsymbol{C}_b(\mathbb{R}^d)$ consisting of functions which are *uniformly continuous* (and bounded) resp. *decaying at infinity*, i.e.

$$f \in \boldsymbol{C}_0(\mathbb{R}^d) \quad \text{ if and only if } \quad \lim_{|x|\to\infty} |f(x)| = 0.$$

Cub-char **Lemma 2.** *i) Characterization of $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ within $\boldsymbol{C}_b(\mathbb{R}^d)$ (or even $\boldsymbol{L}^\infty(\mathbb{R}^d)$).*

Cub-char0 (5)
$$\|T_x f - f\|_\infty \to 0 \quad \textit{for} \quad x \to 0 \quad \textit{if and only if } f \in \boldsymbol{C}_{ub}(\mathbb{R}^d).$$

ii) $\boldsymbol{C}_0(\mathbb{R}^d)$ is closed subspace of $\big(\boldsymbol{C}_{ub}(\mathbb{R}^d), \|\cdot\|_\infty\big)$.

Proof. The reformulation of the usual $\varepsilon - \delta$-definition into the format of (Cub-char0 5) is left as an important exercise to the interested reader.

So let us only show who this characterization helps us to easily derive the fact that $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ is a closed subspace of $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$. Assume that we have a sequence $(f_n)_{n=1}^\infty$ is a convergent sequence in $\boldsymbol{C}_{ub}(\mathbb{R}^d)$, with limit $f_0 \in \boldsymbol{C}_b(\mathbb{R}^d)$ (we know already that this is a Banach space). So we have to find for any given $\varepsilon > 0$ some $\delta > 0$ such that we can guarantee that

$$\|T_x f_0 - f_0\|_\infty < \varepsilon \quad \text{if only} \quad |x| \le \delta.$$

In order to find such a $\delta > 0$ for given $\varepsilon > 0$ we first choose any index n_0 such that $\|f_0 - f_{n_0}\|_\infty < \varepsilon/3$. Since f_{n_0} belongs to $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ we can find for the positive number $\varepsilon/3 > 0$ some $\delta > 0$ such that $|x| < \delta$ implies that we have $\|T_x f_{n_0} - f_{n_0}\|_\infty < \varepsilon/3$. But by the triangular equation we obtain from this the required estimate:

Cubclosed2 (6)
$$\|T_x f_0 - f_0\|_\infty \le \|T_x(f_0 - f_{n_0})\|_\infty + \|T_x f_{n_0} - f_{n_0}\|_\infty + \|f_{n_0} - f_0\|_\infty < \varepsilon.$$

□

Remark 3. We will use exactly the same reasoning in order to show that the subspace of all bounded measures with continuous translation (to be identified with $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$) is a closed subspace of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

We will use the symbol $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ for the Banach space of all bounded and linear operators on the Banach space $\boldsymbol{C}_0(\mathbb{R}^d)$, endowed with the operator norm (using various symbols, listed in the sequel[6]):

Topninf (7)
$$\|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} = |||T|||_{\,\infty} = \|T\|_\infty := sup_{\|f\|_\infty \le 1} \|Tf\|_\infty.$$

3. Banach algebras of bounded and continuous functions

banalg-de **Definition 5 (Banach algebra).** A Banach space $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a *Banach algebra* if it has a *bilinear multiplication* $(a, b) \to a \bullet b$ (or simply $a \cdot b$ or just ab, this means that it is also associative and distributive) with the extra property that for some constant $C > 0$[7]

Banalgsubm (8)
$$\|a \bullet b\|_{\boldsymbol{A}} \le C \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{A}} \quad \forall a, b \in \boldsymbol{A}.$$

[6]The perhaps most unusual symbol is the triple vertical line for *operator norms*, based on a given norm. So we will measure the size of an operator $T : \boldsymbol{B} \to \boldsymbol{B}$, where $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a normed space by the operator norm, and we write $|||T|||_{\,\boldsymbol{B}}$ in this situation.

[7]Without loss of generality one can assume $C = 1$, because for the case that $C \ge 1$ one moves on to the equivalent norm $\|a\|'_{\boldsymbol{A}} := C \cdot \|a\|_{\boldsymbol{A}}$.

A *subalgebra* $\boldsymbol{B}$ is as a subspace which is an algebra with the multiplication inherited from $\boldsymbol{A}$, i.e. if $\boldsymbol{B} \bullet \boldsymbol{B} \subseteq \boldsymbol{B}$. *Right* and *two-sided ideals* are defined correspondingly[8].

Sometimes Banach algebras allow in addition for some *involution*, i.e. some isometric and linear algebra automorphism $j : \mathbf{b} \to \mathbf{b}^*$ (i.e. with $\|\mathbf{b}^*\|_{\boldsymbol{B}} = \|\mathbf{b}\|_{\boldsymbol{B}}$) and $j(j(\mathbf{b})) = \mathbf{b}$, or $(\mathbf{b}^*)^* = \mathbf{b}$ and $(\mathbf{a} \bullet \mathbf{b})^* = \mathbf{b}^* \bullet \mathbf{a}^*$ for all $\mathbf{a}, \mathbf{b} \in \boldsymbol{B}$. Such Banach algebras are called B^*-*algebras.*

CbC0thm1 **Theorem 1.** *(Banach algebras of continuous functions)*

1. $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *is a Banach algebra with respect to pointwise multiplication, even a* B^*-*algebra. This means that the Banach algebra has an involution, namely* $j : f \mapsto \bar{f}$ *which is linear, isometric, self-inverse (meaning* $\bar{\bar{f}} = f$*) and satisfies* $\|\bar{f}\|_\infty = \|f\|_\infty$, *and compatible with multiplication:* $\overline{fg} = \bar{f}\bar{g}$.
2. $\big(\boldsymbol{C}_{ub}(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *is a closed subalgebra of* $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$.
3. $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *is a closed ideal within* $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$.

Proof. First we show that $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ is a closed subspace of $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$. In fact, let (f_n) be a uniformly convergent sequence in $\boldsymbol{C}_{ub}(\mathbb{R}^d)$, convergent to $f \in \boldsymbol{C}_b(\mathbb{R}^d)$. Then for given $\varepsilon > 0$ there exists n_0 such that $\|f - f_{n_0}\|_\infty < \varepsilon/3$. Since $f_{n_0} \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$ we can find $\delta > 0$ such that $\|T_z f_{n_0} - f_{n_0}\|_\infty < \varepsilon/3$ for $|z| < \delta$. This implies of course for $|z| < \delta$:

nif-uniflim
$$\|T_z f - f\|_\infty \le \|T_z(f - f_{n_0})\|_\infty + \|T_z f_{n_0} - f_{n_0}\|_\infty + \|f - f_{n_0}\|_\infty < 3\varepsilon/3 = \varepsilon. \tag{9}$$

We have to estimate $\|T_x(gf) - gf\|_\infty$, for $|x| \to 0$. If we choose $\delta > 0$ such that both $\|T_x f - f\|_\infty \le \varepsilon'$ and $\|T_x g - g\|_\infty \le \varepsilon'$ for $|x| \le \delta$ we have

$$\|T_x(g \cdot f) - g \cdot f\|_\infty \le \|T_x g \cdot (T_x f - f)\|_\infty + \|(T_x g - g) \cdot f\|_\infty \le 2(\|f\|_\infty + \|g\|_\infty)\varepsilon'.$$

□

Lemma 3. *Characterization of* $\boldsymbol{C}_0(\mathbb{R}^d)$ *within* $\boldsymbol{C}_b(\mathbb{R}^d)$*:* $\boldsymbol{C}_0(\mathbb{R}^d)$ *coincides with the closure of* $\boldsymbol{C}_c(\mathbb{R}^d)$ *within* $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$.

Definition 6. A directed family (a net or sequence) $(h_\alpha)_{\alpha \in I}$ in a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is called a **BAI** (= bounded approximate identity or "approximate unit" for $\boldsymbol{A}$) if

$$\lim_\alpha \|h_\alpha \cdot h - h\|_{\boldsymbol{A}} = 0 \quad \forall h \in \boldsymbol{A}.$$

We need the following definition of a value-preserving dilation operator (the function values are just displaced to new positions by some dilation factor):

Drhodef **Definition 7.** **[Rd-specific!]**The value preserving **dilation operators** are given by:

Drhodef1
$$D_\rho f(z) = f(\rho \cdot z), \ \ \forall \rho \neq 0, \, z \in \mathbb{R}^d \tag{10}$$

For $\rho = -1$ a special symbol will be used:

checkdef0
$$f^{\checkmark}(z) = [D_{-1} f](z) = f(-z), \forall f \in \boldsymbol{C}_0(\mathbb{R}^d), z \in \mathbb{R}^d. \tag{11}$$

It is easy to verify that

Drho2
$$\|D_\rho f\|_\infty = \|f\|_\infty \quad \text{and} \quad D_\rho(f \cdot g) = D_\rho(f) \cdot D_\rho(g). \tag{12}$$

For later use let us mention that the dilation and translation operators satisfy the following commutation relation **[Rd-specific!]**:

-trans-comm
$$D_\rho \circ T_x \; = \; T_{x/\rho} \circ D_\rho \, , \quad \rho > 0, x \in \mathbb{R}^d, \tag{13}$$

[8] Obviously any ideal is a subalgebra.

and in particular

-trans-comm (14)
$$[T_x f]^{\checkmark} = T_{-x} f^{\checkmark}, \quad \text{and} \quad [f^{\checkmark}]^{\checkmark} = f, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d). \tag{14}$$

Proof. For any $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ one has for any $z \in \mathbb{R}^d$:

$$[D_\rho(T_x f)](z) = (T_x f)(\rho z) = f(\rho z - x) = f(\rho(z - x/\rho)) = (D_\rho f)(z - x/\rho) = [T_{x/\rho}(D_\rho f)](z).$$

□

Theorem 2. *$\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ is a Banach algebra with bounded approximate units. In fact, any family of functions $(h_\alpha)_{\alpha\in I}$ which is uniformly bounded, i.e. with*

$$|h_\alpha(x)| \le C < \infty \qquad \forall x \in \mathbb{R}^d, \forall \alpha \in I$$

and satisfies

$$\lim_\alpha h_\alpha(x) = 1 \quad \textit{uniformly over compact sets}$$

constitutes a BAI (the converse is true as well).

[Rd-specific!]*: In particular, one obtains a BAI by stretching any function $h_0 \in \boldsymbol{C}_0(\mathbb{R}^d)$ with $h_0(0) = 1$, i.e. by considering the family $D_\rho h_0(g)(t) := h_0(\rho \cdot t)$, for $\rho \to 0$.*

The following elementary lemma is quite standard and could in principle be left to the reader. Probably it should be placed in the appendix. However, it is quite typical for arguments to be used repeatedly throughout these notes and therefore we state it explicitly.

Lemma 4. *Assume that one has a bounded net $(T_\alpha)_{\alpha\in I}$ of operators from a Banach space $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to another normed space $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$, such that $T_\alpha \to T_0$ strongly, i.e. for any finite set $F \subset \boldsymbol{B}^{(1)}$ and $\varepsilon > 0$ there exists an index α_0 such that for $\alpha \succeq \alpha_0$ one has $\|T_\alpha f - T_0 f\|_{\boldsymbol{B}^2} < \epsilon$ for all $f \in F$, then one has uniform convergence over compact subsets $M \subset \boldsymbol{B}^1$.*

Proof. The argument is based on the usual compactness argument. Assuming that $|||T_\alpha||| \le C_1$ for all $\alpha \in I$ we can find some finite set $f_1, \dots f_K \in M$ such that balls of radius $\delta = \varepsilon/(3C_1)$ around these points cover M. According to the assumption (and the general properties of nets) one finds α_0 such that

$$\|T_\alpha f_j - T_0 f_j\|_{\boldsymbol{B}^{(2)}} < \varepsilon/3, \quad \text{for } j = 1, 2, \dots K.$$

Consequently one has for any $f \in M$ and suitable chosen index j (with $\|f - f_j\|_{\boldsymbol{B}^{(1)}} < \delta$):

comp-convg (15)
$$\|T_\alpha f - T_0 f\|_{\boldsymbol{B}^{(2)}} < \|T_\alpha(f - f_j)\|_{\boldsymbol{B}^{(2)}} + \|T_\alpha f_j - T_0 f_j\|_{\boldsymbol{B}^{(2)}} + \|T_0(f_j - f)\|_{\boldsymbol{B}^{(2)}}. \tag{15}$$

Since the operator norm of T_0 is also not larger than C_1 (! easy exercise) this implies

comp-convg2 (16)
$$\|T_\alpha f - T_0 f\|_{\boldsymbol{B}^{(2)}} \le 2C_1 \|f - f_j\|_{\boldsymbol{B}^{(1)}} + \|T_\alpha f_j - T_0 f_j\|_{\boldsymbol{B}^{(2)}} \le 2C_1\delta + \varepsilon/3 < \varepsilon. \tag{16}$$

□

total-test1 *Remark* 4. In fact, a similar argument can be used to verify the w^*-convergence of a bounded net of operators, by verifying the convergence only for f from a total subset within its domain. In fact, using linearity implies that convergence is true for linear combinations of the elements for such a set, and by going to a limit one obtains convergence for arbitrary elements in $\boldsymbol{B}^{(1)}$.

Obviously the above argument applies to nets of bounded linear functionals as well (choosing simply $\boldsymbol{B}^{(2)} = \mathbb{C}$).

Lemma 5. [Rd-specific!] *[Group of dilation operators]*
The family of operators $(D_\rho)_{\rho>0}$ *is a family of isometric isomorphisms on* $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. *Moreover, the mapping* $\mathbb{R}^+ \to \mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ *given by* $\rho \mapsto D_\rho$, *is a group homomorphism from the multiplicative group of positive reals into the isometric linear operators on* $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, *i.e. one has*

dilcomut1
$$D_{\rho_2} \circ D_{\rho_1} = D_{\rho_1 \cdot \rho_2} = D_{\rho_1} \circ D_{\rho_2} \tag{17}$$

It is however not continuous in the operator sense, since $|||D_{\rho_1} - D_{\rho_2}|||_{\boldsymbol{L}^\infty} \geq 1$ *for* $\rho_1 \neq \rho_2$, *but it is strongly continuous, i.e. for any* $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ *one has:*

Drhostrongc
$$\lim_{\rho \to 1} \|\mathrm{D}_\rho f - f\|_\infty = 0. \tag{18}$$

Proof. The lemma contains a number of useful, but easy to verify claims. The fact that the operator norm of two operators of two different dilation operators does not tend to zero for $\rho_1 \to \rho_2$ follows from the observation that for any such pair of dilation parameters there exist (in fact many) functions $k \in \boldsymbol{C}_c(\mathbb{R}^d), \|k\|_\infty = 1$, which small support such that their dilates have disjoint support. Note $\mathrm{supp}\,(k) = K$ then $\mathrm{supp}(\mathrm{D}_\rho k) = K/\rho$, so it is enough to choose K such that $\rho_2 K \cap \rho_1 K = \emptyset$.

The positive statement (about strong continuity) follows from the observation that it is a consequence of the *uniform continuity* of $k \in \boldsymbol{C}_c(\mathbb{R}^d)$, plus the usual approximation argument for the general case $f \in \boldsymbol{C}_0(\mathbb{R}^d)$. □

Remark 5. One can make similar claims about general dilations, based on matrices, including *anisotropic dilations.* Given $\boldsymbol{A} \in \mathcal{M}_{d,d}(\mathbb{R})$ the generalized dilation [9]

dilatmatr1
$$\mathrm{D}_{\boldsymbol{A}}(f)(\mathbf{z}) := f(\boldsymbol{A} * \mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^d. \tag{19}$$

Then again

DrhostrongM
$$\|\,\mathrm{D}_{\boldsymbol{A}}\, f - f\| \to \mathbf{0} \text{ if } \boldsymbol{A} \to \mathbf{Id_d} \text{ in } \mathcal{M}_{d,d}(\mathbb{R}). \tag{20}$$

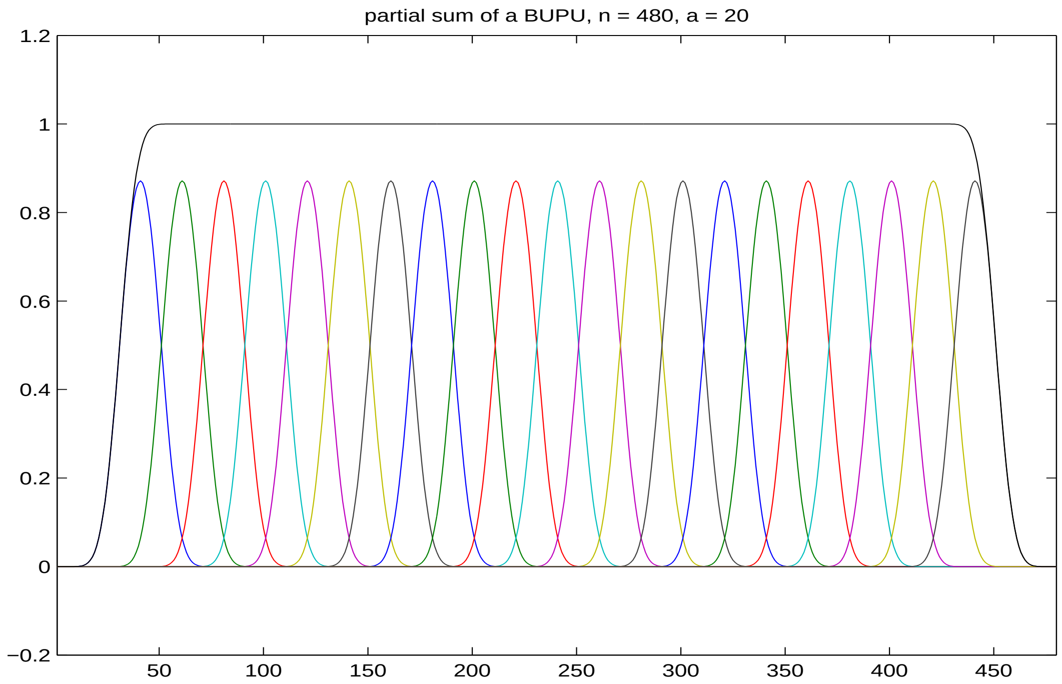
-approxunit
Lemma 6. *Let* I *be the family of all compact subsets of* $\mathbb{R}^d$, *and define* $K \succeq K'$ *if* $K \supseteq K'$. *If we choose for every such* $K \subset\subset \mathbb{R}^d$ [10] *a plateau function* p_K *such that* $0 \leq p(x) \leq 1$ *on* $\mathbb{R}^d$ *and* $p_K(x) \equiv 1$ *on* K. *Then* $(p_K)_{K \in I}$ *constitutes a BAI for* $\boldsymbol{C}_0(\mathbb{R}^d)$.

Proof. First we have to show that the "direction" of I is reflexive ($K \supseteq K$ and transitive, i.e. $K_1 \supseteq K_2$ and $K_2 \supseteq K_3$ obviously implies $K_1 \supseteq K_3$). Finally the key property for the index set of a net is easily verified: Given two "indices" K_1, K_2 the set $K_0 := K_1 \cup K_2$ is an element of I, i.e. a compact set, with $K_0 \succeq K_i$ for $i = 1, 2$. Hence $(p_K)_{K \in I}$ is in fact a net.

In order to verify the BAI property let $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ and $\varepsilon > 0$ be given. Then - by definition of $\boldsymbol{C}_0(\mathbb{R}^d)$ there exists some $R > 0$ such that $|x| > R$ implies $|f(x)| \leq \varepsilon$. Then one has $|f(x) \cdot p(x) - f(x)| = |(1 - p(x))||f(x)| \leq |f(x)| \leq \varepsilon$ for $|x| \geq R$. On the other hand $|f(x) \cdot p(x) - f(x)| = |f(x) \cdot 1 - f(x)| = 0$ for $|x| \leq R$, as long as p is a plateau function equal to 1 on the compact set $K_0 = B_R(0)$ [11], i.e. as long as $p = p_K$ has an index with $K \succeq K_0$. Altogether $\|f \cdot p - f\|_\infty \leq \varepsilon$ □

The family of all partial sums of BUPUs (defined below) will satisfy the conditions described in Lemma plat-approxunit 6. The following figure illustrates the situation for $\mathscr{G} = \mathbb{R}$ (the partial sum adds up to a nice *plateau function* with compact support, covering some region of interest.

[9]We write $\boldsymbol{A} * \mathbf{z}$ for matrix-vector multiplication, following MATLAB^{TM}.
[10]We write $K \subset\subset \mathbb{R}^d$ to indicate that K is a *compact* subset of $\mathbb{R}^d$.
[11]$B_R(0)$ denotes the ball of radius R around 0.



char-C0inCb **Lemma 7.** *Another characterization of* $\boldsymbol{C}_0(\mathbb{R}^d)$ *within* $\boldsymbol{C}_b(\mathbb{R}^d)$*:*
$h \in \boldsymbol{C}_b(\mathbb{R}^d)$ belongs to $\boldsymbol{C}_0(\mathbb{R}^d)$ if and only if

char-C01
$$\lim_{\alpha} \|h_\alpha \cdot h - h\|_\infty = 0 \tag{21}$$

for one (hence all) BAIs for $\boldsymbol{C}_0(\mathbb{R}^d)$ (as described above).

Remark 6. It is a good exercise that such a BAI acts *uniformly* on compact subsets, i.e. for any relatively compact set M in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ one finds: Given $\varepsilon > 0$ one can find some α_0 such that for $\alpha \succeq \alpha_0$ implies

char-C02
$$\|h_\alpha \cdot h - h\|_{\boldsymbol{B}} \leq \varepsilon \quad \forall h \in M. \tag{22}$$

The following simple lemma will be useful later on (characterization of LTIS, i.e. *linear time-invariant systems*, also called TILS, i.e. *translation-invariant linear systems* for $d = 2$ we can talk about linear space-invariant operators, as they are used for image processing applications):

homog-BF1 **Lemma 8.** *A function $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ belongs to $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ if and only if the (non-linear) mapping $z \mapsto T_z f$ is continuous from $\mathbb{R}^d$ into $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$. In fact, such a mapping is continuous at zero if and only if it is uniformly continuous.*

Proof. It is clear that continuity is a necessary condition (and we have already seen that it is equivalent to uniform continuity for $f \in \boldsymbol{C}_b(\mathbb{R}^d)$). Conversely assume continuity at zero, i.e. $\|T_z f - f\|_\infty < \varepsilon$ for sufficiently small z ($|z| < \delta$). Then one derives continuity at x as follows:

$$\|T_{x+z} f - T_x f\|_\infty = \|T_x(T_z f - f)\|_\infty = \|T_z f - f\|_\infty < \varepsilon$$

implying uniform continuity of the discussed mapping. [12] □

[12] The proof gives an argument, that a homomorphism from an Abelian topological group G ($\mathbb{R}^d$ in our case) into a group of operators on a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (here $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$), is strongly continuous, i.e. satisfies the discussed continuity property analogue to $f \mapsto T_z f$, is continuous from G into $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ if and only it is continuous at zero (the identity element of G).

Remark 7. One can say, that the mapping $x \mapsto T_x$ from $\mathbb{R}^d$ into $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ is a representation of the Abelian group $\mathbb{R}^d$ on the Banach space $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, which by definition means that the mapping is a homomorphism between the additive group $\mathbb{R}^d$ and the group of invertible (even isometric) operators on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. The additional property that $z \mapsto T_z f$ is continuous for $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ is referred to as the *strong continuity* of this representation. The same mapping into the larger space $\mathcal{L}(\boldsymbol{C}_b(\mathbb{R}^d))$ would not be strongly continuous, because for $f \in \boldsymbol{C}_b(\mathbb{R}^d) \setminus \boldsymbol{C}_{ub}(\mathbb{R}^d)$ this mapping fails to be continuous (cf. below)

Remark 8. One can derive from the above definition that the mapping $(x, f) \to T_x f$, which maps $\mathbb{R}^d \times \boldsymbol{C}_0(\mathbb{R}^d)$ into $\boldsymbol{C}_0(\mathbb{R}^d)$ is continuous with respect to the product topology (Exercise).

Definition 8. We denote the dual space of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ with $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Sometimes the symbol $\boldsymbol{M}_b(\mathbb{R}^d)$ is used in order to emphasize that one has "bounded" (regular Borel) measures.

The *Riesz Representation Theorem* (see e.g. [4], p.112, as well as many other sources, probably also [5], or [1]) provides the justification for this definition and establishes the link to the concept explained in measure theory. In that case the action of μ on the test function is of course written in the form

meas-act
$$\mu(f) = \int_{\mathbb{R}^d} f(t) d\mu(t). \tag{23}$$

In the classical case of functionals on $\boldsymbol{C}(I)$, where $I = [a, b]$ is some interval, one can describe these integrals using Riemann-Stieltjes integrals. They make use of functions F of bounded variation. The distribution function [13] F is connected with the *measure* μ, defined on the σ-algebra of Borel sets in I via

dist-meas
$$F(x) = \int_a^x d\mu(x); \qquad \int_I f(x) d\mu(x) = lim_{\delta \to 0} \sum_i f(\xi_i)[F(x_i) - F(x_{i-1})]. \tag{24}$$

The original approach to $\big(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\big)$ was within this context: A closed subspace of the linear space of all functions of bounded variation $BV(\mathbb{R})$ is the space of *absolutely continuous functions* (cf. historical notes, in particular see the article of 1929 by Plessner [54], where he shows that these are exactly the elements within BV which have continuous translation, hence are approximated using e.g. convolution by nice summability kernels.[14])

In this context also the "main theorem of analysis" (popularly known by the claim: taking the anti-derivative F is the an operation which is - up to additive constants - an operation which is inverse to the differentiation operator): A function is the distribution function of some function in $\boldsymbol{L}^1$ if and only if it is absolutely continuous. In this case one can be assured that $F'(x)$ (taken in the *classical sense*) exists almost everywhere, that the resulting function is measurable and in fact in $\boldsymbol{L}^1$, and the absolutely continuous function F can be recovered (up to an additive constant) from its derivative by taking the indefinite integral (area function).

Of course the norm (measurement of the *size of total variation*), the so called BV-norm of F and the norm of the linear functional turn out to be the same. Moreover, there is natural

[13] This use of the word distribution is quite different from the use of distributions in the sense of generalized functions, as it is used in the rest of these notes.

[14] It is also interesting to see how most of the still relevant facts concerning Fourier series, Lebesgue integral and so on appear already in a rather clear form in those early papers and books on the subject. However, this was before the age of Banach space theory and in fact not at all referring to abstract Hilbert space theory, as it is seen nowadays as a corner stone. For further comments see the [forthcoming] section on historical notes.

way to split a function F of bounded variation into a difference of two non-decreasing functions, intuitively the “increasing” and the “decreasing part”[15]. The sum of these two parts (as opposed to the difference) is a monotonous function which induces therefore a non-negative measure of the same total variation, which is called $|\mu|$ in the general measure theory (Bochner's decomposition), the absolute value of the measure μ (just a symbolic operation). Normally the (non-linear) mapping $\mu \mapsto |\mu|$ requires quite some measure theory (see e.g. [1]). [Nov.2013:] There is a way to provide a proof of this decomposition using the approximation by discrete measures, in combination with the “obvious” decomposition of a real-valued discrete measure into a positive and a negative a part (splitting the non-zero coefficients into two groups: the positive and the negative ones). The proof (the author just has some handwritten notes) is however technically somewhat involved and will be given in a separate publication. It goes beyond the scope of these lecture notes, and will not be needed elsewhere.

Remark 9. As a compromise between the simple Riemannian integral and the high-level Lebesgue integral (which is *complete* with respect to the $\boldsymbol{L}^1$-norm!) people sometimes restrict their attention to the closure of the step functions in the sup-norm (this is a space containing both the step functions and the continuous functions), if we work over an interval or bounded subset of $\mathbb{R}^d$. One could use the closure of the simple (i.e. defined by indicator functions of $d-$dimensional boxes or cubes) in the sense of the Wiener norm of $\boldsymbol{W}(\boldsymbol{L}^1, \boldsymbol{\ell}^\infty)(\mathbb{R}^d)$ (see Sections 9 and 28 below).

EXAMPLES: point measures $\delta_0 : f \mapsto f(0)$, $\delta_x : f \mapsto f(x)$, or integrals over bounded sets: $f \mapsto \int_a^b f(x)dx$, or more generally $\mu_k : f \mapsto \int_{\mathbb{R}^d} f(x)k(x)dx$, where integration is taken in the sense of Riemannian (or Lebesgue) integrals, for $k \in \boldsymbol{C}_c(\mathbb{R}^d)$. For the setting of locally compact groups $\mathscr{G}$ one will use the *Haar measure* for the definition of such integrals.
Note: the norm of $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ is of course just the functional norm, i.e.

$$\|\mu\|_{\boldsymbol{M}} := \sup_{\|f\|_\infty \le 1} |\mu(f)| = \sup_{\|f\|_\infty = 1} |\mu(f)|.$$

EXERCISE: $\|\delta_t\|_{\boldsymbol{M}} = 1$, and more generally:

Theorem 3. *For any finite linear combinations of Dirac-measures, i.e. for* $\mu = \sum_{k\in F} c_k \delta_{t_k}$ *(where F is some finite index set and we assume that one has the natural representation, with* $t_k \neq t_{k'}$*) one has* $\|\mu\|_{\boldsymbol{M}_b} = \sum_{k\in F} |c_k|$.

Proof. The upper estimate is simple. Applying $\sum_{k\in F} c_k \delta_{t_k}$ to f one gets the estimate

$$|\sum_{k\in F} c_k f(t_k)| \le \sum_{k\in F} |c_k|\,|f(t_k)| \le \|f\|_\infty \sum_{k\in F} |c_k|.$$

Conversely, one cannot get a better estimate, because it is possible for any sequence (t_k) of distinct points and corresponding complex coefficients to find a continuous functions (e.g. a linear combination of sufficiently narrow triangular functions) such that $\|f\|_\infty = 1$ and $f(t_k) = c_k/|c_k|$ (whenever $c_k \neq 0$). But then

$$\sum_{k\in F} |c_k f(t_k)| = \sum_{k\in F} |c_k|,$$

and the proof is complete. □

[15]Think of a hiker how walks up and down in the mountains and reports on the total height of ascent and descent part.

iracnonconv **Corollary 1.**

$$\|\delta_x - \delta_y\|_{\boldsymbol{M}_b(\mathbb{R}^d)} = 2 \quad \textit{whenever} \quad x \neq y.$$

In particular, $x_n \to x_0$ *does NOT imply* $\delta_{x_n} \to \delta_{x_0}$ *in the norm of* $\boldsymbol{M}_b(\mathbb{R}^d)$.

In contrast, one has of course $\delta_{x_n}(f) \to \delta_{x_0}(f)$*, since* $f(x_n) \to f(x_0)$ *for* $f \in \boldsymbol{C}_b(\mathbb{R}^d)$.

A simple lemma helps us to identify the closed linear subspace of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ generated from the (linear) subspace of finite-discrete measures. We will call it the space of *discrete measures.*

Lemma 9. *Let* V *be a linear subspace of a normed space* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. *Then the closure of* V *coincides with the space obtained by taking the absolutely convergent sequences in* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *with elements from* V*:*

bs-convser1

$$Abs(\boldsymbol{B}) := \{x = \sum_n v_n, \ \textit{with} \quad \sum_n \|v_n\|_{\boldsymbol{B}} < \infty\} \tag{25}$$

Proof. It is obvious that $Abs(\boldsymbol{B}) \subseteq V^-$, the closure of V in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Conversely, any element $x = lim_{k\to\infty} v_k$ can also be written as a limit of a (sub)sequence with $\|x - v_k\|_{\boldsymbol{B}} < 2^{-k}$. In fact, one can choose by induction a strictly increasing subsequence (v_{n_k}) of the original sequence such that this is valid (and relabel the elements afterwards), and may therefore be rewritten as a telescopic sum, with $y_1 = v_1$, $y_{n+1} = v_{n+1} - v_n$. .... ☐

sconv-dense *Remark* 10. A simple but powerful variant of the above lemma is obtained if one allows the elements v_n being only taken from a dense subset of V (density of course in the sense of the $\boldsymbol{B}-$norm. Since such a set has the same closure this is an immediate corollary from the above lemma.

As a consequence (of the last two results) one finds that the closed linear subspace generated by the finite discrete measures coincides with absolutely convergent series of Dirac measures:

Definition 9 (Bounded discrete measures)**.**

$$\boldsymbol{M}_d(\mathbb{R}^d) = \{\mu \in \boldsymbol{M}(\mathbb{R}^d) : \mu = \sum_{k=1}^{\infty} c_k \delta_{t_k} \ \textit{s.t.} \ \sum_{k=1}^{\infty} |c_k| < \infty\}$$

The elements of $\boldsymbol{M}_d(\mathbb{R}^d)$ are called the *discrete measures*, and thus we claim that they form a (proper) subspace of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

The following statement will be understandable once the concept of tightness is formally introduced (see Definition 34):

Definition 10. A sequence of measures $(\mu_n)_{n\geq 1}$ is *Bernoulli convergent* to some $\mu_0 \in \boldsymbol{M}(\mathbb{R}^d)$ according to Bochner ([7, p.15), if it is bounded in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ and if

$$\mu_n(f) \to \mu_0(f) \quad \text{for all} \quad f \in \boldsymbol{C}_b(\mathbb{R}^d).$$

CbExtMb02 **Lemma 10.** *The action of any* $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ *extends in a natural way to all of* $\boldsymbol{C}_b(\mathbb{R}^d)$*, and the extended functional (which we simply denote by the same symbol) has the same norm as element of* $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)'$. *The convolution operator* C_μ *extends to a bounded linear operator on* $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ *of norm* $\|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)}$*, commuting with translation.*

Proof. We make use of Thm. (4). For any given BUPU Ψ we can define the action of μ as the sum of the action of all the local pieces $\mu\psi_i, i \in I$ (and of course this definition is independent of Ψ, we leave this verification to the reader), we simply set

measCbact2

$$\mu(h) := \sum_{i\in I} \mu(\psi_i \cdot h), \quad h \in \boldsymbol{C}_b(\mathbb{R}^d). \tag{26}$$

In order to check that the action of μ on $\boldsymbol{C}_b(\mathbb{R}^d)$ is well defined we have to introduce the function

$$\psi_i^* := \sum_{j:\psi_j\cdot\psi_i\neq 0} \psi_j, \quad i \in I.$$

As a finite sum of terms (the number if neighbors is uniformly bounded) also the supports of the family $\Psi^* := (\psi_i^*)_{i\in I}$ is of uniform size. Moreover it is clear that one has for all $i \in I$:

$$\psi_i = \psi_i \cdot 1 = \psi_i \cdot \sum_{j\in I} \psi_j = \psi_i \cdot \psi_i^*.$$

Obviously the sum well defined, since $\mu(\psi_i \cdot h) = (\mu \cdot \psi_i)(\psi_i^* \cdot h)$, and all these functions $\psi_i^* \cdot h$ belong to $\boldsymbol{C}_c(\mathbb{R}^d) \subset \boldsymbol{C}_0(\mathbb{R}^d)$. Finally thanks to formula (abscónvmbo 29) we have

MbActCb02 (27)
$$\sum_{i\in I} |(\mu\psi_i)(\psi_i^* \cdot h)| \leq \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}_b} \|\psi_i^* \cdot h\|_\infty \leq \|\mu\|_{\boldsymbol{M}_b} \|h\|_\infty, \tag{27}$$

so the new functional is also bounded by $\|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)}$. It is clear that the new functional is linear and extends the given on $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$, and hence the two norms are equal.

In the next step we have to verify that the convolution operator by C_μ also extends to $\big(\boldsymbol{C}_b(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$. The extension to the space $\big(\boldsymbol{C}_{ub}(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$. is in fact easy to verify, using the argument just given and the preservation of uniform uniform continuity of translation invariant linear operators already used in the proof of the representation theorem for TLIS.

Now we want to argue that the pointwise defined result of the convolution operator, namely $C_\mu(h)(x) = \mu(T_x h^{\checkmark})$ gives a *continuous function*, even if h just belongs to $\boldsymbol{C}_b(\mathbb{R}^d)$.

Since continuity is a local property, and since we have reduced the problem to measures of the form $\mu\,\psi_i$. It is enough to show that $\psi_j \cdot C_{\mu\,\psi_i}(h)$ is a continuous function, because then

$$\left(\sum_{j\in I} \psi_j\right) \cdot C_{\mu\,\psi_i}(h) = C_{\mu\,\psi_i}(h)$$

is a continuous function. Recall that $\mathrm{supp}(\psi_j) \subset B_\delta(x_j)$, and let us assume for simplicity that we have $\delta \leq 1$. Since it is easy to verify that $\mathrm{supp}(C_{\mu\,\psi_i})(\psi_l f) \subset B_2(x_i + x_l)$ it is clear that we can ignore most of the terms $\psi_j f$, because they will provide only contributions outside of the support of ψ_j. In fact, for any fixed $i \in I$ the family

$$F_{i,j} := \{l \,|\, B_2(x_i + x_l) \cap B_1(x_j) \neq \emptyset\} = \{l \,|\, x_l \in B_3(x_j - x_i)\}, \quad i, j \in I.$$

is finite (even uniformly bounded in number), hence we can say that

$$(C_{\mu\,\psi_i}(h)) \cdot \psi_l = C_{\mu\,\psi_i} \cdot \left(\sum_{l\in F_{i,j}} h\psi_l\right) \cdot \psi_i.$$

But this auxiliary expression $h_{i,j} := \sum_{l\in F_{i,j}} h\psi_l$ is a function in $\boldsymbol{C}_c(\mathbb{R}^d)$ and hence $C_{\mu\,\psi_i}(h_{i,j})$ is in $\boldsymbol{C}_c(\mathbb{R}^d)$ as well. This completes our proof. $\square$

BernConv1 **Lemma 11.** *[Bernoulli convergence for tight sequences] Assume that $(\mu_n)_{n\geq 1}$ is a bounded and tight sequence in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, such that $\mu_0 = w^*{-}lim_n \mu_n$. Then μ_n is Bernoulli convergent to μ_0.*

The proof if this claim is a good exercise and thus is left to the interested reader. It is using the fact that the action of bounded measures (originally only on $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ extends in a natural way to all of $\boldsymbol{C}_b(\mathbb{R}^d)$).

Another variation of the theme is provided by the next proposition.

wststrop1 **Proposition 1.** *Assume that* $(\mu_n)_{n\geq 1}$ *is a bounded and tight sequence in* $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. *Then* $\mu_0 = w^* - lim_n\, \mu_n$ *if and only if*

$$\mu_n * f \to \mu_0 * f, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d),$$

uniformly over compact sets.

Proof. Of course we make use of the details provided by Thm. (characterization-TLIS 7) which will be given below.

That the convergence of operators (in the strong operator topology) implies the converge of the associated measure in the w^*- sense follows easily from equation (impuls-def 43), but OVERALL WE HAVE TO MOVE THIS STATEMENT TO A LATER POSITION.

□

We claim, however, that the finite discrete measures form a w^*-dense subspace of $\boldsymbol{M}(\mathbb{R}^d)$. Before doing this we will define the adjoint of the pointwise multiplication within $\boldsymbol{C}_0(\mathbb{R}^d)$ on its dual space, i.e. how to define $h \cdot \mu$ (for $h \in \boldsymbol{C}_0(\mathbb{R}^d)$ or even $h \in \boldsymbol{C}_b(\mathbb{R}^d)$).

nctimesmeas **Definition 11.** $\mu \cdot h(f) := \mu(h \cdot f), \quad h \in \boldsymbol{C}_b(\mathbb{R}^d), f \in \boldsymbol{C}_0(\mathbb{R}^d).$

easmodlemma **Lemma 12.** *For* $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ *and* $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ *one has*

ncmeasnorms
$$\|h \cdot \mu\|_{\boldsymbol{M}} \leq \|h\|_\infty \|\mu\|_{\boldsymbol{M}}. \tag{28}$$

In fact, the mapping from the pointwise algebra $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ *into the algebra of pointwise multiplication operators* $M_h : \mu \mapsto h \cdot \mu$ *is in fact an isometric embedding.*

Proof. The statements are more an exercise in terminology than a deep mathematical statement and are therefore left to the interested reader. □

Remark 11. Being an adjoint mapping it is also w^*-w^*--continuous, i.e. for any bounded and w^*-convergent net $(\mu_\alpha)_{\alpha\in I}$ with limit μ_0 and any fixed $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ on has

$$h \cdot \mu_\alpha \to h \cdot \mu_0.$$

Since "ordinary functions" (e.g. the space $\boldsymbol{C}_c(\mathbb{R}^d)$) form a w^*-dense subspace of $\boldsymbol{M}_b(\mathbb{R}^d)$ one can show that the definition of the pointwise product is the *uniquely defined* w^*-w^*--continuous extension of the ordinary concept of *pointwise products* for continuous functions.

For this reason we will introduce a simple tool, the so-called BUPUs, the "bounded uniform partitions of unity". For simplicity we only consider the regular case, i.e. BUPUs which are obtained as translates of a single function:

regBUPUdef **Definition 12.** A (countable) *family* $\Phi = (T_\lambda \varphi)_{\lambda\in\Lambda}$, where φ is a compactly supported function (i.e. $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$), and $\Lambda = A(\mathbb{Z}^d)$ a lattice in $\mathbb{R}^d$ (for some non-singular $d \times d$-matrix) is called a **regular BUPU** if

$$\sum_\lambda \varphi(x - \lambda) \equiv 1.$$

We say that $\mathrm{diam}(\Phi) \leq \gamma$ if $\mathrm{supp}(\varphi) \subset B_\gamma(0)$.

The regular BUPUs are sufficient for our purposes. They are a special case for a more general concept of (unrestricted) BUPUs:

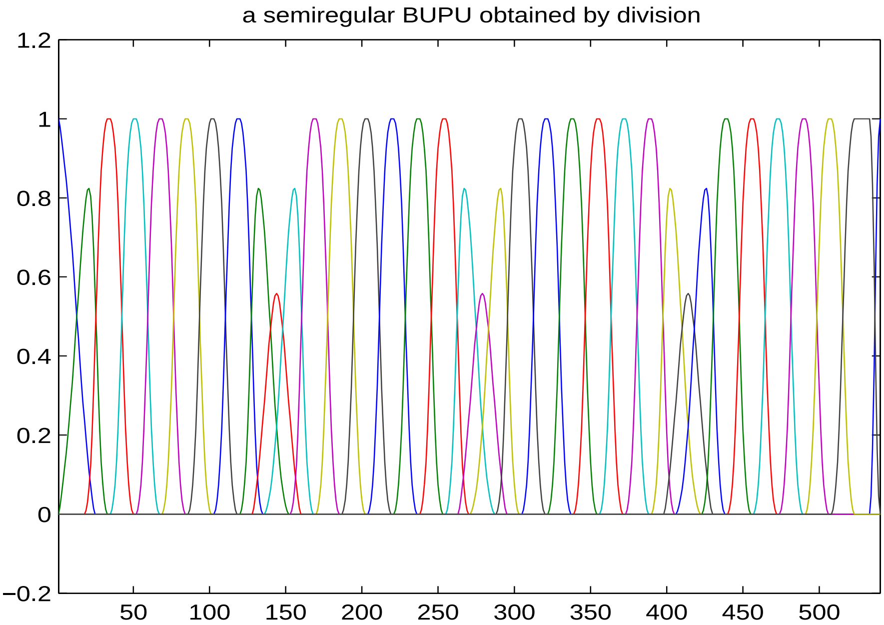
BUPU-gen0 **Definition 13.** A BUPU, a so-called **b**ounded **u**niform **p**artition of **u**nity[16] in some Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ of continuous functions on $\mathscr{G}$ is a family $\Psi = (\psi_i)_{i\in I}$ of non-negative functions on $\mathscr{G}$, if the following set of conditions is satisfied:

(1) There exists some *relatively compact* neighborhood U of the identity element of the group $\mathscr{G}$ that for each $i \in I$ there exists $x_i \in \mathscr{G}$ such that $\mathrm{supp}(\psi_i) \subseteq x_i + U$ for all $i \in I$;
(2) The family Ψ is bounded in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, i.e. there exists $C_{\boldsymbol{A}} > 0$ such that $\|\psi_i\|_{\boldsymbol{A}} \le C_{\boldsymbol{A}}$ for all $i \in I$;
(3) The family of supports $(x_i + U)_{i\in I}$ is *relatively separated*, i.e. the number of intersecting neighbors is uniformly bounded in the sense that there exists some finite number $C_0 < \infty$ such that

$$\#\{j \,|\, (x_i + U) \cap (x_j + U) \neq \varnothing\} \le C_0;$$

(4) $\sum_{i\in I} \psi_i(x) \equiv 1$.

Occasionally we will refer to U as the *size of the* BUPU. Any constant C_A above is just an upper bound for the bounded set $\{\psi_i, i \in I\}$ (the best constant could be called the $\boldsymbol{A}$-norm of Ψ) and C_0 is a kind of *overlapping constant* of the family. Note that it follows from (3) that for each point the sum is finite, so even if one does not make any positivity assumption on the functions ψ_i one has no convergence problems because sums are finite at every point.



Let $\Psi = (\psi_i)_{i\in I}$ be a BUPU, i.e. a bounded uniform partition of unity.

[16]Here the uniformity of the partition is referring to the *uniform size* of the partition, while the boundedness, resp. the more precisely the $\boldsymbol{A}$-boundedness refers to the boundedness of the family in the Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$. Of course $C_{\boldsymbol{A}}$ could be chosen to be just $\sup_{i\in I} \|\psi_i\|_{\boldsymbol{A}}$. Of course for e.g. for C^∞-BUPUs there could be various different constants, depending on the choice of $\boldsymbol{A}$, e.g. $\boldsymbol{A} = \boldsymbol{C}^{(k)}$

asabsdecomp **Theorem 4.** *Let $\Psi = (\psi_i)_{i\in I}$ be a non-negative BUPU. Then*[17]

absconvmbo
$$\|\mu\|_{\boldsymbol{M}} = \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}}, \tag{29}$$

hence in particular $\mu = \sum_{i\in I} \mu\psi_i$ is absolutely convergent for $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$.

Proof. The estimate $\|\mu\|_{\boldsymbol{M}} \le \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}}$ is obvious, by the triangular inequality of the norm and the completeness of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

In order to prove the opposite inequality (and in fact the finiteness of the series on the right hand side) it will be sufficient to go for the estimate $\|\mu\|_{\boldsymbol{M}} + \varepsilon \ge \sum_{i\in F} \|\mu\psi_i\|_{\boldsymbol{M}}$ for an arbitrary given $\varepsilon > 0$ and an appropriately chosen sufficiently large finite set $F \subset I$.

Recalling that $\|\mu\|_{\boldsymbol{M}_b} = \sup_{\|f\|_\infty \le 1} |\mu(f)|$ it is enough to show that for every $\varepsilon > 0$ there exists some $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\|f\|_\infty \le 1$ such that $|\mu(f)| \ge \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}} - 3\varepsilon$.

For the given $\varepsilon > 0$ we choose first a sequence $\varepsilon_i > 0$ such that $\sum_{i\in I} \varepsilon_i < \varepsilon$. By the definition of $\|\mu\psi_i\|_{\boldsymbol{M}}$ we can find $f_i \in \boldsymbol{C}_0(\mathbb{R}^d)$ with $\|f_i\|_\infty = 1$, such that $|\mu\psi_i(f_i)| = |\mu(\psi_i f_i)| > \|\mu\psi_i\|_{\boldsymbol{M}} - \varepsilon_i$. Without loss of generality (by changing the phase of f_i if necessary) we can assume that $\mu(\psi_i f_i)$ is real-valued and in fact non-negative, i.e. absolute values can be omitted.

Hence for any finite set $F \subset I$ a function $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ can be defined by $f := \sum_{i\in F} f_i\psi_i$ we have $|f(t)| \le \sum_{i\in F} \|f_i\|_\infty |\psi_i(t)| \le \sum_{i\in F} \psi_i(t) = 1$. Since μ is a linear functional we have therefore positivity of $\mu(f)$ due to

$$\mu(f) = \sum_{i\in F} \mu(\psi_i f_i) = \sum_{i\in F} \mu(\psi_i f_i) > \sum_{i\in F} (\|\mu\psi_i\|_{\boldsymbol{M}} - \varepsilon_i) \ge \sum_{i\in F} \|\mu\psi_i\|_{\boldsymbol{M}} - \varepsilon.$$

The proof is finished by observing that the estimate ensures (absolute) convergence of the sum $\sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}}$ and therefore one can choose F_0 such that $\sum_{i\in I\setminus F_0} \|\mu\psi_i\|_{\boldsymbol{M}} < \varepsilon$ and hence for a suitably defined $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\|f\|_\infty = 1$ one has

$$|\mu(f)| \ge \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}} - 2\varepsilon,$$

and the proof is complete. ∎

lemeastight **Corollary 2.** *Every measure μ is a limit of its finite partial sums. Hence the compactly supported measures*[18] *are dense in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. In particular, $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is an essential Banach module over $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ with respect to pointwise multiplications. One possible approximate unit consists of the families ψ_J, where the index J is running through the finite subsets of I and is given as $\psi_J = \sum_{i\in J} \psi_i$.*

Later on we will discuss (at an abstract level) that a set shows "uniform approximability" by some approximation process (approximate unit in a Banach algebra) if and only if one specific approximating process is working well. See for details in the section on Banach modules.

One can easily verify by the same arguments as above that one obtains an equivalent norm on $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ if one considers the expression $\sum_{i\in I} \|h_i \cdot \mu\|_{\boldsymbol{M}}$, for some family of non-negative functions with $0 < \delta_0 \le \sum_{i\in I} h_i(x) \le C_1 < \infty, \forall x \in \mathbb{R}^d$.

[17] The reader should not worry about the order in which we write a pointwise product. Strictly speaking we might prefer to use the symbol $\psi_i \cdot \mu$ for what is written below as $\mu\psi_i$.

[18] A formal discussion of the support of a measure and then of a distribution is postponed to a later section. Let us take it as a (possible) terminology for now: We will say that "a measure has compact support" (you may think of the property of "support-boundedness" being introduced before the more involved notion of a support is provided) if there exists some $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that μ with $\mu = \mu \cdot k$.

Definition 14. For any BUPU Φ the *Spline-type* or *Quasi-interpolation operator* Sp_Φ can be defined on the space of continuous functions on $\mathbb{R}^d$ via

SpPhidef
$$f \mapsto Sp_\Phi(f) := \sum_{\lambda \in \Lambda} f(\lambda)\phi_\lambda \tag{30}$$

PERHAPS BETTER/MORE CONSISTENT:
For any BUPU Ψ the *Spline-type Quasi-Interpolation operator* Sp_Ψ is defined on the space of continuous functions as follows:

SpPsidef
$$f \mapsto \mathrm{Sp}_\Psi(f) := \sum_{i \in I} f(x_i)\psi_i. \tag{31}$$

Lemma 13. *The operators Sp_Ψ map continuous functions into continuous functions, and are in fact a uniformly bounded family of operators on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ and $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ respectively. Moreover $\|\mathrm{Sp}_\Psi(f) - f\|_\infty \to 0$ for $|U| \to 0$ whenever f is uniformly continuous, i.e. $f \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$.*

For the proof of these statements one can/should use the following pointwise estimates:

sc-estimspl
$$|(\mathrm{Sp}_\Psi\, g - g)(x)| \,\leq\, \mathrm{osc}_\delta(g)(x), \,\forall x \in \mathbb{R}^d. \tag{32}$$

where the local (δ-) oscillation of a function g is given by

oscdef
$$\mathrm{osc}_\delta(g)(x) := sup_{|h|\leq\delta}\{|g(x) - g(x-h)|\}. \tag{33}$$

Of course the sup can be replaced by a max if g is a continuous function!

Proof. Note that we only have to do a pointwise estimate between $\mathrm{Sp}_\Psi\, g(x)$ and $g(x) = \sum_{i\in I}\psi_i(x)g(x)$, where $\mathrm{supp}(\psi_i) \subseteq B_\delta(x_i)$. So let us fix $x \in \mathbb{R}^d$ for a moment. Then

$$|\,\mathrm{Sp}_\Psi\, g(x) - g(x)| \leq \sum_{i\in I} |g(x_i) - g(x)| \cdot |\psi_i(x)|$$

has to be estimated, and of course the sum can be reduced to the subset $I(x) \subset I$ given by $I(x) := \{i \in I \,|\, \psi_i(x) \neq 0\}$. For $i \in I(x)$ however we have $x \in \mathrm{supp}(\psi_i)$ hence $|x - x_i| \leq \delta$, because we assume that Ψ is a δ-BUPU. Hence for each such $i \in I(x)$ we have $|g(x_i) - g(x)| \leq \mathrm{osc}_\delta(g)(x)$. This implies the desired estimate:

$$|\,\mathrm{Sp}_\Psi\, g(x) - g(x)| \leq \sum_{i\in I(x)} \mathrm{osc}_\delta(g)(x)\psi_i(x) \leq \mathrm{osc}_\delta(g)(x),$$

summarized in argument-free form as

$$|\,\mathrm{Sp}_\Psi\, g - g| \leq \mathrm{osc}_\delta(g)$$

□

or in short, a pointwise estimate of the form (see below!)

osc-estim1b
$$[(\mathrm{D}_\Psi f - f) * g] \,\leq\, [|f| * \mathrm{osc}_\delta(g)]. \tag{34}$$

They can be verified using the observation that (again inequalities to be understood in the pointwise sense!)

osctransl
$$\mathrm{osc}_\delta(T_x g) = T_x(\mathrm{osc}_\delta\,(g)) \quad \text{and} \quad \mathrm{osc}_\delta(\breve{g}) = (\mathrm{osc}_\delta\,(g))\breve{}\quad, \quad \breve{g}(x) = g(-x). \tag{35}$$

The family of Spline-type operators forms a *net* of operators, indexed by the neighborhood system of the identity element. In other words, we consider some $U_1 - BUPU\Psi^{(1)}$ to be superior to some $U_2 - BUPU\Psi^{(2)}$ if $U_1 \subset U_2$ (i.e. if it is associated to the smaller neighborhood of the identity element). It is also clear that in this way the index set (namely the neighborhood system of the neutral element of the group) is oriented in terms of "concentration", and consequently the observation can just be reformulated by saying:

The bounded net of operators $(\mathrm{Sp}_\Psi)_U$ is strongly convergent to the identity operator on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$.

Its adjoint operator, which we will call *discretization* operator, denoted by D_Ψ, maps bounded measures into discrete measures.

The concrete form of the dual operator can be obtained from the following reasoning:

SPpsi-dual0

$$\mathrm{Sp}_\Psi{}'(\mu)(f) = \mu(\mathrm{Sp}_\Psi(f)) = \mu\left(\sum_{i\in I} f(x_i)\psi_i\right) = \sum_{i\in I}\mu(\psi_i)f(x_i) = \sum_{i\in I}\mu(\psi_i)\delta_{x_i}(f), \tag{36}$$

hence we can make the following definition

DPsi-def1 **Definition 15.** $\mathrm{D}_\Psi(\mu)$ *(or better)* $\mathrm{D}_\Psi\mu := \sum_{i\in I}\mu(\psi_i)\delta_{x_i}$.

It is a good exercise to verify the following statements:

Lemma 14. *For $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ the sum defining $Sp_\Phi f$ is (unconditionally) norm convergent in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ (even finite at each point), and $\|Sp_\Phi(f)\|_\infty \le \|f\|_\infty$, i.e. Sp_Φ is a linear and non-expansive mapping on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. In particular, the family of operators $(Sp_\Phi)_\Phi$, where Φ is running through the family of all (regular) BUPUs of uniform size (that means that the support size of ϕ with $\|\phi\|_\infty$ is limited) is uniformly bounded on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$.* [19] *Moreover, these spline-type quasi-interpolants are norm convergent to f as $|\Phi|$ (the maximal diameter of members of Φ) tends to zero. In other words, we claim: For every $\varepsilon > 0$ and any finite subset $F \subset \boldsymbol{C}_0(\mathbb{R}^d)$ there exists $\delta > 0$ such that $\|Sp_\Phi f - f\|_\infty < \varepsilon$ if only $|\Phi| \le \delta_0$.*

[20]

For every BUPU Φ we denote the adjoint mapping to Sp_Φ by D_Φ: Discretization operator to the partition of unity Φ. It maps $\boldsymbol{M}(\mathbb{R}^d)$ into itself (in a linear way), and since Sp_Φ is non-expansive the same is true for D_Φ

Again: one might prefer to write D_Ψ !

We will allow to apply operators of this kind also to families of operators, i.e. we will shortly write $D_\rho\Phi$ for the family $(D_\rho(T_\lambda\varphi))_{\lambda\in\Lambda}$. Since D_ρ preserves values of functions (it moves the values via stretching or compression to "other places"), hence $D_\rho\mathbf{1}_{\mathbb{R}^d} = \mathbf{1}_{\mathbb{R}^d}$[21]. Consequently Φ is a (regular) BUPU if and only if $D_\rho\Phi$ is a BUPU (for some, hence all) $\rho > 0$.

In order to understand the engineering terminology of an "impulse response" uniquely describing the behavior of a linear time-*invariant* system let us take a quick look at the situation over the group $G = \mathbb{Z}_n$, the cyclic group (of complex unit roots of order n). Obviously $c_0(\mathbb{Z}_n)$ is just $\mathbb{C}^n$, and the (group-) translation is just cyclic index shift (mod n).

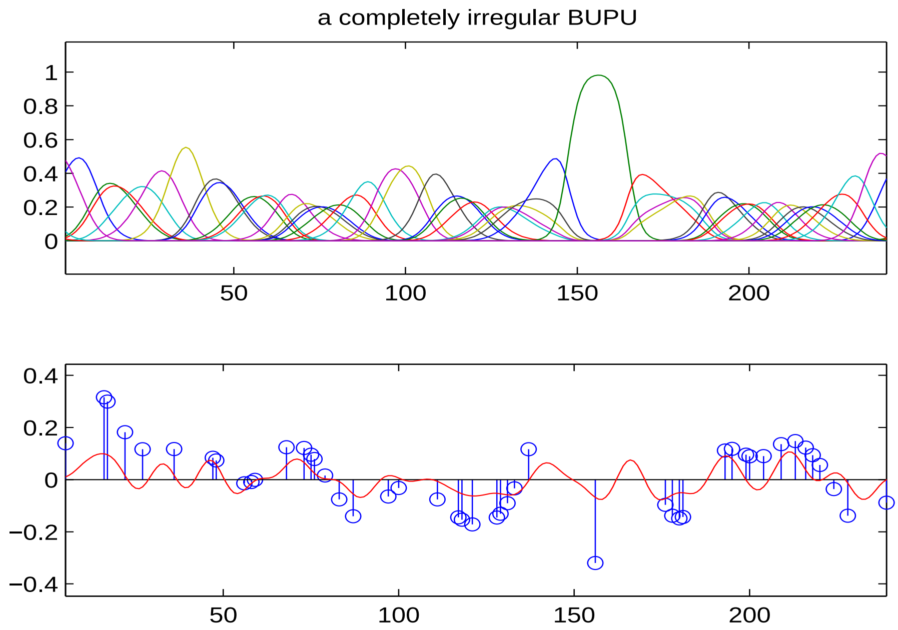
Cn-TILS **Lemma 15.** *A matrix A represents a "translation-invariant" linear mapping on $\mathbb{C}^n$ if and only if it is circulant, i.e. if it is constant along side-diagonals (we will also call such matrices "convolution matrices");*

Proof. We can start with the "first unit vector", which is mapped onto some column vector in $\mathbb{C}^n$ by the linear mapping $x \mapsto A * x$. Since we can interpret all further vectors as the image of the other unit vectors, but these are obtained from the first unit vector by cyclic shift, we

[19] these statements have to be simplified

[20] In some cases it might be of interest to look at BUPUs which preserve uniform continuity, i.e. which have the property that $\mathrm{Sp}_\Psi(f) \in \boldsymbol{C}_{ub}(G)$ for any $f \in \boldsymbol{C}_b(G)$. This is certainly the case if one has a regular BUPU, i.e. a BUPU which is generated from translate of a single, or perhaps a finite collection of "building blocks".

[21] We use the symbol $\mathbf{1}_M$ to indicate the *indicator function* of a set, which equals 1 on M and zeros elsewhere.



see immediately that the columns of the matrix are obtained by cyclic shift of the first column, hence the matrix A has to be circulant (in the cyclic sense).

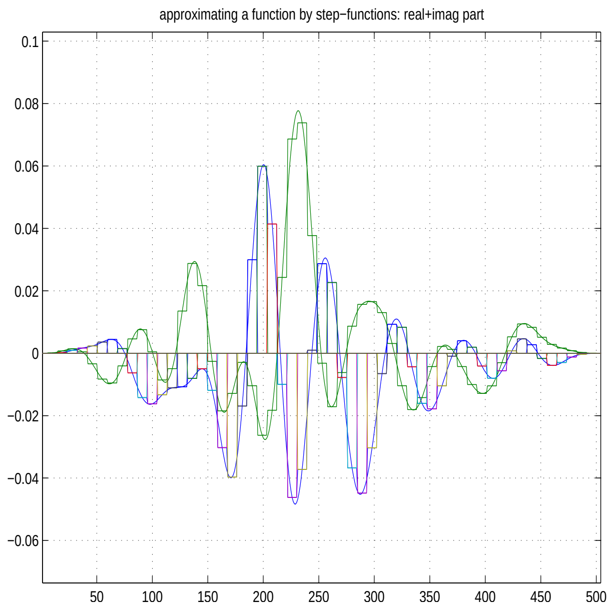
Usually engineers call the first column of this matrix, which is the output corresponding to an "impulse like" input (the first unit vector) the *impulse response of the translation invariant linear system* A.

Since the converse is easily verified we leave it to the interested reader. In fact, it may be interesting to verify the translation invariance by deriving first for a general matrix action A the action of $T_{-1} \circ A \circ T_1$ and to observe subsequently that this operation does not change circulant matrices. □

For the "continuous domain" (i.e. for linear systems over $\mathbb{R}$ resp. over $\mathbb{R}^d$) one has to invoke functionals in order to be able to "represent" the translation invariant systems. We give the details below.

Nevertheless we would like to recall the usual argument (and later how it should be modified). Standard textbooks explain the behavior of a TILS (a translation invariant linear system[22]) by the following illustration. On decomposes, resp. approximates the given input signal (here a smooth, well localized complex-valued function, with real part plotted in blue, imaginary part in green) by a step function, or in other words, a superposition of shifted rectangular functions. Since the linear system $T : f \mapsto T(f)$ (input-output relation) is translation invariant, the output of such a box function is some output signal, which one has to store, say $g = T(box)$. Then the output is a linear combination of shifted copies of g (same shifts with the same coefficients/amplitudes).

[22]Systems are mathematically speaking just linear operators, or input-output relations which satisfy the so-called *superposition principle* $T(f+g) = T(f) + T(g)$, which practically equivalent with linearity of T. Often the domain of T remains vague, i.e. it may change, or the information about output depends on the information of the input. Engineers often use box-diagrams, viewing T as a "black box" with an input arrow and a corresponding output arrow



The argument then continues with the "observation" that one obtains, by making the rectangles shorter and shorter (suitably normalized, such that they all have area one) an **impulse** and the corresponding limit of the output results $T(g)$ "tend to some **impulse response** " function (or object, in whatever sense), so that one can explain the behavior of a linear system by translating the impulse response (with amplitudes coming from the input signal f). In quasi-mathematical writing one could say: We start from the so-called *sifting property* (see the literature of signal processing)

`sifting`
$$f = \int_{\mathbb{R}^d} f(x)\delta_x dx = \int_{\mathbb{R}^d} f(x)T_x(\delta_0)dx \tag{37}$$

to the realization of a TILS T by

`Tilsift1`
$$Tf = \int_{\mathbb{R}^d} f(x)T_x[T(\delta_0)]dx. \tag{38}$$

Writing $h := T(\delta_0)$ for the impulse response function (assuming it is a function) we get

`TILSsift2`
$$Tf(z) = \int_{\mathbb{R}^d} f(x)\, T_x h(z)dx = \int_{\mathbb{R}^d} h(z-x)f(x)dx, \tag{39}$$

which is exactly the well-known formula for convolution of functions, which at first sight appears to justify the reasoning.

We go on (following the engineers arguments) to say, that it is plausible that this calculus is valid, because the step-functions plotted in the above figure "tend to the limiting" function, and therefore (by the continuity of the system) one can expect that also the output is convergent to $T(f)$, the output to the true (smooth, well decaying) function f used as input.

This is a nice plausibility explanation, but it leaves more things open than providing explanations. In which sense do we have convergence (uniform convergence, e.g. if the input function is uniformly continuous). But what are the assumption on the TILS. Is it mapping bounded continuous functions into bounded continuous functions? Is it what engineers would call a BIBOS system (bounded input to bounded output system)? And how can one understand that the limit of the output signals to smaller and smaller box-functions (centered at zero, $\boldsymbol{L}^1$-norm $= 1$) exists, and in which sense?? But if the domain are the bounded and (uniformly?) continuous stepapprox1

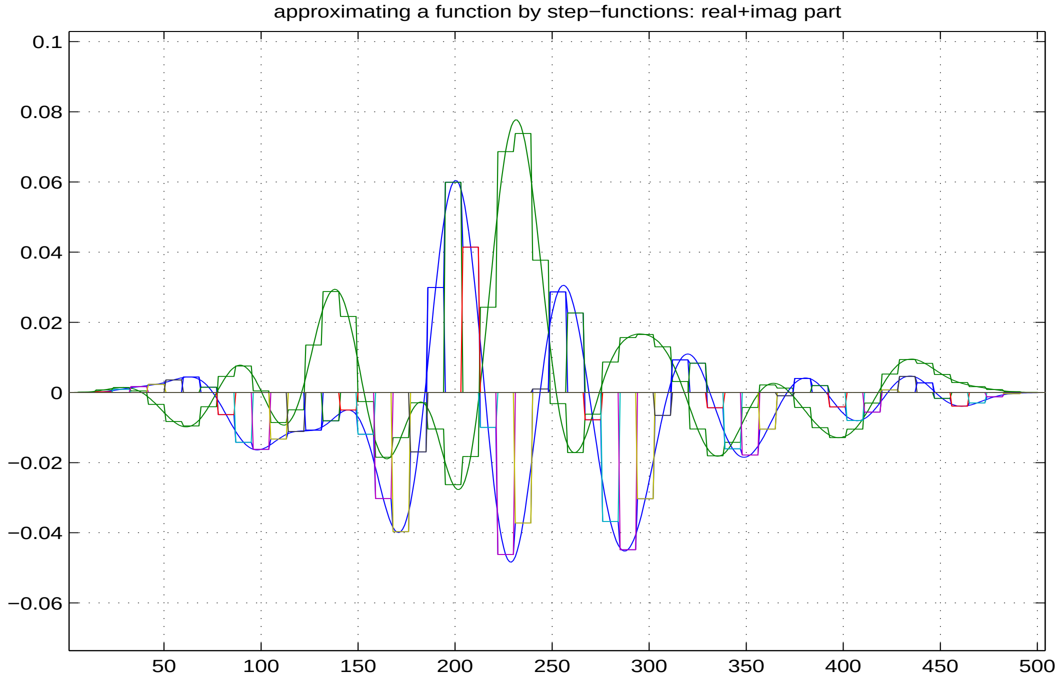


FIGURE 1. A typical illustration of an approximation to the input signal of a translation invariant linear system, preparing for the use of the Dirac impulse as limit of the rectangular functions. But in which sense are these step function convergent to the input signal f, and how are the steps determined?

functions, how can we be sure that the step-functions (approximating them) are also in the domain of the operator (after all they are *discontinuous*!).

Just to illustrate the situation by a plot, which is similar to those found in most books on engineering applications let us look at a step-function approximation of some decent (bounded and decaying) continuous function f. Those books would argue: Feeding the step-function approximation into the system (instead of the continuous function itself) will (but based on which reasoning?) give a very similar output, so that it is enough to know the output for the compressed box function in order to predict the output $T(f)$. The typical picture in stanard text books looks like Fig. stepapprox1 3 below (cf. [**?**], Sec.1.4 introducing the "Faltungsintegral"). I.Sandberg has argued (receiving a lot of sceptical voices within the engineering community) that there is a scandal in the literature ([64, 65]), but how can we get out of this trap?

We will use a slightly different model, by assuming that TILS are defined on the Banach space $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and bounded there. Since $\boldsymbol{C}_c(\mathbb{R}^d)$ is a dense subspace this really means that we only assume that TILS are defined for compactly supported and continuous functions, but with the extra property of boundedness, i.e. that there exists some $C = C(T) > 0$ such that

BIBOS-est1
$$\|T(f)\|_\infty \le C\|f\|_\infty, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d). \tag{40}$$

We will see that in this way the above reasoning can be made sharp. Moreover, in order to approximate within the space $\boldsymbol{C}_0(\mathbb{R}^d)$ it is better to use for the approximation piecewise linear functions, which in fact are superpositions of shifted triangular functions (which in turn form the basis for spline-functions of order 2 resp. degree 1). We will see that under these

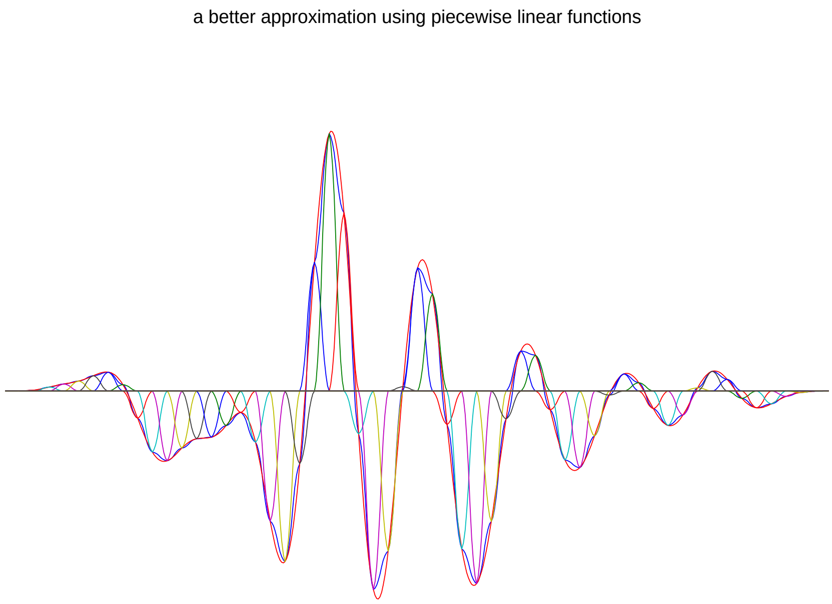


FIGURE 2. The piecewise linear functions are special cases of our approximations of the form $\mathrm{Sp}_\Psi(f)$ and thus converge to f in the uniform norm. Hence by the BIBOs assumption we can also ensure that the output $T(\mathrm{Sp}_\Psi(f))$ will also converge to $T(f)$ uniformly.

circumstances the convolution is with respect to some measure, which in turn is the limit of elementary (discrete) measures.

The common part for both viewpoints is the observation, that for a TILS T the image of a function of the form $\sum_{i\in I} c_i T_{x_i} h$ under T obviously will be of the form

$$T\left(\sum_{i\in I} c_i T_{x_i} h\right) = \sum_{i\in I} c_i T_{x_i} T(h).$$

where $T(h)$ is the approximate impulse response of T.

We may explain, in which way the action of $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ extends in a very natural way from the action only on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ (original domain) to all of $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$.

Given $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ and some BUPU Ψ, let us recall that we have (according to Thm. measabsdecomp 4) following equation absconvmbo 29:

absconvmbo1 (41)
$$\|\mu\|_{\boldsymbol{M}} = \sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}},$$

and hence we can put for $h \in \boldsymbol{C}_b(\mathbb{R}^d)$:

$$\mu(h) := \sum_{i\in I} (\psi_i\mu)(h) = \sum_{i\in I} \mu(\psi_i \cdot h),$$

which is well defined and can be estimated via

$$|\mu(h)| \le \sum_{i\in I} |\mu(\psi_i \cdot h) \le \sum_{i\in I} \|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)} \sup_{i\in I} \|\psi_i\|_\infty \|h\|_\infty \le \|\mu\| \|h\|_\infty.$$

Of course it is not difficult to verify that this definition does *not depend* on the particular choice of the BUPU and that, for a uniformly bounded net (f_α) of functions in $\boldsymbol{C}_b(\mathbb{R}^d)$ it suffices to have uniform convergence over compact sets in order to guarantee that $\mu(f_\alpha) \to \mu(f_0)$. This argument will help us to show that $\hat{\mu}$ is a continuous function on the dual group!

ompact-CORd **Theorem 5.** *A bounded and closed subset $M \subset \boldsymbol{C}_0(\mathbb{R}^d)$ is compact if and only if it is uniformly tight and uniformly equicontinuous* [23]*, i.e. if the following conditions are satisfied:*

- *for $\varepsilon > 0$ there exists $\delta > 0$ such that $|y| \leq \delta \Rightarrow \|T_y f - f\|_\infty \leq \varepsilon \quad \forall f \in M$;*
- *for $\varepsilon > 0$ there exists some $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|f - kf\|_\infty \leq \varepsilon, \quad \forall f \in M$;*

Proof. It is obvious that finite subsets $M \subset \boldsymbol{C}_0(\mathbb{R}^d)$ have these two properties, and it is easy to derive these two properties for compact sets by the usual approximation argument.

So we have to show the converse. We observe first that $\|pf - f\|_\infty \to 0$, if p is a sufficiently large plateau-function. The set $\{pf \mid f \in M\}$ is still equicontinuous (cf. the proof, that $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ is a Banach algebra with respect to pointwise multiplication). We may assume that p has compact support. Then we apply a (sufficiently) fine BUPU to ensure that $\|pf - Sp_\Psi(pf)\|_\infty \leq \varepsilon$. Since p has compact support only finitely many terms make up $Sp_\Psi(pf)$, i.e. one can approximate by finite linear combinations of the elements of Ψ. In other words, up to some $\varepsilon > 0$ one can approximate the elements of M by the elements of a bounded set in a finite-dimensional vector space, which is of course (by the usual Heine-Borel argument) a relatively compact set.

Since a set which can be approximated arbitrarily well by a relatively compact set is relatively compact itself (think of the corresponding ε-coverings with the finite covering property!) the claim follows[24]. □

In the literature (see e.g. [13], p.175) the so-called Arzela-Ascoli Theorem is the prototype characterizing relative compactness in $(\boldsymbol{C}(X), \|\cdot\|_\infty)$ over compact domains.

ArzAscoThm **Theorem 6.** *[Arzela-Ascoli Theorem] Let X be a compact (and hence completely regular) Hausdorff topological space, and M a bounded and closed subset of $(\boldsymbol{C}(X), \|\cdot\|_\infty)$. Then M is compact if and only if it is equicontinuous, which means: For every $\varepsilon > 0$ and $x_0 \in X$ there exists some neighborhood U_0 of x_0 such that*

$$|f(z) - f(x_0)| \leq \varepsilon, \quad \forall f \in M.$$

Just in order to demonstrate that the technical details are in fact not very deep here let us formulate the statement:

Lemma 16. *Given $\varepsilon > 0$ and a set H in some metric space, we denote by H_ε the ε-hull of H, defined as $H_\varepsilon := \bigcup_{h \in H} B_\varepsilon(h)$.*

If there is a sequence of finite sets H^n and a sequence ε_n such that $H \subset H^n_{\varepsilon_n}$ for all $n \geq 1$, with $\varepsilon_n \to 0$ for $n \to \infty$, then H is a precompact set.

Proof. Given $\varepsilon > 0$ we have to show that H has a finite covering of balls of radius ε. For this purpose let us choose n_0 such that $\varepsilon_{n_0} < \varepsilon$. Using now that $H \subset H^{n_0}_{\varepsilon_{n_0}}$ we find that H covered by a finite collection of balls of radius ε[25]. □

[23]The concept of pointwise equicontinuity of functions could be formulated, but is rarely relevant, hence in many cases on just speaks of *equicontinuity* of M.

[24]In this part of the argument we make use of the fact that in a complete metric space relative compactness (resp. the assumption that a set has compact closure) and praecompactness (for every $\varepsilon > 0$ there exists a finite covering with balls of radius ε) are equivalent properties.

[25]It would in fact be sufficient that H_n is a precompact set for every $n \geq 1$.

ef-BIBOTLIS **Definition 16.** The Banach space of all "translation invariant linear systems" (TLIS) on $\boldsymbol{C}_0(\mathbb{R}^d)$ is denoted by[26]

$$\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d)) = \{T : \boldsymbol{C}_0(\mathbb{R}^d) \to \boldsymbol{C}_0(\mathbb{R}^d), \text{ bounded, linear : } T \circ T_z = T_z \circ T, \forall z \in \mathbb{R}^d\}$$

Remark 12. [27] It is easy to show that $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ is a closed subalgebra of the Banach algebra of $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ (in fact it is even closed with respect to the strong operator topology), hence it is a Banach algebra of its own right (with respect to composition as multiplication). We will see later that it is in fact a commutative Banach algebra.

Definition 17. recall the notion of a FLIP operator: $\check{f}(z) = f^{\checkmark}(z) = f(-z)$.

Given $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ we define the *convolution operator* C_μ by: $C_\mu(f)(z) := \mu(T_z f^{\checkmark})$.

The reverse mapping R recovers a measure $\mu = \mu_T$ from a given translation invariant system T via $\mu(f) := T(f(z))(0)$.

zation-TLIS **Theorem 7.** *[Characterization of LTISs on $\boldsymbol{C}_0(\mathbb{R}^d)$]*
There is a natural isometric isomorphism between the Banach space $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$, endowed with the operator norm, and $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, the dual of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, by means of the following pair of mappings:

(1) *Given a bounded measure $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ we define the operator C_μ (to be called convolution operator with convolution kernel μ later on) via:*

onv-measdef
$$C_\mu f(x) = \mu(T_x f^{\checkmark}). \tag{42}$$

(2) *Conversely we define $T \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ the linear functional $\mu = \mu_T$ by*

impuls-def
$$\mu_T(f) = [T f^{\checkmark}](0). \tag{43}$$

The claim is that both of these mappings: $C : \mu \mapsto C_\mu$ and the mapping $T \mapsto \mu_T$ are linear, non-expansive, and inverse to each other. Consequently they establish an isometric isomorphism between the two Banach spaces with

LTIS-isomet
$$\|\mu_T\|_{\boldsymbol{M}} = \|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} \quad \text{and} \quad \|C_\mu\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} = \|\mu\|_{\boldsymbol{M}}. \tag{44}$$

This is a rather important theorem, allowing to define convolution, and also to derive the convolution theorem, not just in an $\boldsymbol{L}^1$-context, but for general measures!

Proof. The proof has a number of partial steps carried out in the course. Some of them are quite elementary (e.g. that the resulting objects are really linear systems resp. linear functionals, or that the mappings C and its inverse are linear mappings) and are left to the interested reader. We concentrate on the most interesting parts.

First of all it is easy to check that the definition of C_μ really defines an operator on $\boldsymbol{C}_0(\mathbb{R}^d)$, with the output being a bounded function.[28] Since

conv-bd1
$$|C_\mu(f)(x)| \le \|\mu\|_{\boldsymbol{M}} \|T_x f^{\checkmark}\|_\infty = \|\mu\|_{\boldsymbol{M}} \|f\|_\infty, \tag{45}$$

and hence the operator norm of C_μ[29] satisfies $\|C_\mu\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} \le \|\mu\|_{\boldsymbol{M}(\mathbb{R}^d)}$.

[26] The letter $\mathcal{H}$ in the definition refers to *homomorphism* [between normed spaces], while the subscript G in the symbol refers to "commuting with the action of the underlying group $G = \mathbb{R}^d$ realized by the so-called regular representation, i.e. via ordinary translations.

[27] Sometimes we will write $[T, T_z] \equiv 0$ in order to express the commutation formula using the commutator symbol, and the "$\equiv$"–symbol to express that this relation holds true $\forall z \in \mathbb{R}^d$.

[28] Engineers talk of BIBOS, i.e. Bounded Input [gives] Bounded Output Systems.

[29] Of course we think of the "convolution by the measure μ, in conventional terms, and this is also the reason for using the symbol C.

Furthermore it commutes with translations, since $D_{-1}(T_z f) = T_{-z} D_{-1} f = T_{-z}\check{f}$

onv-commut1
$$C_\mu(T_z f)(x) = \mu(T_x T_{-z}\check{f}) = \mu(T_{x-z}\check{f}) = C_\mu(f)(x-z) = T_z C_\mu f(x). \tag{46}$$

It is also easy to check - using this fact - that (uniform) continuity is preserved, since

conv-cont
$$\|T_z(C_\mu(f)) - C_\mu(f)\|_\infty = \|C\mu(T_z f - f)\| \le \|\mu\|_{\boldsymbol{M}} \|T_z f - f\|_\infty \to 0 \quad \text{for} \quad |z| \to 0. \tag{47}$$

Finally we have to verify that the output of $f \to C_\mu(f)$ is not just continuous but also decaying at infinity. Due to the closedness of $\boldsymbol{C}_0(\mathbb{R}^d)$ within $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ it is sufficient to approximate $C_\mu(f)$ for $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ by compactly supported (continuous) functions. This is achieved using Theorem 4 resp. Corollary 3. Given $\varepsilon > 0$ one can choose $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\|\mu - h\mu\|_{\boldsymbol{M}} < \varepsilon$. Then the difference

compapprox
$$\|C_{h\mu} f - C_\mu f\|_\infty \le \|h \cdot \mu - \mu\|_{\boldsymbol{M}} \|f\|_\infty \tag{48}$$

can be made arbitrarily small, while on the other hand

ompsuppconv
$$C_{h\mu} f(x) = h\mu(T_x f^{\checkmark}) = \mu(h \cdot T_x f^{\checkmark}) = 0 \tag{49}$$

if $\operatorname{supp}(h) \cap \operatorname{supp}(T_x f^{\checkmark}) = \operatorname{supp}(h) + (x - \operatorname{supp}(f)) \neq \emptyset$, i.e. if $x \notin \operatorname{supp}(h) + \operatorname{supp}(f)$, since we have assumed that f has compact support. Thus altogether we have found that $f \mapsto C_\mu(f)$ is in $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$, with a control of the operator norm on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ by $\|\mu\|_{\boldsymbol{M}}$.

Let us verify next that the two mappings are inverse to each other. Let us verify first that the system associated with μ_T is the original system, in other words we have to show that $C_{\mu_T} = T$ for all $T \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$. Using the relevant definitions this follows from the following chain of equalities

sysmeassys1
$$[C_{\mu_T}(f)](x) = \mu_T(T_x f^{\checkmark}) = T((T_x f^{\checkmark})^{\checkmark})(0) = T(T_{-x} f)(0) \tag{50}$$

Since T commutes with all the translations we continue by

sysmeassys
$$[C_{\mu_T}(f)](x) = T(T_{-x} f)(0) = T_{-x} T f(0) = T f(0 - (-x)) = T f(x). \tag{51}$$

Since this is valid for all $x \in \mathbb{R}^d$ and $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ one direction is shown. It also shows that *every* $T \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ is a convolution operator by a uniquely determined measure $\mu_T \in \boldsymbol{M}_b(\mathbb{R}^d)$, and that $\|\mu_T\|_{\boldsymbol{M}} = \|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))}$. In fact, since we know already that $\|\mu_T\|_{\boldsymbol{M}} \le \|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))}$ we only have to verify that a strict inequality would imply a contradiction. Thus using (51) we come to the following exact isometry:

$$\|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} = \|C_{\mu_T}\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))} \le \|\mu_T\| \le \|T\|_{\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))}.$$

Let us now look for the converse identity: the measure associated to the convolution operator C_μ associated with some given $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ is just the original measure, since

measysmeas1
$$\mu_{C_\mu}(f) = C_\mu(f^{\checkmark})(0) == \mu(T_0(f^{\checkmark})^{\checkmark}) = \mu(f). \tag{52}$$

which in turn implies that the inverse mapping is surjective, i.e. that the (already known to be isometric) bijection between bounded measures and systems is in fact surjective onto the bounded measures. Thus our proof is complete. □

Note also that there is a matter of consistency to be verified. For "ordinary functions" $g \in \boldsymbol{C}_c(\mathbb{R}^d)$ [30] one can define a unique bounded measure

regbdmeas1
Definition 18. $\mu_g(f) = \int_{\mathbb{R}^d} f(x) g(x) dx$,

[30] Resp. for $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ if one is familiar with Lebesgue integration, but we prefer to avoid this part of analysis, mostly in order to demonstrate that it is not of the importance usual given to it in most books.

3.1. **Many ways to introduce convolutions!** The previous characterization allows to introduce in a natural way a Banach algebra structure on $\boldsymbol{M}(\mathbb{R}^d)$. In fact, given μ_1 and μ_2 the translation invariant system $C_{\mu_1} \circ C_{\mu_2}$ is represented by a bounded measure μ. In other words, we can define a new (so-called) *convolution product* $\mu = \mu_1 * \mu_2$ of the two bounded measures such that the relation (completely characterizing the measure $\mu_1 * \mu_2$)

CONVO1 $$C_{\mu_1 * \mu_2} = C_{\mu_1} \circ C_{\mu_2} \tag{53}$$

TURN THE NEXT STATEMENT INTO A FORMAL DEFINITION

It is immediately clear from this definition that $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is a **Banach algebra with respect to convolution**! Associativity is given for free, but *commutativity of the new convolution is not so obvious* (and will follow only later, clearly as a consequence of the commutativity of the underlying group).

The translation operators themselves, i.e. T_z are elements of $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$, which correspond exactly to the Dirac measures δ_z, $z \in \mathbb{R}^d$, due to the following simple consideration[31].

conv-deltax $$C_{\delta_x} f(z) = \delta_x(T_z f^{\checkmark}) = [T_z f^{\checkmark}](x) = f^{\checkmark}(x - z) = f(z - x) = [T_x f](z), \tag{54}$$

and thus $C_{\delta_x} = T_x$, in particular $C_{\delta_0} = T_0 = Id$, resp.

convDirac0 $$f = \delta_0 * f \quad \text{and} \quad T_x f = \delta_x * f, \forall f \in \boldsymbol{C}_0(\mathbb{R}^d). \tag{55}$$

Remark 13. In the literature there are other, quite non-obvious definitions of the convolution of two bounded measures. As a consequences of this "usual" approach it is required to provide a (non-trivial) proof for the associativity, which is *not necessary* in the case of our definition. In fact, if convolution is defined in an individual manner in the context of generalized functions this associativity rule may even *fail to be valid* (cf. [2], p. 111/112). A similar point arises in H. Shapiro's description of (abstract) *homogeneous Banach spaces* where he also has to assume that the action of (generalized convolution, i.e. the integrated group action) is associative, and he gives a proof for the case of locally integrable functions ([66]).

The classical approach describes the convolution of two measures (for now the order matters) as follows: Given $\mu_1, \mu_2 \in \boldsymbol{M}_b(\mathbb{R}^d)$ the action of $\mu_1 * \mu_2$ on $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ can be characterized by

conv-meas1 $$\mu_1 * \mu_2(f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y + x) d\mu_2(y) d\mu_1(x). \tag{56}$$

Proof. First recall the definition, i.e. that $\mu_1 * \mu_2$ is the linear functional corresponding to $T := C_{\mu_1} \circ C_{\mu_2} \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ satisfies

$$\mu_1 * \mu_2(f) = T(f^{\checkmark})(0) = [(C_{\mu_1} \circ C_{\mu_2})(f^{\checkmark})](0) = [C_{\mu_1}(C_{\mu_2}(f^{\checkmark}))](0) = \mu_1(g),$$

with $g(x) = [C_{\mu_2}(f^{\checkmark})]^{\checkmark}(x) = C_{\mu_2}(f^{\checkmark})(-x) = \mu_2(T_x f)$. Putting everything into the standard notation we have $g(x) = \int_{\mathbb{R}^d} f(y + x) d\mu_2(y)$ which implies the result stated above. □

Remark 14. The different realizations of convolution can conveniently be disguised by using (mostly from now on throughout the rest of the manuscript) the symbol $*$ for convolution, i.e. we write $\mu * f$ instead of $C_\mu(f)$ from now on, or $g*f$ for the convolution of $\boldsymbol{L}^1(\mathbb{R}^d)$-functions using the Lebesgue integral. We have discussed that the reinterpretation of those things in varying contexts is - from a pedantical point of view - a delicate matter, but having discussed this problem in detail for a few cases we leave the rest to the reader if there is interest in the subject. We just want to convince all our readers that e.g.

31 We start writing $\mu * f$ for $C_\mu(f)$, the convolution of μ with f. This is justified by the fact that in the case of double interpretation, e.g. for $\mu = \mu_g$ for some $g \in \boldsymbol{C}_c(\mathbb{R}^d)$ either as a pointwise defined integral or the action of a measure on test functions one can *easily* check that there is consistency among the different view-points. From now on we will not discuss this issue in all details and leave the verification of details to the reader.

associativity or commutativity of convolution is always well justified in the context that we are providing, and that in this sense the user of the calculus does not have to worry about possible inconsistencies.

Although the above result, combined with Fubini's theorem indicates that convolution is *commutative* we do not make use of this fact, because we do not want to invoke results from measures theory at this point. In fact, using an approximation argument (using discrete measures) we can get the result. Looking back we can say however that this is also more or less the way how we prove Fubini, using suitable resummation of suitable Riemannian sums.

3.2. **Convolution and the commutativity question.**

Proposition 2. *For every $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ and $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ one has*

convdiscr1 (57)
$$lim_{|\Psi| \to 0} \|D_\Psi \mu * f - \mu * f\|_\infty = 0.$$

Moreover the limit is uniform for (bounded and) equicontinuous sets $M \subset \boldsymbol{C}_0(\mathbb{R}^d)$.

CompOpenCB1 *Remark* 15. We will demostrate separately (see [29] for details) that we can say something similar for these convolution operators acting on $\boldsymbol{C}_b(\mathbb{R}^d)$. However, in this case we cannot guarantee uniform convergence anymore, but only *uniform convergence over compact sets.*

convDpismu1 *Remark* 16. Although it looks as if the convolution with $D_\Psi \mu$ on f was just the same (by the adjointness of D_Ψ and Sp_Ψ) as convolving μ with $\mathrm{Sp}_\Psi(f)$ (which in turn is close to f), but this is not so. Since the spline quasi-interpolation operators Sp_Ψ does NOT commute with all translations (or the inflection $f \mapsto f^\checkmark$) we have in general
$$\mu(T_x(\mathrm{Sp}_\Psi)^\checkmark) \neq \mu(\mathrm{Sp}_\Psi(T_x f^\checkmark)).$$

siOscEstim1 *Remark* 17. There are different ways to verify this statement. The proof in the course was based on the fact that one can derive (from the definition) first pointwise convergence, hence (by the use of ε-coverings) uniform convergence over compact sets, and finally by the *uniform concentration* (approximation of both μ and f by compactly supported elements in the respective spaces) uniform convergence overall. This is OK but it seems that we can give a better argument, using a pointwise oscillation estimate:

scmeasconv2 (58)
$$[(D_\Psi \mu - \mu) * f](x) \,\leq\, [|\mu| * \mathrm{osc}_\delta(f)](x), \,\forall x \in \mathbb{R}^d.$$

Unfortunately this argument requires (at the moment) the use of a measure $|\mu|$ (with the same norm as μ, i.e. with $\| \, |\mu| \, \|_{\boldsymbol{M}_b} = \|\mu\|_{\boldsymbol{M}_b}$).

Proof. There are different ways to verify this statement. The proof in the course was based on the fact that one can derive (from the definition) first pointwise convergence, hence (by the use of ε-coverings) uniform convergence over compact sets, and finally by the *uniform concentration* (approximation of both μ and f by compactly supported elements in the respective spaces) uniform convergence overall.

Let us try now to outline the details of the proof, we do it in different steps.

(1) First we prove that we have pointwise convergence. This is easy because convolution can be rewritten (pointwise) as the action of the convolving measure on $T_x f^\checkmark$, thus
$$(D_\Psi \mu - \mu)(T_x f^\checkmark) \to 0 \text{ for } |\Psi| \to 0.$$
(2) Using the properties of a net (or sequence) one can easily show that this convergence is uniform over arbitrary finite sets;
(3) As a next step we verify that it is uniform over compact subsets. This is now a general functional analytic argument. Given a compact set $K \subset \mathbb{R}^d$ (or a LCA group) and $\varepsilon > 0$

we find that there is some open (relatively compact) neigborhood U of of the identity such that

$$\|T_x f^{\checkmark} - T_{x+u} f^{\checkmark}\|_\infty = \|f - T_{x+u} f\|_\infty < \varepsilon/(3\|\mu\|_{\boldsymbol{M}})$$

for all $u \in U$. Thus the family $(x+U)_{x\in K}$ is a covering of K by open subsets, and since K is compactwe can find a finite sequence $(x_k)_{k=1}^K$ such that $(x_k+U)_{k=1}^K$ covers all of K. Choosing now $\delta > 0$ in such a way that we can be sure the

$$|(D_\Psi\mu - \mu)(T_{x_k} f^{\checkmark})| < \varepsilon/3 \text{ if } |\Psi| \le \delta$$

we can put things together. For every given $x \in K$ we can find at least one $u \in U$ and x_k in the finite list such that $x = x_k + u$, and thus we can estimate as follows

$$\|T_x f^{\checkmark} - T_{x_k+u} f^{\checkmark}\|_\infty \le \varepsilon/(3\|\mu\|_{\boldsymbol{M}}),$$

and as a consequence

$$|C_\mu(f)(x) - C_\mu(f)(x_k)| \le \|\mu\|_{\boldsymbol{M}} \cdot \varepsilon/(3\|\mu\|_{\boldsymbol{M}}) = \varepsilon/3 \text{ for } x = x_k + u.$$

But since $\|D_\Psi\mu\|_{\boldsymbol{M}} \le \|\mu\|_{\boldsymbol{M}}$ we can argue that in a similar way we also have

$$|C_{D_\Psi\mu}(f)(x) - C_{D_\Psi\mu}(f)(x_k)| \le \varepsilon/3 \text{ for } x = x_k + u.$$

(4) Putting these inequalities together, knowing that one has first to choose the finitely many points (for the estimate which depends only on the required $\varepsilon > 0$ and the known norm of the measure) and *then*, after choosing an appropriate sequence (x_k), and then a common parameter $\gamma \le \delta$ such $|\Psi| \le \gamma$ implies

convestK
$$|C_\mu(f)(x) - C_{D_\Psi\mu}(f)(x)| \le \varepsilon, \text{ for all } x \in K. \tag{59}$$

(5) What we have implies so far is: whenever μ and f have compact support, i.e. are of the form $\mu = \mu \cdot p$ for some $p \in \boldsymbol{C}_c(\mathbb{R}^d)$ and $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ then $\mu * f(x) = 0$ outside the compact set $K = \mathrm{supp}(p) + \mathrm{supp}(f)$. We thus have under this extra assumption, in a more compact form

$$\|C_\mu(f) - C_{D_\Psi\mu}(f)\|_\infty \le \varepsilon, \text{ whenever } |\Psi| \le \gamma, \text{ for } x = x_k + u.$$

(6) Next we show that not only for compactly supported measures or functions f (for such functions it is enough to observe the convergence over some compact set, because the convolution product is zero outside of some compact set!) the convergence is possible, but also for general measure $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ and functions $f \in \boldsymbol{C}_0(\mathbb{R}^d)$, using an approximation argument.
(7) The trick will be to approximate μ by measures of the form $\mu \cdot p$, with $p \in \boldsymbol{C}_c(G)$ (plateau functions with compact support, e.g. a finite partial sum of a BUPU), and in a similar way $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ by some $k \in \boldsymbol{C}_c(G)$ (possibly also of the form $k = p_1 \cdot f$. We need then an estimate of the form

$$\|\mu * f - (\mu \cdot p) * (f \cdot p_1)\|_\infty$$

and a similar estimate with $D_\Psi\mu$, using essentially the boundedness of the bilinear mapping $(\mu, f) \to \mu * f$, in the following manner:

Using the triangle inequality we get the following estimate:

$$\begin{aligned}\|\mu * f - (\mu \cdot p) * (f \cdot p_1)\|_\infty &\le \|(\mu - \mu \cdot p) * (f \cdot p_1)\|_\infty + \|(\mu \cdot p)(f \cdot p_1 - f)\|_\infty \\ &\le \|\mu - \mu \cdot p\|_{\boldsymbol{M}} \|f \cdot p_1\|_\infty + \|\mu \cdot p\|_{\boldsymbol{M}} \|f \cdot p_1 - f\|_\infty.\end{aligned}$$

Given $\varepsilon > 0$, and knowing the constants $\|f \cdot p_1\|_\infty \leq \|f\|_\infty$ and $\|\mu \cdot p\|_{\boldsymbol{M}} \leq \|\mu\|_{\boldsymbol{M}}$ we can of course choose $p, p_1 \in \boldsymbol{C}_c(\mathbb{R}^d)$ in such a way that the overall expression is smaller than $\varepsilon/5$, say.

□

THIS UPCOMING STATEMENT SHOULD BE TURNED INTO A PROPOSITION OR EVEN BETTER THEOREM (:FA HINT!) The internal convolution (this is how we have defined it as a multiplication on the bounded measures, realizing the composition of TILS) is exactly the adjoint action (up to inversion = flip) of measures via convolution operators on the predual of $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, which is $\boldsymbol{C}_0(\mathbb{R}^d)$.

So we have the following situation: On the one hand there exists an external action of a bounded measure μ on the (homogeneous) Banach space $\big(\boldsymbol{C}_0(\mathbb{R}^d),\ \|\cdot\|_\infty\big)$, via the *convolution operator* C_μ, which was used to define the *internal* multiplication which we called convolution. Recall that this has been defined in such a way that one enforces the associativity law $C_{\mu_1} \circ C_{\mu_2} = C_{\mu_1 * \mu_2}$. But since $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ is the dual space of $\big(\boldsymbol{C}_0(\mathbb{R}^d),\ \|\cdot\|_\infty\big)$ we also have the adjoint action, i.e. the operator C'_μ defined as usual by the identity

$$C'_\mu(\nu)(f) = \nu(C_\mu(f)), \quad f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

It is a good exercise to verify that in sloppy terms one has:

rnaladjoint **Theorem 8.** *The operator C'_μ can be described by*

adjconv01 (60) $$\nu \mapsto \mu^{\checkmark} * \nu, \quad \nu \in \boldsymbol{M}_b(\mathbb{R}^d).$$

Remark 18. The fact that the Banach space on which the action of the (then) Banach space $\big(\boldsymbol{C}_0(\mathbb{R}^d),\ \|\cdot\|_\infty\big)$ on which the Banach convolution algebra $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ has the *extra relation* the this Banach algebra by *duality* implies that the action of the algebra on $\big(\boldsymbol{C}_0(\mathbb{R}^d),\ \|\cdot\|_\infty\big)$ has an adjoint action on the dual space. Since we would like to call this action also "convolution" we have to check compatibility of the two operations. In fact, what we observe, convolution *within* $\boldsymbol{M}_b(\mathbb{R}^d)$ turns out to be the adjoint action to convolution combined with inversion (note that this makes the adjoint action covariant (same associativity rule) even in a non-commutative setting as in $\boldsymbol{M}_b(\mathscr{G})$).

flip-convC0 (61) $$(\mu, f) \mapsto \mu^{\checkmark} * f = (\mu * f^{\checkmark})^{\checkmark} \quad \forall \mu \in \boldsymbol{M}_b(\mathbb{R}^d), f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

Preliminary comment: Here it might be the right place to comment on the compatibility of involutions, such as $f \mapsto f^{\checkmark}$ with convolution! (even over groups!)

lipconvmeas (62) $$(\mu_1 * \mu_2)^{\checkmark} = {\mu_2}^{\checkmark} * {\mu_1}^{\checkmark}, \quad \mu_1, \mu_2 \in \boldsymbol{M}_b(\mathscr{G}).$$

Remark 19. First we observe that this involution is 'self-dual' in the following sense: assume that you have a measure, then one can try to define the dual operation on measures, which is then an involution on $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$. On the other hand one can try to apply this involution directly to ordinary functions and then extend it (expecting that it should be w^*-w^*--continuous) to an involution on all of $\boldsymbol{M}_b(\mathbb{R}^d)$.

We are supposed to check the mapping $T'(\mu)$ for $T : f \mapsto f^{\checkmark}$, or $T'(\mu)(f) = \mu(f^{\checkmark})$. For discrete measures it is clear that one has $T'(\delta_x)(f) = \delta_x(f^{\checkmark}(x) = f(-x) = \delta_{-x}(f)$, or $T'(\delta_x) = \delta_{-x}$. In fact, we can write $f^{\checkmark} = D_{-1}(f)$, hence $T' = \mathrm{St}_{-1}$, consistent with our operations.

One can also make a similar considerations for other involutions, e.g. $f \mapsto f^*$, given by $f^*(x) = \overline{f(-x)}$, which is obviously a combination (the order does *not* matter in this case) of the involution $f \mapsto f^{\checkmark}$ and the natural involution on function spaces coming from complex conjugation: $f \mapsto \overline{f}$.

In order to verify this other consistency we can argue as follows: By definition we have

$$C_{\mu_1 * \mu_2} h(z) = C_{\mu_1}(C_{\mu_2} h)(z)$$

for all $h \in \boldsymbol{C}_0(\mathbb{R}^d)$ and $z \in \mathbb{R}^d$, hence for $z = 0$, hence by our conventions

$$(\mu_1 * \mu_2)(h^{\checkmark}) = \mu_1((\mu_2 * h)^{\checkmark}) = \mu_1({\mu_2}^{\checkmark} * h^{\checkmark})$$

or by writing $f = h^{\checkmark} \in \boldsymbol{C}_0(\mathbb{R}^d)$:

dist-conv
$$(\mu_1 * \mu_2)(f) = \mu_1({\mu_2}^{\checkmark} * f) = \mu_1((\mu_2 * f^{\checkmark})^{\checkmark}). \tag{63}$$

This form of defining the convolution of two "abstract objects" is very much like the typical definition of the convolution of distributions. It makes sense as long as the convolution of the functional μ_2 with test functions results in another test function [32], here a member of $\boldsymbol{C}_0(\mathbb{R}^d)$.

Since the algebra $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ is *not* commutative it is not at all clear from this definition why $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_M, *)$ should be a commutative Banach algebra, which is in fact true.

In order to prepare for this statement we have to provide a few more statements.

The compatibility of the (isometric) dilation operators D_ρ with translations, i.e. the rule **[Rd-specific!]**

dil-trans
$$D_\rho T_z = T_{z/\rho} D_\rho \tag{64}$$

makes it possible to define another norm preserving *automorphism* for the Banach algebra $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_M, *)$, given as follows: [33].

Strhodef **Definition 19.** The *adjoint* action of the group $\mathbb{R}^+$ on $\boldsymbol{M}(\mathbb{R}^d)$ is defined as the family of adjoint operators, i.e. $\mathrm{St}_\rho := {\mathrm{D}_\rho}'$ is defined on $\boldsymbol{M}(\mathbb{R}^d)$ via:

def-Stroh
$$\mathrm{St}_\rho \mu(f) := \mu(D_\rho f), \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d),\ \rho \neq 0. \tag{65}$$

We consider this the *mass-preserving* dilation operator (the letters "St" can be read as *stretching*[34] operator). It is easy to check that for $k \in \boldsymbol{C}_c(\mathbb{R}^d) \subset \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d)$ one has:

rodrohcomp1
$$\mathrm{St}_\rho[\mu_k] = \rho^{-d} \cdot \mu_{[D_{1/\rho} k]}, \quad \text{and} \quad \mathrm{D}_\rho[\mu_k] = \rho^d \mu_{[\mathrm{St}_\rho k]}, \tag{66}$$

which justifies the equation

StroDro2
$$\mathrm{St}_\rho = \rho^{-d} \mathrm{D}_{1/\rho} \quad \text{resp.} \quad \mathrm{D}_{1/\rho} = \rho^d\, \mathrm{St}_\rho,\ \rho \neq 0. \tag{67}$$

Later on (for the sampling theorem) we will need two observations, involving the Shah-distribution $\sqcup\!\sqcup := \sum_{n \in \mathbb{Z}^d} \delta_n$, often also called a *Dirac comb* [35].

Note that $\mathrm{St}_\rho(\sqcup\!\sqcup) = \sqcup\!\sqcup_\alpha := \sum_{n \in \mathbb{Z}^d} \delta_{\alpha n}$ and, according to the above formulas $\mathrm{D}_\rho \sqcup\!\sqcup = (1/\rho) \cdot \sqcup\!\sqcup_{1/\rho}$.

The corresponding variant of convergence is then the so-called vague convergence:

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[32] Then the right hand side makes sense and can be used to give the expression $\mu_1 * \mu_2$ a meaning. Note however that this is *dangerous terrain*, because it may happen that such an individually defined convolution between distributions - similar to pointwise products - may turn out to be *non-associative*!

[33] We will see that these two operations on $\boldsymbol{C}_0(\mathbb{R}^d)$ resp. $\boldsymbol{M}(\mathbb{R}^d)$ are adjoint to each other.

[34] In German language we use the words "STrecken" for "stretching" ($\rho > 1$) and "STauchen" for "compression, which occors for $\rho < 1$, so it German the symbol fits even better for this Stauch-Streck operators.

[35] It is of course only an unbounded measure, but not too bad. We will see that it however a bounded linear functional on the so-called Wiener algebra $\boldsymbol{W}(\mathbb{R}^d)$, see Def. WCdefphi 37.

vagconvdef2 **Definition 20.** A sequence (resp. net) of (bounded or unbounded) measures μ_α is convergent in the *vague sense* to the limit measure μ_0 if one has

$$\lim_\alpha \mu_\alpha(k) = \mu_0(k), \quad \forall k \in \boldsymbol{C}_c(\mathbb{R}^d).$$

Remark 20. It is an easy exercise to check that a bounded net in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is vaguely convergent if and only if it is w^*-convergent.

Using this concept we can show

shah-conv **Lemma 17.** *One has the following convergence of measures (unbounded here) in the vague sense:*

- $\lim_{\rho\to\infty} \mathrm{St}_\rho \sqcup\!\sqcup = \lim_{\rho\to\infty} \sqcup\!\sqcup_\rho \; = \; \delta_0;$
- $\lim_{\rho\to\infty} \mathrm{D}_\rho \sqcup\!\sqcup = \lim_{\rho\to\infty} \rho^{-d} \sqcup\!\sqcup_{1/\rho} \; = \; 1.$

Proof. i) Given $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ one finds $R > 0$ such that $k(x) = 0$ if only $|x| \geq R$. It is then clear that $\mathrm{D}_\rho k(z) = 0$ for $|z| > 1$, as long as $\rho > R$, and thus in fact

$$\mathrm{St}_\rho \sqcup\!\sqcup(k) = \sum_{n\in\mathbb{Z}^d} \mathrm{D}_\rho(k) = \delta_0(k) = k(0).$$

ii) As for the second argument it is sufficient to verify that the functional $\mathrm{D}_\rho \sqcup\!\sqcup = \rho^{-d} \sqcup\!\sqcup_{1/\rho}$ represents exactly a typical Riemannian sum applied to $f \in \boldsymbol{C}_c(\mathbb{R}^d)$, namely $\rho^{-d} \sum_{n\in\mathbb{Z}^d} f(n/\rho)$ which is known to tend (for $\rho \to \infty$) to the integral, which itself can be written in the form of an integrator against the constant function $\mathbf{1}$, i.e. the limit is $\int_{\mathbb{R}^d} f(x)\mathbf{1}(x)dx$. This arguments completes the proof. □

Being defined as adjoint operators each of the operators St_ρ is not only isometric on $\boldsymbol{M}(\mathbb{R}^d)$, but also w^*-w^*-continuous on $\boldsymbol{M}(\mathbb{R}^d)$.

StroAltdef **Lemma 18.** *Given $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ the uniquely determined measure corresponding to the operator $D_\rho^{-1} \circ C_\mu \circ D_\rho$ coincides with $\mathrm{St}_\rho\mu$. Also from this characterization the fact that St_ρ defines an automorphism on $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is evident.*

Proof. One has to apply both operators to an arbitrary $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ and verify that the result is the same. □

Remark 21. Reformulated in a more usual way (using convolutions), one has the identity:

stromsconv1 (68) $$\mathrm{St}_\rho\mu * f = {\mathrm{D}_\rho}^{-1}(\mu * \mathrm{D}_\rho(f)),\ f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

We collect the basic facts for this new mapping:

conv-strho (69) $$\mathrm{St}_\rho\mu_1 * \mathrm{St}_\rho\mu_2 = \mathrm{St}_\rho(\mu_1 * \mu_2)$$

Proof. To be provided later. It shows that the two alternative definitions above are indeed equivalent!! □

We have [**Rd-specific!**]

Strho-isom (70) $$\|\mathrm{St}_\rho\mu\|_{\boldsymbol{M}} = \|\mu\|_{\boldsymbol{M}}.$$

While the (mass-preserving) operator St_ρ is well compatible with convolution, the (value-preserving) operators D_ρ is ideally compatible with pointwise multiplications, since obviously

Drho-ptwmul (71) $$\mathrm{D}_\rho(f \cdot g) = \mathrm{D}_\rho f \cdot \mathrm{D}_\rho g.$$

While D_ρ is well adapted to pointwise multiplication and St_ρ to convolution, it is still possible to use D_ρ in a convolution context, observing the following relations:

Drho-conv1 (72)
$$\mathrm{St}_\rho \mu * D_{1/\rho} f = D_{1/\rho}(\mu * f) \quad \text{resp.} \quad \mathrm{St}_{1/\rho}\, \mu * \mathrm{D}_\rho f = \mathrm{D}_\rho(\mu * f).$$

One could state similar formulas for St_ρ with respect to pointwise multiplication.

CHECK FOR DUPLICATES

xuxuxu **Lemma 19.** *The group $\mathbb{R}^+$ is acting strongly continuous on $\boldsymbol{C}_0(\mathbb{R}^d)$ via D_ρ, i.e. the mapping $x \to \mathrm{D}_\rho f$ is a continuous mapping from $\mathbb{R}^+$ to $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big), \forall f \in \boldsymbol{C}_0(\mathbb{R}^d)$.*

Proof. The proof can be done in an elementary way, checking continuity at $\rho = 1$ for $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ (using the uniform continuity of f and the joint compact support of all the functions $(\mathrm{D}_\rho f)_{\rho\in[1/2,2]})$. The general case of $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ follows then by approximation (Try and check the details if you are not sure about the argument!)[36] □

Stroh-map **Lemma 20.** *The mapping $\rho \mapsto \mathrm{St}_\rho$ is continuous from $\mathbb{R}^+$ into $\mathcal{L}(\boldsymbol{M}(\mathbb{R}^d))$, endowed with the strong operator topology. It is also true that the mapping $(f, \rho) \mapsto C_{\mathrm{St}_\rho \mu}(f)$ is continuous from $\boldsymbol{C}_0(\mathbb{R}^d) \times \mathbb{R}^+$ into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. Obviously the operators $(\mathrm{St}_\rho)_{\rho>0}$ form a commutative group of isometric operators on $\boldsymbol{M}(\mathbb{R}^d)$ (but also - since they leave $\boldsymbol{L}^1(\mathbb{R}^d)$ invariant - on $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$).*

stro-group (73)
$$\mathrm{St}_{\rho_1} \circ \mathrm{St}_{\rho_2} = \mathrm{St}_{\rho_1\cdot\rho_2} = \mathrm{St}_{\rho_2} \circ \mathrm{St}_{\rho_1} \qquad \rho_1, \rho_2 > 0.$$

Proof. Proof to be typed later on. The isometric action on any of the $\boldsymbol{L}^p$-spaces results from the transformation rule for integrals, while the composition law follows by easy direction computation.

Among others it is obvious that the property $\|\mathrm{D}_\rho f\|_\infty = \|f\|_\infty$ for all $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ implies $\|\mathrm{St}_\rho \mu\|_{\boldsymbol{M}_b} = \|\mu\|_{\boldsymbol{M}_b}$ for all $\mu \in (\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$.[37] □

trho-deltas (74)
$$\mathrm{St}_\rho \delta_x = \delta_{\rho x},$$

Remark 22. Due to the w^*-density of finite discrete measures in $\boldsymbol{M}(\mathbb{R}^d)$ one can characterize St_ρ as the uniquely determined norm-to-norm continuous and $w^* - w^*$-continuous mapping which is isometric on $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ and satisfies formula (Strho-deltas 74).

Remark 23. There is an interesting connection to probability theory and random variables. For each random variable X with values in $\mathbb{R}^d$ there exists exactly on probability measure μ on $\mathbb{R}^d$ such that for every Borel set $M \subseteq \mathbb{R}^d$ one has
$$E\,(X(t) \in M) = \int_M d\mu(x).$$
In fact, any probability statement about the random variable $X(t)$ can be expressed knowing only μ.

[36] This can be seen as a general fact: The mapping $\rho \to \mathrm{D}_\rho$ is a group representation of $(\mathbb{R}_+, \cdot)$ as a group of isometric transformation on the Banach space $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, i.e. one can use the fact that $\mathrm{D}_{\rho_2} \circ \mathrm{D}_{\rho_1} = \mathrm{D}_{\rho_2\cdot\rho_1}$.

[37] Generally speaking: the adjoint operator to a given isometric operator is again isometric on the dual Banach space.

Remark 24. In this context convolution comes into the game as follows: Assume that X_1, X_2 are two *independent random variables* (this has to be defined and explained separately, just think for now of things which have "nothing to do with each other in terms of probability") with corresponding measures μ_1, μ_2. Then the random variable $X_1 + X_2$ has of course another measure μ associated to it. The claim is then: $\mu = \mu_1 * \mu_2(!)$.

This is easy to understand in the case of constant random variables. If it is 100% certain that $X(t) = x, \forall t \in \Omega$ ad $Y(t) = y, \forall t \in \Omega$, or in other words $\mu_X = \delta_x$ and $\mu_Y = \delta_y$, then it is obvious that $(X+Y)(t) = x + y, \forall t \in \Omega$, hence $\mu_{X+Y} = \delta_{x+y}$ which matches perfectly with the observation that $\delta_{x+y} = \delta_x * \delta_y$.

For discrete random variables, i.e. for random variables taking only a finite number of discrete values with positive probability (such as a dice with $E(D = k) = 1/6$ for $k = 1, \dots, 6$ similar rules apply, because the corresponding discrete measures (in our case $\mu_D = \sum_{k=1}^{6} \delta_k/6$) have to be convolved.

A good exercise is to check what can be said about the probabilities of a sum of two independent dices (wether one has two independent dices thrown in parallel or two independent repetitions of the same dice does not matter!). Obviously there are altogether 36 possibilities, one of them being the pair $(1,1)$ (resp. $(6,6)$). Another three possibilities (giving exactly the sum of 4) are the pairs $(1,3),(2,2),(3,1)$, etc.. Checking this at a formal level reveals the analogy to convolution in a discrete setting, resp. the Cauchy product formula for the multiplication of polynomials (as we learn it at school).

HINT: DISTRIBUTION FUNCTION: $F(z) = \int_{-\infty}^{z} d\mu(t)$.

The usual way to provide visual information about a probability measure is the look at its distribution function, or – in the case of experimental data – some *histogram.* Depending on the available amount of data and the variance of the measure one can have a good representation with a certain number of bins, not too small but also not too big. The law of the large number tells us essentially that we can expect that for a large number of realizations the probability distribution can be "read off from the histogram", or from a different view-point: The histogram will give a realistic picture about the probability distribution, in some discretized way.

Assume that we have a real-valued *random variable* $X : \Omega \to \mathbb{R}$, with a *probability density function* $\mu = p_X$ and *cumulative distribution function* F_X, i.e. with $F_X(x) = P(X \leq x), x \in \mathbb{R}$, hence

$$\int_y^z p_X(x)dx = F_X(z) - F_X(y) = \int \mathbf{1}_{[y,z]}(x)\, d\mu(x).$$

Then of course we expect a histogram to contain good approximative information about integrals over the collection of such integrals, for a collection of intervals $[k/n, (k+1)/n]$, $k \in \mathbb{Z}$, or equivalently, the numerical sequence $\mu(\mathbf{1}_{[k/n,(k+1)/n]})_{k\in\mathbb{Z}}$, i.e. one has information about a collection $\mu(\psi_k)$, where $(\psi_k)_{k\in\mathbb{Z}}$ forms a partition of unity for $\mathbb{R}$ of diamter $1/n$, for some given $n \in \mathbb{N}$. If the range of X is in $[-1,1]$ (for example) it is of course enough to make use of bins supported in that region of $\mathbb{R}$.

We will make use of a so-called BUPUs which will serve a similar purpose: They will allow us to localize information provided by the measure μ, and recollect the global information by summing up all these local pieces. Again one can define BUPUs in the setting of general locally compact (LC) groups G, but for a more convenient presentation of the underlying ideas we will mostly make use of so-called regular BUPUs, because they are easy to generate on $\mathbb{R}^d$ and make proofs much shorter. Nevertheless we take care that the properties that we use are the same ones which by definition are satisfied by the more general (unrestricted) BUPUs.

Remark 25. Note that the central limit theorem can also be reformulated as a statement[38] about $\mathrm{St}_{1/\sqrt{N}}(\mu^{N*})$, which is convergent to the normal distribution if $\mu(\mathbf{1}) = 1$.

There is also a nice compatibility of the Fourier transform with dilations, which in fact provides another argument for the use of two versions of the dilation operators.

THIS PARAGRAPH WILL BE MOVED DOWN BEHIND THE EXPLANATION OF THE FOURIER TRANSFORM, WE LEAVE A FOOTNOTE HERE ONLY!

Fourdil1 **Lemma 21.** $\mathcal{F}(\mathrm{St}_\rho \mu) = \mathrm{D}_\rho(\mathcal{F}\mu), \;\; \rho \neq 0, \quad \mu \in \boldsymbol{M}_b(\mathbb{R}^d)$.

In other words, the Fourier transform (and also its inverse) is an *intertwining operator* for these two types of dilation operators:

ntertwindil (75)
$$\mathcal{F} \circ \mathrm{St}_\rho = \mathrm{D}_\rho \circ \mathcal{F} \quad \text{for } \rho \neq 0.$$

For the Shannon sampling theorem one uses dilated (compressed) interpolators, and convolves with the sampled functions

$$[f \cdot \sqcup\!\sqcup_\rho] * D_{1/\rho}\,\varphi = [f \cdot \mathrm{D}_\rho \sqcup\!\sqcup] * \mathrm{St}_\rho \varphi$$

Observing that $\mathrm{D}_\rho \sqcup\!\sqcup \to \mathbf{1}$ for $\rho \to \infty$ (in the vague sense),hence one may conclude

$$f \cdot \mathrm{D}_\rho \sqcup\!\sqcup \to f \cdot \mathbf{1} = f$$

in the w^*-sense,

$$f = w^* - \lim f \cdot \mathrm{D}_\rho \sqcup\!\sqcup \to f \cdot \mathbf{1} = f$$

and moreover $\mathrm{St}_\rho \varphi \to \delta_0$ if only $\int_{\mathbb{R}^d} \varphi(x)dx = 1 = \hat{\varphi}(1)$, also for $\rho \to \infty$. Altogether we may conclude that (at least for $f \in \boldsymbol{C}_c(\mathbb{R}^d)$):

dilconv04 (76)
$$[f \cdot \mathrm{D}_\rho \sqcup\!\sqcup] * \mathrm{St}_\rho \varphi \to [f \cdot 1] * \delta_0 = f.$$

4. Basic functional analytic considerations

A series of lemmata (lemmas) making use of density.

Lemma 22. *Assume a bounded linear mapping between two Banach spaces is given on a dense subset only, then it can be extended in a unique way to a bounded linear operator of equal norm on the full space.*

Proof. Obviously it is enough to know a bounded, linear mapping T on a dense subset D of a Banach (or in fact just normed) space, in order to observe that for any element v in the domain of T one has a convergent sequence $(d_n)_{n\in\mathbb{N}}$ in D with limit x. Hence $(d_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, so that the boundedness of T implies that $(Td_n)_{n\in\mathbb{N}}$ is a again a Cauchy sequence in the range, hence (by completeness) has a limit. The uniqueness of this extension and the fact that this extension has the same norm, i.e. that

$$|||T||| := \|T\|_{Op} = sup_{\|d\|\leq 1}\|Td\|$$

is also an immediate consequence. $\square$

AIsubspace **Lemma 23. Test of BAI on a dense subspace:**
A bounded family $(e_\alpha)_{\alpha\in I}$ in a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a BAI for $\boldsymbol{A}$ if (only) for some total subset $D \subseteq \boldsymbol{A}$ one has:

totalst1 (77)
$$\|e_\alpha \cdot d - d\|_{\boldsymbol{A}} \to 0 \quad \forall d \in D.$$

[38] It would be interesting to know under which conditions one has relative compactness hence norm convergence of this sequence in some nice, hopefully small function space, e.g. in $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$. E.g. under the assumption that $\mu = g \in \boldsymbol{Q}_s(\mathbb{R}^d)$ for sufficiently large $s > d$, for example.

Proof. Obviously the validity of (77) is necessary. Conversely, it is easy to derive - using the properties of a net - that one has (uniform) convergence for finite linear combinations of elements from D, and then by a density argument one verifies convergence for all elements. [39] In this last step the (a priori) uniform boundedness of the approximate unit plays a crucial role. □

4.1. **Iterated limits of nets and interchange of order.** The following result should be compared with a theorem on *iterated limits* provided in Kelley's book ([48], p.69).

doublenet **Lemma 24.** *Assume that* $(T_\alpha)_{\alpha\in I}$ *and* $(S_\beta)_{\beta\in J}$ *are two bounded nets of operators in* $\mathcal{L}(\boldsymbol{V})$, *which are strongly convergent to limits* T_0, *and* S_0 *resp. i.e.*

$$T_o(\mathbf{v}) = \lim_\alpha T_\alpha(\mathbf{v}) \quad \forall\, \mathbf{v}\in \boldsymbol{V} \quad \textit{and} \quad S_0(\mathbf{w}) = \lim_\beta S_\beta(\mathbf{w}) \quad \forall\, \mathbf{w}\in \boldsymbol{V}.$$

Then the net $(T_\alpha\circ S_\beta)_{(\alpha,\beta)}$ *(with index set* $I\times J$ *and natural order*[40]*) is also strongly convergence, with limit* $T_0\circ S_0$, *i.e. for each* $\mathbf{v}\in\mathbf{V}$ *one has:*

doubnetconv
$$T_o[S_0(\mathbf{v})] = [T_0\circ S_0]\,(\mathbf{v}) = \lim_{\alpha,\beta}\,[T_\alpha\circ S_\beta]\,(\mathbf{v}). \tag{78}$$

In detail: For any $\mathbf{v}\in\boldsymbol{V}$ *and* $\varepsilon>0$ *there exists a pair of indices* $(\alpha_0,\beta_0)\in I\times J$ *such that for every* $\alpha\succeq\alpha_0$ *in* I *and* $\beta\succeq\beta_0$ *in* J *one has*

doubnetest
$$\|T_0(S_0(\mathbf{v})) - T_\alpha(S_\beta(\mathbf{v}))\| \le \varepsilon. \tag{79}$$

In particular we have in the sense of strong limits:

doublenet2
$$T_0\circ S_0 = \lim_\alpha\lim_\beta T_\alpha\circ S_\beta = \lim_\beta\lim_\alpha T_\alpha\circ S_\beta \tag{80}$$

Proof. Note that $|||T_\alpha|||\le C<\infty$. The statement depends on the following estimate

oublenetest
$$\|T_\alpha[S_\beta(\mathbf{v})] - T_0[S_0(\mathbf{v})]\| \le \|T_\alpha[S_\beta(\mathbf{v})] - T_\alpha[S_0(\mathbf{v})]\| + \|T_\alpha[S_0(\mathbf{v})] - T_0[S_0(\mathbf{v})]\| \tag{81}$$

The first expression can be estimated as follows:

doubest1
$$\|T_\alpha[S_\beta(\mathbf{v})] - T_\alpha[S_0(\mathbf{v})]\| \le |||T_\alpha|||\,\|S_\beta(\mathbf{v}) - S_0(\mathbf{v})\| \le C\|S_\beta(\mathbf{v}) - S_0(\mathbf{v})\|, \tag{82}$$

which gets $<\varepsilon/2$ for $\beta\succeq\beta_0$ (chosen for ε/C), while the second term can be estimated by

doubest1
$$\|T_\alpha[S_0(\mathbf{v})] - T_0[S_0(\mathbf{v})]\| \le \varepsilon/2, \tag{83}$$

for any α with $\alpha\succeq\alpha_0$ (choosing $\mathbf{w}=S_0(\mathbf{v})$).

Finally we have to check that the validity of (83) implies that also the iterated limits exist (of course with the same limit). We elaborate on the first iterated limit (namely $\lim_\alpha\lim_\beta$) because the other one works in the same way[41]. So for the situation in (79) we can first fix any α and look at the net $(T_\alpha(S_\beta(\mathbf{v})))_{\beta\in J}$. By assumption the limit $S_0(\mathbf{v}) = lim_\beta S_\beta(\mathbf{v})$ exists. Since T_α (for fixed α first) is a bounded linear operator also the following limits exist and can be estimated by

$$\|T_0(S_0(\mathbf{v})) - T_\alpha(\lim_\beta S_\beta(\mathbf{v}))\| \le \varepsilon, \quad \text{if only} \quad \alpha\succeq\alpha_0.$$

Setting $\mathbf{w}_0 := S_0(\mathbf{v})$ we see that by assumption also $(T_\alpha(\mathbf{w}_0))$ is convergent, and of course the last estimate remains true in the limit (due to the continuity of the norm), hence

$$\|T_0[S_0(\mathbf{v})] - \lim_\alpha\lim_\beta T_\alpha[S_\beta(\mathbf{v})]\| \le \varepsilon.$$

Since $\varepsilon>0$ was arbitrary, equation (80) is valid. □

[39] Of course this is more or less a statement about strong operator convergence for a net of bounded operators.

[40] Of course I and J may have completely different order!

[41] Observe however that the *order* in which the operators are applied does NOT change!!!

Remark 26. Note that only the boundedness of the net (T_α) is only required, not that of (S_β). Moreover, in the case of sequences (instead of nets) the *uniform boundedness principle* can be applied, which tells us that the strong convergence of operators implies the norm convergence of the corresponding sequence. This situation could be formulated as a corollary then, i.e. strong convergence of (T_n) and (S_k) implies

doubesequ
$$\lim_n \lim_k T_n[S_k(\mathbf{v})] = T_0[S_0(\mathbf{v})] = \lim_k \lim_n T_n[S_k(\mathbf{v})] \tag{84}$$

AIoncomp
Lemma 25. One can choose the BAI elements from a dense SUBSPACE . *Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with a bounded approximate unit, and D some dense subset of $\boldsymbol{A}$. Then there exists also approximate units $(d_\alpha)_{\alpha\in I}$ in D (if D is a dense subspace the new family $(d_\alpha)_{\alpha\in I}$ can even be chosen to be of equal norm).*

Proof. One just has to choose d_α close enough to e_α, especially for α large enough (to be expressed properly). By the density of D one can do this. If D is a subspace (not just a subset) one can renormalize the new elements so that they have the same norm in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ as the original elements $(e_\alpha)_{\alpha\in I}$. □

Lemma 26. Uniform action of BAIs on compact subsets *By the definition a family $(e_\alpha)_{\alpha\in I}$ acts pointwise like an identity in the limit case, for each "point". However, the action is even uniformly over finite sets, and hence over compact sets, by approximation (we shall use the acronym unifcomp-convergence for this situation in the future).*

Proof. That one has uniform convergence over finite subsets is easily verified, using the property of a net. The inductive step is based on the following argument: Assume that one has found α_0 such that

$$\|e_\alpha \cdot a_i - a_i\| \le \varepsilon \quad \forall \alpha \succ \alpha_0,\ 1 \le i \le m.$$

Since $\|e_\alpha \cdot a_{m+1} - a_{m+1}\| \le \varepsilon$ for all $\alpha \succ \alpha_1$ we just have to choose some index α_2 with $\alpha_2 \succ \alpha_0$ *and* $\alpha_2 \succ \alpha_1$ (which is possible due to the definition of directed sets). Obviously

$$\|e_\alpha \cdot a_i - a_i\| \le \varepsilon \quad \forall \alpha \succ \alpha_2,\ 1 \le i \le m+1.$$

In order to come up with uniform convergence over compact sets, we use again a typical approximation argument. Given any compact set $M \subseteq \boldsymbol{C}_0(\mathbb{R}^d)$ and $\varepsilon > 0$ we have to find some index α_3 such that

$$\|e_\alpha \cdot a - a\| \le \varepsilon \quad \forall a \in M.$$

Recalling that $(e_\alpha)_{\alpha\in I}$ is bounded, i.e. $\|e_\alpha\| \le C$ for some $C \ge 1$ for all $\alpha \in I$, we may choose some finite subset $F \subseteq M$ such that for any $a \in M$ there exists $a_i \in F$ with $\|a_i - a\| \le \varepsilon/(3C) > 0$ (which is just another positive constant, known once we know C and ε). Hence for any given $a \in M$ one can use one of such elements $a_i \in F$ in order to argue that the triangular inequality implies (adding and subtracting the term $e_\alpha \cdot a_i$).

$$\|e_\alpha \cdot a - a\| \le \|e_\alpha \cdot (a - a_i)\| + \|e_\alpha \cdot a_i - a_i\| + \|a_i - a\|.$$

If we choose now α_3 such that $\|e_\alpha \cdot a_i - a_i\| \le \varepsilon/3 \quad \forall \alpha \succ \alpha_3$ and $a_i \in F$ we obtain altogether (more details are left to the reader, ...),

$$\|e_\alpha \cdot a - a\| \le \varepsilon.$$

□

boundedAI1
Lemma 27. *The following properties are equivalent.*

- *there is a bounded approximate identity in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$;*

- *there exists $C > 0$ such that for every finite subset $F \in \boldsymbol{A}$ and $\varepsilon > 0$ there exists element $h \in \boldsymbol{A}$ with $\|h\| \leq C$, such that*

$$\|h \cdot a - a\| \leq \varepsilon \quad \forall a \in F.$$

The argument to turn this family into a bounded net is the obvious one. One just has to set $\alpha := (F, \varepsilon)$, and defining such a pair "stronger than another pair $\alpha_1 := (F_1, \varepsilon_1)$ if $F_1 \supseteq F$ and $\varepsilon_1 \leq \varepsilon$. It is easy to verify that this defines a directed set, and that the choice $e_\alpha = h$ (corresponding to the pair (K, ε) as describe above) turn $(e_\alpha)_{\alpha \in I}$ into a bounded and convergent net, hence constitutes a BAI for $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$.

Note: In words: The existence of a BAI is equivalent to the existence of a bounded net in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ such that the action of "pointwise multiplication" (each element e_α is identified with the left algebra multiplication operator $a \mapsto e_\alpha \cdot a$) is convergent to the identity operator in the strong operator norm topology (which is just the pointwise convergence of operators).

Note: Sometimes one observes that one has unbounded approximate identities, where the "cost" (i.e. the norm of e_α grows as the required quality of approximation, expressed by the smallness of ε), tends to the ideal limit zero. It makes sense to think of limited costs (given by $C > 0$) for arbitrary good quality of approximation which makes the BAI so useful.

Theorem 9. *(The Cohen-Hewitt factorization theorem, without proof, see [41])*
Let $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with some BAI, then the algebra factorizes, which means that for every $a \in \boldsymbol{A}$ there exists a pair $a', h' \in \boldsymbol{A}$ such that $a = h' \cdot a'$, in short: $\boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{A}$. In fact, one can even choose $\|a - a'\| \leq \varepsilon$ and $\|h'\| \leq C$.

There is a more general result, involving the terminology of **Banach modules**.

BanMod **Definition 21.** A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach module* over a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if one has a bilinear mapping $(a, b) \mapsto a \bullet b$, from $\boldsymbol{A} \times \boldsymbol{B}$ into $\boldsymbol{B}$ with

BanModEst (85) $$\|a \bullet b\|_{\boldsymbol{B}} \leq \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{B}} \quad \forall\, a \in \boldsymbol{A}, b \in \boldsymbol{B},$$

which behaves like an ordinary multiplication, i.e. is associative, distributive, etc.:

BanModAss (86) $$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}.$$

A special case of a Banach module is a Banach module inside the Banach algebra:

BanIdDef **Definition 22.** A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach ideal* in (or within, or of) a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is continuously embedded into $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$, and if in addition BanModEst (85) is valid with respect to the internal multiplication inherited from $\boldsymbol{A}$.

Ex1Banmod **Lemma 28.** *(Ex. to Def. BanMod 21) A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is an (abstract) Banach module over a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$* [42] *if and only if there is a non-expansive (hence continuous) linear algebra homomorphism J from $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ into $\mathcal{L}(\boldsymbol{B})$.*[43]

Proof. For a Banach module the mapping: $a \mapsto J(a) : [b \mapsto a \bullet b]$ defines a linear mapping from $\boldsymbol{A}$ into $\mathcal{L}(\boldsymbol{B})$.

Conversely, one can define a $\boldsymbol{A}$-Banach module structure on $\boldsymbol{B}$ by the definition: $a \bullet b := J(a)(b)$.

[42] $\boldsymbol{A}$ may be commutative or non-commutative, with our without unit.

[43] This viewpoint immediately suggests to discern between general embedding and those having special properties, e.g. the case where J is an injective mapping, i.e. for every $a \in \boldsymbol{A}$ there exists some $b \in \boldsymbol{B}$ such that $a \bullet b \neq 0$. Such modules are called *faithful* (or sometimes *true*) modules.

Without going into all necessary details let us recall that the associativity law

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}$$

is a immediate consequence of the homomorphism property of J: The left hand side $a_1 \bullet (a_2 \bullet b)$ translates into $J(a_1)[J(a_2)b]$, while the right hand side equals $J(a_1 \cdot a_2)(b)$. $\square$

ess-Banmod **Definition 23.** A Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ over some Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is called *essential* if it coincides with the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B} = \{a \bullet b \,|\, a \in \boldsymbol{A}, b \in \boldsymbol{B}\}$. The same terminology applies to Banach ideals.

For a general Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B}$ is denoted by $\boldsymbol{B}_e$ or $\boldsymbol{B}_{\boldsymbol{A}}$. It is called the *essential part* of the Banach module $\boldsymbol{B}$ (with respect to $\boldsymbol{A}$).

Despite the fact that the terminology obviously suggest this result it is something that should be proved (nevertheless we leave the easy proof to the reader):

essmodess1 **Lemma 29.** *For every Banach algebra with $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ with bounded approximate units the essential part of $\boldsymbol{B}$ is an essential submodule of $\boldsymbol{B}$, i.e. a Banach module over $\boldsymbol{A}$. In fact, it contains all other essential submodules over $\boldsymbol{A}$ within $\boldsymbol{B}$.*

4.1.1. *Essential Banach modules and BAIs.* The usual way to define the *essential part* $\boldsymbol{B}_e$ of a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with respect to some Banach algebra action $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \bullet \mathbf{b}$ is the define it as the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B}$ within $\big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$. However it is interesting that this subspace can be shown to have another nice characterization using BAIs (bounded approximate units in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ (if they exist):

sspartchar1 **Lemma 30.**

$$\boldsymbol{B}_e = \{\mathbf{b} \in \boldsymbol{B} \,|\, \lim_{\alpha} \mathbf{e}_\alpha \bullet \mathbf{b} = \mathbf{b}\}$$

where $(\mathbf{e}_\alpha)_{\alpha \in I}$ is one resp. any BAI in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$.

In particular one has: Let $(\mathbf{e}_\alpha)_{\alpha \in I}$ and $(\mathbf{u}_b)_{\beta \in J}$ be two bounded approximate units (i.e. bounded nets within $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$wa acting in the limit almost like an identity in the Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$. Then one has

BAIequiv2 (87)
$$\lim_{\alpha} \mathbf{e}_\alpha \bullet \mathbf{b} = \mathbf{b} \Leftrightarrow \lim_{\beta} \mathbf{u}_\beta \bullet \mathbf{b} = \mathbf{b}. \tag{87}$$

Although this relation follows immediately from the above characterization it is instructive to look at a direct proof of this equivalence.

Proof. Assume that $(\mathbf{e}_\alpha)$ acts *'properly'* on a given element $\mathbf{b} \in \boldsymbol{B}$, i.e. we assume that the left hand side of BAIequiv2 (87) is valid. We want to estimate the action of $\mathbf{u}_\beta$ (for "good indices" $\beta \succeq \beta_0$) on the same given element $\mathbf{b} \in \boldsymbol{B}$.

So for any given (but then fixed) $\varepsilon > 0$ we can fix beforehand and for convenience some constants: $\gamma := \varepsilon/(3\|\mathbf{b}\|_{\boldsymbol{B}})$, $C_0 = \sup_{\beta \in J} \|\mathbf{u}_\beta\|_{\boldsymbol{A}}$ and $C_1 := 1 + C_0$. We can then *fix*[44] some α_0 such that $\|f - \mathbf{e}_{\alpha_0} \bullet f\|_{\boldsymbol{B}} < \varepsilon/2C_1$.

Since $(\mathbf{u}_\beta)$ is a BAI for $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ we can find β_0 such that for $\beta \succeq \beta_0$ one has $\|\mathbf{u}_\beta \bullet \mathbf{e}_{\alpha_0} - \mathbf{e}_\alpha\| < \gamma$ and hence

$$\|\mathbf{u}_\beta \bullet \mathbf{e}_{\alpha_0} \bullet \mathbf{b} - \mathbf{e}_{\alpha_0} \bullet \mathbf{b}\|_{\boldsymbol{B}} \leq \|\mathbf{u}_\beta \bullet \mathbf{e}_{\alpha_0} - \mathbf{e}_{\alpha_0}\|_{\boldsymbol{A}} \cdot \|\mathbf{b}\|_{\boldsymbol{B}} < \gamma \cdot \|\mathbf{b}\|_{\boldsymbol{B}} = \varepsilon/3.$$

[44]For the *subsequent* estimates it will be important to fix α, despite the fact that we could take any α in the first upcoming step!

But also $\mathbf{u}_\beta \bullet \mathbf{b}$ is not far away since

$$\|\mathbf{u}_\beta \bullet \mathbf{e}_{\alpha_0} \bullet \mathbf{b} - \mathbf{u}_\beta \bullet \mathbf{b}\|_{\boldsymbol{B}} \leq \|\mathbf{u}_\beta\|_{\boldsymbol{A}} \cdot \|\mathbf{e}_{\alpha_0} \bullet \mathbf{b} - \mathbf{b}\|_{\boldsymbol{B}} \leq C_0 \cdot \|\mathbf{e}_{\alpha_0} \bullet \mathbf{b} - \mathbf{b}\|_{\boldsymbol{B}}.$$

Putting all this together one obtains as an estimate for $\|\mathbf{b} - \mathbf{u}_\beta \bullet \mathbf{b}\|_{\boldsymbol{B}} \leq$

$$\leq \|\mathbf{b} - \mathbf{e}_{\alpha_0} \bullet \mathbf{b}\|_{\boldsymbol{B}} + \|\mathbf{e}_{\alpha_0} \bullet \mathbf{b} - \mathbf{u}_\beta \bullet \mathbf{e}_{\alpha_0} \bullet \mathbf{b}\|_{\boldsymbol{B}} + \|\mathbf{u}_\beta \bullet \mathbf{e}_{\alpha_0} \bullet \mathbf{b} - \mathbf{u}_\beta \bullet \mathbf{b}\|_{\boldsymbol{B}} \leq$$

$$\leq \varepsilon/3 + (1 + C_0)\|\mathbf{u}_\beta \bullet \mathbf{e}_{\alpha_0} \bullet \mathbf{b}\|_{\boldsymbol{B}} \leq \varepsilon/3 + \varepsilon/2 < \varepsilon, \quad \forall \beta \succeq \beta_0$$

as was requested, hence $\lim_\beta \|\mathbf{u}_\beta \bullet \mathbf{b} - \mathbf{b}\|_{\boldsymbol{B}} = 0$. □

The notion of "essential Banach modules" is of course trivial in case the Banach algebra $\boldsymbol{A}$ has a unit element which is mapped into the identity operator, i.e. if there exists $u \in \boldsymbol{A}$ such that $u \cdot a = a \, \forall a \in \boldsymbol{A}$ and also $u \bullet b = b$, $\forall b \in \boldsymbol{B}$.

Lemma 31. *Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with BAIs $(e_\alpha)_{\alpha \in I}$. Then a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is essential if and only if*

essmod-BAI (88) $$\|e_\alpha \bullet b - b\|_{\boldsymbol{B}} \to 0 \quad \forall b \in \boldsymbol{B}$$

Hence relation (88) holds true for one such BAI if and only if it is true for every BAI in $\boldsymbol{A}$.

As already mentioned earlier it is interesting to consider bounded subsets $M \subset \boldsymbol{B}$ with the property that the convergence in (88) is taking place in a uniform manner. Let us formalize this in a definition:

unifapprdef **Definition 24.** A bounded subset $M \subset \boldsymbol{B}$ within an (essential) Banach module $\boldsymbol{B}$ over a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ with bounded approximate units is called *uniformly approximable* of the following situation occurs: For every bounded approximate unit (e_α) in $\boldsymbol{A}$ one has: Given $\varepsilon > 0$ one can find α_0 such that for $\alpha \succeq \alpha_0$ one has

unifapprdef2 (89) $$\|e_\alpha \bullet b - b\|_{\boldsymbol{B}} \leq \varepsilon, \quad \forall f \in M.$$

We have chosen the "practical version" of the definition, instead of the axiomatic/minimalistic, which would be to assume the property (89) only for one (out of many possible) approximate units. As we show next that would not make a difference (and since the above choice is more useful in practice we have given that, and thus the subsequent lemma just provides a sufficient condition).

onetoallBAI **Lemma 32.** *Assume that in the situation described in Def. 24 the condition (89) is satisfied for only one specific BAI (e_α). Then it is also valid for any other approximate unit $(u_\beta)_{\beta \in J}$.*

Maybe we should think about other choices of symbols, e.g. u_α, v_β?

Proof. Proof to be given here! Or just reference to Lemma 30? To be checked.

Assume that the condition of uniform approximability is satisfied and $(u_\beta)_{\beta \in J}$ is some other BAI for $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$. Since M is bounded we can fix $C := \sup_{m \in M} \|b\|_{\boldsymbol{B}}$.

Given $\varepsilon > 0$ we can thus find α_0 such that

$$\|e_\alpha \bullet b - b\|_{\boldsymbol{B}} \leq \varepsilon/2 \quad \forall \alpha \succeq \alpha_0.$$

Choosing β_0 suitably one has

$$\|u_\beta \bullet e_{\alpha_0} - e_{\alpha_0}\|_{\boldsymbol{A}} \leq \varepsilon/(2C)$$

for all $\beta \succeq \beta_0$, and consequently

$$\|u_\beta \bullet e_{\alpha_0} \bullet b - e_{\alpha_0} \bullet b\|_{\boldsymbol{B}} < \varepsilon/2 \quad \forall b \in M.$$

□

dual-alg **Definition 25.** For any Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ over the Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ the dual space$(\boldsymbol{B'}, \|\cdot\|_{\boldsymbol{B'}})$ can be turned naturally into a $\boldsymbol{A}$-Banach module via the action

dual-alg1
$$[a \bullet \sigma](b) := \sigma(a \cdot b), \quad \forall a \in \boldsymbol{A}, b \in \boldsymbol{B}, \sigma \in \boldsymbol{B'}. \tag{90}$$

Note that the dual space of a *left* Banach module is a *right* Banach module (over the same Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$), which means that one has the "right" associativity law:

$$a_2 \bullet (a_1 \bullet \sigma) = (a_1 \bullet a_2) \bullet \sigma.$$

Of course the change of order is irrelevant in the case of commutative Banach algebras, but it has to be taken into account for the general case. Maybe it appears as more natural to write the action on the dual space from the right. Then we have the following alternative definition:

Most likely this first version will be eliminated later, because:

dual-algalt **Definition 26.** For any Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ over the Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ the dual space$(\boldsymbol{B'}, \|\cdot\|_{\boldsymbol{B'}})$ can be turned naturally into a $\boldsymbol{A}$-Banach module via the action

ual-algalt2
$$[\sigma \bullet \mathbf{a}](\mathbf{b}) := \sigma(\mathbf{a} \cdot \mathbf{b}), \quad \forall \mathbf{a} \in \boldsymbol{A}, \mathbf{b} \in \boldsymbol{B}, \sigma \in \boldsymbol{B'}. \tag{91}$$

Remark 27. This way of writing is most natural if we consider the situation familiar from linear algebra! Choose $\boldsymbol{A} = \mathcal{M}_{n,n}(\mathbb{R})$, the Banach algebra of $n \times n$-matrices over $\mathbb{R}$, acting on $\boldsymbol{B} = \mathbb{R}^n$. We know that in this case the dual space is, by definition the space $\mathcal{L}\mathbb{R}^n, \mathbb{R}$ which - by general linear algebra reasoning - can be identified with the set of $1 \times n$-matrices, which we would normally view as $\mathbb{R}^n$, but considered as the space of real row-vectors. The $\bullet$-operation of $\mathbf{a} \in \mathcal{M}_{n,n}(\mathbb{R})$ is then in both cases through matrix multiplication.

While the formula (26) is a definition it is easy to check that (due to the associativity law for matrix multiplication the dual action of $\mathbf{a}$ on a row vector $\sigma \in \boldsymbol{B'}$ is just through (right) matrix multiplication, since one has

$$(\sigma * \mathbf{a}) * \mathbf{b} = \sigma * (\mathbf{a} * \mathbf{b}), \quad \mathbf{a} \in \mathcal{M}_{n,n}(\mathbb{R}), \mathbf{b} \in \mathbb{R}^n.$$

The "natural" interpretation of the action as described in the first variant (90) would be to identify $\mathcal{L}\mathbb{R}^n, \mathbb{R}$ again with $\mathbb{R}^n$, viewed now as a system of column vectors (via the transposition mapping, or alternatively by choosing the unit vectors as a basis) and then have $\mathcal{M}_{n,n}(\mathbb{R})$ act on those column vectors by matrix multiplication with the *transpose matrix* from the right. Of course, these two viewpoints describe (isometrically) isomorphic (right) Banach module action, thanks to the well known linear algebra formula

$$(\mathbf{h} * \boldsymbol{C})^t = \boldsymbol{C}^t * \mathbf{h}^t, \quad \mathbf{h} \in \mathbb{R}^n, \boldsymbol{C} \in \mathcal{M}_{n,n}(\mathbb{R}).$$

DiracMult **Lemma 33.** *(Ex. to Def. (25)) Multiplication of a Dirac measure is realized as scalar multiplication of this Dirac measure by the point value of the continuous function h at that point:*

Diracmult1
$$h \cdot \delta_x = \delta_x \cdot h = h(x)\delta_x, \quad h \in \boldsymbol{C}_b. \tag{92}$$

For *convenience* we will write $h \cdot \mu$ instead of the abstract symbol $\bullet$ in order to indicate pointwise multiplication between functionals μ and functions $h \in \boldsymbol{C}_0(\mathbb{R}^d)$.

Theorem 10. *The Banach space $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is an essential Banach module over $\boldsymbol{C}_0(\mathbb{R}^d)$ with respect to the natural (dual) action of pointwise multiplication.*

Proof. In fact we will show much more: for any BUPU $\Phi = (\phi_i)$ one has

$$\mu = \sum_{i \in I} \mu \cdot \phi_i$$

as absolutely convergent sum in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, hence

artsumconv1
$$\|\mu - \mu \cdot \sum_{i\in F} \phi_i\|_{\boldsymbol{M}} \to 0 \quad \text{for} \quad F \,\text{finite},\, F \uparrow I. \tag{93}$$

□

Psipartsum1 **Definition 27.** It will be convenient to use for any set $J \subseteq I$ the symbol ψ_J for the function $\psi_J := \sum_{i\in J} \psi_i$. For the choice $F = F(i) = \{j | \psi_j \cdot \psi_i \neq 0\}$ we write ψ_i^* for $\psi_{F(i)}$.

A very useful relation which will be used later on is:

platrepr
$$\psi_i \cdot \psi_i^* = \psi_i \quad \text{for all} \quad i \in I. \tag{94}$$

Relation (93) above is thus equivalent to

$$\lim_{F\to I} \|\mu \cdot \psi_F - \mu\|_{\boldsymbol{M}} = 0$$

which of course translates into the more explicit variant:

Given $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ and $\varepsilon > 0$ as well as some BUPU Ψ there exists a finite set $F_0 \subset I$ such that for $F \supset F_0$ on has:

$$\|\mu - \mu \cdot \psi_F\|_{\boldsymbol{M}} < \varepsilon.$$

Another useful observation is this one:

measactloc **Lemma 34.** *Let $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ and $\varepsilon > 0$ be given. Then there exists $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\|k\|_\infty \leq 1$ such that $\mu(k) \geq 0$ and $\mu(k) \geq \|\mu\|_{\boldsymbol{M}}(1-\varepsilon)$.*

Proof. Using the definition of the functional norm and the density of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, one only has to take into account phase factors and multiply, if necessary, k by $|\mu(k)|/\mu(k)$, which does not change $\|k\|_\infty$[45]. □

eascompsupp **Corollary 3.** *Every $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ is the limit of "compactly supported" measures of the form $(\psi_F \cdot \mu)$, where F is running through the finite subsets of I.*

We will derive therefrom that one can also "integrate" arbitrary elements $h \in \boldsymbol{C}_b(\mathbb{R}^d)$.

integrCbRd1 **Lemma 35. Integration of bounded functions against bounded measures**
For any $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ and $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ the net $(p_K \cdot \mu)(h) = \mu(p_K \cdot h)$ is a Cauchy-net in $\mathbb{C}$. Therefore it makes sense to define $\mu(h) = \lim_K \mu(p_K \cdot h)$. It is clear that in this way μ extends in a unique way to a bounded linear functional on $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$, and that the norm of this extension equals $\|\mu\|_{\boldsymbol{M}}$.

The above statement does of course not imply that the dual space of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ and that of $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ are equal, because functionals on $\boldsymbol{C}_b(\mathbb{R}^d)$ which are equal on $\boldsymbol{C}_0(\mathbb{R}^d)$ need not be equal on $\boldsymbol{C}_b(\mathbb{R}^d)$.

`Alternative formulation`

integrCbRd2 **Lemma 36.** *Let $(p_\beta)_{\beta\in J}$ be any bounded approximate identity for the (pointwise) Banach algebra $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. Then for any $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ and $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ the net $\mu(p_\beta \cdot h)$ is a Cauchy-net in $\mathbb{C}$. Therefore it makes sense to define $\mu(h) = \lim_K \mu(p_\beta \cdot h)$. It is clear that in this way μ extends in a unique way to a bounded linear functional on $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$, i.e. the extension does not depend on the specific BAI (p_β), and and that the norm of this extension equals $\|\mu\|_{\boldsymbol{M}}$.*

[45] See the embedding of $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$ into $\boldsymbol{M}_b$ given below for more details.

Remark 28. MAYBE TO BE MOVED ELSEWHERE? For any bounded set of convolution operators (resp. translation invariant systems) the image of a (bounded and) equicontinuous set is again equicontinuous. This is clear since $T_y(Tf) - Tf = T(T_y f - f)$, which tends to zero uniformly if f is chosen from an equicontinuous set.

$$\|T_y(Tf) - Tf\| \leq \|T\|\|(T_y f - f)\|_\infty \leq C\|T_y f - f\|_\infty \to 0.$$

Remark 29. If the net of bounded measures is (bounded and) tight, then it is an exercise to show, that it is *vaguely convergent* [46] if and only if it is w^*-convergent, resp. if and only if $\mu_\alpha(h) \to \mu_0(h)\ \forall h \in \boldsymbol{C}_b(\mathbb{R}^d)$. It is a bit more work (but still an exercise) to check out that this "pointwise convergence" even takes place *uniformly over compact subsets* of $\boldsymbol{C}_b(\mathbb{R}^d)$. Since obviously the collection $\{\chi_s, s \in K\}$ is a compact subset of $\boldsymbol{C}_b(\mathbb{R}^d)$ for any compact subset $K \subset \mathbb{R}^d$) (as the continuous image of a compact set) this implies that in this case one obtains uniform convergence of the Fourier (Stieltjes) transform of measures over compact sets.

It may be a good exercise to consider the question, whether (resp. why probably not) the same statement fails to be true for non-tight sets. What about δ_{x_n} with $|x_n| \to \infty$.

Remark 30. (MAYBE MISPLACED REMARK)
Recall that the "mass-preserving" stretching/compression operator St_ρ can be extended to $\boldsymbol{M}_b(\mathbb{R}^d)$ by the definition **[Rd-specific!]**

$$\mathrm{St}_\rho\, \mu(f) := \mu(\mathrm{D}_\rho f).$$

Check that one has $\mathrm{St}_\rho\, \delta_x = \delta_{\rho x}$, and that $lim_{\rho\to 0}\, \mathrm{St}_\rho\, \mu = \mu(1)\delta_0$. In fact, for each $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ one has: $\mathrm{D}_\rho f$ is uniformly bounded with respect the sup-norm for $\rho \to 0$ one has: $f(x) \to f(0)$, uniformly over compact sets. Since we can approximate the measure μ (by localizing it) to a measure with compact support its action is defined on sufficiently large compact sets, where $\mathrm{D}_\rho f$ is like the constant function $f(0) = \delta_0(f)$, while the action on $Const \equiv 1$ is denoted by $\mu(1)$.

One can use this fact to find out that it is even possible to extend the convolution operators $f \mapsto C_\mu f$ to all of $\boldsymbol{C}_b(\mathbb{R}^d)$ (still with the equality of operator norm on $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ with the functional norm of μ), and with the property that the operators arising in such a way commute with translations. However, by means of the Hahn-Banach theorem one can construct translation invariant *means* on $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ which in turn allow to construct bounded linear operators on $\boldsymbol{C}_b(\mathbb{R}^d)$ which commute with all the translation operators without being of the form of a convolution by some bounded measure. In fact, those operators are non-zero operators on $\boldsymbol{C}_b(\mathbb{R}^d)$, but they map all of $\boldsymbol{C}_0(\mathbb{R}^d)$ onto the zero function. It is also not much more than a simple exercise to find out (using the characterization of $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ within $\boldsymbol{C}_b(\mathbb{R}^d)$ given early on) to check that any operator on $\boldsymbol{C}_b(\mathbb{R}^d)$ commuting with translations will map $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ into itself (in fact, this argument was used at the beginning of the identification theorem.)

We are now in the position to define the **Fourier transform** of a bounded measure. In the classical literature this is often referred to as the Fourier-Stieltjes transform of a measure, because it can be carried out technically over $\mathbb{R}$ using Riemann-Stieltjes integrals. Such a R-St-integral is the difference of two R-St-integrals with respect to bounded, non-decreasing "distribution" functions F. So in such a definition the ordinary Riemannian sum is replaced by a sum of the same form, but instead of the "natural length" of the interval $[a, b]$, which is $|b - a|$, one uses the length in the sense of F which is $F(b) - F(a)$.

[46] This means that it is convergent in the $\sigma(\boldsymbol{M}_b(\mathbb{R}^d), \boldsymbol{C}_c(\mathbb{R}^d))$-topology, resp. $\mu_\alpha(k) \to \mu_0(k) \quad \forall k \in \boldsymbol{C}_c(\mathbb{R}^d)$.

aracterdef1 **Definition 28.** A *character* on a group $\mathscr{G}$ is a continuous function from a topological group $\mathscr{G}$ into the torus group $\mathbb{U} = \{z \in \mathbb{C} \,|\, |z| = 1\}$. In other words, χ is a character if

chardef0
$$\chi(x+y) = \chi(x) \cdot \chi(y) \quad \forall x, y \in \mathscr{G}. \tag{95}$$

Definition 29. The set of all character is called the *dual group*, because those *characters* form an Abelian group under pointwise multiplication (!Exercise!).

The *neutral element* in this group is of course the *trivial* character $\chi_0(t) \equiv 1$. Consequently, since $|\chi(x)| = 1$ one has $\overline{\chi(x)} = 1/\chi(x)$ for all $x \in \mathscr{G}$, as inverse element.

We write $\widehat{\mathscr{G}}$ for the dual group [47] corresponding to $\mathscr{G}$.

Theorem 11. *In the case of $(\mathbb{R}^d, +)$ the dual group consists of the characters χ of the form $\chi_s : t \mapsto exp(2\pi i s \cdot t)$, with $s \in \mathbb{R}^d$. Due to the exponential law the pointwise multiplication of characters turns into addition of the parameters s (describing the "frequency content" of χ_s).*

Moreover, one can show that the convergence of a sequence s_n in $\mathbb{R}^d$ is equivalent to the uniform convergence over compact sets, i.e. the metric topology on $\mathbb{R}^d$ and the compact-open topology on $\widehat{\mathbb{R}^d}$ (of characters) turn this identification into an isomorophism of (locally compact) topological groups.

Proof. We omit the proof here, indicating that it is found in various books on the field (e.g. [17]). It is not part spelled out in Reiter's book [55] or Rudin's classical book [63], if I remember correctly [to be verified!]. In any case it can be reduced (by induction) to the case $d = 1$, cf. Lemma X below.

Typically one verifies first that the continuity (in fact even measurability is sufficient!) of a given character χ implies in fact continuous differentiability and from that the validity of a simple ordinary differential equation. From this it can be concluded that $\chi(t) = exp(\alpha t)$, for some $\alpha \in \mathbb{C}$. But since χ also has to be bounded the exponent has to be purely imaginary! $\square$

Remark 31. We have two remarkable properties, all resulting from the well-known (and all-important) exponential law:

expo-law
$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}, \quad z_1, z_2 \in \mathbb{C}. \tag{96}$$

which on the one hand (obviously) gives

expo-law1
$$\chi_s(t_1 + t_2) = \chi_s(t_1) \cdot \chi_s(t_2), \quad t_1, t_2 \in \mathbb{R}, \tag{97}$$

but also on the other hand

expo-law2
$$\chi_{s_1}(t) \cdot \chi_{s_2}(t) = \chi_{s_1+s_2}(t), \quad \text{resp.} \quad \chi_{s_1} \cdot \chi_{s_2} = \chi_{s_1+s_2}. \tag{98}$$

which corresponds to the fact that each function χ_s defines a character, while on the other hand (98) [expo-law2] implies that the abstract multiplication in $\hat{G}$ is isomorphic to the usual group operation (namely addition) in $(\mathbb{R}, +)$. Corresponding laws are valid for $\mathbb{R}^d$ by induction, but also in the context of general LCA groups!

Definition 30. The Fourier transform[48] of $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ is defined by

meas-FT-def
$$\hat{\mu}(s) = \mu(\chi_{-s}) = \mu(\overline{\chi_s}). \tag{99}$$

[47] Often the group law written as addition in order to emphasize that it is a (obviously) a *commutative* group!

[48] Using the classical terminology, e.g. of Bochner, one often talks about the Fourier Stieltjes transform of the measure. This has to do with the classical view-point, which does not talk about abstract measures but uses instead the corresponding distribution function $F_\mu(z) = \int_{-\infty}^{z} d\mu(z)$ and realized integrals using Riemann-Stieltjes sums and their limits, cf. 24 [dist-meas] above.

Remark 32. Later on, with the help of distributions (resp. the Banach Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$) we will show that this is a special case of the more general approach.

[Rd-specific!]

There is also a nice compatibility between pure frequencies and dilations. It will be more convenient to use the dilation using the D_ρ-operators, since they move pure frequencies into new frequencies because this dilation operator does NOT change the values of a function (all in the unit circle).

dilpurfr1 **Lemma 37.**

$$\mathrm{D}_\rho \chi_s = \chi_{s/\rho}, \quad \rho \neq 0, s \in \mathbb{R}^d.$$

The proof is left to the readers as an easy exercise!

Note that $\chi_s \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$ and that $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ can be applied according to Lemma 35. integrCbRd1

The argument is based on the following general result:

onvmeasfcts **Proposition 3.** *Assume that a (bounded) and tight net (μ_α) is w^*-convergent to μ_0 and that (h_β) is a bounded net in $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ which is uniformly convergent to h_0 over compact subsets. Then for every $\varepsilon > 0$ there exist indices α_0 and β_0 such that for every $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$*

convmeas2 (100)
$$|\mu_\alpha(h_\beta) - \mu_0(h_0)| \leq \varepsilon.$$

which is expressed equally by the formulation

$$\lim_{\alpha,\beta} \mu_\alpha(h_\beta) = \mu_0(h_0).$$

Proof. Given $\varepsilon > 0$ we recall that $\|h_\beta\|_\infty \leq C_0$ and $\|\mu_\alpha\|_{\boldsymbol{M}} \leq C_1$ for all α. By the tightness of the net (μ_α) we can find some plateau-like function $p \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\|p\|_\infty \leq 1$ such that $\|\mu_\alpha - p \cdot p \cdot \mu_\alpha\|_{\boldsymbol{M}} < \varepsilon {C_0}^{-1}$ for all α. Writing $K := \mathrm{supp}(p)$ we can find some index β_0 such that $|h_\beta(x) - h_0(x)| \leq \varepsilon / C_1$ for all $x \in K$ and all $\beta \geq \beta_0$. Altogether we then have

$$|\mu_\alpha(h_\beta) - \mu_0(h_0)| \leq$$

$$|\mu_\alpha(h_\beta) - p \cdot \mu_\alpha(h_\beta)| + |\mu_\alpha(p \cdot h_\beta) - \mu_\alpha(p \cdot h_0)| + |\mu_\alpha(p \cdot h_0) - \mu_0(p \cdot h_0)| + |p \cdot \mu_0(h_0) - \mu_0(h_0)| \leq$$

$$|[\mu_\alpha - p \cdot \mu_\alpha](h_\beta)| + |\mu_\alpha(p \cdot (h_\beta - h_0))| + |[\mu_\alpha - \mu_0](p \cdot h_0)| + |[p \cdot \mu_0 - \mu_0](h_0)| \leq$$

$$\|\mu_\alpha - p \cdot \mu_\alpha\|_{\boldsymbol{M}} \|h_\beta\|_\infty + \|\mu_\alpha\|_{\boldsymbol{M}} \|p \cdot (h_\beta - h_0)\|_\infty + |[\mu_\alpha - \mu_0](p \cdot h_0)| + \|p \cdot \mu_0 - \mu_0\|_{\boldsymbol{M}} \|h_0\|_\infty \leq$$

$$\varepsilon \cdot C_0 + C_1 \cdot \|p \cdot (h_\beta - h_0)\|_\infty + |[\mu_\alpha - \mu_0](p \cdot h_0)| + \varepsilon C_0 \leq C_2 \cdot \varepsilon.$$

for some constant $C_2 > 0$, and for all $\alpha \succ \alpha_0, \beta \succ \beta_0$. WELL: WE HAVE TO CHECK THE DETAILS Note that the w^*-convergence of the net (μ_α) implies that $\|\mu_0\|_{\boldsymbol{M}} \leq \liminf_\alpha \|\mu_\alpha\|_{\boldsymbol{M}}$. The first and last part is due to the tightness and boundedness, independent from the , while the is due to uniform convergence of (h_β) over $K = \mathrm{supp}(p)$ and the third term is getting small due to the w^*-convergence of μ_α to μ_0. ☐

Corollary 4. *Given $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$. Then the family $\{\widehat{\mathrm{D}_\Psi \mu}, |\Psi| \leq 1\}$ is equicontinuous, i.e. for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

ererer (101)
$$|\widehat{\mathrm{D}_\Psi \mu}(s) - \widehat{\mathrm{D}_\Psi \mu}(s-h)| \leq \varepsilon \quad \textit{for all} \quad |h| \leq \delta, \quad |\Psi| \leq 1.$$

: THIS IS A GENERAL FACT: TIGHTNESS IN THE TIME-DOMAIN IMPLIES EQUICONTINUITY IN THE FOURIER DOMAIN (AND IN FACT VICE VERSA)!

MATERIAL INSERTED FROM THE VERSION OF 2013 iemStiLemma **Proposition 4** (Riemann-Stieltjes transform properties)**.** *The Fourier transform is a linear and non-expansive mapping from* $\boldsymbol{M}_b(\mathbb{R}^d)$ *into* $\big(\boldsymbol{C}_{ub}(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$[49].

Proof. The uniform continuity results from the essential concentration of bounded measures over compact sets, the "usual rule" $T_s\mathcal{F}(\mu) = \mathcal{F}(M_s\mu)$., and the properties of characters.

In fact, given $\varepsilon > 0$ (alternative typing: $\varepsilon > 0$, and $\chi_0 \in \hat{G}$, we can find a compact subset $Q \subseteq G$ such that $\|\mu - \psi_Q\mu\|_{\boldsymbol{M}} < \varepsilon/3$. Hence there is a neighborhood W of the identity in $\hat{G}$ such that for all $\chi \in \chi_0 + W$ one has $|\chi(y) - \chi_0(y)| = |\chi \cdot \overline{\chi_0}(y) - 1| < \varepsilon/3$ for all $y \in Q$, by the definition of neighborhoods in $\hat{G}$ (compact open topology), and the fact that the quotient $\chi/\chi_0 (= \chi \cdot \overline{\chi_0})$ belongs to W. Expressed differently we have $\|\psi_Q(\chi_0 - \chi)\|_\infty < \varepsilon/3$. Altogether we have

$$|\hat{\mu}(\chi_0) - \hat{\mu}(\chi)| \leq |\mathcal{F}(\mu - \mu\psi_Q)(\chi_0)| + |\mathcal{F}(\mu\psi_Q)(\chi_0 - \chi)| + |\mathcal{F}(\mu\psi_Q - \mu)(\chi)|$$

and consequently

$$|\hat{\mu}(\chi_0) - \hat{\mu}(\chi)| \leq 2\|\mu - \mu\psi_Q\|_{\boldsymbol{M}} + \varepsilon/3 \leq \varepsilon.$$

□

Although little can be said about the connection between w^*-convergence (in $\boldsymbol{M}(\mathbb{R}^d)$) and pointwise convergence "on the Fourier transform side" in general one has the following useful fact:

wsttoFOUR1 **Lemma 38.** *Let* (μ_α) *be a* w^**-convergent and tight net in* $\boldsymbol{M}_b(\mathbb{R}^d)$*, with* $\mu_0 = w^* - lim_\alpha \mu_\alpha$*. Then we have* $\widehat{\mu_\alpha}(s) \to \widehat{\mu_0}(s)$*, uniformly over compact subsets of* $\widehat{G}$*.*

Proof. Pointwise convergence of the Fourier transform is a consequence of the fact that due to the tightness of the whole family the action can be reduced to the case of measures with compact joint support. SHOULD BE MORE EXPLICIT!. □

DpsiFourCO1 **Corollary 5.** *Consider the family* $\mathrm{D}_\Psi\mu$ *with* $|\Psi| \leq 1$*. Then their Fourier (Stieltjes) transforms are uniformly convergent to* $\hat{\mu}$ *over compact sets of* $\mathbb{R}^d$*. They are uniformly bounded as well as equi-continuous on* $\mathbb{R}^d$*.*

The basic relation

deltFdelt (102) $$\delta_y(\widehat{\delta_x}) = e^{-2\pi i x\cdot y} = \delta_x(\widehat{\delta_y}).$$

BETTER FORMULATION REQUIRED
deltFdelt
For any fixed BUPU Φ and Ψ (102) implies the following relation, observing that the convergence of the possible infinite sequence of the resulting discrete measures is absolute (hence in norm, in $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$).

measFmeas3 (103) $$\mathrm{D}_\Psi\mu(\widehat{D_\Phi\nu}) = D_\Phi\nu(\widehat{\mathrm{D}_\Psi\mu}) \quad \text{for any pair} \quad \mu, \nu \in \boldsymbol{M}_b(\mathbb{R}^d) \text{ and any BUPUs } \Phi, \Psi.$$

convmeasfcts
By linearity and then passing to the limit (using Proposition 3 below) we can derive from the following important *Fundamental Formula for the Fourier Stieltjes Transform*:

measFmeas (104) $$\nu(\hat{\mu}) = \mu(\hat{\nu}) \quad \text{for any pair} \quad \mu, \nu \in \boldsymbol{M}_b(\mathbb{R}^d).$$

[49] In the same way as the Fourier transform from $\boldsymbol{L}^1(\mathbb{R}^d)$ into $\boldsymbol{C}_0(\mathbb{R}^d)$ is injective, but not surjective we also have here (for "local reasons") that the mapping is injective but not surjective. Only for non-negative measures, i.e. if $f \geq 0$ implies $\mu(f) \geq 0$ then we have $\hat{\mu}(0) = \mu(1) = \|\mu\|_{\boldsymbol{M}}$ and thus $\|\hat{\mu}\|_\infty = \|\mu\|_{\boldsymbol{M}}$ in this case. This applies in particular to probability measures, with $\|\hat{\mu}\|_\infty = 1 = \|\mu\|_{\boldsymbol{M}}$

For integrable functions this relation turns into what H. Reiter has called the *Fundamental Relation of the Fourier Transform*

L1FourL1 (105) $$\int_{\mathbb{R}^d} f(t)\hat{g}(t)dt = \int_{\mathbb{R}^d} g(z)\hat{f}(z)dz, \quad \text{for all} \quad f, g \in \boldsymbol{L}^1(\mathbb{R}^d).$$

Of course it can be verified directly, using Fubini's theorem, since $(x,y) \to f(t)g(z) \in \boldsymbol{L}^1(\mathbb{R}^{2d})$.

As an immediate consequence of identity (104) we can derive the uniqueness theorem for Fourier Stieltjes transforms. If $\hat{\mu}$ is the zero-function, we can apply $\nu = \mu_g$, for any possible $g \in \boldsymbol{C}_c(\mathbb{R}^d)$ (or $\boldsymbol{L}^1(\mathbb{R}^d)$). Hence $\mu(\hat{g}) = 0$ for every $\hat{g} \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$. Since $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is dense in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ with respect to the $\|\cdot\|_1$-norm this implies that μ is the zero functional.

Note that an easy way to show that $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is dense in $\boldsymbol{C}_0(\mathbb{R}^d)$ is based on the following sequence of observations:

(1) $\mathcal{F}\boldsymbol{L}^1(\mathbb{R})$ contains the triangular function Δ, which can be viewed as B-spline of order 1 (piecewise linear function) and whose Fourier transform is the (non-negative and even) function $sinc^2$;
(2) the shifted triangular functions $T_n\Delta$ forms a BUPU (bounded in $\big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$, since translation corresponds to modulation in $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$ and this is isometric)
(3) for $d > 1$ one obtains BUPUs by looking at tensor products of $1D$-Δ-functions. Moreover one has $\|f \otimes g\|_1 = \|f\|_1\|g\|_1$ and

tensorFour (106) $$\mathcal{F}(f \otimes g) = \mathcal{F}(f) \otimes \mathcal{F}(g),$$

based on the following convention (using the *tensor product symbol*):

tensordef0 (107) $$f \otimes g(x,y) := f(x) \cdot g(y), \quad x, y \in \mathbb{R}^d,$$

(4) Since $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$ is isometrically dilation invariant under $\mathrm{St}_\rho, \rho > 0$ we find that $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is isometrically dilation invariant under the family $\mathrm{D}_\rho, \rho > 0$, hence one obtains arbitrary fine BUPUs of the form $\mathrm{D}_\rho T_n\Delta, n \in \mathbb{Z}^d, \rho > 0$;
(5) for $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ one has convergence of the corresponding spline-type approximations with respect to the $\|\cdot\|_\infty$-norm, i.e. $\|Sp_{\rho\Delta}k - k\|_\infty \to 0$.

Using the last lemma we can derive the formula (104) from (102) via the following relation: *Fundamental Relation for the Fourier-Stieltjes transform*:

measFmeas2 (108) $$\mu(\hat{\nu}) = \nu(\hat{\mu}) \quad \text{for} \quad \mu, \nu \in \boldsymbol{M}_b(\mathbb{R}^d).$$

then follows using exactly Proposition 3 and Lemma 38.

convFTmeas **Theorem 12.** *The FT on $\boldsymbol{M}_b(\mathbb{R}^d)$ is injective, and turns convolution into pointwise multiplication, i.e. in fact, it is a homomorphism of Banach algebras. This also implies (once more) that convolution is commutative (because obviously pointwise multiplication is a commutative operation).*

The compatibility with convolution is an easy exercise for discrete measures, and can be transferred to the general case using a weak-star argument. Recall again that w^*-convergence of bounded nets of measures implies pointwise convergence of their Fourier transforms.

Recall that a net (or just sequence) of measures $(\mu_\alpha)_{\alpha\in I}$ is w^*-convergent to μ_0 if and only if one has: for any $f \in \boldsymbol{C}_0$ and $\varepsilon > 0$ one can find some index α_0 such that $\alpha \succeq \alpha_0$ implies:

$$|\mu_\alpha(f) - \mu_0(f)| \leq \varepsilon.$$

There is an alternative way of proving commutativity of convolution. It is easy to see that the convolution of (finite) discrete measures is commutative, and the general case follows from this (by approximation in the strong operator topology).

We still have to give the *ordinary Fourier inversion formula using summability kernels.* This Fourier inversion is a direct consequence of the
Fundamental Relationship for the Fourier transform of $\boldsymbol{L}^1(\mathbb{R}^d)$-functions (we give a proof in the Euclidean context, making use of the fine properties of the dilation operators):

undamFour02

$$\int_{\mathbb{R}^d} \hat{f}(y)g(y)dy \quad = \quad \int_{\mathbb{R}^d} f(x)\hat{g}(x)dx. \tag{109}$$

A paper giving a short proof based on Poisson's formula is [60]: Alain Robert: A short proof of the Fourier inversion formula, which can be said to be based on the Fourier invariance of the Shah-distributions (Dirac comb) (probably in [61] or [62]).

iFourInvers **Theorem 13.** *For $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$* [50] *one has the pointwise (a.e.) relation:*

ourinvForm1

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(s)e^{2\pi i x\cdot s}ds. \tag{110}$$

Proof. There are of course many proofs to this results, most of them involving some kind of summability kernels, helping to derive (110) from (105). These arguments typically differ in the way how auxiliary functions g (and the corresponding $\hat{g}$) are chosen in order to achieve the desired goal.

FourinvForm1L1FourL1

One option is to make use of the **Gauss function**, given by

Gaussdef00

$$g_0(x) = e^{-\pi x^2} \quad \text{with} \quad \hat{g}_0(0) = \|g_0\|_{\boldsymbol{L}^1(\mathbb{R}^d)} = 1 = \|g_0\|_{\boldsymbol{L}^2(\mathbb{R}^d)}\,. \tag{111}$$

We are going to prove this relationship for an individual point $x_0 \in \mathbb{R}^d$. In order to obtain the point value on the left hand side one would like to put $\hat{g} = \delta_{x_0}$, and correspondingly one may expect that $g = \chi_s$ could be the pure frequency (since we know that $\widehat{\chi_s} = \delta_s$), but this is not directly available in the $\boldsymbol{L}^1$-context.

Therefore we look at a w^*-convergent family, tending to [INCOMPLETE] $\square$

There is also another possibility that would allow us to verify that there is a dense subspace in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ which is Fourier invariant and for which the Fourier inversion formula is valid, namely the so-called exotic Banach space described in a joint paper with G. Zimmermann ([35]).

NOT SURE WHETHER THIS FORMULA HASN'T BEEN GIVEN ELSEWHERE

[Rd-specific!]

FTdilcomp **Lemma 39.** *For any $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ and $\rho \neq 0$ one has*

$$\mathcal{F}(\mathrm{St}_\rho \mu) = \mathrm{D}_\rho \mathcal{F}\mu.$$

4.2. A short proof of the Fourier Inversion Theorem.

Gauß function:

$$g_0(x) = e^{-\pi x^2} \quad \text{with} \quad \|g_0\|_{\boldsymbol{L}^2(\mathbb{R}^d)} = 1\,.$$

Show $\widehat{g_0} = g_0$ (proof: GZ)

Note that the $\boldsymbol{L}^1$-normalized dilation operator D^1_ρ is denoted by St_ρ in the notes of hgfei, (where "St" stands for stretching, or Stauch-Streck operator in German).

INVERSION THEOREM: for $f \in \boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d)$ with $\widehat{f} \in \boldsymbol{L}^1$.

[50] There are of course many $\boldsymbol{L}^1$-functions, such as the BOX-function, whose FT, namely the SINC function, is *not* in $\boldsymbol{L}^1(\mathbb{R}^d)$.

Remark 33. One can then argue separately that $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$ already implies that there is exactly one *continuous* representative in the class representing f as member of $\boldsymbol{L}^1(\mathbb{R}^d)$ (defined as space of equivalence classes of measurable functions), to which the inversion formula converges, but this is outside of the scope of these notes.

$$\begin{aligned} f(x) &= \lim_{\rho\to\infty} \int f(t)\,[\mathrm{T}_x\,\mathrm{D}^1_\rho\,g_0](t)\,dt \\ &= \lim_{\rho\to\infty} \int \widehat{f}(t)\,[\mathrm{M}_x\,\mathrm{D}^0_{1/\rho}\,g_o](t)\,dt = \int \widehat{f}(t)\,e^{2\pi i t x}\,dt \end{aligned}$$

i.e. Inversion theorem. Thus $\boldsymbol{A}\cap\boldsymbol{L}^1 = \{f \in \boldsymbol{L}^1 : \widehat{f} \in \boldsymbol{L}^1\}$

ALTERNATIVE writeup:

Combing the formula (which is a simple consequence of Fubini's theorem)

$$\int_{\mathbb{R}^d} f(y)\hat{g}(y)dy = \int_{\mathbb{R}^d} \hat{f}(z)g(z)dz$$

for any pair of integrable functions $f, g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with the fact that, due to the properties of the Fourier transform we have

$$\mathcal{F}(M_x \mathrm{D}_\rho h) = T_x \mathrm{St}_\rho \hat{h}, \quad x \in \mathbb{R}^d, \rho > 0,$$

specialized to $h = g_0$ and $\hat{g} = T_x \mathrm{St}_\rho h$ we get, at least (exactly) for $f \in \boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d)$:

ourInvFast2

$$f(x) = \lim_{\rho\to\infty} \int f(t)\,[\mathrm{T}_x\,\mathrm{St}_\rho g_0](t)\,dt = \lim_{\rho\to\infty} \int \widehat{f}(t)\,[\mathrm{M}_x\,\mathrm{D}_\rho g_o](t)\,dt = \int \widehat{f}(t)\,e^{2\pi i t x}\,dt \tag{112}$$

A detailed proof of the Fourier invariance of the Gauss function can be found in Example 1.3.3 of [4]. For a technical report on this see [75] (Technical Report by G. Zimmermann).

Material on Banach Modules

The Banach module is called "true" if the mapping J described above is injective.

If one only has a continuous (but not necessarily non-expansive algebra homomorphism J) one can replace the norm on $\boldsymbol{A}$ by another equivalent norm (just some constant multiple of the original one) in order to ensure this (harmless) extra property.

Recall the notions of *weak topology* on any Banach space (such as $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$, and the *$w^*$-topology*) on any dual space, such as $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

Theorem 14. *A sequence (or indeed a bounded net) of functions in $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ is weakly convergent if and only if it is pointwise convergent (while in contrast norm-convergence means uniform convergence over $\mathbb{R}^d$).*

Proof. Since the Dirac measures are specific linear functionals on $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ weak convergence of a sequence (f_n) in $\boldsymbol{C}_0(\mathbb{R}^d)$ implies $f_n(x) = \delta_x(f_n) \to \delta_x(f_0) = f_0(x)$ for any $x \in \mathbb{R}^d$. Conversely, the possibility of approximating a general measure in a bounded way by linear combinations of Dirac measures implies that pointwise convergence indeed implies weak convergence. If one goes into the details of the proof the boundedness of the set of approximating measures as well as the boundedness of the sequence (resp. a net (f_α)) □

Remark 34. For *equicontinuous* families one can show that weak (or pointwise) convergence is equivalent to "uniform convergence over compact set". A net (f_α) is weakly convergent if and only if it is UCOCS, etc. dots

Remark 35. How can we characterize w^*-convergence in $\boldsymbol{M}(\mathbb{R}^d)$? (for bounded sets): cf. Bernoulli convergence, resp. *vague convergence* ($= \sigma(\boldsymbol{C}_c(G), \boldsymbol{R}(G))$) using standard topological constructions in functional analysis), pointwise convergence of the STFT or of the Fourier transform etc...

A neighborhood of $\mathbf{0} \in \boldsymbol{R}(\mathscr{G})$ is then given by a finite subset $F \subset \boldsymbol{C}_c(G)$ and $\varepsilon > 0$, via

vagneigh1 (113)
$$U(F, \varepsilon) := \{\nu \,|\, |\nu(k)| < \varepsilon, \forall k \in F\}$$

Alternative description of "multipliers" on $\boldsymbol{C}_0(\mathbb{R}^d)$ resp. translation invariant BIBOS (bounded input bounded output systems, with the property of mapping $\boldsymbol{C}_0(\mathbb{R}^d)$ into itself).

cterizeHGC0 **Theorem 15.** *Let H be any w^*- total subset of $\boldsymbol{M}(\mathbb{R}^d)$, and assume that a bounded linear operator $T \in \mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ commutes with the action of H, i.e. that the commutators $[C_h, T] \equiv 0$ for all $h \in H$. Then $T \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$.*

Note that in the original definition the set H was just the set of convolution operators by Dirac measures $\delta_x, x \in \mathbb{R}^d$ (or at least from some dense subset).

UPCOMING MATERIAL:

Embedding of test functions into $\boldsymbol{M}(\mathbb{R}^d)$ (over groups this requires the use of the [invariant] Haar measure, which indeed is a linear functional on $\boldsymbol{C}_c(G)$). Compatibility of operators which are now available on both the functions and the measures (resp. functionals). E.g. we can now do an internal convolution of functions (viewed as bounded measures) or an external action (one is acting as a bounded measure, the other is consider as the $\boldsymbol{C}_0(\mathbb{R}^d)$ element on which the action takes place). Associativity of convolution in the most general situation (also of course **commutativity**, etc.).

Further notes:

$\boldsymbol{C}_{ub}(\mathbb{R}^d)$ is a (closed) subspace of the dual of $\boldsymbol{L}^1(\mathbb{R}^d)$. This is easily verified directly (without using the fact that the full dual space can be identified with $\big(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\big)$. In fact one just has to verify first that for $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ one finds that $\sigma_h : f \mapsto \int_{\mathbb{R}^d} f(t)h(t)dt$ is a bounded linear functional on $\boldsymbol{L}^1(\mathbb{R}^d)$ and that $\|\sigma_h\|_{\boldsymbol{L}^{1\prime}} = \|h\|_\infty$, and thus consequently $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ is isometrically embedded into the dual of $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$. Similar statements can be made for general groups, just by replacing the Lebesgue integral by the Haar measure (which may be viewed simply as a translation invariant, continuous linear functional on $\boldsymbol{C}_c(\mathbb{R}^d)$)[51].

Hence it carries a $\sigma(\boldsymbol{C}_{ub}(\mathbb{R}^d), \boldsymbol{L}^1(\mathbb{R}^d))$ topology which can be shown to be equivalent (at least on bounded sets!?) to the uniform convergence over compact sets DETAILS MISSING, SHOULD BE EXPLAINED!

STATEMENT: Every $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ is a limit (in the sense of uniform limit over compact sets) of a *bounded* sequence of functions from $\boldsymbol{C}_0(\mathbb{R}^d)$ resp. even from $\boldsymbol{C}_c(\mathbb{R}^d)$. In fact, on can take the sequence $p_n \cdot f$, where (p_n) is a BAI for $\boldsymbol{C}_0(\mathbb{R}^d)$ consisting of (increasing) plateau functions.

[51] In fact, it automatically extends, due to its translation invariance to the Banach space, in fact the pointwise algebra $\big(\boldsymbol{W}(\mathscr{G}), \|\cdot\|_{\boldsymbol{W}(\mathscr{G})}\big)$, the Wiener algebra over $\mathscr{G}$, see Def. 37 below. WCdefphi

EXTENSION PRINCIPLE. Let (p_n) be as above, and $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ and $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ be given. Then the sequence $\mu(p_n \cdot f)$ is a Cauchy sequence, hence convergent in $\mathbb{C}$. In fact, the limit is the same for any other BAI in $\boldsymbol{C}_0(\mathbb{R}^d)$. Therefore it makes sense to define $\mu(f) := lim_{n\to\infty} \mu(p_n f)$.

FourStdef5 *Remark* 36. This will be important to define the Fourier Stieltjes transforms for bounded measures, i.e. for $\hat{\mu}(s) = \mu(\chi_{-s})$ later on! We will also have then to verify that this definition is consistent with the definition of the generalized Fourier transform at the level of $\boldsymbol{S}_0'(\mathbb{R}^d)$ or $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$.

From the above arguments it is also quite clear that the Fourier Stieltjes transforms for the discrete approximations $D_\Psi \mu$ converge pointwise to the Fourier-Stieltjes transform of μ. We formulate it as lemma.

Lemma 40. *We have for each $s \in \widehat{\mathbb{R}^d}$ and $\mu \in \boldsymbol{M}(\mathbb{R}^d)$:*

$$\lim_{\mathrm{diam}(\Psi)\to 0} \widehat{D_\Psi \mu}(s) = \hat{\mu}(s).$$

Proof. We play the usual trick. In order to show that for a given point $s \in \widehat{\mathbb{R}^d}$ (in the "frequency domain", so to say) we want to make sure that

$$|\widehat{D_\Psi \mu}(s) - \hat{\mu}(s)| \le \varepsilon \quad \text{for} \quad \mathrm{diam}(\Psi) < \delta,$$

we first start out with the observation that there exists some plateau function $p \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\|p\|_\infty = 1$ such that $\|(1-p) \cdot D_\Psi \mu\|_{\boldsymbol{M}} < \varepsilon/2$ for all Ψ, and also (in fact as a consequence) $\|(1-p) \cdot \mu\|_{\boldsymbol{M}} < \varepsilon/3$.

Then observing that $p \cdot \chi_{-s} \in \boldsymbol{C}_c(\mathbb{R}^d)$ we have by the establised w^*-convergence the convergence of

$$D_\Psi \mu(p \cdot \chi_{-s}) \to \mu(p \cdot \chi_{-s}),$$

or in other words for a well-chosen $\delta > 0$ we have: $\mathrm{diam}(\Psi) < \delta$ implies

$$|(D_\Psi \mu - \mu)(p \cdot \chi_{-s})| \le \varepsilon/3.$$

Combining these estimates we get

$$|\widehat{D_\Psi \mu}(s) - \hat{\mu}(s)| \le |D_\Psi \mu((1 - p \cdot (\chi_{-s})| + ...$$

and consequently the estimate

$$|\widehat{D_\Psi \mu}(s) = \hat{\mu}(s)| \le \|D_\Psi \mu \cdot (1-p)\|_{\boldsymbol{M}} \|\chi_{-s}\|_\infty + |(D_\Psi \mu - \mu)(p \cdot \chi_{-s})| + \|\mu \cdot (p-1)\|_{\boldsymbol{M}} \|\chi_{-s}\|_\infty.$$

Since $\|\chi_r\|_\infty = 1$ for any $r \in \mathbb{R}^d$ we have altogether obtained the desired estimate, whenever $\mathrm{diam}(\Psi) \le \delta$. □

DPsifour2 *Remark* 37. Probably the result could be obtained easier by the fact that the uniform continuity of χ_s, for any given s and in fact for any compact range of paramters $s \in \mathbb{R}^d$ implies that

$$\mathrm{Sp}_\Psi(\chi_{-s}) \to \chi_{-s} \quad \mathrm{in}\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big).$$

convopcomm **Lemma 41.** *The convolution operators form a (commutative!) Banach algebra of operators. It turns out that the characters can be identified with the joint eigenvectors for this whole class of operators. Indeed, we have $C_\mu(\chi_s) = \hat{\mu}(s)\chi_s$ for any $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ and any character χ_s on $\mathbb{R}^d$.*

Proof. The claim is valid for $\mu = \delta_x$, for any $x \in \mathbb{R}^d$, and hence for finite linear combinations of measures and also immediately for discrete measures (due to a simple approximation argument). However, it is enough to use the fact that one has Bernoulli convergence of $\mathrm{D}_\Psi \mu$ to μ (i.e. w^*-convergence by a tight net of discrete, bounded measures with limit μ).

Maybe even more convincing is the following observation. Due to the *exponential law* one has

$$\chi_s{}^{\checkmark}(z) = \chi_{-s}(z) \quad \text{and} \quad T_z(\chi_{-s}{}^{\checkmark}) = \chi_s(z) \cdot \chi_{-s}$$

and therefore

FourConvEig
$$\text{“}\mu * \chi_s(z)\text{''} = C_\mu(\chi_s)(z) = \mu(T_z(\chi_{-s}{}^{\checkmark})) = \mu(\chi_{-s}) \cdot \chi_s(z) = \hat{\mu}(s)\chi_s(z), \tag{114}$$

or expressed more compactly: χ_s is an *eigenvector* for C_μ with *eigenvalue* $\hat{\mu}(s)$:

eigenCmu1
$$C_\mu(\chi_s) = \hat{\mu}(s) \cdot \chi_s. \tag{115}$$

So in some sense the Fourier (Stieltjes) transform is the description of a commutative Banach algebra of operators (e.g. on $\boldsymbol{L}^2(\mathbb{R}^d)$, or in our case at the moment $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)\big)$ via their (joint) eigen-vectors. $\square$

Although the following result will be a consequence of basic facts about a more general (distributional) Fourier transform let us claim (and verify) the injectivity of the Fourier Stieltjes transform just defined:

FourStinj **Proposition 5** (Uniqueness of Fourier Stieltjes transform)**.** *Let $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ be given, and assume that $\hat{\mu}(\chi) \equiv 0$. Then $\mu = 0$ in $\boldsymbol{M}_b(\mathbb{R}^d)$. Consequently the bounded linear mapping $\mu \to \hat{\mu}$ from $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ into $\big(\boldsymbol{C}_{ub}(\mathbb{R}^d), \|\cdot\|_\infty\big)$ is injective.*

Proof. We use the density of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ first, which is a consequence of the fact that $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is a Banach algebra (due to the convolution theorem) which is of course closed under conjugation ($h = \hat{f}$ ..) and point separating, hence it is dense as a consequence of the Stone-Weierstrass approximation theorem. Being a subspace of $\boldsymbol{M}_b(\mathbb{R}^d)$ we also know that we can approximate any $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ in the w^*-sense by discrete measures, in such a way that the corresponding Fourier Stieltjes transforms (they are trigonometric polynomials) converge to $\hat{g}$ uniformly over compact sets. $\square$

5. Identifying “ordinary functions with functionals”

There is a lot to be said about ordinary functions and generalized functions: above all one has to be aware that most of the things which can be done in one or the other way for ordinary, at least for nice functions, can be done for “generalized functions” in the same way, e.g. they can be translated, multiplied by (decent) continuous functions, they can be dilated, or more generally transformed, e.g. be rotated, and often they have a Fourier transform of the same type. It is possible to determine their support there, and there are also convolutions and other things. In all these situations there should not be any difference whether and when one likes to switch from one consideration to another one. It is like multiplying with rational numbers and/or with real numbers. What is the quotient of πp over πq, i.e. of two irrational numbers, if p, q are non-zero integers. But it is enough to know that we can rewrite it (without doing the actual computation) to

$$\frac{p\pi}{q\pi} = \frac{p}{q}.$$

and if we are asked to compute the inverse of this number we just write q/p (without any computation). OF COURSE we could have evaluated (well: just approximately) the two irrational numbers and then apply a numerical method to compute their quotient, and the result would have been (at least up to the level of precision used) more or less the same. ALSO this mini-example (or ex tempore) should be elaborated a little bit more, cf. my “story of $1/\pi^2$.

There is a natural way to identify "ordinary functions" (say $k \in \boldsymbol{C}_c(\mathbb{R}^d)$) with linear functionals $\mu \in \boldsymbol{M}(\mathbb{R}^d)$, by the following trick: Given

cttomeasure

$$\mu = \mu_k, \qquad \text{resp.} \qquad \mu(f) = \int_{\mathbb{R}^d} f(x)k(x)dx \tag{116}$$

It is important to note that this is indeed an injective (linear) mapping from $\boldsymbol{C}_c(\mathbb{R}^d)$ into $\boldsymbol{M}_b(\mathbb{R}^d)$. As we will see it is isometric, if we impose on $\boldsymbol{C}_c(\mathbb{R}^d)$ the usual $\boldsymbol{L}^1$-norm.

This is also possible over general locally compact Abelian groups, but requires the existence of the Haar measure (we will not go into this direction, a good explanation is given in Deitmar's book [17]).

Lemma 42. *The mapping $k \to \mu_k$ described above defines an isometric embedding from $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$ into $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Hence we may identify the closure of $\boldsymbol{M}_{C_c} = \{\mu_k \,|\, k \in \boldsymbol{C}_c(\mathbb{R}^d)\}$ with the completion of the normed space $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$.*

isomL1emb **Proposition 6** (Functions to Measures)**.** *There is a natural,* isometric *embedding of $(\boldsymbol{C}_c(G), \|\cdot\|_1)$ into $\boldsymbol{M}_b(\mathscr{G})$, given by*

$$k \mapsto \mu_k : \mu_k(f) = \int_G f(x)k(x)dx.$$ [52]

Proof. It is clear that each μ_k is a bounded linear functional on $\boldsymbol{C}_0'(G)$ and that the mapping $k \mapsto \mu_k$ is linear and nonexpansive, since evidently for each $f \in \boldsymbol{C}_0(G), k \in \boldsymbol{C}_c(G)$ one has:

$$|\mu_k(f)| = \left| \int_G f(x)k(x)dx \right| \leq \int_G |f(x)||k(x)|dx \leq \|f\|_\infty \|k\|_1.$$

The converse is a bit more involved. In principle one would choose, for the case of a real-valued function k a function $f \in \boldsymbol{C}_0(G)$ which is a minimal (but continuous!) modification of the signum function, which turns (when integrated against k) the negative parts into positive parts, thus turning $\mu_k(f)$ in a good approximation of $\|k\|_1$.

To be more formal let us consider $f \in \boldsymbol{C}_c(G)$, let us consider for any $\eta > 0$ the "essential" support $K_\eta := \{z \in G \,|\, |k(z)| \geq \eta\}$. Then K_η is a compact set and we can find a continuous function [53] h_η with values in $[0,1]$ such that $h_\eta(z) = 1$ on K_η and with support of h_η within (the interior) of $K_{\eta/2}$. The function $f_\eta(x) := h_\eta(x)|k(x)|/k(x)$ is then well defined (because $k(x) \neq 0$ for any point in the support of h_η, and $\|f_\eta\|_\infty \leq 1$). We observe that

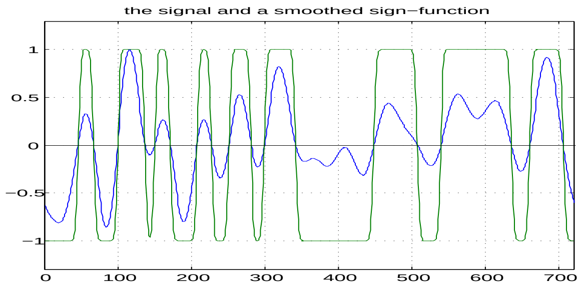
$$\mu_k(f) = \int_G f_\eta(x)k(x)dx = \int_G |k(x)|h_\eta(x)dx.$$

It remains to verify that this tends to $\|k\|_1$ for $\eta \to 0$.

---

[52] The integration is with respect to the Haar measure on the group G.

[53] The existence of h_η is guaranteed by Tietze's theorem, one of the important theorems concerning locally compact, hence completely regular topological spaces. It helps to avoid the potential problem of a phase discontinuity, i.e. problems with the continuity of $k(x)/|k(x)|$ near the zeros of k.



Writing K_0 for supp(k) this is a direct consequence of the following estimate

$$\int_G |k(x)(1-h_\eta(x))| \leq \int_{K_0} |k(x)|(1-h_\eta(x))dx \leq Vol(K_0)^{54} \cdot \|k(1-h_\eta)\|_\infty \to 0.$$

$\square$

Notation: $\boldsymbol{M}_{cs}$ = continuous shift.

Li-Def **Definition 31.** We define $\boldsymbol{L}^1(\mathbb{R}^d)$ as

$$\boldsymbol{M}_{cs} := \{\mu \in \boldsymbol{M}_b(\mathbb{R}^d) \,|\, \|T_x\mu - \mu\| \to 0 \quad \text{for} \quad x \to 0.\}$$

within $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

Mcsclosed1 **Lemma 43.** *$\boldsymbol{M}_{cs}(\mathbb{R}^d)$ is a closed ideal within $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$.*

Following the Riemann-Lebesgue Lemma we derive: REPETITION?

Four-fund0 **Theorem 16** (Fundamental Theorem for Fourier Analysis)**.** *Let $f, g \in \boldsymbol{L}^1(\mathbb{R}^d)$ be given. Then*

nd-Fourier0 (117)
$$\int_{\mathbb{R}^d} f(t)\hat{g}(t)dt = \int_{\mathbb{R}^d} \hat{f}(x)g(x)dx$$

[54] $Vol(K_0)$ stands for the Haar measure of the set K_0, but the $\|\cdot\|_1$ of a plateau-function $p(x)$ with p with $p(x) \cdot k(x) = k(x)$ would do. In fact, the "measure of K_0, i.e. $Vol(K_0) = \mu(K_0)$ can be shown to be equal to the infimum over all those $\|\cdot\|_1$-norms.

Proof. The Fourier transforms $\hat{f}$ and $\hat{g}$ are bounded and continuous, hence both integrands are in $\boldsymbol{L}^1(\mathbb{R}^d)$. The relation (fund-Fourier0 117) then follows via Fubini's theorem (at (*)), since $e^{2\pi i t\cdot x} f(t)g(x) \in \boldsymbol{L}^1(\mathbb{R}^{2d})$:

$$\begin{aligned}\int_{\mathbb{R}^d} f(t)\hat{g}(t)dt &= \int_{\mathbb{R}^d} f(t)\left(\int_{\mathbb{R}^d} e^{-2\pi i x\cdot t} g(x)dx\right)\\ &\underset{(*)}{=} \int_{\mathbb{R}^d} g(x)\left(\int_{\mathbb{R}^d} e^{-2\pi i x\cdot t} f(t)dt\right)dx \qquad (118)\\ &= \int_{\mathbb{R}^d} f(x)\hat{g}(x)dx.\end{aligned}$$

□

Of course one has to justify this definition, by recalling that the usual definition of $\boldsymbol{L}^1(\mathbb{R}^d)$ based on Lebesgue's integrability criterion provides us with a Banach space (of equivalence classes of measurable functions, identifying two functions if they are equal almost everywhere), which contains $\boldsymbol{C}_c(\mathbb{R}^d)$ as a dense subspace (cf. more or less any book on measure theory for details on this matter: in fact, it is sufficient to approximate - in the $\boldsymbol{L}^1$-norm - indicator functions of parallel-epipeds by continuous functions with compact support, i.e. something like a trapezoidal function sufficiently close to a "box-car"-function in the 1-dimensional case).

It is also of interest to introduce **the concept of a support to measures**, in a way which is compatible with the notion supp(k) for $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ given above:

eas-support **Definition 32.** A point x does *not* belong to the support of a measure $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ if there exists some $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $k(x) = 1$, **but nevertheless** $k \cdot \mu = 0$. (!) The complement of this set is denoted by supp(μ).

Lemma 44. .

- supp(μ) *is a closed subset of* $\mathbb{R}^d$*,*
- *there is consistency with the concept already defined for* $k \in \boldsymbol{C}_c(\mathbb{R}^d)$*, in other words:* supp$(k)$ *(in the old sense) coincides with* supp(μ_k)*, just defined.*
- *For a discrete measure the support is given by the closure of the union of all points involved, i.e. for* $\mu = \sum_{k=1}^\infty c_k \delta_{t_k}$ *we have*[55] supp$(\mu) = \left(\bigcup_k t_k\right)^-$*, where the bar denotes "closure".*
- *The notion of support is compatible with pointwise products: i.e. for any* $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ *on has* supp$(h\mu) \subseteq$ supp$(h) \cap$ supp(μ) *(as for functions).*
- *consequently every* $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ *can be approximated by compactly supported measures of the form* $p \cdot \mu$*, with* supp$(p \cdot \mu) \subset$ supp(μ)*.*

We have the following equivalent description of the supp(μ) (which is actually the usual definition):

eas-charact **Lemma 45.** *The following properties are equivalent:*

- $z \in$ supp(μ)*;*
- *for any* $\varepsilon > 0$ *there exists some* $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ *with* supp$(h) \subseteq B_\epsilon(z)$ *with* $\mu(h) \neq 0$.
- supp(μ) *coincides with the intersection of all supports of functions* p *such that* $p \cdot \mu = \mu$.

The following results indicate the w^*-continuity of the concept of a support.

wstsuppconv **Lemma 46.** *Assume that* $\mu_0 = w^* - lim_\alpha \mu_\alpha$*, then* supp$(\mu_0) \subseteq \bigcap_\alpha$ supp(μ_α).

[55]Assuming of course the canonical representation of μ, with $t_k \neq t_l$, for $k \neq l$ and $c_k \neq 0$.

The notion of support is also compatible with convolution: [56]

onv-support **Lemma 47.** *For* $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ *and* $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ *one has*

conv-supp
$$\mathrm{supp}(\mu * f) \subseteq \mathrm{supp}(\mu) + \mathrm{supp}(f). \tag{119}$$

Lemma 48. *Assume that* $\mu_0 = lim_{w^*}\mu_\alpha$. *Then also for any BUPU* Ψ *the family* $D_\Phi\mu_\alpha$ *is* w^**-convergent to* $D_\Phi\mu_0$. *Even more, the family* $D_\Phi\mu_\alpha$ *is uniformly tight and* w^**-convergent to* μ_0 *as* $|\Phi| \to 0$.

Finally we claim that the family $D_\Phi(p_K\mu_\alpha)$, *where* p_K *runs through the family of all plateau functions (with* $K \to \mathbb{R}^d$*), satisfies the same relation. Note that the resulting measures are in fact finite discrete measures.*[57]

It is an important and not completely trivial claim that a measure supported within the zero-set of a continuous function the action is zero. More precisely:

upinzeroset **Lemma 49.** $\mathrm{supp}(\mu) \subseteq \{x \,|\, h(x) = 0\}$ *implies* $\mu(h) = 0$.

Proof. It is enough to verify that any function $h \in \boldsymbol{C}_b(\mathbb{R}^d)$ can be approximated in the sup-norm (not so in other norms, like the $\mathcal{F}\boldsymbol{L}^1$-norm!) by other functions which vanish near the zero-set of h. This can be achieved by multiplying h with a plateau-type function which vanishes on $\{x \,|\, |h(x)| \leq \varepsilon/3\}$ and has a plateau on the set $\{x \,|\, |h(x)| \geq 2\varepsilon/3\}$. etc. ...

Alternatively, one may first show that it is no loss of generality to assume that h has compact support and thus that $h \in \boldsymbol{C}_{ub}(\mathbb{R}^d)$. In this case one can replace h by $Sp_\Phi h$, for sufficiently fine Φ, and then discard all the elements of the form $h(x_j)\phi_j$ with $|h(x_j)| \leq \varepsilon/3$. The effect is practically the same. $\square$

[56] We probably still have to take care of the notion of support for the case that the measure does not have compact support.

[57] Alternatively one could use only finite subfamilies from the partition of unity, or put p_k on the outside, i.e. write $p_k \cdot D_\Phi\mu_\alpha$. The consequences remain the same for all of these variants!

6. Basic properties of $L^1(Rd)$

We have defined $\boldsymbol{L}^1(\mathbb{R}^d)$ as the closure of $\boldsymbol{C}_c(\mathbb{R}^d)$ (identified via $k \mapsto \mu_k$ with a subspace of $\boldsymbol{M}(\mathbb{R}^d)$) in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Since this is a Banach space, it is a Banach space itself, and identical with the (abstract) completion of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\boldsymbol{M}(\mathbb{R}^d)$.

The next theorem gives us more information about the containment of $\boldsymbol{L}^1(\mathbb{R}^d)$ in $\boldsymbol{M}(\mathbb{R}^d)$:

L1-Basic **Theorem 17.** *(Basic properties of $\boldsymbol{L}^1(\mathbb{R}^d)$)*

- *$\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$ is a closed ideal within $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. It is a Banach algebra with a BAI (bounded approximate identity, the so-called Dirac sequences).*
- *$\boldsymbol{L}^1(\mathbb{R}^d)$ can be characterized as the closed subspace of with continuous shift, i.e. a bounded measure μ is of the form $\mu = \mu_g$, resp. $\mu(f) = \int_{\mathbb{R}^d} f(x)g(x)dx$ if and only if $\|T_x\mu - \mu\|_{\boldsymbol{M}} \to 0$ for $x \to 0$.*
- *$\boldsymbol{L}^1(\mathbb{R}^d)$ is w^*-dense in $\boldsymbol{M}(\mathbb{R}^d)$, in fact, for every $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ there exists a tight (hence bounded) sequence (f_k) in $\boldsymbol{L}^1(\mathbb{R}^d)$, with f_k resp. $\mu_{f_k} \to \mu$ in the w^*-topology.*

Proof. The main arguments are the identification of the "internal convolution" within $\boldsymbol{M}(\mathbb{R}^d)$ with the usual convolution formula

conf-CcRd (120)
$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy, \quad \text{for} \quad f, g \in \boldsymbol{C}_c(\mathbb{R}^d).$$

but also the external action of $\boldsymbol{M}(\mathbb{R}^d)$ on the homogeneous Banach space

$$\boldsymbol{M}(\mathbb{R}^d)_e = \{\mu \,|\, \|T_x\mu - \mu\|_{\boldsymbol{M}} \to 0 \text{ for } x \to 0.\}$$

The typical bounded approximate units are of the form $(\mathrm{St}_\rho g)_{\rho>0}$, for an arbitrary $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\hat{g}(0) = \int_{\mathbb{R}^d} g(t)dt = 1$.

It is easy to verify that this net is tight and tends to δ_0 in the w^*-topology. In a similar way one can approximate a finite and discrete measure by a linear combination of such Dirac sequences. Since the Dirac measures form a total subset in $\boldsymbol{M}(\mathbb{R}^d)$ with respect to the w^*-topology the w^*-density of $\boldsymbol{L}^1(\mathbb{R}^d)$ in $\boldsymbol{M}(\mathbb{R}^d)$ is established. ☐

MAYBE A STATEMENT TO BE PLACED ELSEWHERE: JAN 2014

For any $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ the compressed versions of h, i.e. the net $(\mathrm{St}_\rho h)_{h \to 0}$ is convergent to a multiple of δ_0. For most applications h is normalized such that $\int_{\mathbb{R}^d} h(t)dt = 1$ (and for the proof it is enough to work with this normalization).

deltboxdil1 **Lemma 50.** *For any $h \in \boldsymbol{L}^1(\mathbb{R}^d)$ one has*

$$w^*\text{-}\lim_{\rho \to 0} \mathrm{St}_\rho\, \mu_h = \left(\int_{\mathbb{R}^d} h(t)dt\right) \cdot \delta_0$$

Proof. To be given later!? It should be enough to verify the claim[58] for $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} h(t)dt = \hat{h}(0) = 1$. ☐

Bounded measures operate not only on $\boldsymbol{C}_0(\mathbb{R}^d)$ but also on any homogeneous Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. The proof of this fact is essentially based on the idea that it is enough to establish this fact for bounded discrete measures (which is easy) and then show that for any sequence of discretizations of a given measure (where the diameter of the support of the corresponding BUPUs shrinks to zero) generates a sequence μ_k which is uniformly tight and bounded, but also produces a Cauchy sequence in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ of the form $(\mu_k * f)$, for any given $f \in \boldsymbol{B}$. Obviously

[58] !?why, there is some approximation to be done, which - if done properly - still requires some "epsilontic", i.e. fiddling around with fractions of ε.

it makes sense to define the limit (which does not depend on the choice of discretizations via BUPUs) by $\mu * f$ (although it is formally a new operation, and the "star" just introduced should be distinguished for a little while from the "known" star which denotes convolution within $\boldsymbol{M}(\mathbb{R}^d)$): $\mu * f \in \boldsymbol{B}$ and NORMS TO-BE-DONE

Let us recall that the index set of a "net" is a directed set.

`http://de.wikipedia.org/wiki/Gerichtete_Menge`

Definition 33. A set is called a *directed* set (we use the symbol $\preceq$) if it is a relation on a set X satisfying the following three axioms, namely *reflexivity, transitivity and joint majorant property*:

- $x \preceq x$, $\forall x \in X$; (reflexivity);
- $(x \preceq y) \wedge (y \preceq z) \Rightarrow (x \preceq z)$;
- $\forall x, y \in X \, \exists z \in X$ with $x \preceq z$ and $y \preceq z$.

Dirac-Conc **Lemma 51.** *A bounded net of functions $(h_\alpha)_{\alpha \in I}$ is a BAI for $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ if the following property is satisfied: For every $\varepsilon > 0$ there exists some α_0 such that for $\alpha \succ \alpha_0$ one has:*

Dirac-eps
$$\left|\int_{B_\varepsilon(0)} h_\alpha(t)dt - 1\right| \leq \varepsilon \quad \text{and} \quad \int_{|x|\geq\varepsilon} |h_\alpha(t)|dt \leq \varepsilon. \tag{121}$$

Proof. Argument: Due to the density of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\boldsymbol{L}^1(\mathbb{R}^d)$ one can reduce the discussion to functions $k \in \boldsymbol{C}_c(\mathbb{R}^d)$, i.e. it is enough to show that $h_\alpha * k \mapsto k$ for any $k \in \boldsymbol{C}_c(\mathbb{R}^d)$. The second condition allows to restrict the attention to a net with common compact support K. Consequently one has $h_\alpha * k(x) \neq 0$ only for $x \in K + \mathrm{supp}(k)$. Furthermore we obtain $h_\alpha * k(x) = \int_{\mathbb{R}^d} h_\alpha(y)k(x-y)dy \mapsto$ MORE TO BE DONE TOMORROW!! $\square$

Remark 38. Of course one can also consider $\boldsymbol{M}(\mathbb{R}^d)$ as a Banach module over the Banach algebra $\boldsymbol{L}^1(\mathbb{R}^d)$ (with respect to convolution). Then the $\boldsymbol{L}^1(\mathbb{R}^d)$-essential part of $\boldsymbol{M}(\mathbb{R}^d)$ is equal to $\boldsymbol{L}^1(\mathbb{R}^d)$ itself.

On the other hand we can consider $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ as a Banach module over $\boldsymbol{L}^1(\mathbb{R}^d)$ (again with respect to convolution), and then this is an essential Banach module.

7. Tight subsets

WARNING: THE CONCEPT OF TIGHTNESS HAS BEEN USED ALREADY ON AND OFF!

A given $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ is of course "essentially concentrated" on a compact set (and uniformly small outside a sufficiently large compact set, by definition). We also have shown that a functional $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ is having most of its "mass" sitting within a compact set, while its action outside of this compact set is small. Indeed, since for any BUPU Φ we have $\mu = \sum_{iinI} \phi\mu$ as absolutely convergent sum the tails μ minus a large partial sum is small in the $\boldsymbol{M}(\mathbb{R}^d)$-sense.

Next we want to extend this "concentration over compact sets" concept to general bounded subsets of $\boldsymbol{M}(\mathbb{R}^d)$ (and later other functional spaces):

ght-measdef **Definition 34.** A bounded subset $W \subset \boldsymbol{M}(\mathbb{R}^d)$ is called (uniformly) tight if for every $\varepsilon > 0$ there exists $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|\mu - k \cdot \mu\|_{\boldsymbol{M}} \leq \varepsilon$ for all $\mu \in W$.

In a similar way we define tightness in $\boldsymbol{C}_0(\mathbb{R}^d)$:

Definition 35. A bounded subset $H \subset \boldsymbol{C}_0(\mathbb{R}^d)$ is called (uniformly) tight if for every $\varepsilon > 0$ there exists $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|h - k \cdot h\|_\infty \leq \varepsilon$ for all $h \in H$.

Note: For a general $\boldsymbol{C}_0(\mathbb{R}^d)$ module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ one can define tightness as follows:

Definition 36. A bounded subset $H \subset (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called (uniformly) tight if for every $\varepsilon > 0$ there exists $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|h - k \cdot h\|_\infty \leq \varepsilon$ for all $h \in H$.

The concept of *tightness* plays a big role in the characterization of *relatively compact subsets* in function spaces and hence is also important for the description of compact operators.

Theorem 18. *Assume that W is a tight set within $\boldsymbol{M}(\mathbb{R}^d)$ and that H is a tight subset within $\boldsymbol{C}_0(\mathbb{R}^d)$. Then $W * H = \{\mu * h \,|\, \mu \in W, h \in H\}$ is a tight subset in $\boldsymbol{C}_0(\mathbb{R}^d)$.*

cf. the "compactness paper" p.307 (bottom):
`http://univie.ac.at/nuhag-php/bibtex/open_files/fe84_compdist.pdf`

Proof. Indeed, for any plateau-function τ which satisfies $\tau(x) \equiv 1$ on $\mathrm{supp}(k^1)+\mathrm{supp}(k^2)$, hence the following estimate holds:

$$(1-\tau)(f^1 * f^2) = (1-\tau)(f^1 * f^2 - f^1k^1 * f^2k^2)$$

$$(1-\tau)(\mu * f) = (1-\tau)(\mu * f - \mu k^1 * f k^2)$$

Applying norms to both sides and using the triangle inequality we obtain the following estimate in the sup-norm:

$$\|(1-\tau)(\mu * f)\| = \|(1-\tau)(\mu * f - \mu k^1 * f k^2)\| \leq$$

$$\leq \|(1-\tau)\| \|(1-k^1)\mu * f\| + \|k^1 \mu * (1-k^2) f\| \leq$$

$$\leq \|(1-\tau)\| \|(1-k^1)\mu\|_{\boldsymbol{M}} \|f\| + \|k^1\| \|\mu\|_{\boldsymbol{M}} \|(1-k^2) f\|.$$

□

8. The Fourier transform for *LiRd*

The Fourier transform maps $\boldsymbol{M}(\mathbb{R}^d)$ into $\boldsymbol{C}_{ub}(\mathbb{R}^d)$. It will be seen as a non-expansive Banach algebra homomorphism from the closed ideal $\boldsymbol{L}^1(\mathbb{R}^d)$ into the closed ideal $\boldsymbol{C}_0(\mathbb{R}^d)$ of $\boldsymbol{C}_{ub}(\mathbb{R}^d)$ (this result is usually known as Riemann-Lebesgue Lemma).

The range of the Fourier transform is a dense subalgebra (closed under complex conjugation), due to the "locally compact version" of the Stone-Weierstrass theorem.

Recall the standard version of the Stone-Weierstrass theorem:

eierstr-thm **Theorem 19.** *Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra within $\boldsymbol{C}(X)$, where X is some compact topological space. Then $\boldsymbol{A}$ is dense with respect to the uniform norm if $\boldsymbol{A}$ contains the constant functions, is closed under conjugation, and separates points, i.e. , if for any pair of points $x_1, x_2 \in X$, with $x_1 \neq x_2$ there exists some $f \in \boldsymbol{A}$ such that $f(x_1) \neq f(x_2)$.*

Since $\mathcal{F}\boldsymbol{L}^1$ does not have a unit (for pointwise multiplication), due to the fact that $\boldsymbol{L}^1(\mathbb{R}^d)$ does not contain a unit (as the unit with respect to convolution is the Dirac measure δ_0, which *cannot* be approximated in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ from within $\boldsymbol{C}_c(\mathbb{R}^d)$), one has to modify the above result to the locally compact case, by "adding" the constant functions, and replacing $\mathbb{R}^d$ by its Alexandroff (one-point) compactification X of $\mathbb{R}^d$. Indeed, $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ can be identified with a closed subalgebra of all continuous functions vanishing "at infinity". In fact, if $C_0 + h$ in $\boldsymbol{C}(X)$ is approximated by a sequence of the form $C_n + f_n$, then $|C_n - C_0| \to 0$ for $n \to \infty$, so $\|f_n - h\|_\infty$ for $n \to \infty$ (details left to the reader).

FL1-algprop **Lemma 52.** *$\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ is a Banach algebra with respect to the standard norm $\|\hat{f}\|_{\mathcal{F}\boldsymbol{L}^1} := \|f\|_1$ for $f \in \boldsymbol{L}^1(\mathbb{R}^d)$, which is closed under translation, modulations and dilations (in fact, with continuous dependence on the shift resp. dilation parameters). It is also closed under complex conjugation.*

Proof. All these properties follow from the algebraic properties of the Fourier transform (intertwining of modulation and translation operators), and interwining of St_ρ and D_ρ operators, as well as $f \mapsto f^*$ with $f^*(x) = \overline{f(-x)}$, which intertwines with ordinary conjugation of $\hat{f}$. □

For $\mathbb{R}^d$ one can of course use alternative arguments, e.g. by observing that certain functions, such as the Schwartz functions, belong to the Fourier algebra, so there is enough richness in the function space $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ in order to show the density of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ in $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, but the above argument applies to general LCA groups.

FliR-dens1 **Lemma 53.** *Let $k \in \boldsymbol{C}_c(\mathbb{R})$ and Ψ the family of B-splines of any fixed order $s \in \mathbb{N}$ (hence a BUPU of size $s/2$). Then for every $\delta > 0$ the spline-approximation $Sp_{\delta\Psi}k$ is a finite sum of shifted B-spline functions, hence a function having $(s-1)$ continuous derivatives. Also for $s \geq 2$ the B-splines are continuous and belong to $\mathcal{F}\boldsymbol{L}^1(\mathbb{R})$, because their Fourier transforms are of the form sinc^s. In particular, $\boldsymbol{C}^m$ as well as $\mathcal{F}\boldsymbol{L}^1$ are dense subspaces of $\boldsymbol{L}^1(\mathbb{R})$ (and other $\boldsymbol{L}^p$-spaces, for $p < \infty$).*

On of the central statements concerning the Fourier transform is *Plancherel's theorem*, stating that the Fourier transform can be considered as a unitary linear automorphism of the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ onto itself. This is in complete analogy to the statement that for the case of $\mathbb{C}^n$ the Discrete Fourier transform (often realized in the form of the FFT) is a change of bases from the orthonormal basis of unit vectors to the orthogonal system of pure frequencies. Since the vectors representing the pure frequencies (which are exactly the joint eigenvalues to all the translation operators) are of absolute value one, they are all of norm $\sqrt{n}$ the inverse FFT is essentially the conjugate (transpose) of the Fourier matrix, with a compensating factor of the form $1/n$. The advantage of our normalization in the continuous case (with the factor 2π as part of the exponent) has the advantage that the inverse Fourier transform will come in the form

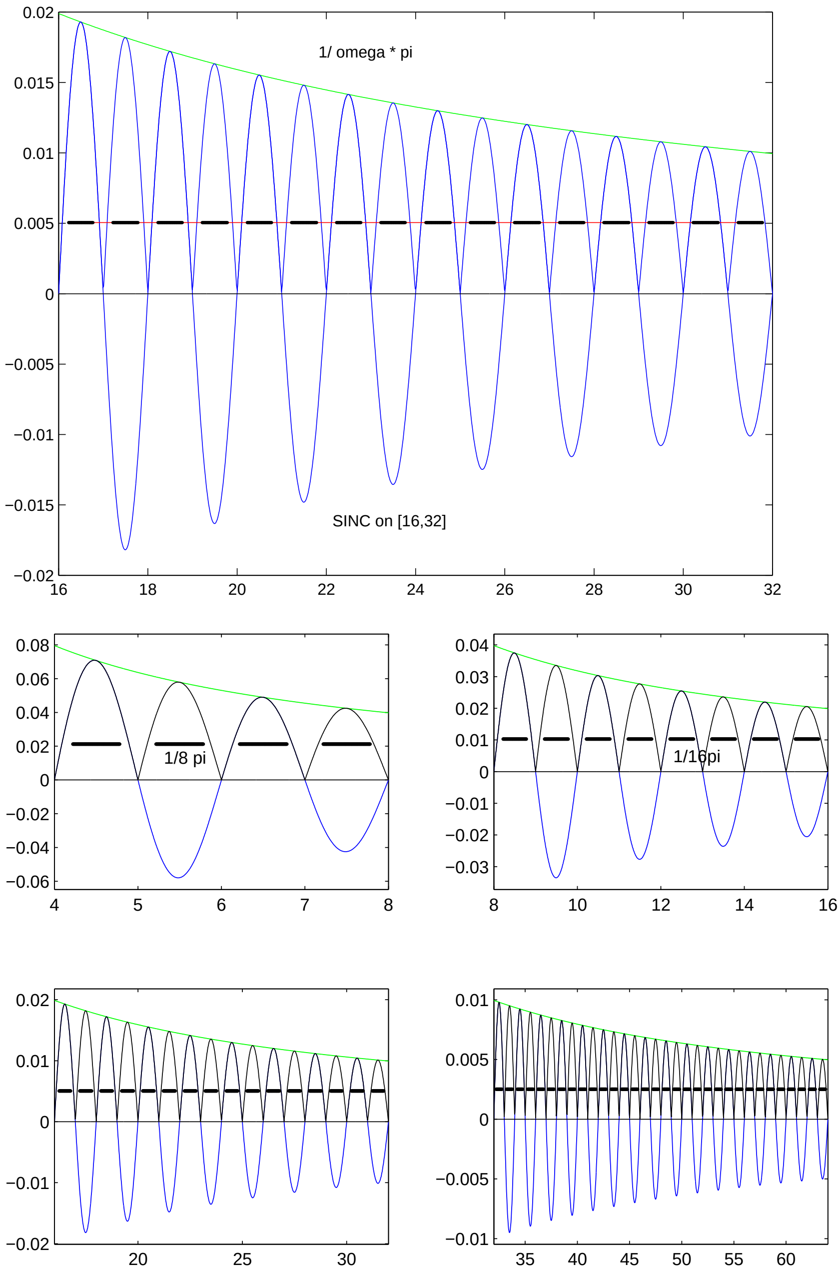
inv-Fourdef (122)
$$h(t) = \int_{\mathbb{R}^d} \hat{f}(s) e^{2\pi i t \cdot s} dt$$

There is a lot to be said about the Fourier inversion. Above all it has to be noted that it is *wrong* to view it as a formula, allowing to recover the function $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ from its Fourier transform pointwise, simply because there is no guarantee that the Fourier transform of such a function should be integrable. The box-car function $\mathbf{1}_{[-1/2,1/2]}$ is a typical example, whose Fourier transform is the so-called SINC-function, defined as

SINCdef1 (123)
$$\operatorname{sinc}(t) := \frac{\sin \pi t}{\pi t}, \quad t \in \mathbb{R}.$$

This is where classical analysis is bringing in a lot of Fourier analysis.

As an illustration for the poor decay of the SINC-function (implying sinc $\notin \boldsymbol{L}^1(\mathbb{R})$!), let us have a look at the graph of this function:



Since the integral definition of the FT and its inverse do not apply to general functions $\hat{f} \in \mathcal{F}\boldsymbol{L}^1$, part of the discussion of the **Fourier-Plancherel Theorem** is concerned with technical questions around problems of this kind (how to overcome lack of integrability, e.g.by applying so-called *summability* methods, which are a generalization of the idea of an infinite integral, taken as limit of finite integrals).

Lemma 54. *It is enough to verify that for some dense subspace $\boldsymbol{B}$ of $\boldsymbol{L}^2(\mathbb{R}^d)$ within $\boldsymbol{L}^1(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{C}_0(\mathbb{R}^d)$ one can find that the mapping $f \mapsto \hat{f}$ is well defined and isometric, and with dense range, in order to be able to extend the "classical" Fourier transform and its inverse (given by the integral) to an isometry from $\boldsymbol{L}^2(\mathbb{R}^d)$ onto itself.*

Proof. Since we assume that $\boldsymbol{B} \subseteq \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ one can claim that the direct and the inverse Fourier transform given via (absolutely convergent) Riemannian integrals is valid. For the rest one only has to show that for an arbitrary $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ and any sequence f_n, with $f_n \in \boldsymbol{B}$ with $\|f - f_n\|_2 \to 0$ for $n \to \infty$ one finds that $\|\hat{f} - \hat{f}_n\|_2 = \|f - f_n\|_2 \to 0$, so by the completeness of $\boldsymbol{L}^2(\mathbb{R}^d)$ the Cauchy sequence $\hat{f}_n$ must have a limit, which may be denoted (by a so-called abuse of language) as the Fourier transform $\hat{f}$ of f. The extended (still isometric) mapping has dense range according to our assumptions, and therefore the same argument can be applied to the inverse Fourier transform in order to realize that the extended mapping (often called the *Fourier-Plancherel* or just *Plancherel* transform defines an isometric automorphism of $\boldsymbol{L}^2(\mathbb{R}^d)$. Due to the polarization identity

$$\langle f, g \rangle = \sum_{k=0}^{3} i^k \|f + i^k g\|^2.$$

such a mapping also preserves scalar products in general. □

For the proof of Plancherel's theorem one may use $\boldsymbol{B} = \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ or the linear span of all the time-frequency shifted and dilated version of a Gauss function (we do not require any norm on $\boldsymbol{B}$). Ideally one can or should use a space of functions which is invariant under the Fourier transform. Details have been given in the FA (= functional analysis) course WS0506 by HGFei.

Of course the extended Fourier transform is still compatible with convolution resp. pointwise multiplication. In other words, convolution on one side of the FT goes into pointwise product on the other side (and vice versa). As a consequence one obtains a characterization of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$: A function belongs to $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ if and only if it can be written as a convolution product of two functions in $\boldsymbol{L}^2(\mathbb{R}^d)$, i.e. $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ if and only if there exist two functions $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$ such that $h = f * g$. The direct direction (i.e. convolution products are in $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$) is easy, because their Fourier transforms give a function which is a pointwise product of two $\boldsymbol{L}^2$-functions, and hence according to the Cauchy-Schwartz inequality $\hat{h} = \hat{f} \cdot \hat{g} \in \boldsymbol{L}^2 \cdot \boldsymbol{L}^2 \subseteq \boldsymbol{L}^1$, or equivalently $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.

The easiest argument for the converse is again on the Fourier transform side. Write $\hat{h} \in \boldsymbol{L}^1(\mathbb{R}^d)$ as a pointwise product of two $\boldsymbol{L}^2$-functions. If $\hat{h}$ was non-negative there is a natural solution to this problem, just take $\sqrt{\hat{h}}$. If $\hat{h}$ is a complex-valued function, one can apply this trick only to $|\hat{h}|$, and can assign the phase factor to one of the two non-negative square roots (details are left to the reader).

We are deriving (a weak form) of the Fourier inversion theorem from (measFmeas 104).

ier_invers2 **Theorem 20.** *Assume $f \in \boldsymbol{L}^1 \cap \boldsymbol{C}_0(\mathbb{R}^d)$ with (the additional assumption that) $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$* [59] *then for every $t \in \mathbb{R}^d$ one has:*

$$f(t) = \int_{\mathbb{R}^d} e^{2i\pi st} \hat{f}(s) ds.$$

Proof. Recall that we have the consistency of the Fourier transform for $\boldsymbol{L}^1(\mathbb{R}^d)$-functions f with the corresponding measure μ_f, since

$$\hat{f}(s) = \int_{\mathbb{R}^d} f(t) \overline{\chi_s(t)} dt$$

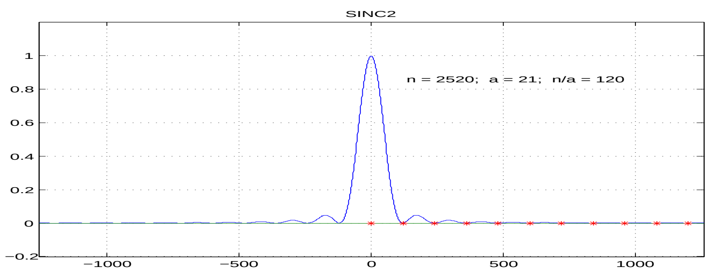
[59] Because $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{C}_0(\mathbb{R}^d)$ we could express this a bit more symmetric by assuming that $f \in \boldsymbol{L}^1 \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$: Note that this is a Fourier invariant Banach space, and even a Banach algebra, both with respect to pointwise multiplication and convolution!

In order to retrieve the inversion formula we just have to use (measFmeas 104) with the following choices: □

FTmeasinj **Lemma 55.** *$\hat{\mu} = 0$ implies $\mu = 0$.*

Proof. If $\hat{\mu} = 0$, we find from (measFmeas 104) that $\mu(\hat{\nu}) = 0$ for every $\nu \in \boldsymbol{M}_b(\mathbb{R}^d)$, in particular $\mu(\hat{f}) = 0$ for all $f \in \boldsymbol{L}^1(\mathbb{R}^d)$. Since $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is a dense subspace of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ this implies that $\mu = 0$. □

Arguments why $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is dense in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$: either using Stone-Weierstrass, or by approximation of any $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ by piecewise linear functions, and the observation that these functions are sums of triangular functions, which in return belong to the *Fourier algebra* because they correspond to convolution squares of rectangular functions and hence their Fourier transforms are (up to dilation and phase factors) just squared SINC-functions, i.e. of the form $\hat{\Delta}(s) = [sin(\pi s)/(\pi s)]^2$. **[Rd-specific!]**



A more direct way of verifying (also relying directly on the Weierstrass approximation theorem) can be formulated as follows

FourStinj3 **Lemma 56.** *The Fourier-Stieltjes transform $\mu \to \hat{\mu}$, given by $\hat{\mu}(s) = \mu(\chi_s)$ for $s \in \mathbb{R}^d$ is injective.*

Proof. We start with the observation that $\hat{\mu} = 0$ is equivalent to the statement that $\mu(t) = 0$ for any so-called *trigonometric polynomial*, i.e. any function which is a finite linear combination of pure frequencies: $t(x) = \sum_{k=1}^n c_k \chi_{s_k}$. On the other hand, due to the density of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\boldsymbol{C}_0(\mathbb{R}^d)$ we can assume that a non-zero functional μ would allow some $k \in \boldsymbol{C}_c(G)$ with $|\mu(k)| \geq \gamma > 0$. By choosing a sufficiently large plateau function p with $\|p\|_\infty = 1$ (in principle a sufficiently large partial sum of any given BUPU would do) we can have the following two facts at the same time:

$$k = p \cdot k \quad \text{and} \quad \|\mu - \mu \cdot p\|_{\boldsymbol{M}} < \gamma/(3\|k\|_\infty).$$

Choosing next (based on the Weierstrass approximation theorem) some trigonometric polynomial t such that

$$\|(k-t)\cdot p\|_\infty \leq \gamma/(3\|\mu\|_{\boldsymbol{M}})$$

(meaning: good uniform approximation only over the support of p; k will be zero outside anyway, but the trigonometric polynomial will be periodic and thus repeat with some large period outside, somewhere). Putting things together we reach a contradiction, because the following estimate would have to be valid, in contradiction to our assumption $|\mu(k)| = \gamma$:

$$|\mu(k)| \leq |\mu(k) - mp(k)| + |\mu(p\cdot(k-t))| \leq \|\mu - \mu\cdot p\|_{\boldsymbol{M}}\|k\|_\infty + \|\mu\|_{\boldsymbol{M}}\|p\cdot(k-t)\|_\infty < 2/3\gamma.$$

□

9. Wiener's algebra $W(R^d)$

algebraSecR

Because it can be defined without the existence of a Haar measure the following space plays an important role within Harmonic Analysis. We define $\boldsymbol{W}(\mathscr{G})$ as follows:

Because it can be defined without the existence of a Haar measure the following space plays an important role within Harmonic Analysis. We define $\boldsymbol{W}(\mathscr{G})$ as follows;

Then we define $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ as follows:

WCdefphi **Definition 37.** Let φ be any non-zero, non-negative function in $\boldsymbol{C}_c(\mathbb{R}^d)$.

WCdefphi1 (124)
$$\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1) = \{\, f \in \boldsymbol{C}_0(\mathbb{R}^d) \,|\, \exists (c_k)_{k\in\mathbb{N}} \in \boldsymbol{\ell}^1, (x_k)_{k\in\mathbb{N}} \,\text{in}\, \mathbb{R}^d, |f(x)| \leq \sum_{k\in\mathbb{N}} c_k\varphi(x-x_k)\,\} \tag{124}$$

We define

$$\|f\|_{\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{L}^1)(\mathbb{R}^d)} = \|f\|_{\boldsymbol{W}} := \inf\{\|\mathbf{c}\|_{\boldsymbol{\ell}^1} = \sum_{k\in\mathbb{N}} |c_k|\,\}$$

where the infimum is taken over all "admissible dominations" of f as in (WCdefphi1 124).

MAYBE STILL SOME MATERIAL MISSING

It is obvious that $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ is continuously embedded into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ [60] since $\|f\|_\infty \leq (\sum_{k\in\mathbb{N}}|c_k|)\|\varphi\|_\infty$. By a similar argument we have a continuous embedding of $\big(\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\big)$ into $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$ or in any other $\boldsymbol{L}^p$-space, for $1 \leq p \leq \infty$.

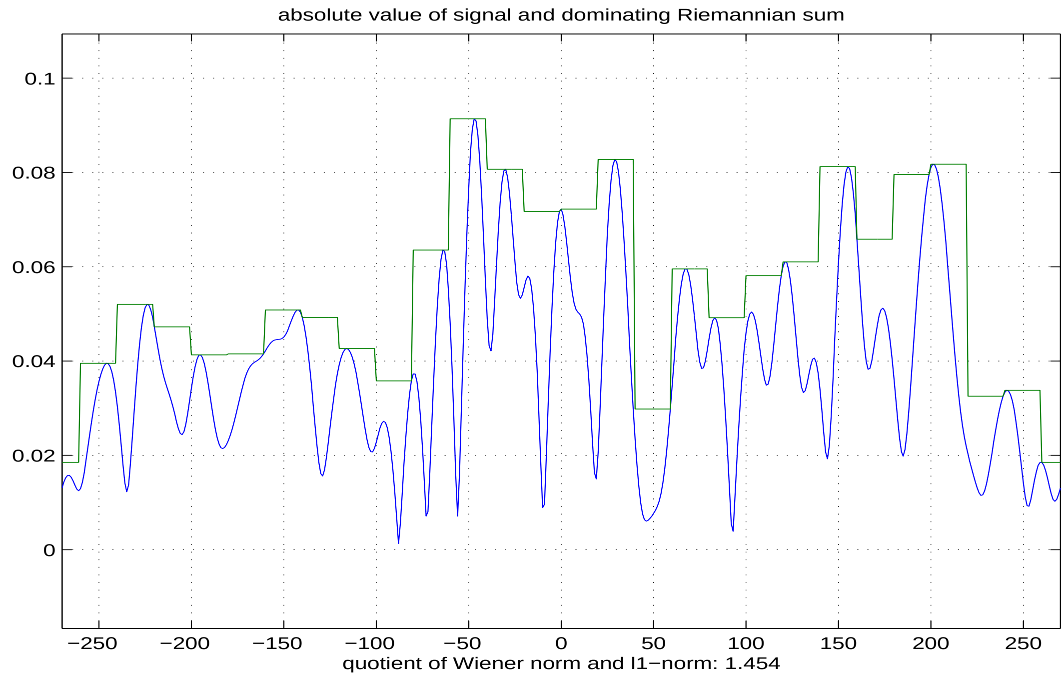
A convenient characterization of Wiener's algebra over $\mathbb{R}^d$ can be given using (arbitrary, non-negative, regular) BUPUs (see Def. regBUPU ??).

iener-BUPU2 **Lemma 57.** *Given a regular BUPU Ψ. Then a continuous function belongs to $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ if and only if the decomposition $f = \sum_{\lambda\in\Lambda}\psi_\lambda \cdot f$ is absolutely convergent, i.e. if and only if $\|f\|_{\boldsymbol{W},\Psi} = \sum_{\lambda\in\Lambda}\|\psi_\lambda\cdot f\|_\infty < \infty$. Moreover, every such norm $\|f\|_{\boldsymbol{W},\Psi}$ is equivalent to the one given above (in the definition).*

The proof is left as an exercise to the reader. Of course any continuous function f with the expression $\|f\|_{\boldsymbol{W},\Psi}$ finite only countably many non-zero contributions for $\lambda_n, n \geq 1$ and thus essentially has a representation as in Def. WCdefphi 37, with $f_n = T_{\lambda_n} f\psi_{\lambda_n}$.

As a consequence of this observation (fix the BUPU and read the proof with two different functions φ_1 resp. φ_2 used in the definition of the space) one obtains that the space is independent from the chosen (non-zero) function $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$. This could have been verified also

[60] by the same argument $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ is also contained in many other Banach spaces of functions with the property that translations are isometric and that the space is solid.



directly, because for every two such functions it is always possible to dominate one by a finite sum of translates of the other (based on a simple compactness argument: any such function is obviously bounded away from zero on some open set).

The classical approach to the Wiener algebra uses the BUPU consisting of shifted indicator functions of the form $(\mathbf{1}_{[\mathbf{n},\mathbf{n+1})})_{\mathbf{n}\in\mathbb{Z}}$. The resulting sum of local maxima is then equal to the upper Riemannian sum for the integral for any $k \in \boldsymbol{C}_c(\mathbb{R})$. This can be illustrated as follows.

Wiener-Conv **Lemma 58.** *$\big(\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\big)$ is a Banach algebra with respect to convolution. In fact it is a Banach ideal (and even a so-called Segal algebra in the sense of H. Reiter) within $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$.*

Proof. Since every element in $\big(\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\big)$ is an absolutely convergent sum of elements which are (up to translation) all bounded and have common compact support (within $\mathrm{supp}(\psi)$) it is enough split a convolution product of $g \in \boldsymbol{L}^1(\mathbb{R}^d) \subseteq \boldsymbol{M}_b(\mathbb{R}^d)$ with $f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$ into little blocks and estimate

eas-WRdconv
$$\|\mu\psi_k * f\psi_n\|_\infty \leq \|\mu\psi_k\|_{\boldsymbol{M}} \cdot \|f\psi_n\|_\infty. \tag{125}$$

Assuming that $\mathrm{diam}(\Psi) \leq \gamma$ for some $\gamma > 0$ we see that the support of these convolution products belongs are of size at most 2γ. Thus for any non-negative $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\varphi(x) \geq 1$ on $B_{2\gamma}(0)$ we have

omin-atoms1
$$|\mu\psi_k * f\psi_n(x)| \leq \|\mu\psi_k\|_{\boldsymbol{M}} \|f\psi_n\|_\infty T_{x_{n,k}}\varphi(x). \tag{126}$$

and hence for some constant $C > 0$ (by Theorem measabsdecomp 4 and Lemma Wiener-BUPU2 57) we have

$$\|\mu * f\|_{\boldsymbol{W}} \leq \sum_k \sum_n \|\mu\psi_k\|_{\boldsymbol{M}} \|f\psi_n\|_\infty \leq C\|\mu\|_{\boldsymbol{M}} \|f\|_{\boldsymbol{W}}.$$

$\square$

Remark 39. An alternative method to proof the $\big(\boldsymbol{W}(\boldsymbol{C}_0,\boldsymbol{L}^1)(\mathbb{R}^d),\, \|\cdot\|_{\boldsymbol{W}}\big)$ is a *Banach convolution module*, i.e. a Banach module over $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$ with respect to convolution as abstract multiplication is to verify that it is a *homogeneous Banach space*. Any such homogeneous Banach space is even a Banach convolution module over the Banach algebra $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ (with convolution), as well as an *essential Banach convolution module* over $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$.

Remark 40. Yet another method to control convolution products of Wiener amalgam spaces is provided by the paper [11]. Essentially, instead of using the domination property using a sufficiently wide plateau-function in the last step of the proof of Lemma Wiener-Conv 58 above they use the fact that one can split the convolution product into a controlled number of building blocks and thus only has to rearrange the sums. This method would be easy in the current situation but requires and extra argument for more general Wiener amalgam spaces.

Similar statements can be made for other function spaces, in particular for the pointwise algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}}) = \big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$. This is a special case of a Segal algebra (see next section). It has been introduced around 1979 as $\big(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\big)$ in [22] (the zero in the subscript stands for the *minimality* of this space). Meanwhile (see [58]) this space is called *Feichtinger's algebra*.

Within the family of Wiener amalgam spaces (defined via BUPUs, which are bounded in the Fourier Algebra $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$) we can give the following definition:

SO-WFLili **Definition 38.** Let $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda}$ be any bounded BUPU in $\big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$.

$$\boldsymbol{S}_0(\mathbb{R}^d) := \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^1)(\mathbb{R}^d).$$

One of the first and most important properties is the invariance of the space under the Fourier transform! We derive it from the convolution properties of Wiener amalgam spaces.

CHECK CONSISTENCY OF ARGUMENT!

SO-FOURinv **Theorem 21.**

$$\mathcal{F}[\boldsymbol{S}_0(\mathbb{R}^d)] = \boldsymbol{S}_0(\mathbb{R}^d),$$

with equivalence of norms[61].

Moreover, both the time-shift operators and the modulation operators act uniformly bounded on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$, *and furthermore* $\|T_x f - f\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \to 0$ *for* $x \to 0$ *as well as* $\|M_s f - f\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \to 0$ *for* $s \to 0$, *for each* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$.

As a consequence one has $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d)$ *and* $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d)$

Proof. First the uniform boundedness of both time and frequency shift operators has to be considered (should be considered an exercise). In fact, it is enough to verify it on atoms (i.e. on function of the form $f\varphi_n$), and there the properties of a BUPU in combination with the algebra properties come into play.

Since obviously the compactly supported (partial sums) elements are dense in $\boldsymbol{S}_0(\mathbb{R}^d)$. But over compact sets the $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$-norm and the $\boldsymbol{S}_0(\mathbb{R}^d)$-norms are equivalent, and hence to continuous dependencies claimed above are valid.

As a further consequence by (vector-valued) integration we get $\boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d)$, and hence in particular $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)$, with $\|g * f\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \le \|g\|_1 \|f\|_{\boldsymbol{S}_0}$, for all $g \in \boldsymbol{L}^1, f \in \boldsymbol{S}_0$.

Since by the definition the decomposition $f = \sum_{\lambda\in\Lambda} \psi_\lambda \cdot f$ is an absolutely convergent series in $\big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$ it follows that $\hat{f} = \sum_{\lambda\in\Lambda} \widehat{\psi_\lambda \cdot f}$ is absolutely convergent in $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$.

[61] Later on we will see that with a specific norm one can make the Fourier transform an isometric automorphism on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$.

But we want more. The series should also be convergent in the (much smaller) Banach space $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^1)(\mathbb{R}^d)$. In order to do this we take *any* function in $\boldsymbol{S}_0(\mathbb{R}^d)$ (e.g. $SINC.^2$) such that $\hat{g}$ belongs to $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ and has compact support (e.g. the Vallee DePoussin kernel), and such that $\psi \cdot \hat{g} = \psi$, hence

$$T_\lambda(\psi \cdot \hat{g}) = T_\lambda\psi \cdot T_\lambda\hat{g} = T_\lambda\psi \cdot \widehat{M_\lambda g}$$

and consequently we can derive

$$\|\widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{S}_0(\mathbb{R}^d)} = \|T_\lambda\hat{g}\cdot\widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \leq \|M_\lambda g\|_{\boldsymbol{S}_0(\mathbb{R}^d)}\|\widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{L}^1(\mathbb{R}^d)} = \|g\|_{\boldsymbol{S}_0(\mathbb{R}^d)}\|\widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{L}^1(\mathbb{R}^d)}.$$

and consequently

$$\|\hat{f}\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \leq \sum_{\lambda\in\Lambda} \|\widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \leq \|g\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \sum_{\lambda\in\Lambda} \|\widehat{\psi_\lambda \cdot f}\|_{\boldsymbol{L}^1(\mathbb{R}^d)} \leq \|g\|_{\boldsymbol{S}_0(\mathbb{R}^d)} \cdot \|f\|_{\boldsymbol{S}_0(\mathbb{R}^d)}.$$

□

10. The Segal algebra $S_O(Rd)$ and Banach Gelfand triples

There is quite a large number of talks by the author reporting on the idea of Banach Gelfand Triples, see

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There are different ways of defining  $\SORd = \WFLili$  !
% should be renamed % {\tt  WFLili instead of  WFlili.  }
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We have now a Banach space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ which is Fourier invariant, contained (continuously embedded, and densely) in $\left(\boldsymbol{L}^2(\mathbb{R}^d),\, \|\cdot\|_2\right)$. Together with the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ we will have a convenient setting for a theory of generalized Fourier transforms. The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is also isometrically invariant under TF-shifts, as well as the various involutions (such as $f \mapsto \overline{f}$ or $f \mapsto f^*$ or $f \mapsto f^{\checkmark}$), but also (not-isometrically[62] invariant under the dilation operators (of course both D_ρ and St_ρ)

11. The inverse STFT

Part of this material is from Severin Bannert's master thesis ([3]). An upcoming article in the Notices of the AMS ([32]) will provide a good summary of this part of Harmonic Analysis. Already now [14] can serve as a good introduction to Banach Gelfand triples.

[62] Asymptotic behavior of the dilation operators on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ has been studied by Cordero et. al., see [**?**, 15].

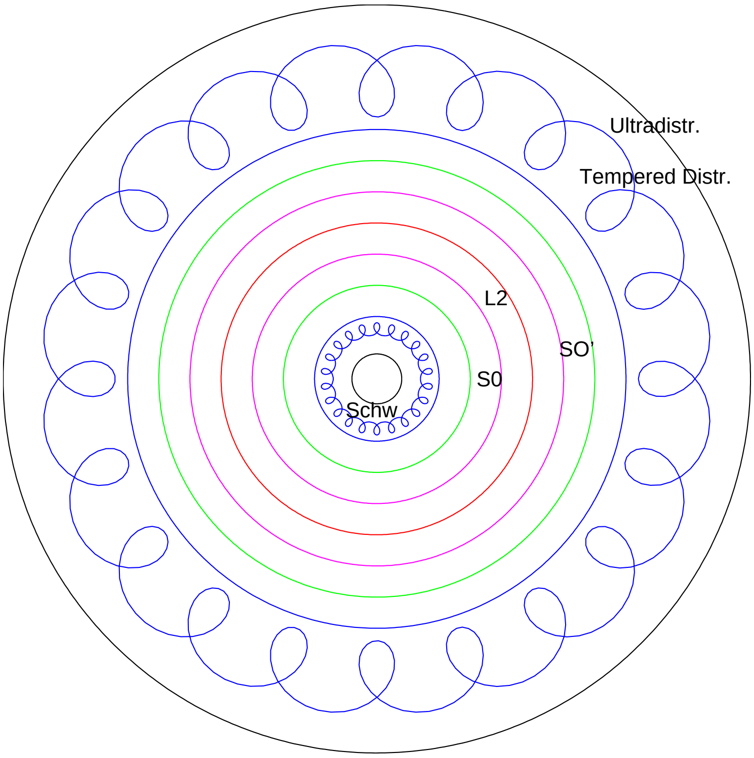

FIGURE 3. Spaces of ultradistributions, PDF: SOPLOT

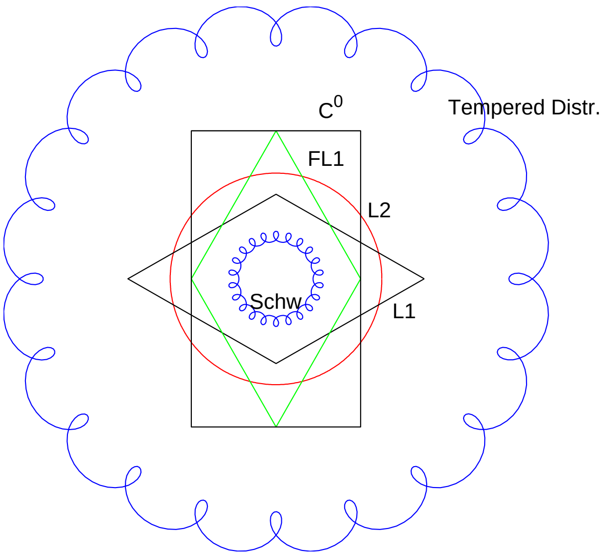

FIGURE 4. The usual space for the Fourier transform, PDF: SOPLOT2

Definition 39. The *Short Time Fourier Transform (STFT)* of a function $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ with respect to a window[63] $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is defined for $(x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ via

$$\begin{aligned} V_g f(x,\omega) &= \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}dt \\ &= \int_{\mathbb{R}^d} f(t)\overline{M_\omega T_x g(t)}dt \;=\; \langle f, M_\omega T_x g\rangle. \end{aligned}$$

In other words: the STFT describes the similarity (correlation) between the given signal (which is to be analyzed) with all the possible time-frequency shifted copies of the template signal g. There are many good reasons (mostly practical ones) to think of g as a symmetric and smooth, real valued function which is well concentrated around the origin.

Theorem 22. *(Orthogonality relations of the STFT)*
Let $f_1, f_2, g_1, g_2 \in \boldsymbol{L}^2(\mathbb{R}^d)$, then

th-rel-stft

$$\langle V_{g_1} f_1, V_{g_2} f_2\rangle = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}. \tag{127}$$

In particular $V_{g_k} f_k \in \boldsymbol{L}^2(\mathbb{R}^{2d})$ for $k = 1, 2$.

Proof. See [39], p. 42. ☐

The orthogonality relations immediately lead to the following corollary.

Corollary 6. *(Moyals Formula) Let $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$, then*

eq:moyals

$$\|V_g f\|_{\boldsymbol{L}^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} = \|g\|_{\boldsymbol{L}^2(\mathbb{R}^d)}\|f\|_{\boldsymbol{L}^2(\mathbb{R}^d)} \tag{128}$$

Remark 41. If in particular $\|g\|_2 = 1$ (for example if g is the normed Gaussian), then (128) says that the STFT is an isometry, $V_g : \boldsymbol{L}^2(\mathbb{R}^d) \to \boldsymbol{L}^2(\mathbb{R}^{2d})$, and thus injective, i.e each $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is uniquely determined by its STFT. Next we will find a way to reconstruct f from its STFT.

With the previous results we have the right tools at hand to formulate and proof the inversion formula of the STFT.

Theorem 23. *(The Inversion formula of the STFT)*
Let $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ with $\|g\|_{\boldsymbol{L}^2} = 1$, then for $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ we have the following identity (to be understood in the weak sense):

eq:inv-stft

$$f = \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} V_g f(x,\omega) M_\omega T_x g \, d\omega dx \tag{129}$$

Proof. Since $\|g\|_2 = 1$, (127) implies that

$$\langle V_g f_1, V_g f_2\rangle = \langle f_1, f_2\rangle \qquad \forall f_1, f_2 \in \boldsymbol{L}^2(\mathbb{R}^d),$$

which leads to

:Vg*Vgf1-f2

$$\langle V_g^* V_g f_1, f_2\rangle = \langle f_1, f_2\rangle, \tag{130}$$

which in turn implies $V_g^* V_g = \mathrm{id}$, where V_g^* denotes the adjoint operator of V_g. What we have to show now is that[64]

*F-inv-stft

$$V_g^* F = \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} F(x,\omega) M_\omega T_x g \, dx d\omega, \quad \text{for} \quad F \in \boldsymbol{L}^2(\mathbb{R}^{2d}). \tag{131}$$

[63]The word *"window"* is used with the understanding that the multiplication of the signal f to be analyzed by the typically compactly supported function g localizes the function. Often g is assumed to be symmetric and real-valued, centered at zero, maybe a plateau-function, or a B-spline. For this reason the STFT is also often called the *sliding window Fourier transform*.

[64]Recall that V_g^* is the adjoint of the operator $V_g : \boldsymbol{L}^2(\mathbb{R}^d) \to \boldsymbol{L}^2(\mathbb{R}^{2d})$.

This is done by the following computations:

$$\begin{aligned}
\langle V_g f, F\rangle &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_g f(x,\omega)\overline{F(x,\omega)}\,dxd\omega \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(t)\overline{T_x g(t)}e^{-2\pi i t\omega}dt\right)\overline{F(x,\omega)}\,dxd\omega \\
&= \int_{\mathbb{R}^d} f(t)\overline{\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x,\omega)M_\omega T_x g(t)\,dxd\omega\right)}dt \\
&= \langle f, V_g^* F\rangle.
\end{aligned}$$

Hence, by setting $F = V_g f$ and with the help of eq:Vg*Vgf1-f2 (130) the proof is complete. $\square$

Remark 42. We can omit the assumption that $\|g\|_{\boldsymbol{L}^2} = 1$ in the previous theorem. The inversion formula then reads

$$f = \frac{1}{\|g\|_2^2}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} V_g f(x,\omega)M_\omega T_x g\,dxd\omega.$$

About the actual convergence of the integral (describing the adjoint linear mapping, applied to F) one has to say: it is only defined in the "weak sense" (i.e. it is understood in a kind of symbolic sense in general). For $F \in \boldsymbol{L}^1(\mathbb{R}^{2d})$, specifically for $F = V_g(f)$ for some $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ the convergence in the spirit of a Riemannian sum is taking place in the $\boldsymbol{S}_0$-sense, hence also uniformly, in $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$ as well as pointwise (cf. [71]).

$$f(t) = \frac{1}{\|g\|_2^2}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} V_g f(x,\omega)M_\omega T_x g(t)\,dxd\omega,$$

where in fact it would be enough to use an absolutely convergent (improper) Riemannian integral with respect to $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$. For *good windows* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ one even has norm convergence (but to my knowledge not necessarily pointwise convergence) for $f \in \boldsymbol{L}^2(\mathbb{R}^d)$.

The next object to be introduced here is the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, consisting of all bounded linear functionals on the Banach space $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$.

12. The distribution space $S_0'(R^d)$

In the same way as we extend, as needed, the field $\mathbb{Q}$ of rational numbers to the field of real numbers (in order to gain *completeness* in the metric sense, because within $\mathbb{R}$ every *Cauchy sequence* is *convergent* in $\mathbb{R}$) and afterwards embed $\mathbb{R}$ into the even larger field of complex numbers $\mathbb{C}$, in order to gain more freedom in computing object (e.g. write every polynomial in a unique way as a product of linear factors!) we are going to *enlarge the vector space objects* which we are able to handle from the collection of ordinary functions (such as $\boldsymbol{C}_c(G), \boldsymbol{L}^2(G), \boldsymbol{L}^1(G)$, in fact up to equivalence, identifying two measurable functions if they are equal almost everywhere) to a space of "generalized functions", also called distributions[65].

There are certainly different choices (how large the space of generalized functions should be) and how general, perhaps complicated the corresponding theory of generalized functions should be chosen. From a didactical point of view one will certainly choose the setting in accordance with the background of the readers or students attending such a course, but also what the purpose is.

For example, it is save to say that in the context of PDE (partial differential equations, where one needs the concept of a "generalized or weak solution") the space of tempered distributions $\boldsymbol{S}'(\mathbb{R}^d)$ is the right setting, which is the dual of L. Schwartz's space $\boldsymbol{S}(\mathbb{R}^d)$ of rapidly decreasing functions. As we will see, the dual space of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is well suited for pratically all the questions arising in standard engineering applications as well as for the treatmen of questions of *abstract* (and as the author tries to promote) *conceptual harmonic analysis* (see [28]).

Of course one could go on and use even larger spaces, such as the space of *ultra-distributions* introduced by Björck ([6]).

There are different ways to generate such larger objects. The more intuitive one (comparable with the idea of *completion of metric spaces* which can be used to obtain, i.e. somehow *construct* $\mathbb{R}$ from $\mathbb{Q}$ by adding *limits of non-convergent Cauchy-sequences* is unfortunately the more cumbersame, and despite its simple "start" requiring rathe elaborate and lengthly (see the books of Lighthill ([50]) or Jones ([44] for such an approach). Obviously this approach has not followed many followers, because it brings some restrictions and even the question whether the objects obtained in this way are the same as the ones described in other books and papers concerning distributions has to be addressed if one tries to make use of this theory.

So in effect the established way to create various spaces of *generalized functions* is to define them as dual spaces, i.e. as a space of linear functionals on some topological vector space of test functions. In other words, one starts by describing a vector space of "nice" functions (typically well concentrated, maybe with some smoothness), with ordinary pointwise addition and scalar multiplication, but also a natural notion of convergence (i.e. a topology, compatible with the vector space structure, maybe even metrizable, turning the vector space into a complete metric vector spaces, as in the case of the Schwartz space $\boldsymbol{S}(\mathbb{R}^d)$). At the beginning of these notes we have chosen to work with $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ (even a Banach space), and in some sense $\boldsymbol{M}_b(\mathbb{R}^d)$ is the first space of "generalized functions" that we have met so far.

Another important point in this context it the ability to consider ordinary functions as generalized functions, i.e. the possibility of creating (a natural) identification of "ordinary" functions with generalized functions. Such a situation is well known from probability: Whenever we have a continous real-valued random variable (e.g. the normal distribution) with a density $g(t)$ we

[65]This wording should not be confused with the same vocabulary arising in probability theory. A probability measure is certainly describing the distribution of values of a random variable, and is at the same time a bounded measure, hence a distribution (or order zero), but this duplication of terminology is more or less pure coincidence and should not confuse the reader!

identify this function with the corresponding *probability distribution* or probability measure (as we prefer to call it here), with

$$\mu_g(A) = \int_A g(t)dt = \int_{\mathbb{R}} \mathbf{1}_A(t) \cdot g(t)dt,$$

or viewed as a functional on $\boldsymbol{C}_0(\mathbb{R})$:

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$$\mu_g(f) = \int_{\mathbb{R}} f(t) \cdot g(t)dt, \tag{132}$$

which is the same as **integration against** $g(t)$.

So in the same way as $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ is *embedded* (in fact even isomerically) into $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ in this way, we want to think of a natural embedding (in fact the same) as a space of test functions always being embedded into its dual space.

There are other considerations which may influence the choice of spaces: Whenever the space of test functions is sufficiently rich (one has enough compactly supported functions) one can define the support of a distribution ($\mathrm{supp}(\sigma)$), which is of course compatible with the usual operations (e.g. shift or dilation). So, even if it is *typically NOT possible* to talk about the function values of a distribution, there it is still possible to talk about the subset of the domain on which it is defined where its "action takes place" (namely the complement of the open domain on which the linear functional acts in a *trivial way*, i.e. just like the zero functional).

Thus the question is: How large should the space of test functions be chosen and what kind of topology should one take. It is plausible that a small space will go along with a fine topology, hence the collection of convergent sequences will be reduced more and more, the smaller the space of test functions is and the finer the topology is. This goes hand in hand, especially for Banach spaces of test functions one can show that the natural embedding of a small space of test functions is always a bounded linear mapping (at least of convergence implies local $\boldsymbol{L}^1$-convergence, by the closed graph theorem). We also find it preferable to have a simple (single) norm and thus a Banach space of test functions (namely $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$) and not just a topology arising from a countable collection of seminorms, simply because it makes it more difficult and cumbersome to describe convergence.

Thus it seems that one should take the space of test functions rather small, but there are limitations to this. For example, it is a bad idea to consider spaces of test functions consisting of (restrictions from $\mathbb{C}$ to $\mathbb{R}$) of analytic functions, because none of them will have compact support.

Also one of the evident advantages of $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ (but also of $\boldsymbol{S}_0(\mathbb{R}^d)$ or $\boldsymbol{S}_0(\mathscr{G})$) is its natural behaviour under the Fourier transform. With the definition of $\boldsymbol{S}_0(\mathscr{G})$ over LCA groups one can show that $\mathcal{F}(\boldsymbol{S}_0(\mathscr{G}) = \boldsymbol{S}_0(\widehat{\mathscr{G}})$.

Depending on the approach taken (either via a kind of completion process or by duality methods) the definitions of extended operations look a bit different, but one can certainly argue, that a careful realization of any such process leeds to the same concept (e.g. the generalization of translation, dilation, convolution, the Fourier transform, etc.), i.e. the the *only natural* extension one should consider. In fact, it will be always a kind of w^*-continuous extension of procedures known to work well on the space of test functions.

There are different ways of doing this. If one thinks of the "large" space of "generalized objects" as the set of abstract elements giving limits to (kind of) Cauchy-sequenes then one has to show that the linear operators in mind are compatible with this procedure of completion. Given such a sequence (or in fact a net) representing such an abstract element one has to show that after applying the operator (defined on the space of test functions) one has again a well defined sequence of test function, which is showing the same Cauchy-type behaviour, thus defininig another element (the target for the extended operator). Of course one has to verify that different representative sequences (or nets) in the domain of the operator result in equivalent sequences in the target domain, i.e. that the mapping is well defined (and preserves equivalence classes, making up the elements of the abstract completion). While such an approach appears to be very natural and elementary turns out to be rather cumbersome in practice and overall the burden of book-keeping outweights the perceived simplicity of the approach at first sight.

For this reason the much more elegant (but at the beginning more abstract looking) approach *using dual pairs of Banach spaces* together with the w^*-convergence of sequences (and/or nets) is preferred in most cases. We also will follow this method (but will try a bit later to indicate that the two approaches are in fact equivalent and lead to the same extensions).

In fact, a combination of such approaches is perhaps most intuitive: Given the bounded linear operator T (say the Fourier transform) on $\boldsymbol{S}_0(\mathbb{R}^d)$ we can extend it by duality to all of $\boldsymbol{S}_0'(\mathbb{R}^d)$ (with the usual formula xxx), and then check easily that this extended mapping is w^*-w^*--continuous. Hence the action on a given distribution $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ can be realized by saying: Whatever regularizing net/sequence of operators is applied to σ (with $\sigma = w^*\text{-}\lim_\alpha A_\alpha(\sigma)$), then we will have that

$$T(\sigma) = w^*\text{-}\lim_\alpha T(A_\alpha(\sigma)).$$

Of course the choice of this regularizing net of operators may depend on the given σ, i.e. the observed deficiencies of σ (with respect to the property of belong to the space of test function).

In our setting one can say, whenever it comes to the inversion of the Fourier transform, defined on $\boldsymbol{L}^1(\mathbb{R}^d)$: The range contains only functions within $\boldsymbol{C}_0(\mathbb{R}^d)$ (according to the Riemann Lebesgue Lemma), and in fact the local structure of $\mathcal{F}\boldsymbol{L}^1$ and $\boldsymbol{S}_0$ coincide, and $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is in the pointwise multiplier space of $\boldsymbol{S}_0(\mathbb{R}^d)$, *but not every function* from $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is also in $\boldsymbol{L}^1(\mathbb{R}^d)$, leave alone within $\boldsymbol{S}_0(\mathbb{R}^d)$. Hence one has to just pull functions from $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$ which is easily done by means of summability kernels, all of which belong to $\boldsymbol{S}_0(\mathbb{R}^d)$ (cf. papers by F. Weisz) and which converge to the constant 1, uniformly over compact sets, which is enough to guarantee w^*-convergence (within the dual pairing $(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{S}_0(\mathbb{R}^d))$).

Exercise: Given a bounded sequence (h_n) in $\big(\boldsymbol{C}_b(G), \|\cdot\|_\infty\big)$. If $h_n(x) \to h_0(x)$ uniformly over compact subsets then $h_0 = w^*\text{-}\lim h_n$ in $\boldsymbol{S}_0'(\mathbb{R}^d)$ (viewed as distributions). In fact a partial converse is true. A bounded sequence $(\sigma_n)_{n=1}^\infty$ is w^*-convergent in $\boldsymbol{S}_0'(\mathbb{R}^d)$ if and only if every convolution with some fixed $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ results in a net $(\sigma_n * g)_{n=1}^\infty$ which has this property: It converges uniformly over compact sets to the limit $\sigma_0 * g \in \boldsymbol{C}_b(\mathbb{R}^d)$.

For functions $f \in \boldsymbol{L}^1(\mathbb{R}^d)$, in fact for bounded measures $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$ one has already good enough decay, so that only smoothing is required. Given any approximate unit (bounded with respect to the $\boldsymbol{L}^1(\mathbb{R}^d)$-norm) inside of $\boldsymbol{S}_0(\mathbb{R}^d)$ (e.g. the net $(\mathrm{St}_\rho g_0)_{\rho \to 0}$, with $\int_{\mathbb{R}^d} g_0(t)dt = 1$) one has a w^*-approximation of μ (or in fact an $\boldsymbol{L}^1$-approximation of f in case $\mu = \mu_f$, i.e. for absolutely continuous measures on the group). For $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ or even $h = \widehat{\mu}$ for some $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ one has good enough local properties (in fact, they are within the pointwise multiplier algebra of $\boldsymbol{S}_0(\mathbb{R}^d)$) so that any *pointwise approximate unit* in $\boldsymbol{S}_0(\mathbb{R}^d)$ will serve as *regularizing operator*. The classical cases where such a situation arises is the Fourier inversion problem. Due to the problem that in many cases (and certainly in the case of Fourier Stieltjes transforms)

Note that one has - for almost trivial reasons, bases on the symmetry of the concept of the Banach Gelfand triple - with respect to the Fourier transform: A family is a pointwise regularizer (in the sense of summability kernels) if and only if its Fourier transformed version is a Dirac-like smoothing operator. One just has to recall that in the current context the approximate units act via pointwise convolution or via convolution respectively, and that the Fourier transforms intertwines these two operators.

In fact, one has a similar property property with respect to the Shah-distribution (or Dirac comb). Pointwise multiplication of a bounded function h gives a translation bounded measure $\sum_{\lambda\in\Lambda} h(\lambda)\delta_\lambda$, which obviously belongs to $\boldsymbol{S}_0'(\mathbb{R}^d)$, which has a Fourier transform. Since this is a discrete measuere supported by the lattice Λ it is clear that its Fourier transform is a distribution in $\boldsymbol{S}_0'(\mathbb{R}^d)$ which is periodic with respect to the dual (or orthogonal) lattice $\Lambda^\perp$.

On the other hand, viewed as a distribution $h \in \boldsymbol{C}_b(\mathbb{R}^d) \subset \boldsymbol{S}_0'(\mathbb{R}^d)$ has a Fourier transform $\hat{h}$, and one may expect that this Fourier transform just has to be periodized with respect to $\Lambda^\perp$, i.e. it is supposed to be of the form $\sum_{\Lambda^\perp} T_{\lambda^\perp}\hat{h}$. But in which sense is this periodization possible. Since both factors in this in this convolution product, namely $\sqcup\!\sqcup_{\Lambda^\perp}$ and $\hat{h}$ may be "real distributions" their convolution product may not be defined in the usual way, and one has to verify in which sense the convolution product can be taken. And again: If $\hat{h}$ was in $\boldsymbol{L}^1(\mathbb{R}^d)$ (or even more general a bounded measure), then one could view itself as an absolutely convergent sum of its pieces $\mu_l amp$, by just using a regular BUPU over $\Lambda^\perp$ in order to cut the measure into pieces, and use the fact that $\sum_{\lambda^\perp\in\Lambda^\perp} \psi_{\lambda^\perp}\mu$ is abolutely convergent with

$$\|\mu\| = \sum_{\lambda^\perp\in\Lambda^\perp} \|\psi_{\lambda^\perp}\mu\|.$$

(cf. formula XXX)

But in fact, $\hat{h}$ might be a measurable function which is compactly supported but *not locally integrable* (at one single point) and in this case the periodization would make perfect sense, using the pointwise addition.

On the other hand it may be quite possible (details have to be worked out, June 2015) to find an example where the function h is made in such a way that despite the fact that the pseudo-measure is defined by a pointwise function (in this case with large support, so that pointwise infinite sums have to be considered) one has a problem in defining the periodization in the usual pointwise sense.

NOTE: Finally these remarks have become far to long and to "exhaustive" (for the poor readers).

Material for course October 2006, Double modules

Given a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ over a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ with bounded approximate units we define the *essential part* $\boldsymbol{B_A}$ and the *relative completion* of $\boldsymbol{B}$ with respect to $\boldsymbol{A}$, which is given as $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{B})$. Note that this is again in a natural way a Banach module (with respect to the operator norm) over the original Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, and that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ can be mapped into this Banach module in a natural way as a closed subspace (at least of $\boldsymbol{B} = \boldsymbol{B_A}$ and if $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ has bd. approx. units).

Thus we have altogether the following chain of embeddings of closed subspaces:

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$$\boldsymbol{B_A} \subset \boldsymbol{B} \subset \boldsymbol{B_A}. \tag{133}$$

Just as one non-trivial example we have for $\boldsymbol{B} = \big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ with the Banach algebra $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$ acting via convolution:

essrelemb1

$$\boldsymbol{C}_{ub}(\mathbb{R}^d) \subset \boldsymbol{C}_b \subset \boldsymbol{L}^\infty. \tag{134}$$

Proof. Since $\big(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\big)$ is a dual space (namely the dual space to $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$) the relative completion of $\big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ cannot be larger than $\boldsymbol{L}^\infty(\mathbb{R}^d)$.
(I guess one has to give a more detailed argument here! hgfei Dec. 2013) $\square$

ARGUMENT: One has to identify each element $b \in \boldsymbol{B}$ with the operator $T_b \in \mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B})$ obtained by something like the "right regular representation", i.e. the operator $T_b : a \mapsto a \bullet b$. It is always true that $\|T_b\|_{op} \leq \|b\|_{\boldsymbol{B}}$, and by applying T_b to the elements of some bounded approximate unit in $\boldsymbol{A}$ one finds the converse estimate, i.e. $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ can be identified with a closed subspace of all bounded linear operators from $\boldsymbol{A}$ to $\boldsymbol{B}$.

Note that one should not forget that one has to impose the "natural" $\boldsymbol{A}$-module structure on $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}_1, \boldsymbol{B}_2)$, before making the claim that the $\boldsymbol{A}$-module $\boldsymbol{B}$ can be embedded via an $\boldsymbol{A}$-module homomorphism (embedding) into the larger $\boldsymbol{A}$-module $\boldsymbol{B}^{\boldsymbol{A}}$.

Exercise: For the case of the pointwise algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}}) = \big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ one finds that $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{A}) = \big(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\big)$ in a "natural way. Note that the "identity operator always belongs to $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A}, \boldsymbol{A})$ (obviously it commutes with any other operator), and therefore the "enlargement" from $\boldsymbol{A}$ to $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{A})$ also implies the adjunction of a unit element to the Banach algebra $\boldsymbol{A}$, but typically much more than this. So in a way the (in this case isometric) embedding of $\boldsymbol{C}_0(\mathbb{R}^d)$ into $\boldsymbol{C}_b(\mathbb{R}^d)$ (both with the sup-norm) can be seen as an embedding of $\boldsymbol{A} = \boldsymbol{C}_0(\mathbb{R}^d)$ into the maximal algebra "with the same norm" (and a unit).

Plancherel's theorem can be used (we skip those details) to identify $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^2, \boldsymbol{L}^2)$ (or equivalently the set of all bounded linear operators from $\boldsymbol{L}^2(\mathbb{R}^d)$ into $\boldsymbol{L}^2(\mathbb{R}^d)$ which commute with translation) with $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{L}^2, \boldsymbol{L}^2)$ (for $\boldsymbol{A} = \mathcal{F}\boldsymbol{L}^1$, i.e. the operators from $\boldsymbol{L}^2(\mathbb{R}^d)$ into $\boldsymbol{L}^2(\mathbb{R}^d)$ which commute with pointwise multiplications with elements from $\boldsymbol{A} = \mathcal{F}\boldsymbol{L}^1$, or equivalently, just with the multiplication with pure frequencies resp. characters on $\mathbb{R}^d$, which are the functions $x \mapsto e^{2\pi i s\cdot x}$. These are again pointwise multiplication operators, and it is not hard to find out that a pointwise multiplier h of $\boldsymbol{L}^2(\mathbb{R}^d)$ is has to be a measurable function which is essentially bounded, i.e. $h \in \boldsymbol{L}^\infty(\mathbb{R}^d)$.

Let us just sketch the basic idea behind this fact:

Lemma 59. *Assume that $\boldsymbol{B}$ is a Banach module with respect to a pointwise Banach algebra $\boldsymbol{A}$ (and assume that $\boldsymbol{A}_c(\mathbb{R}^d) = \boldsymbol{C}_c(\mathbb{R}^d) \cap \boldsymbol{A}$ is dense in $\boldsymbol{A}$), and assume that $\boldsymbol{A} \cap \boldsymbol{B}$ contains arbitrary large plateau-functions, i.e. with the property, that for each compact set $K \subseteq \mathbb{R}^d$ there exists some $q \in \boldsymbol{B} \cap \boldsymbol{A}$ such that $q(x) \equiv 1$ on K. Then the elements in $\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}, \boldsymbol{B})$ are pointwise multipliers with suitable functions h which belong locally to $\boldsymbol{A}$.*

Proof. TO BE GIVEN LATER ON! □

ptwmultalg1 **Lemma 60.** *Assume that an operator S on some function space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ satisfies the property $S(h \cdot f) = h \cdot S(f)$ for a sufficiently rich family of pointwise multipliers h on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Then $S(f) = h_0 \cdot f$ for some function h_0.*

Proof. Note that in the case $\mathscr{G} = \mathbf{Z}_N$ the claim is simply: Assume a linear mapping from $\mathbb{C}^N$ into itself, represented by a matrix, commutes with pointwise multiplication with unit vectors then it must be a diagonal matrix. In fact, such linear mappings do not increase the support of a function. In fact, assume that some coordinates of f are zero, then f does not change of it is multiplied with the sequence h taking the value one on those specific coordinates. On the other hand after pulling out of the argument one finds that also $h \cdot S(f) = S(h\dot{f}) = S(f)$ must be zero at the same coordinated, which easily implies that $N \times N$ matrix representing the linear mapping $f \mapsto S(f)$ must be a diagonal matrix, resp. S must be a multiplication operator (with $h_0 = \mathrm{diag}(S)$).

Let us now do the continuous version of the argument. For simplicity we assume that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ allows pointwise multiplications by the elements of some nice, regular Banach algebra (such as $\big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$), and that we can make use of BUPUs, i.e. we have a bounded family (of pointwise multipliers on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$) such that $\mathbf{1} = \sum_{\lambda \in \Lambda} \psi_\lambda$. Using (as usual now) we have $\psi = \psi \cdot \psi^*$, hence also $f = \sum_{\lambda\in\Lambda} \psi_\lambda \cdot \psi_\lambda^* f$, hence (assuming now that the series is norm convergent in $\boldsymbol{L}^2$ or $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$), for example, and setting $h_0 := \sum_{\lambda\in\Lambda} S(\psi_\lambda)$. Note that the sequence is pointwise well convergent, since $\psi_\lambda^* \cdot \psi_\lambda = \psi_\lambda$ implies the $\mathrm{supp}(S\psi_\lambda) \subseteq \mathrm{supp}(\psi_\lambda^*)$, but only finitely many of them overlap! On the other hand one has

ointw-mult1
$$Sf = S\left(\sum_{\lambda\in\Lambda} \psi_\lambda \cdot f\right) = \sum_{\lambda\in\Lambda} S(f \cdot \psi_\lambda) = f \cdot \sum_{\lambda\in\Lambda} S(\psi). \tag{135}$$

□

For us an important Banach algebra is $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$, endowed with convolution as multiplication (which is commutative, due to the commutativity of addition in $\mathbb{R}^d$). It does not have any units, but we have shown earlier (?) that $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$ has bounded approximate units. Typically such a family is obtained by taking any sequence f_n (or net) of functions, e.g. in $\boldsymbol{C}_c(\mathbb{R}^d)$, with $\int_{\mathbb{R}^d} f_n(x)dx = 1$ and "shrinking support". This can be obtained by choosing the shape of f_n arbitrarily, but assuming that $f_n(x) = 0$ for $|x| \geq \delta_n$ for some null-sequence $\delta_n \to 0$ for $n \to \infty$. Alternatively one compresses a given function $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ or even in $\boldsymbol{L}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} f_n(x)dx = 1$, and chooses $f_n = St_{\rho_n} f$, for some sequence $\rho_n \to 0$ for $n \to \infty$. The choice $f(x) = e^{-\pi x^2}$ is a popular choice (which also shows that it is not important for f to be compactly supported).

We will see shortly that $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1) = \mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1, \boldsymbol{L}^1)$ can be identified with the space $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ of all bounded measures (resp. with $\big(\boldsymbol{C}_0{}'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}}\big)$). This result is called "Wendel's theorem" ([49,72]). It has of course two parts: First of all one has to show that convolution operators induced by elements from $\boldsymbol{M}_b(\mathbb{R}^d)$ leave the closed subspace $\boldsymbol{L}^1(\mathbb{R}^d)$ invariant. In the second part one has to verify that any abstract (bounded and linear) operator on $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ commuting with all the translation must be such a convolution operator.

Wendel-thm1 **Theorem 24.** *The space of $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1, \boldsymbol{L}^1)$ all bounded linear operators on $\boldsymbol{L}^1(G)$ which commute with translations (or equivalently: with convolutions) is naturally and isometrically identified with $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$.*

Proof. For the first part we have to verify that $\boldsymbol{L}^1(\mathbb{R}^d)$ is a closed ideal of $\boldsymbol{M}_b(\mathbb{R}^d)$, i.e. that $\boldsymbol{M}_b(\mathbb{R}^d) * \boldsymbol{L}^1(\mathbb{R}^d) \subseteq \boldsymbol{L}^1(\mathbb{R}^d)$. One way to do that is to check that $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ is a *homogeneous Banach space*, i.e. to show that the group $\mathbb{R}^d$ acts in a continuous and isometric way on $\boldsymbol{L}^1(\mathbb{R}^d)$. This means that we have to verify the following conditions: $\|T_x f\|_1 = \|f\|_1$ for all $x \in \mathbb{R}^d$ and all $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ (hence all $f \in \boldsymbol{L}^1(\mathbb{R}^d)$) and also

ontLi-shift (136) $$lim_{x\to 0}\|T_x f - f\|_1 = 0 \quad \text{for} \quad x \to 0,$$

for any $f \in \boldsymbol{C}_c(\mathbb{R}^d)$, hence (by approximation for any $f \in \boldsymbol{L}^1(\mathbb{R}^d)$).

That $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ (viewed as a Banach algebra with respect to convolution) is acting boundedly on *any homogeneous Banach space* will be discussed separately.

Alternatively one can even describe $\boldsymbol{L}^1(\mathbb{R}^d)$ as the subset of all bounded measures which have continuous translation, in other words, one can show (see a classical paper by Plessner, 1929) that $\|T_x\mu - \mu\|_{\boldsymbol{M}_b} \to 0$ for $x \to 0$ implies that μ is an "absolutely continuous" measure, i.e. belongs to $\boldsymbol{L}^1(\mathbb{R}^d)$.

The argument for this result typically relies on a compactness argument (w^*- compactness of the unit ball in the dual Banach space $\boldsymbol{M}_b(\mathbb{R}^d) = \left(\boldsymbol{C}_0{}'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}}\right)$). One applies the given operator $T \in \mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1)$ to any *Dirac- sequence* f_n which forms a bounded approximate unit in $\boldsymbol{L}^1(\mathbb{R}^d)$. By the boundedness of (f_n) and the operator T the image sequence $T(f_n)$ is also bounded in $\boldsymbol{L}^1(\mathbb{R}^d)$, hence in the larger (dual) space $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}))$. By the w^*-compactness of bounded balls in this space we obtain a w^*-convergent subnet, with some limit $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$. It remains to show that this limit is inducing the operator, i.e. one has to verify that $T = C_\mu$. □

Corollary 7. *In the situation described in Wendel's theorem we have: Every multiplier of $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ has the property that the restriction to $\boldsymbol{L}^1 \cap \boldsymbol{C}_0(G)$ (endowed with the natural norm $\|f\|_S := \|f\|_1 + \|f\|_\infty$) is also a multiplier on that space, which in turn is dense both $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ as well as $\left(\boldsymbol{C}_0(G), \|\cdot\|_\infty\right)$. The same is true for multipliers on $\left(\boldsymbol{C}_0(G), \|\cdot\|_\infty\right)$.*

Also the *converse* is true, at least for $G = \mathbb{R}^d$: every multiplier on the Segal algebra $\boldsymbol{L}^1 \cap \boldsymbol{C}_0(G)$. A proof should be found in Larsen's book [49], Corollary 3.5.2.

Material of Nov. 9-th (! given below) is partially covered by the paper "Banach spaces of distributions having two module structures J. Funct. Anal. (1983)" ([9]). The main result of this paper is a "chemical diagram" that can be attached to each of the spaces in >>> standard situation:

Another Segal algebra is $\boldsymbol{L}^1 \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ with the natural norm $\|f\|_S := \|f\|_1 + \|\hat{f}\|_1$.

Some comments on the classical Riemann-Stieltjes integral (in German)

`http://de.wikipedia.org/wiki/Stieltjes-Integral`
`http://de.wikipedia.org/wiki/Beschr%C3%A4nkte_Variation`
`http://de.wikipedia.org/wiki/Absolut_stetig`
`http://de.wikipedia.org/wiki/Satz_von_Radon-Nikodym`
`http://de.wikipedia.org/wiki/Lebesgue-Integral`

Some of the material in this course has been already given in courses in Heidelberg (1980), Maryland (1989/90) or at the university of Vienna in the last 20 years.

The material concerning the Segal algebra $\boldsymbol{S}_0(G)$ is going back to various original publications by the author, see for example [22], where this particular Segal algebra has been introduced and where it is shown that it is the *minimal* TF-isometric homogeneous Banach space (and many other properties). The role of the dual space has been described already in [21] (both papers downloadable from the NuHAG site). The *double module view-point* is described in much detail in [9] (Banach spaces of distributions having two module structures, J. Funct. Anal.). A detailed account of notions of *compactness* (and also a clean description of tightness, etc.) is given in [23]. The first atomic characterization of modulation spaces ($\boldsymbol{S}_0(\mathbb{R}^d)$ is among them) has been given at a conference in Edmonton in 1986 (published then in [24]).

There are many places where especially the role of the Segal algebra $\boldsymbol{S}_0(G)$ for the discussion of basic questions in Gabor Analysis has been described. The very first systematic discussion as been probably given in the Chapter by Feichtinger and Zimmermann in the first Gabor book of 1998 ([34]). Another relevant paper is the one by Feichtinger and Kaiblinger ([30]) where it is shown, that (in the $\boldsymbol{S}_0(\mathbb{R}^d)$ context) the dual window is depending continuously on the lattice constants in the case of Gabor frames resp. Gabor Riesz bases.

Preview: In order read about Gabor multipliers the best source is probably the survey article in the second (blue) Gabor book, “Advances in Gabor Analysis”, by Feichtinger and Nowak ([33]).

General references are: Hans Reiter’s book on Harmonic Analysis (including a very fine and compact introduction to Integration Theory over Locally Compact Groups, but without the proof of the existence of the Haar measure) [55]. An updated version (edited by his former PhD student Ian Stegeman is [58]). The book describes (see also [56]) the concepts of Segal algebras (such as $S_0(G)$), and Beurling algebras $\boldsymbol{L}^1_w(G)$ (with respect to multiplicative weights). Both books are available in the NuHAG library.

A nice introduction into ”abstract harmonic analysis” is that of Deitmar ([17]), and of course always Katznelson (starting with classical Fourier series, but also talking about the Gelfand theory for commutative C^*-algebras) is [47]. It also contains the first “propagation” of the concept of *homogeneous Banach spaces*. A similar unifying viewpoint is taken in the book of Butzer and Nessel ([12]) and of course several of the books of Hans Triebel (see BIBTEX collection and book-list). These two mathematicians can also be seen as pioneers of interpolation theory and the so-called theory of function spaces. (Abstract) Homogeneous Banach spaces are also treated in the Lecture Notes by H. S. Shapiro ([66]), having approximation theoretic questions in mind (he makes the associativity of the action of bounded measures an axiom, obviously because he could proof it only in concrete cases).

Solid Banach spaces of function (under the name of *Banach function spaces*) appear in the work of Zaanen: [73] (or [74]), both books should be available in the NuHAG library (Oskar-Morgenstern-Platz 1, Level 5, Room 1.531)

A very good source to learn about Besov spaces is Jaak Peetre’s book entitled “New Thoughts on Besov Spaces” ([53]).

Banach modules: Rieffel's work [59] [some tex-nical problem with BIBTEX]

Generally interesting references about "mathematics and signal processing": Richard Holmes: [43]

TOPIC: **Standard Spaces**

What are standard spaces?? Banach spaces of functions or distributions which are useful for "daily use", and which have properties typically available on concrete situations. In fact most often results are presented in the literature for $\boldsymbol{L}^p$-space, although they apply to much wider classes of function spaces.

But what could be a sufficiently large family (in order to capture a large variety of concrete examples), which still allows to prove an extensive list of interesting properties. Well, we think that such space should (among others) allow sufficiently many *regularization operators*, by convolution) but also *localization* (by pointwise multiplication). Which kind of objects do we want: Banach spaces of continuous functions? Banach spaces of locally (Lebesgue-) integrable functions? Banach spaces of Radon measures, or (tempered?) distributions? Should we allow even ultra-distributions? Wishes: We would like to be able to do functional analysis, so with each spaces we would like to have the dual space in the same family (as long as it can be viewed as a Banach space of distributions, hence only if it can be completely characterized by the sum of the local actions).

With each space the Fourier transform of the space should be in the same family, etc. etc.

Formal suggestion in this direction is given by the following definition, which describes at "reasonable" generality a family of Banach spaces which is *not* restricted to Banach spaces of functions, because such a family will typically *not* be closed under duality (an exception is the family of $\boldsymbol{L}^p$-classes, but already the dual space of $\boldsymbol{C}_0(\mathbb{R}^d)$ contains discrete measures which are not represented by (integrals against measurable) functions.

ardspacdef0

Definition 40. A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *(restricted) standard space* if

(1) $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ (continuous embeddings);
(2) $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$, with $\|h \cdot f\|_{\boldsymbol{B}} \leq \|h\|_{\mathcal{F}\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}$ for $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B}$;
(3) $\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{B} \subseteq \boldsymbol{B}$ with $\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}$; for $g \in \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B}$;

Remark 43. The main idea behind this specific definition (also the reason why it is called for a while the "*restricted standard situation*" is the fact that the fact that $\boldsymbol{S}_0(\mathbb{R}^d)$ and its dual $\boldsymbol{S}_0'(\mathbb{R}^d)$, or that the whole Banach algebra $\boldsymbol{L}^1(\mathbb{R}^d)$ is acting on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ via convolution, can be seen as a matter of convenience. In this way we avoid a number of technical conditions involving weights and still have a fairly large collection of examples available. We will be able to demonstrate the roles of pointwise multiplication and convolutive action in the present context, and it will be easy for the reader to generalize the observations to more general situations.

It is clear that almost all the spaces used "normally" in Fourier analysis are such "standard spaces" (to be defined properly next). It is sufficient that a space (of locally integrable functions or Radon measures) is isometrically invariant under the time-frequency shifts $\pi(\lambda) = M_\omega T_t$ for $\lambda = (t, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and that e.g. the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is contained in $\boldsymbol{B}$ as a dense subspace, to ensure that the above conditions are satisfied. Let us formulate this claim as a lemma:

andardonRd1 **Lemma 61.** *Assume that* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *is a Banach space of locally integrable functions on* $\mathbb{R}^d$ *such that* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ *is contained in* $\boldsymbol{B}$ *as a dense subspace and that* $\|M_\omega T_t f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ *for all* $\lambda = (t,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. *Then it is a standard space.*

A typical alternative view in the context of $\mathcal{G} = \mathbb{R}^d$ is the following setting:

Definition 41 (convenient description)**.** .
A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *tempered standard space* on $\mathbb{R}^d$ if

(1) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ (continuous embeddings);
(2) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$
(3) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) * \boldsymbol{B} \subseteq \boldsymbol{B}$

Aside from the fact that one needs some functional analytic argument in order to establish the equivalence between this "convenient" and another more technical one (which is however what one needs in order to make those concepts useful). It will be convenient for this purpose to make use of polynomial (submultiplicative) weights w_s, given by $w_x : x \mapsto (1+|z|^2)^{s/2}$:

Definition 42 (technical definition)**.** A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *tempered standard space* on $\mathbb{R}^d$ if

(1) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ (continuous embeddings);
(2) There $s \geq 0$ such that $\boldsymbol{L}^1_{w_s}(\mathbb{R}^d)$ acts on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ by convolution and $\|g * f\|_{\boldsymbol{B}} \leq C_s \|g\|_{1,w_s} \|f\|_{\boldsymbol{B}}$; for $g \in \boldsymbol{L}^1_{w_s}$, $f \in \boldsymbol{B}$, for some $C_s > 0$;
(3) $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}$ and there exists some constant $\|h \cdot f\|_{\boldsymbol{B}} \leq \|\widehat{h}\|_{1,w_r} \|f\|_{\boldsymbol{B}}$;

Remark 44. The typical examples of *reduced standard spaces* arise from Banach spaces of say tempered distributions which are isometrically invariant under TF-shifts, i.e. which satisfy

$$\|\pi(t,\omega)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}} \quad \forall f \in \boldsymbol{B}.$$

and which contain $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ or $\boldsymbol{S}_0(\mathbb{R}^d)$ as a dense subspace. In fact, in such a case on can argue that the isometric invariance of the space implies that the continuity of the mapping (t,ω) into $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ resp. $\boldsymbol{S}_0(\mathbb{R}^d)$ implies that one can extend the strong continuity to all of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, in other words, one obtains a so-called *time-frequency homogeneous Banach space* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ in this case, and the mapping (t,ω) to $\pi(t,\omega)(f)$ is continuous for *every* $f \in \boldsymbol{B}$.

One can also discuss from a technical side the need of assuming that the embedding from $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ into $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ should be a continuous one with respect to the occurring natural topologies. In fact, it should be enough, for example, to verify that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ itself is continuously embedded into the space of all locally integrable functions and that for each norm convergent sequence (f_n) in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ there exists a subsequence (f_{n_k}) which is pointwise convergent almost everywhere. etc. ...

Standard spaces are also described within the TEXBLOCK system (part of the NuHAG DB system):

`http://www.univie.ac.at/nuhag-php/tex-blocks/search.php?id=19`

Starting from the observation that for any of the module actions, arising from the pointwise algebra $\boldsymbol{A} = \mathcal{F}\boldsymbol{L}^1$ and the convolutional Banach module structure over $\boldsymbol{L}^1$ one can build two types of completions and also two typos of "essential parts". We will write $\boldsymbol{B}^{\boldsymbol{A}}$ for the

$\boldsymbol{A}-$completion of $\boldsymbol{B}$, and $\boldsymbol{B}_{\boldsymbol{A}}$ for the essential part with respect to the pointwise module action. It is easy to verify that an element is in $\boldsymbol{B}_{\boldsymbol{A}}$ if and only if it can be approximated by elements with compact support, or if and only if any bounded sequence of plateau-like functions (forming a bounded approximate unit in $\mathcal{F}\boldsymbol{L}^1$) acts as approximation to the identity operator on the given element.

Analogously we define the completion and the essential part with respect to the Banach algebra $\boldsymbol{L}^1$. Since the action of this algebra usually comes from the group action (by translation), we will use the symbols $\boldsymbol{B}^{\mathcal{G}}$ and $\boldsymbol{B}_{\mathcal{G}}$.

Combining those four operations in a serial way we can come up with a large number of new spaces, derived from any of those spaces. Since the operations of completion and essential part with respect to the *same* algebra action are canceling each other (similar to the operation of taking a closure resp. the interior of a "nice set") we can concentrate in our discussion on "mixed series", such as: ${\boldsymbol{B}_{\mathcal{G}}}^{\boldsymbol{A}}{}_{\mathcal{G}}$, or even longer chains of operations of a similar kind.

The result that has been derived in [9] can be summarized in the following way: JUST the LAST OCCURRENCE of each algebra operation counts, i.e. the last occurrence of the symbol $\mathcal{G}$ and the last occurrence if $\boldsymbol{A}$. So we have ${\boldsymbol{B}_{\mathcal{G}}}^{\boldsymbol{A}}{}_{\mathcal{G}} = {\boldsymbol{B}^{\boldsymbol{A}}}_{\mathcal{G}}$ or ${\boldsymbol{B}_{\mathcal{G}}}^{\boldsymbol{A}}{}_{\mathcal{G}\boldsymbol{A}} = \boldsymbol{B}_{\mathcal{G}\boldsymbol{A}}$.

The most important spaces in this family are the minimal space, which turns out to be the "double essential part", where the order of algebra operations does *not* play a role anymore. It coincides with the closure of the test functions in the standard spaces. So we have

doubesspart
$$\boldsymbol{B}_{\boldsymbol{A}\mathcal{G}} = \overline{\boldsymbol{S}_0}^{\boldsymbol{B}} = \boldsymbol{B}_{\mathcal{G}\boldsymbol{A}} \tag{137}$$

On the other hand here is a (single) double completion, which coincides with the w^*-relative completion of $\boldsymbol{B}$ within $\boldsymbol{S}_0'$ (details to be presented at another time).

doublcompl
$$\boldsymbol{B}^{\boldsymbol{A}\mathcal{G}} = \widetilde{\boldsymbol{B}} = \boldsymbol{B}^{\mathcal{G}\boldsymbol{A}} \tag{138}$$

where we define the vague or w^*-relative completion of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ in $\boldsymbol{S}_0'$ as follows:

rel-complet
$$\widetilde{\boldsymbol{B}} = \{\sigma \in \boldsymbol{S}_0' \,|\, \sigma = w^* - lim_\alpha f_\alpha, \sup_\alpha \|f_\alpha\|_{\boldsymbol{B}} < \infty\} \tag{139}$$

The infimum over all the bounds $\sup_\alpha \|f_\alpha\|_{\boldsymbol{B}}$ makes $\widetilde{\boldsymbol{B}}$ into a standard space, which contains $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ as a closed subspace.

Example:
Starting from $\boldsymbol{C}_0(\mathbb{R}^d)$ one can find that is relative completion is just $\big(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\big)$.

A WIKIPEDIA contribution:

http://en.wikipedia.org/wiki/Harmonic_analysis
http://en.wikipedia.org/wiki/Tempered_distribution
http://en.wikipedia.org/wiki/Fourier_analysis
http://en.wikipedia.org/wiki/Discrete-time_Fourier_transform
http://en.wikipedia.org/wiki/Colombeau_algebra

Next we will show that the convolution action of bounded discrete measures on a *homogenous Banach space* can be extended to all of the measures in order to generate an action of $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ on such a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.

meas-homBR0 **Theorem 25** (Bounded Measures act on Homogeneous Banach Spaces)**.** .

LATER on we will relabel the symbols and write $\mu \bullet_\pi f$ instead of $\mu \bullet_\rho f$!

Assume that $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a homogeneous Banach-space with respect to some "abstract" group action ρ, i.e. we assume that $x \to \rho(x)f$ is continuous from $\mathscr{G}$ into $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, isometric in the sense that $\|\rho(x)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ for all $x \in \mathscr{G}$ and $f \in \boldsymbol{B}$, and that $\rho(x_1x_2) = \rho(x_1)\rho(x_2)$. Of course we can define $\mu \bullet_\rho f$ for any discrete measure, hence for the family $\mathrm{D}_\Psi \mu$ as the (unique) limit of this Cauchy net in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. Then one has

meas-onB1
$$\|\mu \bullet_\rho f\|_{\boldsymbol{B}} \leq \|\mu\|_{\boldsymbol{M}} \|f\|_{\boldsymbol{B}}, \quad \textit{for all } f \in \boldsymbol{B}. \tag{140}$$

In fact, $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ becomes a Banach module over $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ in this way.

Proof. We are going first to define the action of $\mu \in \boldsymbol{M}_b(\mathscr{G})$ on an individual element $f \in \boldsymbol{B}$, by verifying that the net

$$\mathrm{D}_\Psi(\mu) \bullet_\rho f := \sum_{i \in I} \mu(\psi_i)\rho(x_i)f$$

is convergent, as $|\Psi| \to 0$.

The idea is to consider the action of $D_\Psi \mu$ on f as a Riemann-type sum for the Banach-space valued integral of $x \to \rho(x)f$, usually written as $\int_{\mathscr{G}} \rho(x)f(x)d\mu(x)$. Therefore it is natural to check that the action of bounded discrete measures is OK (this is an easy consequence of the assumptions) and then to compare two such expressions, namely $D_\Psi(\mu)\bullet_\rho f$ and $D_\Phi(\mu)\bullet_\rho f$ by making use of their joint refinement, constituted by the (double indexed family) $(\psi_i\phi_j)$.

Let us first estimate the norm of $\mathrm{D}_\Psi(\mu)\bullet_\rho f$. Using the isometry of the action of ρ on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ one has, independently from Ψ:

crmeasconv1
$$\|\mathrm{D}_\Psi(\mu)\bullet_\rho f\|_{\boldsymbol{B}} \leq \sum_{i \in I} |\mu(\psi_i)| \|\rho(x_i)f\|_{\boldsymbol{B}} \leq \|f\|_{\boldsymbol{B}} \sum_{i \in I} |\mu(\psi_i)| \leq \|\mu\|_{\boldsymbol{M}} \|f\|_{\boldsymbol{B}}. \tag{141}$$

Assume next that there are two families $\Psi = (\psi_i)_{i \in I}$ and $\Phi = (\phi_j)_{j \in J}$ are given, with central points $(x_i)_{i \in I}$ and $(y_j)_{j \in}$. Then we can define the *joint refinement* $\Psi - \Phi$ as the family $(\psi_i\phi_j)_{(i,j) \in I \diamond J}$, where we can agree to call $I \diamond J$ the family of all index pairs such that $\psi_i \cdot \phi_j \neq 0$ (because all the other products are trivial and should be neglected). In fact, if both Ψ and Φ are sufficiently "fine" BUPUs one has: [66]

mann-estim1
$$\begin{aligned} \|D_\Psi \mu \bullet_\rho f - D_\Phi \mu \bullet_\rho f\|_{\boldsymbol{B}} = \sum_{(i,j) \in I \diamond J} \|\rho(x_i)f - \rho(y_j)f\|_{\boldsymbol{B}} |\mu(\psi_i\phi_j)| \leq \\ \sup_{(i,j) \in I \diamond J} \|\rho(x_i)[f - \rho(y_j - x_i)f]\|_{\boldsymbol{B}} \sum_{(i,j) \in I \diamond J} \|(\psi_i\phi_j)\mu\|_{\boldsymbol{M}} \leq \varepsilon \|\mu\|_{\boldsymbol{M}}, \end{aligned} \tag{142}$$

if only Ψ resp. Φ are fine enough. Due to the completeness of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ one finds that there is a uniquely determined limit, which we will call $\mu \bullet_\rho f$. It is then obvious that

astf-estim1
$$\|\mu \bullet_\rho f\|_{\boldsymbol{B}} = \lim_{|\Psi| \to 0} \|D_\Psi \mu \bullet_\rho f\|_{\boldsymbol{B}} \leq \limsup \|D_\Psi \mu\|_{\boldsymbol{M}} \|f\|_{\boldsymbol{B}} = \|\mu\|_{\boldsymbol{M}} \|f\|_{\boldsymbol{B}}. \tag{143}$$

[66] Using that $\psi_i = \sum_{j \in j} \psi_i\phi_j$, hence $\sum_{(i,j) \in I \diamond J} \psi_i\phi_j \equiv 1$ and $\sum_{(i,j) \in I \diamond J} \|(\psi_i\phi_j)\mu\|_{\boldsymbol{M}} = \|\mu\|_{\boldsymbol{M}}$.

Of course it remains to show that the action defined in this way is associative, i.e. that

assocat
$$(\mu * \mu_2)\bullet_\rho f = (\mu_1)\bullet_\rho(\mu_2\bullet_\rho f), \quad \forall \mu_1, \mu_2 \in \boldsymbol{M}_b(\mathbb{R}^d), f \in \boldsymbol{B}, \tag{144}$$

but his is clear because the associativity is valid for the discrete measures $D_\Psi \mu$ and $D_\Phi \mu$. [67] $\square$

Remark 45. In the derivation above we have used the isometric property and the fact that $\rho(x_1 x_2) = \rho(x_1) \circ \rho(x_2)$. It would have been no problem if this identity was only true "up to some constant of absolute value one", i.e. if one has a *projective representation* of $\mathscr{G}$ only, such as the mapping $\lambda = (t, \omega) \mapsto \pi(\lambda) = M_\omega T_t$ from $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ into the unitary operators on the Hilbert space $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$, which is one of the key players in *time-frequency analysis.*

For the next step we need a simple observation from abstract Hilbert space theory.

Lemma 62. *Assume that a (complex) linear mapping between two Hilbert space over the complex numbers, $\mathscr{H}_1 \to \mathscr{H}_2$ is isometric, i.e. satisfies*

isom-embed1
$$\|T(h)\|_{\mathscr{H}_2} = \|h\|_{\mathscr{H}_1} \quad \forall h_1 \in \mathscr{H}_1. \tag{145}$$

Then the adjoint mapping $T' : \mathscr{H}_2 \to \mathscr{H}_1$ is the inverse on the range, i.e. one has $T'(Tf) = f \quad \forall h_1 \in \mathscr{H}_1$.

Proof. The claim follows from the fact that an isometric embedding also preserves scalar products, as a consequence of the polarization identity

larization
$$\langle f, g\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle f + {}^k g, f + i^k g\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|f + i^k g\|^2. \tag{146}$$

Hence

isom-adjinv
$$\langle T'(Tf), g\rangle = \langle Tf, Tg\rangle = \langle f, g\rangle \quad \forall f, g \in \mathscr{H}_1. \tag{147}$$

Since this is true for every $g \in \mathscr{H}_1$ the required claim is valid. Usually one says that $T'(h_2)$ is defined in the weak sense for $h_2 \in \mathscr{H}_2$, through the identity

weak-Tadj
$$\langle T' h_2, h_1\rangle_{\mathscr{H}_1} = \langle h_2, T(h_1)\rangle_{\mathscr{H}_2}, \quad h_1 \in \mathscr{H}_1, h_2 \in \mathscr{H}_2. \tag{148}$$

$\square$

Application: $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ is defined via its STFT: $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ if and only if $V_{g_0} f \in \boldsymbol{L}^1(\mathbb{R}^{2d})$, where g_0 is the Gauss-function (or any other nonzero Schwartz-function). Since $f \mapsto V_g f$ is isometric (assuming that $\|g_0\|_2 = 1$) we have according to the above lemma the (weak) reconstruction formula

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_{g_0} f(\lambda) \pi(\lambda) g_0 \, d\lambda,$$

but if $V_{g_0} f \in \boldsymbol{L}^1(\mathbb{R}^{2d}) \subset \boldsymbol{M}(\mathbb{R}^{2d})$ then we have

$$f = V_{g_0} f \bullet_\pi g_0$$

[67] Note that H.S.Shapiro (cf. [66]) is making this associativity an extra axiom, apparently because he could not proof it directly for technical reasons, because he defines the action of the bounded measures on an "abstract homogeneous Banach space". H.C. Wang exhibits in [70] an example of what he calls a *semi-homogeneous Banach space* (without strong continuity of the action of $\mathscr{G}$ on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which does not allow the extension to all of the bounded measures. Indeed, it is a Banach space of measurable and bounded functions on $\mathbb{R}$ which is non-trivial, but which does not contain any non-zero *continuous* function! The example was suggested to him in a correspondence by the author of these notes.

in the spirit of the above abstract statement (for $\rho = \pi$). It follows that one has for every TF-homogeneous Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. for every Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ such that $\|\pi(\lambda)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}$ and $\|\pi(\lambda)f - f\|_{\boldsymbol{B}} \to 0$ for $\lambda \to 0$:

so-minimal1
$$\|f\|_{\boldsymbol{B}} = \|V_{g_0}f \bullet_\pi g_0\|_{\boldsymbol{B}} \le \|V_{g_0}f\|_{\boldsymbol{L}^1(\mathbb{R}^{2d})}\|g_0\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{S}_0}\|g_0\|_{\boldsymbol{B}}. \tag{149}$$

FROM NOW ON WE MAY USE $\bullet_\pi$ INSTEAD OF $\bullet_\rho$. ADJUSTMENT WILL BE REALIZED LATER ON!

Proof. The proof relies on the fact that for any net of convergent nets Ψ_β (all sufficiently "fine") the overall family $(D_{\Psi_\beta}\mu_\alpha)_{\alpha,\beta}$ is bounded and uniformly tight! [68] Moreover it is clear that for each fixed β the net $D_{\Psi_\beta}\mu_\alpha$ is w^*-convergent to $D_{\Psi_\beta}\mu_0$. Given the tightness of the family only a finite number of indices of the family $(\psi_i^\beta)_{i\in I}$ is relevant for the convergence, hence $\mu_\alpha(\psi_i^\beta) \to \mu_0(\psi_i^\beta)$ implies that

$$D_{\Psi_\beta}\mu_\alpha \bullet_\pi f \to D_{\Psi_\beta}\mu_0 \bullet_\pi f.$$

FURTHER DETAILS HAVE TO BE CHECKED IN A CLEAN FORM LATER ON!

aussdensapp

Lemma 63. *The collection of all dilated and shifted Gaussians is dense in most of the function spaces, including* $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$, $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$, $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$, *but also in Wiener's algeba* $(\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}})$.

Proof. The proof is quite easy observing first that we have $\|\mathrm{St}_\rho g * f - f\|_p \to 0$ for every $f \in (\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ resp. $f \in \boldsymbol{C}_0(\mathbb{R}^d)$, endowed with the sup-norm $\|f\|_\infty$. But for any fixed $\rho > 0$ the convolution product itself can be approximated by expressions of the form $\mathrm{St}_\rho g * D_\Psi(f)$. In any case we can start our considerations (without loss of generality) from some $f \in \boldsymbol{C}_c(\mathbb{R}^d)$.

Hence by first choosing $\rho_0 > 0$ small enough, and subsequently (depending on the smoothness of $\mathrm{St}_\rho g$) a BUPU which is fine enough (suitable choice of $\delta_0 > 0$) such that one has altogether

approxgaus
$$\|f - \mathrm{St}_\rho g * D_\Psi f\|_\infty \le \|f - \mathrm{St}_\rho g * f\|_\infty + \|\mathrm{St}_\rho g * (f - D_\Psi f)\|_\infty. \tag{150}$$

Hence by *first* choosing $\rho > 0$ and then (for fixed $\rho > 0$ the appropriate size of Ψ, i.e. $\delta > 0$ one can make the difference arbitrarily small. □

Since the finite linear combinations of such Gaussians are dense in all the space mentioned one can use them. For example, it is quite easy to derive the validity of the Fourier inversion formula on these functions. In fact it is better to use not only shifted and dilated, but also the modulated version of the Gauss function. Since the Gauss-function $g_0(t) := e^{\pi|t|^2}$ is invariant under the Fourier transform (cf. [75]) the rules for the intertwining properties between dilation, modulation and shifting implies that the Fourier inversion is valid for all linear combinations of such functions, and in fact certain infinite sums (as long as they are convergent). Such functions make up a so-called exotic Banach space ([35] gives details).

DPsi-tight1

Lemma 64. *Let Ψ run through the family of all BUPUs of a given maximal size Q, e.g. with all the function (ψ_i) constituting Ψ (whatever Ψ is used) satisfying with supp$(\psi_i) \subseteq x_i + Q$, for any such ψ and suitably chosen points (x_i), for some fixed compact set Q. Then for any $\mu \in \boldsymbol{M}_b(\mathscr{G})$ on has: $(D_\Psi\mu)_\Psi$ is a tight set.*[69]

[68] It is a good exercise to check the technical details yourself!

[69] In the Euclidean case one will of course talk of BUPUs of size $R > 0$ if the set Q can be chosen to be the closed ball of radius $B_R(0)$ around zero. Hence the assumption is simply that $\mathrm{supp}(\psi_i) \subseteq B_R(x_i)$ for some $x_i \in \mathbb{R}^d$.

Proof. We can start from any fixed BUPU Φ, and recall that $\sum_j \|\phi_j \cdot \mu\|_{\boldsymbol{M}} \leq \|\mu\|_{\boldsymbol{M}}$.

We can use this to approximate μ by a partial sum, i.e. by choosing for $\varepsilon > 0$ a finite set $F \subset J$ can be found such that

$$\| \sum_{j \in F} \phi_j \cdot \mu - \mu\|_{\boldsymbol{M}} = \sum_{j \notin F} \|\phi_j \mu\|_{\boldsymbol{M}} \leq \varepsilon.$$

or equivalently, writing $\Phi_F := \sum_{j \in F} \phi_j$:

$$\|\Phi_F \cdot \mu - \mu\|_{\boldsymbol{M}} \leq \varepsilon.$$

Let now $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ by any plateau-function such that $h(x) \equiv 1$ on some neighborhood of $\operatorname{supp}(\Phi_F)$. More precisely, given the uniform size of the BUPUs Ψ to be considered, we can have $h \cdot \psi = \psi$ for all indices $i \in I$ such that $p \cdot \psi \neq 0$.

Then for any BUPU $\Psi = (\psi_i)_{i \in I}$ of size Q $\|h(D_\Psi \mu) - D_\Psi \mu\|_{\boldsymbol{M}}$ can be controlled in a uniform way[70].

ARGUMENTS:

ss-estimat1 (151) $$|\mu(\phi)| \leq \|\phi \cdot \mu\|_{\boldsymbol{M}}, \quad \forall \phi \in \boldsymbol{C}_b(\mathbb{R}^d), \mu \in \boldsymbol{M}(\mathbb{R}^d),$$

because in case $\phi \in \boldsymbol{C}_c(\mathbb{R}^d)$ one can choose any other function $\phi^* \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|\phi^*\|_\infty = 1$ $\phi^*(y) = 1$ on $\operatorname{supp}(\phi)$ and hence

abs-estim2 (152) $$\phi^* \cdot \phi = \phi \Rightarrow |\mu(\phi)| = |\mu(\phi^* \phi)| \leq |[\phi \cdot \mu](\phi^*)| \leq \|\phi \cdot \mu\|_{\boldsymbol{M}},$$

thus completing the argument. The general case follows from this using the standard reduction to compactly supported measures[71].

Given F we can find some $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $h(x) \in [0,1]$, hence with $\|h\|_\infty = 1$, such that $h(x) \equiv 1$ on $\bigcup_{j \in F} \operatorname{supp}(\phi_j)$. (due to the limited size of the support this is a compact set which can be choosen independently from the concrete choice of Ψ). Consequently

Dpsi-estim3 (153) $$\|h \cdot D_\Psi \mu - D_\Psi \mu\|_{\boldsymbol{M}} \leq \| \sum_{\{i \,|\, \psi_i \cdot p = 0\}} \mu(\psi_i) \delta_{x_i} \|_{\boldsymbol{M}} = \sum_{\{i \,|\, \psi_i \cdot p = 0\}} |[\mu - p\mu](\psi_i)| \leq \|\mu - p\mu\|_{\boldsymbol{M}} \leq \varepsilon.$$

□

Requiring the argument that one has for any $\mu \in D : /NuHAGALL/NuHAGPDFs/$:

$$\|\mu \bullet_\rho f - D_\Psi \mu \bullet_\rho f\|_{\boldsymbol{B}} \leq \varepsilon \|\mu\|_{\boldsymbol{M}}$$

depending only on the element $f \in \boldsymbol{B}$ and the level of "refinement" of Ψ but NOT on the individual choice of μ. □

Next we want to show that there is an important form of continuity from in this action from $\boldsymbol{M}(\mathscr{G}) \times \boldsymbol{B} \to \boldsymbol{B}$, with respect to the w^*-topology on $D : /NuHAGALL/NuHAGPDFs/$.

wst-to-norm **Theorem 26.** *Assume that $(\mu_\alpha)_{\alpha \in I}$ is a bounded and tight net of bounded measures, which is w^*-convergent to some limit measure $\mu_0 \in \boldsymbol{M}(\mathbb{R}^d)$. Then for every $f \in \boldsymbol{B}$:*

st-to-norm1 (154) $$\|\mu_\alpha \bullet_\rho f - \mu_0 \bullet_\rho f\|_{\boldsymbol{B}} \to 0 \quad \text{for} \quad \alpha \to \infty.$$

---

[70] Despite the fact that $p \cdot D_\Psi \mu$ is not equal to $\mathrm{D}_\Psi(p\mu)$

[71] to carry out the details is in fact a fairly good exercise, showing how well the technical details have been understood!

Proof. One can verify this relation by observing that the family $\mathrm{D}_\Psi\mu_\alpha$ running through any net of BUPUs with $|\Psi| \leq 1$ and $\alpha \in I$ is a tight family, hence up to some $\varepsilon > 0$ one can replace the given net (μ_α) by a family of measures with joint compact support. Hence all the BUPUs will consist of finitely many discrete measures, and in particular $\|\mathrm{D}_\Psi\mu_\alpha \bullet_\rho f - \mathrm{D}_\Psi\mu_0 \bullet_\rho f\|_{\boldsymbol{B}} \to 0$ for any $f \in \boldsymbol{B}$, for fixed Ψ. ☐

Corollary 8. *For every homogeneous Banach space* $\big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$ *one has: For every* $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ *and every* $f \in \boldsymbol{B}$*:* $\mu * f$ *is the limit finite linear combinations translates of* f*. In particular one has: Given* μ *and* f *and* $\varepsilon > 0$ *there exists a finite sequence* $(x_i)_{i\in F}$ *and a finite sequence of complex coefficients* $(c_i)_{i\in F}$ *such that*

approxtrans
$$\|\mu * f - \sum_{i\in F} c_i T_{x_i} f\|_{\boldsymbol{B}} \leq \varepsilon. \tag{155}$$

Proof. We just have to remember that the discrete bounded measures $D_\Psi\mu$ resp. their partial sums form a tight family of measures, which is w^*-convergent towards μ. Since each of the approximating measures $D_\Psi\mu$ is of the form $\sum_{i\in I}\mu(\psi_i)\delta_{x_i}$ and can be approximated (even in the norm of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$) by finite sums we just have to put $c_i = \mu(\psi_i)$ and observe that

$$\left(\sum_{i\in F} c_i\delta_{x_i}\right) * f = \sum_{i\in F} c_i T_{x_i} f.$$

☐

Remark 46. A more precise way of expressing what is going on is the use of suitable indexing, allowing to express that it is enough that "sufficiently many" elements from a "sufficiently fine" BUPU Ψ have to be used. Let us choose as index set pairs of the form (K, δ), with $\delta > 0$ and K a compact subset of $\mathbb{R}^d$. They have a partial order, with $(K_1, \delta_1) \succeq (K_2, \delta_2)$ if $K_2 \supseteq K_1$ and $\delta_2 \leq \delta_1$. Then we say that $\sum_{i\in F} c_i\delta_{x_i}$ has the index (K, δ) if Ψ is a δ-BUPU, and $F \supseteq \{i \in I \,|\, x_i \in K\}$. One could also talk of a local BUPU, by assuming (slightly differently, but technically equivalent) that $\sum_{i\in F}\psi(x) \equiv 1$ on K.

In this setting one can say: For every $\mu, f, \varepsilon > 0$ there exists a pair (K_o, δ_0) such that for all local BUPUs which satisfy at least $\sum_{i\in F}\psi(x) \equiv 1$ on K and are at least δ_0-fine, one has

nvtransapp2
$$\|\mu * f - \sum_{i\in F} \mu(\psi_i) T_{x_i} f\|_{\boldsymbol{B}} < \varepsilon. \tag{156}$$

The point of the last remark is that the user may even choose the points by himself, and then suitable coefficients can be found (by making use of an BUPU adapted to the given set), cf. [26] (??).

13. Sobolev spaces, derivatives in $L^2(Rd)$

Sobolev spaces and the relation with the Fourier transform: Relate the classical concept of derivatives or multiple derivatives (as done originally by Sobolev) to the properties of their Fourier transform.

iffer-Four1 **Lemma 65.** *Assume that $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ has continuous partial derivatives. Then their Fourier transforms coincide with $\hat{f}$, multiplied with essentially the coordinate functions.*

Proof. We can restrict the attention to the case $d = 1$. Considering the fact that $f' \in \boldsymbol{C}_c(\mathbb{R}^d)$ by assumption we can guarantee that the difference quotients $\frac{1}{h}(T_h f - f)(x)$ converge *uniformly* to f', hence in the $\boldsymbol{L}^1$-norm.

Consequently their Fourier transforms are uniformly convergent to the Fourier transforms of f', using the mean-value theorem.

diff-four2
$$f(x+h) - f(x) = f'(\xi) \cdot h, \quad \text{for some} \quad \xi \in (x, x+h). \tag{157}$$

Since translation by h goes into multiplication with characters the rule follows by going to the limit

ff-fourier1
$$\frac{1}{h}(e^{2\pi i s h} - 1) \to 2\pi i s \cdot e^{2\pi i s t}, \tag{158}$$

$\square$

Definition 43. A strictly positive and (without loss of generality) continuous function w is called *submultiplicative* (resp. a *Beurling weight*) if it satisfies

Beurlingdef
$$w(x+y) \leq w(x) \cdot w(y) \quad \forall x, y \in G. \tag{159}$$

Examples: $w_s(x) = (1 + |x|)^s$ for $s \geq 0$. It is trivial (Ex!) to prove it for $s = 1$ and then by taking the $s - th$ power it is still valid! The corresponding weighted $\boldsymbol{L}^2$ on $\mathbb{R}^d$ is denoted by $\boldsymbol{L}^2_{w_s}(\mathbb{R}^d)$.

It is easy to verify that this implies a pointwise estimate (using $w(x) \leq w(x-y) \cdot w(y)$ in the convolution integral):

conv-weight
$$|(f * g) \cdot w| \leq |f|w \, * \, |g|w \tag{160}$$

In fact, we have for every fixed $x \in G$:

convwest1
$$f * g(x)w(x) \leq \int_G |g(x-y)||f(y)|w(x)dy \leq \int_G |g(x-y)|w(x-y)|f(y)|w(y)dy. \tag{161}$$

As an immediate consequence the corresponding weighted $\boldsymbol{L}^1$-space $\boldsymbol{L}^1_w$ is a Banach algebra with respect to convolution, a so-called *Beurling algebra* (cf. [58]).

Another important class are the so-called WSA weights:

WSA-wgt **Definition 44.** A strictly positive and continuous weigth w is called WSA (*weakly subadditive*) if there is some constant $C > 0$ such that for all x, y:

WSA-def
$$w(x+y) \leq C(w(x) + w(y)). \tag{162}$$

It implies in a completely similar manner another useful pointwise estimate:

conv-WSA
$$|(f * g) \cdot w| \leq C(|f|w * |g| + |f| * |g|w) \tag{163}$$

recall that $\boldsymbol{L}^2_w := \{f \mid fw \in \boldsymbol{L}^2\}$, with the natural norm $\|f\|_{2,w} := \|fw\|_2$.

Using equation conv-WSA 13 and the fact that $\boldsymbol{L}^1 * \boldsymbol{L}^2 \subseteq \boldsymbol{L}^2$ (with corresponding norm inequalites)

$$\|g * f\|_2 = \|\hat{g} \cdot \hat{f}\|_2 \leq \|\hat{g}\|_\infty \cdot \|\hat{f}\|_2.$$

Theorem 27. *Let w be a WSA weight. $\boldsymbol{L}^1 \cap \boldsymbol{L}^2_w$ is a Banach algebra with respect to convolution. In particular, $\boldsymbol{L}^2_w$ is a Banach algebra with respect to convolution if $1/w \in \boldsymbol{L}^2$.*

Proof. The $\boldsymbol{L}^1$-norm is no problem, so we only have to find for $f, g \in \boldsymbol{L}^1 \cap \boldsymbol{L}^2_w$ an estimate for
conv-WSA $\|(f * g)w\|_2$, which can be obtained from the estimate (). ☐

Argument: then $\boldsymbol{L}^2_w = \boldsymbol{L}^1 \cap \boldsymbol{L}^2_w$ with equivalence of the natural norms associated with these spaces!

Since the Fourier transform maps $\boldsymbol{L}^1$ into $\boldsymbol{C}_0$ (according to the Riemann Lebesgue Lemma) it follows that under the same conditions a Sobolev space is embedded into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$. Let us write $\mathcal{H}_s(\mathbb{R}^d)$ for $\mathcal{F}^{-1}\left(\boldsymbol{L}^2_w(\mathbb{R}^d), \|\cdot\|_{2,w}\right)$, with norm
$\|f\|_{\mathcal{H}_s} := \|\hat{f} \cdot w\|_2$.

For integer values of $s \geq 0$ one can show that this space coincides with the space of all $\boldsymbol{L}^2-$ elements which have (in the sense of distributions!) derivatives of order up to s, i.e. the (within the theory of distributions well defined objects f, f', f" etc. up to order s are regular distributions which can be represented by $\boldsymbol{L}^2$-functions, or equivalently: smoothing the distribution f one has uniform control of those derivatives in the classical sense independent of the order of regularization, e.g. by convolution with a very narrow Gauss function);

obolev-emb0 **Corollary 9.** *For $s > d/2$ the Sobolev space $\mathcal{H}_s$ is continuously embedded into $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$, hence also into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$.*

Proof. Observing that $1/w_s \in \boldsymbol{L}^2(\mathbb{R}^d)$ if (and only if) $w > d/2$ on finds that $\boldsymbol{L}^2_w(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^1(\mathbb{R}^d)$ via Cauchy-Schwartz, writing $h \in \boldsymbol{L}^2_w(\mathbb{R}^d)$ as a product of hw with $1/w$:

LtwinLi
$$\|h\|_1 = \|hw \cdot 1/w\|_1 \leq \|hw\|_2 \cdot \|1/w\|_2. \tag{164}$$

Applying the Fourier transform on both sides we obtain that $\mathcal{H}_s(\mathbb{R}^d) \hookrightarrow \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$, with the natural norm estimates. ☐

Note that the above argument shows that $\mathcal{H}_s(\mathbb{R}^d)$ is not only continuously embedded into $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$ (according to Plancherel, since $\boldsymbol{L}^2_w(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^2(\mathbb{R}^d)$), but also in $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, hence into $\boldsymbol{L}^2 \cap \boldsymbol{C}_0(\mathbb{R}^d)$ with the natural norm (sum of the two norms). Using the so-called theory of Wiener amalgam spaces on can show that $\mathcal{H}_s(\mathbb{R}^d) \hookrightarrow \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^2)(\mathbb{R}^d)$, by the argument that $\boldsymbol{L}^2_w(\mathbb{R}^d) = \boldsymbol{W}(\boldsymbol{L}^2, \boldsymbol{\ell}^2_w) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \boldsymbol{\ell}^2_w) \hookrightarrow \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \boldsymbol{\ell}^1)$ (using again Cauchy Schwartz in the last inclusion), which by a variant of the Hausdorff-Young inequality for Wiener amalgam spaces ([25]) gives

$$\mathcal{H}_s(\mathbb{R}^d) = \mathcal{F}^{-1}\,\boldsymbol{L}^2_w \hookrightarrow \mathcal{F}^{-1}\,\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \boldsymbol{\ell}^1) \hookrightarrow \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^2) \hookrightarrow \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^2)(\mathbb{R}^d),$$

which is a space strictly contained in $\boldsymbol{L}^2 \cap \boldsymbol{C}_0(\mathbb{R}^d)$.

Note that $\mathcal{H}_s(\mathbb{R}^d)$ is not only a Banach space with respect to the natural norm $\|f\|_{\mathcal{H}_s} := \|\hat{f}w\|_2$, but even a Hilbert spaces, because the norm obviously comes from the scalar product

HS-scalprod
$$\langle f, g\rangle_{\mathcal{H}_s} := \langle \hat{f}w, \hat{g}w\rangle_{\boldsymbol{L}^2} = \int_{\mathbb{R}^d} \hat{f}(s)\,\overline{\hat{g}(s)}\; w^2(s)ds. \tag{165}$$

Hs-RKHS **Corollary 10.** *$\mathcal{H}_s(\mathbb{R}^d)$ is a reproducing kernel Hilbert space, i.e. for each $t \in \mathbb{R}^d$ the Dirac measure $\delta_x : f \mapsto f(x)$ is a continuous linear functional on the Hilbert space $\mathcal{H}_s(\mathbb{R}^d)$. The kernel $K(x,y)$, consisting of functions $K(\cdot,y) = k_x(y)$ in $\mathcal{H}_s(\mathbb{R}^d)$ with*

Hs-RKHSphi

$$f(x) = \langle f, k_x \rangle_{\mathcal{H}_s} \quad \textit{for all} \quad x \in \mathbb{R}^d, f \in \mathcal{H}_s \tag{166}$$

is obtain as collections of shifts $T_x\varphi$, with $\varphi = \mathcal{F}^{-1}(1/w^2)$.

Proof. Since $\mathcal{H}_s(\mathbb{R}^d)$ is continuously embedded into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ it is clear that the family of point measures acts even *uniformly bounded* on $\mathcal{H}_s(\mathbb{R}^d)$. Hence we only have to prove the explicit representation of δ_0 on $\mathcal{H}_s(\mathbb{R}^d)$ and then (using the definitions) the covariance of the situation: shifting the point evaluation (δ_0 to δ_x) corresponds to shift the generator representing φ. We get the representation using the Fourier inversion formula (it is easy to recognize that we do not need that f itself is in $\boldsymbol{L}^1(\mathbb{R}^d)$, the inverse Fourier transform a priori defined as an $\boldsymbol{L}^2$-FT has the usual form as integral if $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$).

s-delta0rep

$$f(0) = \int_{\mathbb{R}^d} \hat{f}(s)ds = \int_{\mathbb{R}^d} \hat{f}(s)w(s) \cdot \frac{1}{w^2}(s)w(s)ds =: \int_{\mathbb{R}^d} \hat{f}(s)\,\overline{\hat{\varphi}(s)}\; w^2(s)ds, \tag{167}$$

if we put $\varphi = \mathcal{F}^{-1}(1/w^2)$.

Now we can apply the shift invariance of the scalar product (exercise):

S-scalshift

$$\langle T_x f, T_x h \rangle_{\mathcal{H}_s} = \langle f, h \rangle_{\mathcal{H}_s} \quad \text{for all} \quad f, h \in \mathcal{H}_s(\mathbb{R}^d), \tag{168}$$

because we know that translation goes to modulation on the Fourier transform side, but having the same modulations within a scalar products means (due to the fact that one is taking a conjugation in the scalar product) that it is unchanged. Technically speaking one could argue that translation goes into modulation, but for *all weights* modulations are unitary on the corresponding weighted $\boldsymbol{L}^2$-spaces. □

Another interesting embedding reads as follows:

ob-decay-S0 **Theorem 28.** *For $s > d$ the intersection of $\mathcal{H}_s(\mathbb{R}^d)$ with $\boldsymbol{L}^2_{w_s}(\mathbb{R}^d)$ with its natural norm (sum of the two norms) is continuously (and densely) embedded into $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$.*

Remark 47. Cf. the work of K. Gröchenig: there are also non-symmetric conditions and even conditions with weighted $\boldsymbol{L}^p-$norms on the time side and other weighted $\boldsymbol{L}^q$-conditions on the Fourier transform side which can be used to show that function with a certain amount of both time- and frequency concentration (as expressed by these conditions) are necessarily lying within $\boldsymbol{S}_0(\mathbb{R}^d)$. Of course one can easily adapt those conditions to conditions which are sufficient conditions for other modulation spaces, especially to be within Shubin classes $\boldsymbol{Q}(\mathbb{R}^d)$, or modulation spaces $\boldsymbol{M}^1_w(\mathbb{R}^d)$.

details are in [19], and about WSA functions in the work of Brandenburg [8], online at
`http://univie.ac.at/nuhag-php/bibtex/open_files/1407_9550001.pdf`

For other purposes we mention also the concept of *moderate weights*
COMES TWICE!

A weight function v on $\mathbb{R}^{2d}$ is called *submultiplicative*, if

submultdef1

$$v(z_1 + z_2) \leq v(z_1)v(z_2), \tag{169}$$

for all $z_1 = (x_1, \xi_1), z_2 = (x_2, \xi_2) \in \mathbb{R}^{2d}$.

A weight function w on $\mathbb{R}^{2d}$ is v-moderate if

$$w(z_1 + z_2) \leq cv(z_1)w(z_2)(*) \tag{170}$$

for all $z_1 = (x_1, \xi_1), z_2 = (x_2, \xi_2) \in \mathbb{R}^{2d}$.

Two weights w_1 and w_2 are equivalent, written $w_1 \asymp w_2$, if

$$C^{-1} w_1(z) \leq w_2(z) \leq C w_1(z) \tag{171}$$

for all $z = (x, \xi) \in \mathbb{R}^{2d}$ and some positive constant C.

For references see [19, 20, 40].

14. Some pointwise estimates

Pointwise estimates: Convolution preserves monotonicity We have to define[72] $|\mu|$ for a given measure, and have to show that $\| \, |\mu| \, \|_{\boldsymbol{M}} = \|\mu\|_{\boldsymbol{M}}$ for each $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$.

point-conv
$$|\mu * f| \leq |\mu| * |f|, \quad \forall \mu \in \boldsymbol{M}_b(\mathbb{R}^d), f \in \boldsymbol{C}_b(\mathbb{R}^d). \tag{172}$$

point-conv1
$$|f| \leq |g| \Rightarrow |\mu| * |f| \leq |\mu| * |g| \tag{173}$$

osc-estim1a
$$[(\mathrm{D}_\Psi f - f) * g](x) \; \leq \; [|f| * \mathrm{osc}_\delta(g)](x), \, \forall x \in \mathbb{R}^d. \tag{174}$$

or in short, a pointwise estimate of the form

sc-estim1ba
$$[(\mathrm{D}_\Psi f - f) * g] \; \leq \; [|f| * \mathrm{osc}_\delta(g)]. \tag{175}$$

MAIN ESTIMATE

ain-estim11
$$|\mathrm{D}_\Psi \mu * f - \mu * f| \leq |\mu| * |\,\mathrm{Sp}_\Psi f - f| \leq |\mu| * \mathrm{osc}_\delta f \tag{176}$$

if $\mathrm{diam}(\psi) \leq \delta$.
It relies on a couple of "simple" estimates, such as

$$(\mathrm{osc}_\delta f)^{\checkmark} = \mathrm{osc}_\delta(f^{\checkmark})$$

and

$$\mathrm{osc}_\delta(T_x f) = T_x(\mathrm{osc}_\delta f)$$

Obviously

$$|\,\mathrm{Sp}_\Psi f(x) - f(x)| \; \leq \; \mathrm{osc}_\delta f(x), \, \forall x \in \mathbb{R}^d.$$

Moreover the fact that the discretization operator $\mathrm{D}_\Psi : \boldsymbol{M}_b(\mathbb{R}^d) \mapsto \boldsymbol{M}_b(\mathbb{R}^d)$ is the adjoint of the spline operator $\mathrm{Sp}_\Psi : \boldsymbol{C}_0(\mathbb{R}^d) \mapsto \boldsymbol{C}_0(\mathbb{R}^d)$, implies also that we have:

conv-Dpsi
$$\mathrm{D}_\Psi \mu * f = \mu * \mathrm{Sp}_\Psi f, \qquad \mu \in \boldsymbol{M}_b(\mathbb{R}^d), \, f \in \boldsymbol{C}_0(\mathbb{R}^d). \tag{177}$$

Lemma 66. *If $f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)$ then also $\mathrm{osc}_\delta f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)$.*

Lemma 67. *A function $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ belongs to $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)$ if and only if $f \in \boldsymbol{L}^p(\mathbb{R}^d)$ and $\mathrm{osc}_\delta f \in \boldsymbol{L}^p(\mathbb{R}^d)$.*

[72]This is absolutely a non-trivial task and has to be done carefully. One technical way, which does not require the use of measure theoretical arguments, is to define $|\mathrm{D}_\Psi(\mu)| := \sum_{i \in I} |\mu(\psi_i)| \delta_{x_i}$ and verify than (this is the difficult part) that this tends to a limit in the w^*-sense!

15. Discretization and the Fourier transform

TEST: We shall define here $\Sha_a := \sum_k \delta_{ak}$ and $\Sha^a = \frac{1}{a}\sum_n \delta_{\frac{n}{a}}$. Then $\mathcal{F}\Sha_a = \Sha^a$. In fact, one has for $a = 1$ according to Poisson's formula $\mathcal{F}\Sha_1 = \Sha^1$, and the general formula follows from this by a standard dilation argument: Mass preserving compression St_ρ is converted into "value-preserving" dilation D_ρ on the Fourier transform side, and $\mathrm{D}_\rho \Sha^1 = \Sha^{1/\rho}$.

Let us put a few observations of importance at the beginning of this section:

- The periodic and discrete (unbounded) measures are exactly those which arise as periodic repetitions of a fixed finite sequence of the form $\sum_{k=0}^{N-1} a_k \delta_k$.
- The Fourier transform of such a sequence can be calculated directly using the FFT
- for any (sufficiently nice) function f (e.g. $f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{L}^1)(\mathbb{R}^d)$) one has for $b = 1/a$:

frepper1
$$\mathcal{F}\big[\Sha_{aN} * (\Sha^a \cdot f)\big] = \Sha^{Nb} \cdot (\Sha_b * \hat{f}) = \Sha_b * (\Sha^{Nb} \cdot \hat{f}) \tag{178}$$

The last step in the proof of formula frepper1 178 is easily verified directly: sampling and periodization commute if (and only if) the periodization constant (bN in our case) is a multiple of the sampling period (in our case b).

The question of approximately obtaining the continuous Fourier transform $\hat{f}$ of a "nice function" f from the FFT of it's sampled version can be derived from this fact.

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

Given $h > 0$ and some prescribed function ψ on $\mathbb{R}^d$, such as a cubic B-spline, the quasi-interpolation $Q_h f = Q_h^\psi f$ of a continuous function f on $\mathbb{R}^d$ is defined by

eqdefQh
$$Q_h f(x) = \sum_{k\in\mathbb{Z}} f(hk)\psi(x/h - k), \qquad x \in \mathbb{R}^d. \tag{179}$$

For suitable ψ, this formula describes an approximation to f from its samples on the fine grid $h\mathbb{Z}^d \subset \mathbb{R}^d$.

theoS0 **Theorem 29.** *Assume that $\psi \in \boldsymbol{S}_0(\mathbb{R}^d)$ satisfies $\sum_{k\in\mathbb{Z}^d} \psi(x-k) \equiv 1$, i.e. that the family $(T_k\psi)_{k\in\mathbb{Z}^d}$ forms a partition of unity. Then for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ we have $\|Q_h f - f\|_{\boldsymbol{S}_0} \to 0$ as $h \to 0$.*

Note, that under the same restrictions on ψ one also has convergence of the quasi-interpolation scheme in the Fourier algebra $\mathcal{F}\boldsymbol{L}^1$ for all $f \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$. Consequently one has $Q_h^*\sigma \to \sigma$ in the weak*-sense for each $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$. But $Q_h^*\sigma = \sum_k \sigma(\psi_k)\delta_{hk}$. Hence the discrete measures are w^*-dense in $\boldsymbol{S}_0'(\mathbb{R}^d)$.

Comment on the consistency of distributional Fourier transform with the classical one (defined on $\boldsymbol{L}^1(\mathbb{R}^d)$, using the Lebesgue integration formula).

For any Fourier invariant space of test functions (such as $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ or the Schwartz space of *rapidly decreasing functions* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$) one defines a *generalized Fourier transform* by the formula

Definition 45.

Four-def2 (180)
$$\mathcal{F}\sigma = \hat{\sigma} : (\hat{\sigma})(f) := \sigma(\hat{f})$$

The above formula can then be interpreted as a consistency relation. Write σ_g and $\sigma_{\hat{g}}$ for the distributions generated by the functions $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ and $\hat{g} \in \boldsymbol{C}_0(\mathbb{R}^d)$ respectively [73] then the formula (fund-Fourier0 117) tell us that

$$\widehat{\sigma_g} = \sigma_{\hat{g}}.$$

see e.g. [37], Lesson 17, page 156.

Material inserted 03.06.2017, TUM

$$TEST: \quad \sqcup\!\sqcup\!\sqcup \quad \sqcup\!\sqcup\!\sqcup_\rho$$

$$\sqcup\!\sqcup\!\sqcup(f) = \sqcup\!\sqcup * (\sqcup\!\sqcup \cdot f) = \sqcup\!\sqcup \cdot (\sqcup\!\sqcup * f)$$

with the important property that

$$\mathcal{F}(\sqcup\!\sqcup\!\sqcup(f)) = \sqcup\!\sqcup\!\sqcup(\mathcal{F}f), \quad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

or equivalently

$$\mathcal{F} \circ \sqcup\!\sqcup\!\sqcup = \sqcup\!\sqcup\!\sqcup \circ \mathcal{F}$$

In our code for `gaussnk.m` we use exactly this idea:

```
g=zeros(n,1); t=(0:n-1).';
p=sqrt(n); s = 1/sqrt(n);
for j=-3:3;  % rough periodization and
  g=g+exp(-pi*(t*s+j*p).^2);  % sampling at rate s
end;
g=g(:)'/norm(g);  %normalization norm(g) == 1
```

We also need the dilated version, which I would like to call $\sqcup\!\sqcup\!\sqcup_\rho$: The definition goes as follows (using the two dilation operators that I have in my lecture notes):

$$\sqcup\!\sqcup\!\sqcup_\rho(f) := \mathrm{St}_\rho \sqcup\!\sqcup * (\mathrm{D}_\rho \sqcup\!\sqcup \cdot f).$$

We will be interested in $\rho \to 0$, e.g. $\rho = 1/\sqrt{N}$ (if we want to obtain finite signals on $\mathbb{Z}_N$) or $\rho = 1/N^2$ if we want to have a sampling rate of $1/N$ and periodization by N, and thus obtain N^2 discrete values. So in principle if $(1/\rho)^2 \in \mathbb{N}$ we also have

ampperiod00 (181)
$$\mathrm{St}_\rho \sqcup\!\sqcup * (\mathrm{D}_\rho \sqcup\!\sqcup \cdot f) = \mathrm{D}_\rho \sqcup\!\sqcup \cdot (\mathrm{St}_\rho \sqcup\!\sqcup * f)$$

and consequently we have again

$$\sqcup\!\sqcup\!\sqcup_\rho \circ \mathcal{F} = \mathcal{F} \circ \sqcup\!\sqcup\!\sqcup_\rho$$

[73] so the space of test functions must be contained in $\boldsymbol{L}^1(\mathbb{R}^d)$ to ensure that $\boldsymbol{C}_0(\mathbb{R}^d)$-functions define continuous linear functionals.

In other words, we can apply the ordinary Fourier transform first on a function, OR alternatively apply the DFT/FFT on the finite sequence describing its periodized and sampled version, but these operators intertwine.
Spline-type quasiinterpolation is then what you call regaining the continuous function from a discrete sequence, which gives a continuous and periodic function which then by windowing has to be localized, but all the can be described well with these symbols.

,

16. Quasi-Interpolation

The piecewise linear interpolation operator for data available on the lattice of integers $\mathbb{Z}$, say $(c_k)_{k\in\mathbb{Z}}$, can be described as a sum of shifted triangular functions $\Delta(0) = 1, \Delta(k) = 0$ for $k \notin \mathbb{Z}$. Hence it can be written as a convolution product of the form

$$\big(\sum_{k\in\mathbb{Z}} c_k\delta_k\big) * \Delta.$$

It is easy to show that the resulting sum (the interpolant) belongs to $\boldsymbol{L}^p(\mathbb{R})$ if the sequence $\mathbf{c}$ is from $\boldsymbol{\ell}^p(\mathbb{Z})$. But this is true for much more general functions than then triangular function. It suffices to have $\varphi \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R})$ in order to find out that $\sum_{k\in\mathbb{Z}} T_k\varphi$ belongs to $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R})$ for $\mathbf{c} \in \boldsymbol{\ell}^p(\mathbb{Z})$. In fact, this assumption implies $\sum_k c_k\delta_k \in \boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^p)$ and hence the convolution relations for Wiener amalgam spaces imply:

$$f = \sum_{k\in\mathbb{Z}} T_k\varphi = \big(\sum_{k\in\mathbb{Z}} c_k\delta_k\big) * \varphi \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}).$$

As a consequence f is a continuous function and can be sampled, e.g.over the integers, but in most cases $f(k)$ will be perhaps close to, but different from the original sequence $(c_k)_{k\in\mathbb{Z}}$, hence the name *quasi-interpolation.* [74]
The so-called quasi-interpolation operators make sense for functions from $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d)$, to choose the appropriate generality from now on. For those functions one can guarantee that for some $C > 0$ and all $p \in [1, \infty]$ one has:

$$\|(f(k))_{k\in\mathbb{Z}^d}\|_p \leq C\|f \,|\, \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)\| \quad \forall f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d).$$

The same is true for any other lattice $\Lambda \lhd \mathbb{R}^d$, with

$$\|(f(\lambda))_{\lambda\in\Lambda}\|_p \leq C_\Lambda\|f \,|\, \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)\| \quad \forall f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d).$$

Hence the operator

$$f \mapsto \sum_{\lambda\in\Lambda} f(\lambda)T_\lambda\varphi$$

is a well defined operator on $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d)$ (even uniformly bounded with respect to the range $p \in [1, \infty]$. We will call such an operator the *quasi-interpolation operator* with respect to the pair (Λ, φ).
Among the quasi-interpolation operators those which arise from BUPUs, i.e. from functions $\varphi \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ satisfying

$$\sum_{\lambda\in\Lambda} T_\lambda\varphi(x) \equiv 1$$

[74] Note that SINC is not covered by this example, although for $p \in (1, \infty)$ it shares more or less all the properties described above.

are the most important ones. We are going to show that quasi-interpolation operators with respect to "fine lattices" Λ are good approximation operators.
The interesting phenomenon is the behaviour of piecewise linear interpolation over lattices of the form $\alpha\mathbb{Z}^d$, for $\alpha \to 0$.

Let us recall that $(T_k\varphi)_{k\in\Lambda}$ is a BUPU for some $\varphi \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ if and only if $\hat{\varphi}$ is a Lagrange interpolator over the orthogonal lattice $\Lambda^\perp = \{\chi \,|\, \langle\chi,\lambda\rangle \equiv 1 \quad \forall\lambda\in\Lambda\}$, i.e. that

Lagr-intp1
$$\widehat{\varphi}(\lambda') = \delta_{0,\lambda'} \quad \forall\lambda' \in \Lambda^\perp. \tag{182}$$

Proof. We can reinterpret the BUPU condition as $\sqcup\!\!\sqcup_H * \varphi \equiv 1$, which turns into

$$\mathcal{F}(\sqcup\!\!\sqcup_H) \cdot \hat{\varphi} = \mathcal{F}(1) = \delta_0.$$

Since $\mathcal{F}(\sqcup\!\!\sqcup_H) = C_H \sqcup\!\!\sqcup_{H^\perp}$ this condition reduces to (using $f \cdot \delta_x = f(x)\delta_x$):

$$C_H \sqcup\!\!\sqcup_{H^\perp} \cdot \hat{\varphi} = \sum_{h'\in H^\perp} \hat{\varphi}(h')\delta_{h'} = \delta_0,$$

which in turn is true if and only if $\hat{\varphi}(h') = 0$ for $h' \neq 0$ for all $h' \in H^\perp$.

$\square$

Remark 48. The condition described above is invariant with respect to pointwise powers on the Fourier transform side, i.e. $\hat{\varphi}$ satisfies (182) then the same is true for $\hat{\varphi}^2 = \widehat{\varphi * \varphi}$.

The quasi-interpolation operator $Q_{\Lambda,\varphi}$ can thus be described as the mapping

quasint
$$f \mapsto (\sqcup\!\!\sqcup_H \cdot f) * \varphi. \tag{183}$$

Note that this operators is bounded on $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d)$ because
$\sqcup\!\!\sqcup_H \cdot f \in \boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^\infty)(\mathbb{R}^d) \cdot \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d) \subseteq \boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^p)(\mathbb{R}^d)$, hence

$$(f \cdot \sqcup\!\!\sqcup_H) * \varphi \subseteq \boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^p) * \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1) \subseteq \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d).$$

It is of interest to check the behaviour of quasi-interpolation for the lattices $h\mathbb{Z}^d$, with $h \to 0$: Given $h > 0$ and some prescribed function ψ on $\mathbb{R}^d$, such as a B-spline, the quasi-interpolation $Q_h f = Q_h^\psi f$ of a continuous function f on $\mathbb{R}^d$ is defined by

eqdefQh
$$Q_h f(x) = \sum_{k\in\mathbb{Z}} f(hk)\psi(x/h - k), \qquad x \in \mathbb{R}^d. \tag{184}$$

For suitable ψ, this formula describes an approximation to f from its samples on the fine grid $h\mathbb{Z}^d \subset \mathbb{R}^d$.

theoS0 **Theorem 30.** *Assume that $\psi \in \boldsymbol{S}_0(\mathbb{R}^d)$ satisfies $\sum_{k\in\mathbb{Z}^d} \psi(x-k) \equiv 1$, i.e. that the family $(T_k\psi)_{k\in\mathbb{Z}^d}$ forms a partition of unity. Then for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ we have $\|Q_h f - f\|_{\boldsymbol{S}_0} \to 0$ as $h \to 0$.*

Note, that under the same restrictions on ψ one also has convergence of the quasi-interpolation scheme in the Fourier algebra $\mathcal{F}\boldsymbol{L}^1$ for all $f \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.

17. Pseudo-measures and other auxiliary terms

pseudomeas1 **Definition 46.** The space $\mathcal{F}\boldsymbol{L}^\infty$ is defined as the (inverse) image under the (generalized) Fourier transform of $\boldsymbol{L}^\infty(\mathbb{R}^d) = [(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)]'$.

An alternative definition of pseudo-measures (used e.g. in the work of G. Gaudry or R. Larsen) is based on the following characterization:

ourLinfdual **Lemma 68.** *There is a natural identification of $\mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$ with the $[(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})]'$, via $\sigma(\hat{h}) = \sigma_h(\widehat{\sigma}) = \int_{\mathbb{R}^d} \widehat{\sigma}(s)h(s)ds$.*

Even in the context of non-Abelian Groups $\mathscr{G}$ this makes sense, if one uses Eymard's Fourier algebra $\boldsymbol{A}(\mathscr{G})$ ([18]).

18. Advantages of a distributional Fourier Transform

Whereas most books in the field of Fourier analysis describe the Fourier transform at various levels, typically starting from the classical case of the Fourier transform for periodic functions [17, 46]. Sometimes the need of a generalized Fourier transform is motivated by the fact, that certain objects (like the "pure frequencies") do not have a Fourier transform in the usual sense, because first of all the classical Fourier transform is bound to diverge, while on the other hand the Fourier transform (which is in the generalized calculus a Dirac measure) is not an ordinary function, but has to be a kind of generalized function (in fact a bounded measure in that case), cf. [4].
In this section we want to emphasize (by demonstrating the situation, valid even for locally compact groups in full generality, through the example $\mathcal{G} = \mathbb{R}^d$).
Not only does the distributional Fourier transform (and it suffices to know the $\boldsymbol{S}_0'$-theory for this purpose) allow to define the Fourier transform for decaying objects (like functions in any of the $\boldsymbol{L}^p$-spaces), but also for periodic objects (such as periodic functions belonging locally to $\boldsymbol{L}^p$), even with different periods (which brings us already close to the discussion of almost periodic functions).
As a central topic let us therefore discuss the Fourier transform of periodic functions (or measures, or distributions) as the "infinite limit" of it's periodic repetitions. First of let us recall that it is easy to find for each lattice $\Lambda = \boldsymbol{A} * \mathbb{Z}^d$, for some non-singular $d \times d$-matrix $\boldsymbol{A}$ a fundamental domain (equal to $Q = \boldsymbol{A} * [0,1)^d$) and also bounded partitions of unity of the form $(\phi_\lambda)_{\lambda\in\Lambda} = (T_\lambda \varphi)_{\lambda\in\Lambda}$, with $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$ or even in $\mathcal{S}(\mathbb{R}^d)$. For the next lemma we need $\varphi \in \boldsymbol{S}_0(\mathbb{R}^d)$, or even better, in $\mathcal{S}(\mathbb{R}^d)$.

Lemma 69. *A function f (or distribution in $\boldsymbol{S}_0'(\mathbb{R}^d)$)* [75] *is periodic with respect to $\Lambda \lhd \mathbb{R}^d$ if and only if it is of the form*

$$f = \sum_{\lambda\in\Lambda} T_\lambda f_\lambda^\circ, \tag{185}$$

for some compactly supported pseudo-measure $f^\circ \in \mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$.

Proof. If f is Λ-periodic, i.e. if $T_\lambda f = f$ for all $\lambda\in\Lambda$, then we can choose $f^\circ = f\varphi$, for a function φ with compact support, generating a Λ-BUPU as described above, because

$$f = \sum_{\lambda\in\Lambda} T_\lambda(T_{-\lambda}(f\varphi_\lambda)) = \sum_{\lambda\in\Lambda} T_\lambda f^\lambda = \sum_{\lambda\in\Lambda} T_\lambda f^\circ.$$

because $f^\lambda = T_{-\lambda}(f \cdot \varphi_\lambda) = T_{-\lambda} f \cdot T_{-\lambda}\varphi_\lambda = f \cdot \varphi =: f^\circ$ by the periodicity of f.
Conversely, let f° a compactly supported pseudo-measure or even in $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$. Since $\boldsymbol{W}(\boldsymbol{M}, \boldsymbol{\ell}^\infty) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \boldsymbol{\ell}^1) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^\infty) = \boldsymbol{S}_0'$ the partial sums of the periodization are both

[75] A distribution from $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ which is $\Lambda-$periodic for any co-compact lattice Λ in $\mathbb{R}^d$ does in fact belong automatically to $\boldsymbol{S}_0'(\mathbb{R}^d)$!

uniformly in $\boldsymbol{S}_0'$ as well as w^*-convergent, as can been seen from the interpretation

$$\lim_{F\to\Lambda}\sum_{\lambda\in F} T_\lambda f^\circ = \lim_{F\to\Lambda}\left((\sum_{\lambda\in F}\delta_\lambda) * f^\circ\right).$$

In fact, one can reduce the general case of w^*-convergence to the case where (first of all) f° as compact support, but in testing that the action on arbitrary test functions $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ those in $(\boldsymbol{S}_0)_c = \boldsymbol{A}_c(\mathbb{R}^d)$ are sufficient.
PROBABLY SOME MORE DETAILS TO BE GIVEN □

We remark that a compactly supported pseudo-measure has the property that its Fourier transform is indeed a bounded and continuous function. Indeed, we can find some $\varphi \in \boldsymbol{S}_0(\mathbb{R}^d)$ such that $f^\circ \cdot \varphi$, hence $\mathcal{F}(f^\circ \cdot \varphi) = \widehat{f^\circ} * \hat{\varphi} \in \boldsymbol{L}^\infty * \boldsymbol{S}_0 \subseteq \boldsymbol{C}_b(\mathbb{R}^d)$. One can also show that the Fourier coefficients of the periodic version are just the values $\widehat{f^\circ}$ over the orthogonal lattice $\Lambda^\perp$. Let us describe this in detail:

Using now the fact that the distributional FT of $⊔⊔_\Lambda$ coincides with a multiple of $⊔⊔_{\Lambda^\perp}$, i.e. $\mathcal{F}⊔⊔_\Lambda = C_\Lambda ⊔⊔_{\Lambda^\perp}$ in the $\boldsymbol{S}_0'$-sense, we find that for any periodic function or distribution f we have

$$\boxed{\mathtt{FT}} \quad \mathcal{F}f = \mathcal{F}(⊔⊔_\Lambda * f^\circ) = \mathcal{F}(⊔⊔_\Lambda)\cdot \hat{f}^\circ = C_\Lambda ⊔⊔_{\Lambda^\perp}\cdot \hat{f}^\circ. \tag{186}$$

Consequently we have $\mathrm{supp}(\hat{f}) \subseteq \mathrm{supp}(⊔⊔_{\Lambda^\perp}) = \Lambda^\perp$. In fact, we can give a more explicit description of $\hat{f}$: by carrying out the pointwise multiplication $⊔⊔_{\Lambda^\perp}\cdot \hat{f}^\circ$ we find that

$$\hat{f} = \sum_{\lambda^\perp\in\Lambda^\perp} \hat{f}^\circ(\lambda^\perp)\,\delta_{\lambda^\perp}.$$

But for $f^\circ \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$ (by the Hausdorff-Young principle for generalized amalgams) we know that $\hat{f}^\circ \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^\infty) \subset \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^\infty) \subset \boldsymbol{C}_b(\mathbb{R}^d)$. Hence we can even claim that $\hat{f}$ is a sum of Dirac measures located at the points of $\Lambda^\perp$, with a bounded sequence of coefficients in $\boldsymbol{\ell}^\infty(\Lambda)$.
In fact (this has to be shown separately), one can be shown that this is a complete *characterization* all the tempered distributions $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ with $\mathrm{supp}(\sigma) \subseteq \Lambda^\perp$.

Lemma 70. *(Characterization of distributions supported on discrete subgroups)*
A distribution $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ satisfies $\mathrm{supp}(\sigma) \subseteq \Lambda$ *if and only if it is of the form*

$$\sigma = \sum_{\lambda\in\Lambda} c_\lambda \delta_\lambda$$

for some sequence $\mathbf{c} = (c_\lambda)_{\lambda\in\Lambda} \in \boldsymbol{\ell}^\infty(\Lambda)$.

Summarizing we have a mapping from the periodic elements (in $\boldsymbol{S}_0'(\mathbb{R}^d)$) into $\boldsymbol{\ell}^\infty(\Lambda)$, of the form $f \to (\hat{f}^\circ(\lambda))_{\lambda\in\Lambda}$ (which is of course independent of the choice of f°). Assume we have a regular distribution, “coming from” some $f \in \boldsymbol{L}^1(\mathbb{U})$. Let us discuss (for simplicity) the situation for $d = 1$ and $\Lambda = \mathbb{Z}$. Then we have $Q = [0, 1)$ and we can choose $f^\circ = f \cdot \mathbf{1}_Q \in \boldsymbol{L}^1(\mathbb{R})$. Since $\mathbb{Z}^\perp = \mathbb{Z}$ the Fourier transform of a periodic (local) $\boldsymbol{L}^1$-function are a sequence on $\mathbb{Z}$ again: $\hat{f}^\circ(n) = \int_0^1 f(t)e^{-2\pi i t}dt$ which coincides with the usual (“classical”) definition of Fourier coefficients for locally integrable, $\mathbb{Z}$-periodic functions.
Note that in our context local square (or generally p-) integrability corresponds to additional properties on f° which in such a case will belong locally to $\boldsymbol{L}^2(\mathbb{R})$ (resp. $\boldsymbol{L}^p(\mathbb{R})$), resp. to $\boldsymbol{L}^2(\mathbb{U})$ or $\boldsymbol{L}^p(\mathbb{U})$. In our terminology $f^\circ \in \boldsymbol{W}(\boldsymbol{L}^2, \boldsymbol{\ell}^1)(\mathbb{R})$ resp. $\boldsymbol{W}(\boldsymbol{L}^p, \ell^1)(\mathbb{R})$, and again by the

Hausdorff-Young principle we obtain that $\hat{f}^{\circ} \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \boldsymbol{\ell}^\infty)(\mathbb{R})$, or the Fourier coefficients of f resp. the values of $(\hat{f}^{\circ}(n))_{n\in\mathbb{Z}}$ belong to $\boldsymbol{\ell}^{p'}(\mathbb{Z})$.

A topic of interest in connection with standard spaces is the following one, which is based on the fact that for any (restricted) standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ we have the following chain of continuous inclusion:

WB-incl0

$$\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^1) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^\infty) \tag{187}$$

and the fact that we have

WB-incl2

$$\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^p) \hookrightarrow \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^q) \quad \text{if} \quad p \leq q. \tag{188}$$

ow-uppindex **Definition 47.** Given a (restricted) standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ we define the *lower* resp. *upper index* as follows:

low-index

$$\operatorname{lowind}(\boldsymbol{B}) := \sup\{\, p \,|\, \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^p) \hookrightarrow \boldsymbol{B}\} \tag{189}$$

and

upp-index

$$\operatorname{uppind}(\boldsymbol{B}) := \inf\{\, q \,|\, \boldsymbol{B} \hookrightarrow \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^q)\} \tag{190}$$

In most cases the supremum resp. infimum will *not be attained.* However, we will have in any case

lowlequpp

$$\operatorname{lowind}_{\boldsymbol{B}} \leq \operatorname{uppind}_{\boldsymbol{B}} \tag{191}$$

The following condition is in general slightly stronger than the case of equality of indices:

Definition 48. A Banach space is called to be of *global type p* if one has

deftypep

$$\boldsymbol{B} = \boldsymbol{W}(\boldsymbol{B}, \boldsymbol{\ell}^p). \tag{192}$$

Aside from the trivial facts that $\boldsymbol{L}^p(\mathbb{R}^d)$ is of course of type p for any p, one can check that $\boldsymbol{M}_b(\mathbb{R}^d)$ is of type 1, while the usual $\boldsymbol{L}^2$-Sobolev spaces are of type 2. They have therefore been called $\boldsymbol{\ell}^2$-puzzles by P. Tchamitchian in [67,68] (?)
One of the interesting and non-trivial facts (no proof is given here) is that one has (the most interesting perhaps being the case $p = 1$), see [38]:

Lemma 71. *For* $1 \leq p \leq 2$ $\operatorname{lowind}(\mathcal{F}\boldsymbol{L}^p) = p = \operatorname{lowind}(\mathcal{F}\boldsymbol{L}^{p'})$ *while* $\operatorname{uppind}(\mathcal{F}\boldsymbol{L}^p) = p' = \operatorname{uppind}(\mathcal{F}\boldsymbol{L}^{p'})$. *Hence, except for the case $p = 2$ the space $\mathcal{F}\boldsymbol{L}^p$ is never of a particular type.*

Another interesting family of spaces is the set of all "multipliers" on $\boldsymbol{L}^p$, for say $1 \leq p < \infty$, which we denote by $H_{\boldsymbol{L}^1}(\boldsymbol{L}^p, \boldsymbol{L}^p)$, which is indeed another (restricted) standard space. It is well known that it coincides (by duality) with $H_{\boldsymbol{L}^1}(\boldsymbol{L}^{p'}, \boldsymbol{L}^{p'})$, that $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1) = \boldsymbol{M}_b(\mathbb{R}^d)$ and $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^2, \boldsymbol{L}^2) = \mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$. Hence one may *conjecture* that lowind and uppind of these spaces equals p and p', but to my knowledge nothing is known about it for $1 < p < 2$.
Another interesting question could be the upper and lower index for modulation spaces (which can be shown to be of local type $\mathcal{F}\boldsymbol{L}^q$). Since the space $\boldsymbol{M}^p(\mathbb{R}^d) = \boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$ for $p = q$ are Fourier invariant an coincide with $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \boldsymbol{\ell}^p)$ they are clearly of *type* p.

19. Support and Spectrum of distributions

MATERIAL IN PROGRESS

So far we have tried to avoid a detailed discussion of the concept of a support of a generalized function, and even the support of a measure has not been defined.
First of all we have to extend the notion of a support which is clear for continuous functions (e.g. $k \in \boldsymbol{C}_c(\mathbb{R}^d)$) to generalized functions. Clearly this has to be done through the action of a distribution on the corresponding test functions. It is also clear that both for the case of $\boldsymbol{C}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0(\mathbb{R}^d)$ one has *regular* function algebras, i.e. there are functions with arbitrary small support at any position. The approach to the notion of support (as a set of relevant points) is through its complement, namely the (open!) set of irrelevant points:

cospdef01

Definition 49. A distribution or measure σ is acting trivially at a given point $t \in \mathbb{R}^d$ (generally: $t \in \mathscr{G}$) if there exists some $k \in \boldsymbol{C}_0(\mathbb{R}^d)$ (resp. $k \in \boldsymbol{S}_0(\mathbb{R}^d)$), with $k(t) = 1$ but $\sigma \cdot k = 0$, or equivalently $\sigma(k \cdot f) = 0, \forall f \in \boldsymbol{C}_0(\mathbb{R}^d)$ (resp. $\boldsymbol{S}_0(\mathbb{R}^d)$).

The following, slightly more involved formulation may be more intuitive, but perhaps less practical for use (because it talks about a lot of functions instead of a single function with specific properties)

cospalt1

Lemma 72. *A point t does not belong to the spectrum of σ (resp. σ shows "trivial action near t) if there exists some neighborhood U of the point t, e.g. some open ball $U = B_\varepsilon(t)$ such that $\sigma(f) = 0$ for all $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\operatorname{supp}(f) \subset U$.*

Proof. Assume that $k \cdot \sigma = 0$, then it is clear that $0 = (k \cdot \sigma)(h) = \sigma(k \cdot h)$ for any $h \in \boldsymbol{C}_0(\mathbb{R}^d)$. But by continuity of k (and in the case of the space $\boldsymbol{S}_0(\mathbb{R}^d)$ the much deeper result, the so-called Wiener's inversion theorem provides a similar fact, see [58]) one can find some function $\phi \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $k(t) \cdot \phi(t) \equiv 1$ on some small ball $U = B_\varepsilon(t)$. For $f \in \boldsymbol{C}_c(\mathbb{R}^d)$ with small support inside of U we have of course $f = k \cdot \phi \cdot f$, or $f = k \cdot (\phi \cdot f)$, with $\phi \cdot f \in \boldsymbol{C}_c(\mathbb{R}^d)$, and hence $\sigma(f) = 0$ by the observation at the beginning of the proof.
As for the converse.. One may decompose a measure into a sum of "small pieces". I don't think it is easy to describe without BUPUs! Let us make sure that we have a fairly fine BUPU, consisting of plateau-functions, so each ψ_i which is used (with $\sum_{i \in I} \psi(x) \equiv 1$) has a plateau near its center which we denote by x_i. Without loss of generality we can assume that $x_1 = t$ and that $\operatorname{supp}(\psi_1) = 1$ (it would be enough to have it different from zero!). But then

$$\psi_1 \cdot \sigma(f) = \sigma(\psi_1 \cdot f) = 0,$$

by the assumption, since

$$\operatorname{supp}(\psi_1 \cdot f) \subseteq \operatorname{supp}(\psi) \subseteq U.$$

This argument completes the proof. □

It is a good exercise to verify that the set of "points of trivial action" (e.g. for a given bounded measure) does *not depend* on the choice of the underlying algebra of test functions. Whether one assumes k with $k(t) = 1$ to be from $\boldsymbol{C}_c(\mathbb{R}^d)$ or from $\boldsymbol{S}_0(\mathbb{R}^d)$ does not play a role, since on can approximate one by the other (in the sup-norm).

pentrivact1

Lemma 73. *The set of points for which a distribution $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ (resp. : a bounded measure $\mu \in \boldsymbol{M}_b(\mathbb{R}^d)$) shows trivial action is an open set.*

Proof. Assume that t_0 allows for some $k \in \boldsymbol{S}_0(\mathbb{R}^d)$ (we may also assume by density) with compact support, such that $k_0(t_0) = 1$ and $\sigma \cdot k_0 = 0$. Clearly $|k_0(y)| > 0$ for t near t_0. Hence we find that $k := k_0/k_0(y)$ is now a function with $k(y) = 1$, but still $\sigma \cdot k = 0$. □

Now it is time to define the support of a distribution.

suppS0PRd **Definition 50.** The support $\mathrm{supp}(\sigma)$ of $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ is defined as the complement of the set of points of trivial action. Hence it is a closed set.

Since we have now for $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ several possible concepts of supports we have to show that there is consistency:

uppfctdist1 **Lemma 74.** *Given* $k \in \boldsymbol{C}_c(\mathbb{R}^d)$ *we find that the classical support* $\mathrm{supp}(k) := \{t \,|\, k(t) \neq 0\}^-$ *and the distributional support* $\mathrm{supp}(\sigma_k)$ *coincide. Furthermore, the notion of support does not depend on the algebra of test functions used.*

Proof.
- Assume that $t \notin \mathrm{supp}(k)$ (classical). Since $\mathrm{supp}(k)$ is closed, there exists some neighborhood U of t such that $k(z) \equiv 0$ on U. Choosing now any $k \in \boldsymbol{C}_c(\mathbb{R}^d) \cap \boldsymbol{S}_0(\mathbb{R}^d)$ with $k_0(t) = 1$ and $\mathrm{supp}(k_0) \subseteq U$, hence $k \cdot k_0 = 0$, as a function and consequently as a generalized function.
- conversely assume that $t \notin \mathrm{supp}(k)$, i.e. for any neighborhood U of t there exists some point z such that $k(z) \neq 0$. Choosing $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ with small support ...

$\square$

A typical argument is: If we apply a measure μ to a function $f \in \boldsymbol{C}_b(\mathbb{R}^d)$ which is zero on $\mathrm{supp}(\mu)$ then the result should be (and is in fact) zero.
The analysis behind this claim (of course it is enough to verify the result for compactly supported functions $f \in \boldsymbol{C}_c(\mathbb{R}^d)$) is essentially based on the fact that it is possible to approximate any measure (in the w^* sense with respect to $\big(\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'}\big)$) by discrete measures *whose support* is contained in $\mathrm{supp}(\mu)$, i.e. no other Dirac measures than those from the set $\mathrm{supp}(\mu)$ are needed. This is plausible, but not at all trivial, because the standard approximation (e.g. using regular BUPUs) makes of course only use of "nearby Diracs" (this is an easy claim), but not strictly from within the support. So *some modification* is required to ensure this claim.
This

Since generalized functions $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ have a Fourier transform the support of that Fourier transform (so-to-say the "relevant frequencies" contained in σ) is a meaningful concept, which deserves a separate name:

spectrumdef **Definition 51.** For any $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ the *spectrum* of σ is defined as $\mathrm{supp}(\hat{\sigma})$.

Of course it is of interest to characterize the spectrum of σ (in fact it is not easier or more complicated to characterize the spectrum of $h \in L^\infty(\mathbb{R}^d)$!) in several other, alternative ways:

arspectrum1 **Lemma 75.** *A pure frequency χ_s is in the spectrum (more precisely $s \in \mathrm{spec}(\sigma)$) if and only if χ_s is in the w^*-closure of the* **linear span of the translates of** σ *within $\boldsymbol{S}_0'(\mathbb{R}^d)$.*
Alternatively one can say: $s \in \mathrm{spec}(\sigma)$ is equivalent to the existence of some net (g_α) in $\boldsymbol{L}^1(\mathbb{R}^d)$ (possibly unbounded) such that $g_\alpha * \sigma \to \chi_s$ in the w^*-topology.

Part of the Lemma should contain the information that the w^*-closure of the set of all translates (which of course the same as the w^*-closure of all functions elements of the form $\nu * \sigma$, with $\nu \in \boldsymbol{M}_d(\mathbb{R}^d)$ (discrete measures) or on the w^*-closure of the elements of the form $\boldsymbol{L}^1(\mathbb{R}^d) * \sigma$, or even just the w^*-closure of the more decent (because continuous and bounded) functions of the form $\boldsymbol{S}_0(\mathbb{R}^d) * \sigma$ is the same, due to mutual w^*-approximation (details to be done later or separately!)
QUESTION: I think one should state: w^*-closed LINEAR SPAN of the translates (not just of the translates!!!!)
SINCE $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ IS SEPARABLE I AM QUITE SURE THAT ONE CAN EVEN FIND A *sequence* WITH THESE PROPERTIES, FOR EVERY SINGLE $s \in \widehat{\mathbb{R}^d}$
IT IS ALSO POSSIBLE TO REQUEST THAT $k_0 \equiv 1$ NEAR THE POINT t, WITH DIFFERENT ARGUMENTS.
First of all the idea of local (pointwise) inversion comes into play. This is trivial in the case of $k \in \boldsymbol{C}_c(\mathbb{R}^d)$, since the function $1/k(t)$ is obviously continuous in some neighborhood of t and can be extended, with small support.
For the case of $\boldsymbol{S}_0(\mathbb{R}^d)$ this is not trivial anymore and one has to resort to Wiener's inversion theorem, which tells us that for any given $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ with $h(t_0) \neq 0$ there exists some other function $g \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ with $g(s) = 1/h(s)$ near t_0, and since $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is a pointwise algebra the product $h \cdot g \equiv 1$ near t_0.
Alternative approach: Assume that $\sigma \cdot k = 0$ for all functions with sufficiently small support! Then one can smooth them out (resp. integrate or convolve) and get that there are also some functions having small plateaus near t_0.

20. Ideas on BUPUS, Wiener Amalgam and Spline-Type Spaces

BUPUs are a universal tool for many questions in the theory of function spaces, they are quite useful in order to develop concepts for (conceptual) harmonic analysis, and they are crucial for the definition of Wiener amalgam spaces.
See a talk concerning the importance of BUPUs (Oslo, Feb. 2013)
`http://univie.ac.at/nuhag-php/dateien/talks/2484\_oslo13A.pdf`
Here is a short list of properties of these families that make them so important:

- They can be defined over arbitrary locally compact groups $\mathscr{G}$. In fact, there use is implicit in the construction of the Haar measure following Cartan resp. A. Weil;
- By means of BUPUs it is easy to show that the discrete measures are w^*-dense in the space of bounded measures $\boldsymbol{M}_b(\mathscr{G})$.
- Obviously they are extremely useful in defining Wiener amalgam spaces (the discrete description is much more general, at least in order to introduce the spaces, than the "continuous" description);
- Spline-Type are quite important as well; they are obtained as "closed linear span" of a set of function and their translates, within some larger function space, say $\big(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\big)$. If we find a *Riesz projection basis* for such a space than typically the discrete $\boldsymbol{\ell}^p$-norm on the coefficients, and the $\boldsymbol{L}^p(\mathbb{R}^d)$ or also the $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^p)(\mathbb{R}^d)$-norm are equivalent on the corresponding spline-type space.

For most purposes any result that is valid for regular BUPUs on $\mathbb{R}^d$ can be easily transferred to general statements about arbitrary BUPUs over locally compact, or at least locally compact Abelian groups.

21. Banach Gelfand Triples and Regularization Operators

New material of 30.05.2017, TUM, hgfei
The Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ (and in a similar war $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(G)$ over any given LCA group)

22. Fourier Standard Spaces

23. Riesz Bases and Banach Frames

In the terminology introduced in XX this means that **R** is injective, but not surjective, and **C** is a left inverse of **R**. Thus we have the following commutative diagram.

$$
\begin{tikzcd}
\boldsymbol{X} \arrow[d, "\mathbf{P}"'] \arrow[dr, "\mathbf{C}"] & \\
\boldsymbol{X}_0 \arrow[r, shift left, "\mathbf{C}"] & \boldsymbol{Y} \arrow[l, shift left, "\mathbf{R}"]
\end{tikzcd}
$$

24. Historical Notes

This is of course a very subjective area. From the point of view taken in these notes functional analysis can be used (practically speaking: jus the theory of Banach and Hilbert spaces is used, together with the main principles of the theory of Banach spaces, operators, completions and the like, such as the w^*-compactness of the closed unit ball in a dual Banach space $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$).

25. Open Questions, things to do, state May 2017

We have started by defining the internal convolution in $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ via the action of bounded measures by convolution (resp. moving averages) on $\big(\boldsymbol{C}_0(G),\, \|\cdot\|_\infty\big)$, and have shown that this gives an internal, commutative and of course associative (immediately from the setup) "multiplication". But there is also the adjoint action of this bilinear $(\boldsymbol{M}_b(\mathscr{G}), \boldsymbol{C}_0(G))$ mapping, which also maps $\boldsymbol{M}_b(\mathscr{G})$ into $\boldsymbol{M}_b(\mathscr{G})$ and one has to show that both these two actions are the same. On the other hand the internal convolution in $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ restricts to the subspace which we call $\big(\boldsymbol{L}^1(G),\, \|\cdot\|_1\big)$, and thus there is a dual action on the dual space of $\big(\boldsymbol{L}^1(G),\, \|\cdot\|_1\big)$, which is $\big(\boldsymbol{L}^1(G),\, \|\cdot\|_1\big)$, which in turn contains $\big(\boldsymbol{C}_0(G),\, \|\cdot\|_\infty\big)$ as a closed, translation invariant subspace.
Our concern is: are all these possible interpretations compatible?
The answer will be, "YES, of course, and the argument will essentially rely on continuity considerations.
Later on similar questions can be raised for general homogeneous Banach spaces and their duals.
Let us write $*'$ for the adjoint action, thus

$$\mu *' \nu(f) := \nu(\mu * f), \quad \mu, \nu \in \boldsymbol{M}_b(\mathscr{G}), f \in \boldsymbol{C}_0(G).$$

We will verify that $\mu *' \nu = \mu^{\checkmark} * \nu$.
Proof: It will be enough to verify that this claim is valid for the point measures, because in both sides of the claimed identity the output depends continuously (in the w^*-sense) on the input. So let us assume that $\mu = \delta_x$ and $\nu = \delta_y$.
Then

$$\delta_{-x} *' \delta_y(f) \overbrace{adj.action} = \delta_{-x}(C_{\delta_y}(f)) = \delta_{-x}(T_y f) = f(-x-y) = \delta_{-x-y}(f).$$

OLD MATERIAL in words: The argument is essentially of the following nature. First we verify that $\boldsymbol{M} * \boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{L}^1(\mathbb{R}^d)$ (either because $\boldsymbol{L}^1(\mathbb{R}^d)$ is the set of bounded measures with continuous shifts, which is preserved under convolution given as the dual action from the action on $\boldsymbol{C}_0(\mathbb{R}^d)$, lifted to the measures, or just by using the fact that $\boldsymbol{L}^1(\mathbb{R}^d)$ is a homogeneous Banach space, hence $\boldsymbol{M}(\mathbb{R}^d) \bullet \boldsymbol{L}^1(\mathbb{R}^d) \subset \boldsymbol{L}^1(\mathbb{R}^d)$, where the abstract action is just convolution. This action is again by a second adjointness relation lifted to $\big(\boldsymbol{L}^1,\, \|\cdot\|_1\big)$ (which is just the natural continuation of the convolution of ordinary functions to the *dual space* of $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$). The elements satisfying ?? CubinLinfRd ?? can be approximated in the norm by elements of the form $g * h \in \boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{L}^\infty(\mathbb{R}^d) \subset \boldsymbol{C}_{ub}(\mathbb{R}^d)$. Since this is a closed subspace of $\boldsymbol{L}^\infty(\mathbb{R}^d)$ (cf. lemma below) the limit, i.e. h must belong (more precisely: must be represented) by a function in $\boldsymbol{C}_{ub}(\mathbb{R}^d)$.
We need the following basic observation:

Lemma 76. *The embedding $h \to \sigma_h$: given by*

CbtoLinf (193)
$$\sigma_h(f) = \int_{\mathbb{R}^d} f(x)\, h(x) dx$$

from $\big(\boldsymbol{C}_b(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ into the dual space of $\big(\boldsymbol{L}^1,\, \|\cdot\|_1\big)$ (with its natural norm) is isometric:

CbLinfisom (194)
$$\|h\|_\infty = \sup_{\|g\|_1 \leq 1} \left| \int_{\mathbb{R}^d} g(x) h(x) dx \right|.$$

ef_abs_cont **Definition 52** (absolutely continuous, [42], 18.10)**.** Let f be a complex-valued function defined on a subinterval J of $\mathbb{R}$. Suppose that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$(i) \quad \sum_{k=1}^{n} |f(d_k) - f(c_k)| < \epsilon$$

for every finite, pairwise disjoint, family $\{]c_k, d_k[\}_{k=1}^{n}$ of open subintervals of J for which

$$(ii) \quad \sum_{k=1}^{n} (d_k - c_k) < \delta.$$

Then f is said to be absolutely continuous on J.

e_variation **Theorem 31** ([42], 18.12)**.** *Any complex-valued absolutely continuous function f defined on $[a,b]$ has finite variation on $[a,b]$*

_meas_deriv **Theorem 32** ([42], 18.14)**.** *If f is a real-valued, nondecreasing function on $[a,b]$, then f' is Lebesgue measurable and*

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

If g is a complex-valued function of finite variation on $[a,b]$, then $g' \in \boldsymbol{L}^1([a,b])$.

_derivative **Theorem 33** ([42], 18.15)**.** *Let f be an absolutely continuous complex-valued function on $[a,b]$ and suppose that $f'(x) = 0$ almost everywhere in $]a,b[$. Then f is a constant.*

al_calculus **Theorem 34** (Fundamental theorem of the integral calculus for Lebesgue integrals, [42], 18.16)**.** *Let f be a complex-valued, absolutely continuous function on $[a,b]$. Then $f' \in \boldsymbol{L}^1([a,b])$ and*

tal_theorem (195)
$$f(x) = f(a) + \int_a^x f'(t)dt$$

for every $x \in [a,b]$. The converse is true as well.

ef_integral **Theorem 35** ([42], 18.17)**.** *A function f on $[a,b]$ has the form*

$$f(x) = f(a) + \int_a^x \varphi(t)dt$$

for some $\varphi \in \boldsymbol{L}^1([a,b])$ if and only if f is absolutely continuous on $[a,b]$. In this case we have $\varphi(x) = f'(x)$ almost everywhere on $]a,b[$.

comments on absolutely continuous functions, classically

integral_v1 **Theorem 36** ([42], 18.18)**.** *A function f on $\mathbb{R}$ has the form*

$$f(x) = \int_{-\infty}^x \varphi(t)dt$$

for some $\varphi \in \boldsymbol{L}^1(\mathbb{R})$ if and only if f is absolutely continuous on $[-A,A]$ for all $A > 0$, $V_{-\infty}^{\infty} f$ is finite, and $\lim\limits_{x \to -\infty} f(x) = 0$.

hm_part_int **Theorem 37** ([42], 18.19)**.** *Let f, g be functions in $\boldsymbol{L}^1([a,b])$, let*

$$F(x) = \alpha + \int\limits_a^x f(t)dt,$$

and let

$$G(x) = \beta + \int\limits_a^x g(t)dt.$$

Then

$$\int\limits_a^b G(t)f(t)dt + \int\limits_a^b g(t)F(t)dt = F(b)G(b) - F(a)G(a).$$

or_part_int **Corollary 11** ([42], 18.20)**.** *Let f and g be absolutely continuous functions on $[a,b]$. Then*

$$\int\limits_a^b f(t)g'(t)dt + \int_a^b f'(t)g(t)dt = f(b)g(b) - f(a)g(a)$$

splin-abs1

$$| \, |\mathrm{Sp}_\Psi(f)| - |f| \, |(x) \leq |(\mathrm{Sp}_\Psi(f) - f)(x)| \to 0 \quad \text{for} \quad |\Psi| \to 0. \tag{196}$$

It is of course easy to find some function $h \in \boldsymbol{C}_c(\mathbb{R}^d)$ such that $\|h\|_\infty = 1$ and $h(x_i) = |k(x_i)|/k(x_i)$ at all the points where (x_i) is the family of centers of the BUPU Φ (and the condition is of course only applied when $k(x_i) \neq 0$. Then $\mathrm{Sp}_\Psi(|k|) = \mathrm{Sp}_\Psi(hk)$. Alternative: LOOK at the function (to be embedded into $\boldsymbol{C}_0'(\mathbb{R}^d)$) by considering the set where it is strictly positive and strictly negative, with some margin.
ANOTHER strategy (comparable with this one) is to show that $|k|$ can be norm-approximated (uniformly, with fixed compact support) by functions of the form $h \cdot k$, where h is a continuous version of the $sign(k(x))$-function. (can we do it using find BUPUS?, or does one need Tietze-Urysohn?).

LEFT OVER MATERIAL

Kantorovich

Theorem 38. `[Kantorowich Lemma]` *Let T_α be a strongly convergent and bounded sequence of invertible operators between Banach spaces, with limit T_0, which is assumed to be invertible itself. Then the inverse operators are strongly convergent as well if the inverse operators are uniformly bounded. If we consider only sequences this is a criterion, because then the strong convergence of $T_n^{-1}(y) \to T_0^{-1}(y)$ for every y implies uniform boundedness of the sequence T_n^{-1}.*

QUESTION: For tight nets of bounded measures w^*-convergence implies pointwise and uniform over compact convergence of their FTs. But also the converse is true!!!! (Exercise).
[for non-tight families this is not true, just think of the case $\delta_n, n \to \infty$!

An elementary proof showing that the Gauss function $g_0(t) = e^{-\pi|t|^2}$ is mapped itself by the Fourier transform has been given by Georg Zimmermann, see

`http://www.univie.ac.at/NuHAG/FEICOURS/ws0607/efoft.pdf`

26. APPENDIX

Definition 53. A net $\{f_\alpha\}_{\alpha\in A}$ in a Banach space $\mathscr{X}$ is said to be a Cauchy net if for every $\epsilon > 0$, there is a α_0 in A such that $\alpha_1, \alpha_2 \geq \alpha_0$ implies $||f_{\alpha_1} - f_{\alpha_2}|| < \epsilon$.

Proposition 7. *In a Banach space each Cauchy net is convergent.*

Proof. Let $\{f_\alpha\}_{\alpha\in A}$ be a Cauchy net in the Banach space $\mathscr{X}$. Choose α_1 such that $\alpha \geq \alpha_1$ implies $||f_\alpha - f_{\alpha_1}|| < 1$. Having chosen $\{\alpha_k\}_{k=1}^n$ in A, choose $\alpha_{n+1} \geq \alpha_n$ such that $\alpha \geq \alpha_{n+1}$ implies

$$||f_\alpha - f_{\alpha_{n+1}}|| < \frac{1}{n+1}.$$

The sequence $\{f_{\alpha_n}\}_{n=1}^\infty$ is clearly Cauchy and, since $\mathscr{H}$ is complete, there exists f in $\mathscr{H}$ such that $\lim_{n\to\infty} f_{\alpha_n} = f$.
It remains to prove that $\lim_{\alpha\in A} f_\alpha = f$. Given $\epsilon > 0$, choose n such that $\frac{1}{n} < \frac{\epsilon}{2}$ and $\|f_{\alpha_n} - f\| < \frac{\epsilon}{2}$. Then for $\alpha \geq \alpha_n$ we have

$$||f_\alpha - f|| \leq ||f_\alpha - f_{\alpha_n}|| + ||f_{\alpha_n} - f|| < \frac{1}{n} + \frac{\epsilon}{2} < \epsilon.$$

□

ef_conv_sum

Definition 54. Let $\{f_\alpha\}_{\alpha\in A}$ be a set of vectors in the Banach space $\mathscr{X}$. Let $\mathscr{F} = \{F \subset A : F \text{ finite}\}$. If we define $F_1 \leq F_2$ for $F_1 \subset F_2$, then $\mathscr{F}$ is a directed set. For each F in $\mathscr{F}$, let $g_F = \sum_{\alpha\in F} f_\alpha$. If the net $\{g_F\}_{F\in\mathscr{F}}$ converges to some g in $\mathscr{H}$, then the sum $\sum_{\alpha\in A} f_\alpha$ is said to converge and we write $g = \sum_{\alpha\in A} f_\alpha$.

Proposition 8. *If $\{f_\alpha\}_{\alpha\in A}$ is a set of vectors in the Banach space $\mathscr{X}$ such that $\sum_{\alpha\in A} ||f_\alpha||$ converges in the real line $\mathbb{R}$, then $\sum_{\alpha\in A} f_\alpha$ converges in $\mathscr{X}$.*

Proof. It suffices to show, in the notation of Definition def_conv_sum 54, that the net $\{g_F\}_{F\in\mathscr{F}}$ is Cauchy. Since $\sum_{\alpha\in A} ||f_\alpha||$ converges, for $\epsilon > 0$, there exists F_0 in $\mathscr{F}$ such that $F \geq F_0$ implies

$$\sum_{\alpha\in F} ||f_\alpha|| - \sum_{\alpha\in F_0} ||f_\alpha|| < \epsilon.$$

Thus for $F_1, F_2 \geq F_0$ we have

$$\begin{aligned} ||g_{F_1} - g_{F_2}|| &= \left\| \sum_{\alpha\in F_1} f_\alpha - \sum_{\alpha\in F_2} f_\alpha \right\| \\ &= \left\| \sum_{\alpha\in F_1\setminus F_2} f_\alpha - \sum_{\alpha\in F_2\setminus F_1} f_\alpha \right\| \\ &\leq \sum_{\alpha\in F_1\setminus F_2} ||f_\alpha|| + \sum_{\alpha\in F_2\setminus F_1} ||f_\alpha|| \\ &\leq \sum_{\alpha\in F_1\cup F_2} ||f_\alpha|| - \sum_{\alpha\in F_0} ||f_\alpha|| < \epsilon. \end{aligned}$$

Therefore, $\{g_F\}_{F\in\mathscr{F}}$ is Cauchy and $\sum_{\alpha\in A} f_\alpha$ converges by definition. □

Corollary 12. *A normed linear space $\mathscr{X}$ is a Banach space if and only if for every sequence $\{f_n\}_{n=1}^{\infty}$ of vectors in $\mathscr{X}$ the condition $\sum_{n=1}^{\infty} ||f_n|| < \infty$ implies the convergence of $\sum_{n=1}^{\infty} f_n$.*

Proof. If $\mathscr{X}$ is a Banach space, then the conclusion follows from the preceding proposition. Therefore, assume that $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a normed linear space $\mathscr{X}$ in which the series hypothesis is valid. Then we may choose a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} ||g_{n_{k+1}} - g_{n_k}|| < \infty$ as follows: Choose n_1 such that for $i, j \geq n_1$ we have $||g_i - g_j|| < 1$; having chosen $\{n_k\}_{k=1}^{N}$ choose $n_{N+1} > n_N$ such that $i, j > n_{N+1}$ implies $||g_i - g_j|| < 2^{-N}$. If we set $f_k = g_{n_k} - g_{n_{k-1}}$ for $k > 1$ and $f_1 = g_{n_1}$, then $\sum_{k=1}^{\infty} ||f_k|| < \infty$, and the hypothesis implies that the series $\sum_{k=1}^{\infty} f_k$ converges. It follows from the definition of convergence that the sequence $\{g_{n_k}\}_{k=1}^{\infty}$ converges in $\mathscr{X}$ and hence so also does $\{g_n\}_{n=1}^{\infty}$. Thus $\mathscr{X}$ is complete and hence a Banach space. $\square$

27. Fourier Analysis over finite groups approximating LCA groups

On the on hand there is the following *structure theorem* for LCA (*locally compact Abelian*) groups $\mathscr{G}$. Probably details can be found in Hewitt-Ross ([41]).

Astructure1 **Theorem 39.** *Every LCA group can be approximated by elementary LCA groups, which are of the form* $\mathbb{Z}^k \times \mathbb{R}^d \times \cdots$. *More precisely*
For every LCA group $\mathscr{G}$ there exist arbitrary small compact subgroups $K \lhd \mathscr{G}$ and open subgroups $H \lhd \mathscr{G}$ such that H/K is an elementary group.

This can be used to define the *Schwartz-Bruhat* space (the natural generalization of $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ to LCA groups, see [10]), a much more convenient characterization is given by M.S. Osborne in [52].
It is heavily used in [57]. D. Poguntke has shown that $\boldsymbol{\mathcal{S}}(\mathscr{G}) \subset \boldsymbol{S}_0(\mathscr{G})$ for general LCA groups.

28. Wiener's algebra $W(G)$ over LC groups

algebraSecG

Already the construction of the Haar measure (cf. Cartier's proof the existence of an invariant Haar measure, cf. [69], or [17] uses this space implicitly).
The usual definition of Wiener's algebra (as given e.g. in the book of H. Reiter [55]) involves the use of a lattice, i.e. a discrete and cocompact lattice Λ in the group (typically $\mathbb{Z}^d \lhd \mathbb{R}^d$, with $\mathbb{R}^d = \bigcup_{k \in \mathbb{Z}^d} k + Q$ for some (relatively) compact set $Q \subset \mathbb{R}^d$). But it is possible to describe it equivalently in a different way (in fact without using BUPUs). To justify the general definition we first show that it gives the same space in the well-known situation:

WienerCharG **Lemma 77.** *Given any non-zero, non-negative (bump-) function $k_0 \in \boldsymbol{C}_c(\mathbb{R}^d)$ one has: $f \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^d)$ if and only if there exists a sequence $(x_n)_{n \geq 1}$ in $\mathbb{R}^d$ and a sequence of complex numbers $(c_n)_{n \geq 1} \in \boldsymbol{\ell}^1(\mathbb{N})$ such that*

WienerCsum1

$$|f(x)| \leq \sum_{n \geq 1} c_n T_{x_n} k_0(x), \forall x \in \mathbb{R}^d. \tag{197}$$

29. Segal algebras and the Ideal Theorem

Segal-idthm **Theorem 40.** *For any Segal algebra $(\boldsymbol{S}, \|\cdot\|_{\boldsymbol{S}})$ over a LCA group $\mathscr{G}$ the mappings*

ideal-map

$$I \mapsto I_{\boldsymbol{S}} := I \cap \boldsymbol{S} \qquad I_{\boldsymbol{S}} \mapsto (I_{\boldsymbol{S}})^{\boldsymbol{L}^1} \tag{198}$$

are inverse to each other, i.e. they estabish a bijection between the closed ideals $I \subseteq \boldsymbol{L}^1(\mathscr{G})$ and the closed ideals $I_{\boldsymbol{S}}$ of $\boldsymbol{S}$.

Segal algebras (or as I call them *Reiter's Segal algebras*) have been introduced by Hans Reiter in order to study questions of spectral analysis in a unified way. Due the ideal theorem above one can study the existence of sets of spectral synthesis as via Segal algebras.
Recall that for every ideal there is a *cospectrum*, essentially the complement of all spectra of elements from the closed ideal, better directly described as the (closed) set of common zeros (as the intersection of a collection of closed sets it is closed itself).
However, it may happen that an ideal is not completely characterized by its cospectrum (cosp(I)), let us call it E. In this (very special) situation the minimal closed ideal, which can be shown to be the closure of the subspace of all $\boldsymbol{L}^1(\mathbb{R}^d)$-functions having compact support disjointly from E (since they have compact support they also belong to any Segal algebra $\boldsymbol{S}(\mathscr{G})$), and the maximal ideal, which consists simply of all $\boldsymbol{L}^1(G)$-functions vanishing on the set E, will NOT coincide. In this case E is said to be a closed set for which *spectral synthesis fails*, or a set of non-synthesis.

For a long time the existence of such sets of non-spectral synthesis was open, until it was solved by P. Malliavin ([51]). He was able to show that the surface of the unit-ball in $\mathbb{R}^3$, although a smooth set, is not of spectral synthesis. Later on it was shown that essentially in all non-trivial cases sets of non-synthesis exist within LCA groups.
In it's dual form it is even more interesting. Since the orthogonal complement of the maximal ideal mention above is the set of all point evaluations of $\hat{f}$ on $\hat{G} \setminus E$ spectral analysis *is possible* if for every $\sigma \in \boldsymbol{S}_0'(G)$ with $\mathrm{supp}(\sigma) \subseteq E$ on can find a net of discrete measures, **also supported on the !same! set** E, which is w^*-convergent to the given measure.
See also H. Reiter's book: [55] (or the new version [58]) for this subject (Chap. 7!?).
http://en.wikipedia.org/wiki/Irving_Segal

30. Further Reading and articles by the author

Here we point out articles which have a survey character and have been written in recent years. E.g. the article "Choosing Function Spaces in Harmonic Analysis" ([28]). It tries to discuss the standard approach to Fourier Analysis from a critical view-point, emphasizing that not only does the community of analysts dispose currently over a huge collection of function spaces, but also should start asking itself, which spaces are good for which purpose. So while on the one hand giving a long list of construction principles (certainly not an exhaustive one) the article also encourages the discussion of the "information content" of a function space. Of course there will be more examples in the (near?) future and concrete cases, aside from the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.
There is another article in a similar spirit ([36]) which reinforces the concept of "Conceptual Harmonic Analysis" as the meeting point of application oriented and abstract harmonic analysis. It discusses questions such as "How does the finite Fourier analysis" (e.g. based on the FFT available within MATLAB or other mathematical software packages) allow us to "approximte the continuous reality". Which function spaces and which algorithms do we need in order to come up with good solutions to the problems arising in applications? How can we help engineers to use the proper form of mathematics (and is the currently practised form suitable for this purpose: one may have doubts about this!)?

There is also the hint to the NuHAG talk-server, where a long list of talks on various subjects can be found.

http://www.univie.ac.at/nuhag-php/nuhag_talks/
http://www.univie.ac.at/nuhag-php/home/skripten.php

31. The usual theory of L^p-spaces

In the present description **on purpose** the profile of the usual Lebesgue spaces, even of the certainly important spaces $\boldsymbol{L}^1(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$, has been kept low. In fact, we are convinced that the corresponding *norms* are highly relevant for *any* discussion of aspects of Fourier analysis, while the concrete realization is of much lower importance, essentially due to the fact that any two (metric) completions of a given normed space (e.g. of $\boldsymbol{C}_c(G)$ with the norm inherited from the natural embedding of $\boldsymbol{C}_c(G)$ into $\boldsymbol{M}_b(\mathscr{G})$ via the Haar measure) are isometrically isomorphic anyway, so that claims verified in one setting (and the corresponding technology) is valid for any other realization in exactly the same way.[76]
So we emphasize here that our aim was more to show that one can do all the essential parts of Fourier Analysis without using Lebesgue integration (just the existence of the Haar measure is needed!), although admittedly they are quite prominent in all standard text books. Of course we admit, that it is a fact that both for a proper discussion of convolutions as integrals (existing a.e.) and the definition of the Fourier transform, viewed as an *integral transform*, the *Lebesgue integral* is the right integral. But is the Fourier transform an integral transform by definition or by any other principle, or is it just convenient to express it via an integral, if the object to be transformed happens to be an integrable function (this is of course just our viewpoint!).
We would also like to mention a (negative?) side-effect of having this "perfect integral". It is sometimes (at least psychologically) an obstacle to take a wider perspective. It does not help us to recognize that the Fourier transform is mapping which has the property (if extended properly) that it assigns to a pure frequency the corresponding Dirac measure in the Fourier domain, indicating exactly the frequency and the amplitude of this pure frequency (resp. plane wave for $d = 2$, etc.). So the wider and perhaps more natural perspective on the subject is this one (aside from the technical aspect which involves integrals): How can one *describe* the amount of pure frequency in a given signal (*Fourier Analysis*) and can one resynthesize (using the result of the analysis step) the signal or distribution from the information content? What is the most suitable setting for turning such vague wishes into concrete mathematics? What is the level of technicality that one has to go through in order to have both a mathematically correct description of some model, which ideally should be intimately connected with the physical reality to be described. Finally (and this is even the least developed aspect of Fourier Analysis so far) it should be possible to simulate at least parts of on a computer, which of course is only able to handle finite length vectors.
Having found some indication about some mathematical question formulated in the context of time-frequency or Fourier analysis through computation, we come to the question of "how close is that computed result from the ground-trueth" (i.e. the corresponding numbers that show up when the continuous problem under investigation would have been solved exactly, e.g. by analytic methods)? Again we come to the question of norms for function spaces (in which norm should one measure the deviation) and again one has to hope that functional analysis provides a suitable collection of Banach spaces which allows to describe in a good sense the error due to the reduction to the finite setting in a given situation.
Let us come back to our approach to $\boldsymbol{L}^1(\mathscr{G})$:
In our context $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ (resp. $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$) appears as $\boldsymbol{M}_c(\mathscr{G})$, i.e. as a closed subspace of $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$, which (once the Haar integral is established as a functional on

[76] The argument is along the idea, that the representation of the real number system, the completion of the field of rationals $\mathbb{Q}$, endowed with the Euclidean distance, is conveniently identified with the infinite decimal expressions, but Dedekind sections of equivalence classes of Cauchy sequences in $\mathbb{Q}$ are equally valid interpretations.

$\big(\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(G), \|\cdot\|_{\boldsymbol{W}}\big)$ obtained as the closure of $\boldsymbol{C}_c(G)$ (viewed as space of generalized functions, via the identification of $k \in \boldsymbol{C}_c(G)$ with $\sigma_k \in \boldsymbol{M}_b(\mathscr{G})$). In this sense there is also a dual space (which one may call $\big(\boldsymbol{L}^\infty(G), \|\cdot\|_\infty\big)$, but it should be rather identified with a subspace of $\boldsymbol{S}_0'(G)$.

The discussion of the Hilbert space $\big(\boldsymbol{L}^2(G), \|\cdot\|_2\big)$, of course to be identified with the completion of $\boldsymbol{C}_c(G)$, endowed with the usual scalar product arising from the Haar integral (we write $d\mu(x)$, i.e.

scalprodG

$$\langle f, g\rangle := \int_G f(x)\overline{g(x)} d\mu(x). \tag{199}$$

has to be discussed and developed separately, but should be also done within the reality of $\big(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'}\big)$.

Comment of Nov. 2015, ETH

It should be possible to define the $\boldsymbol{L}^p$-spaces as completions of $\boldsymbol{C}_c(G)$ within $\boldsymbol{S}_0'(G)$. In fact, we would not need any Lebesgue integration theory in that form. Certain "objects" would constitute the elements of that *a priori abstract* completion, and the STFT would be extended (of course for $\boldsymbol{S}_0(G)$-windows g only) to these spaces. The details of such an approach have to be written up in the near future (summer 2017, hgfei).

It is also no doubt that Banach spaces, Banach algebras, but also Banach modules and approximate identities provide a set of tools which helps to organize the facts arising in the context of Fourier Analysis. This is way devoted some space to their desription in these notes.

32. Further general functional analytic considerations

this is to be treated like an APPENDIX, collecting the needed material from general functional analysis, as found in most books on the subject, and as needed anyway, also for other parts of analysis.

E.g. bounded linear operators, dual operators, adjoint operators between Hilbert spaces, Riesz representation theorem, Stone-von-Neumann theorem, Gelfand theory of commutative C^*-algebras etc.. Such material is collection in my course notes of my functional analysis course (WS 13/14), see for example

`http://www.univie.ac.at/nuhag-php/login/skripten/data/FAws1314Fei.pdf`

33. Critical comment on some "BAD" applied literature

33.1. Kanwal's book (for physicists).

`Material already inserted as it is here into TBL 620 (cf. 640), TROUBLE!`

34. Comments on the book by Kanwal on Distributions

[45]
page 10: *Example 3.* says the following (citation):
Various books in physics leave the reader with the impression that $\delta(x) = +\infty$ *at* $x = 0$*. The following sequence illustrates that this is* **not always true:** *.... This sequence is shown in Fig. 1.8, (p.11), which shows that* $\lim_m s_m(x) = -\infty$*, although it yields the unit positive charge for large m; indeed, it is a detal sequence, as will now be proved. [The proof actually takes !2 pages].*
As with Example 4. one just has to use the Stretching operator!
p.14: The **sifting property***,*

$$\int_{-\infty}^{\infty} f(x)\delta(x-\xi)dx = f(\xi),$$

is interpreted as the action of the generalized function $\delta(x-\xi)$ *on* $f(x)$*: that is, when* $\delta(x-\xi)$ *acts on a suitably smooth function* $f(x)$*, it sifts out the value* $f(\xi)$ *at* $x = \xi$*.*
p.28: *Alternative terminology for* **Dirac comb**: The *sampling function* or *replicating function* because
$\sqcup\!\sqcup(x)f(x) = \sum_{n=-\infty}^{\infty} f(x)\delta(x-n)$, but should in fact be $\sqcup\!\sqcup(x)f(x) = \sum_{n=-\infty}^{\infty} \mathbf{f}(\mathbf{n})\delta(x-n)$.
see also the comment on the book of Debnath

34.1. Debnath's article on the genesis of convolution.

Note that Zygmund's books almost does not cotain any relevant hint to convolution, while convolution and the Banach algebra $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ (convolution) is seen as the central object of Harmonic Analysis in Reiter's book ([55]).
This are some personal comments by HGFei (31.05.2017) concerning the article [16] which appeared in ANALYSIS in 2012.
The manuscript is more written for engineers than for mathematicians. For example it uses for the description of the Fourier transform this fomula

DebFT1
$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)dx, \quad k \in \mathbb{R}. \tag{200}$$

Convolution is defined (3.11), p.48 as

DebConv1
$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi, \quad x \in \mathbb{R}. \tag{201}$$

Using mostly Fubini's Theorem various properties, including the convolution theorem and the dual convoluion theorem (pointwise multiplication goes to convolution, if the convolution side has two integrable functions!), also *associativity* of convolution (p.51).

34.1.1. *More serious issues.*

(1) In the definition of $\boldsymbol{L}^1(\mathbb{R})$ (p.46) no *measurability* assumption is made, just integrability of $|f(x)|$;

(2) **p.52: In order to prove the Fundamental Relation for the Fourier transform, here called in Thm.3.3 the General Parseval Relation, a formal step is taken.** The identity

xpintdelta1
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(k-l)x}dx = \delta(k-l) \tag{202}$$

This is later explained (p.63) also via the integral representation of the Dirac delta **FUNCTION (!)**

DebDirac
$$\delta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(k-l)x}dx \tag{203}$$

Intuitively it would be OK to think in this way, because of course the FT of δ_0 is just the constant 1.

(3) The estimate $\|f*g\|_1 \leq \|f\|_1\|g\|_1$ is explained as the fact that $(f*g)(x)$ is at least as smooth as the smoother of the two functions f and g. Derivatives are only discussed later!

(4) Reiter's fundamental relation is only proved afterwards (item 9, p.55)

(5) p.63 Formula DebDirac 203 is also claimed to provide a *simple proof (??)* of the Fourier inverse transform!

(6) Another strange argument is the proof of (3.54), p.58: It makes use of an interchange of limits and integration (without justification), and in addition the claim:

ebDiracLimG
$$\lim_{t\to 0+} G_t(x-\xi) = \delta(x-\xi). \tag{204}$$

Here the Gauss kernel (3.53) is used, in (3.56) later on the Abel kernel is used, with a similar argument.

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