1 Functional Analysis WS 13/14, by HGFei

HERE you will find material concerning the course "functional analysis", held during the winter term 2013/14 at the faculty of mathematics, university Vienna.

Material (Skripten) of other/earlier courses by hgfei can be find at this page

`http://www.univie.ac.at/nuhag-php/home/skripten.php`

1.1 Core Material, Ideas, Motivation

The goal of the *functional analysis course* to provide a good general background of the theory of normed spaces and in particular *Banach spaces* and *Banach algebras*, bounded linear operators between such spaces, and in particular those into the field $\mathbb{C}$ resp. $\mathbb{R}$, i.e. the *linear functionals*[1], which constitute the *dual space* $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$. Therefore checking whether a given space is complete with respect to a suitably chosen norm, realizing certain continuous embeddings, establishing the boundedness of some operator or determine the condition number of an isomorphism of two Banach spaces are among the things to be understood and verified in a number of concrete examples.

In addition that course should raise awareness about the differences between finite dimensional and infinite dimensional situations (see subsection in this direction, towards the end of the script).

1.2 Beyond the Scope of this Course

What this course will not provide is e.g. a detailed background concerning topological vector spaces, or the theory of distributions (generalized functions).

Material (Skripten) of other/earlier courses by hgfei can be find at this page

`http://www.univie.ac.at/NuHAG/FEICOURS/ws1314/FAws1314Fei.pdf`

```
later on: http://www.univie.ac.at/nuhag-php/home/skripten.php
See also:
http://www.univie.ac.at/nuhag-php/login/skripten/data/AKFA1213.pdf
```

for details concerning Banach algebras, Fourier transforms etc. (ca. 95pg).

Exercises found at:

`http://homepage.univie.ac.at/monika.doerfler/UEFA13.html`

Certainly a very interesting (new) reference is the book "Linear Functional Analysis" by Joan Cerda (AMS publication, [2]). I do not yet have this book in our library, maybe in the main library.

[1] This is were the name of the theory, "functional analysis" comes from!

Inhaltsverzeichnis

1 Functional Analysis WS 13/14, by HGFei **1**
1.1 Core Material, Ideas, Motivation . . . 1
1.2 Beyond the Scope of this Course . . . 1

2 Summary, Material to Digest **4**
2.1 Core Material, Ideas, Motivation . . . 4
2.2 Beyond the Scope of this Course . . . 4

3 Basic Definitions, Banach Spaces **5**
3.1 Basic Facts concerning Banach spaces . . . 6

4 Bounded Linear Operators and Functionals **7**
4.1 The Big Theorems Concerning Operators . . . 13
4.2 Closed Graph Theorem . . . 14
4.3 The Uniform Boundedness Principle . . . 15
4.4 The Hahn-Banach Theorem . . . 16
4.5 Existence of projections onto closed subspaces . . . 18
4.6 Dual Operators and Annihilators . . . 22
4.7 Examples of dual and adjoint operators . . . 24
4.8 Consequences of the Big Theorems . . . 26

5 Banach Algebras and Banach Modules **28**
5.1 Inversion in Banach Algebras . . . 30
5.2 Examples of bounded linear operators . . . 33
5.3 Banach algebras with convolution . . . 34

6 Different kinds of convergence **37**
6.0.1 Examples of subnets . . . 39
6.0.2 Similarities . . . 40
6.0.3 Dissimilarities . . . 40
6.1 Cauchy nets in Banach Spaces II . . . 41
6.2 Banach's fixed point theorem . . . 43
6.3 Strong Operator Convergence . . . 43
6.4 Iterated limits of nets and interchange of order . . . 45
6.5 Strong operator convergence and dual operators . . . 47
6.6 A number of simple applications in HA . . . 49
6.7 Weak versus w^*-convergence . . . 51
6.8 Characterizing weak convergence in Banach spaces . . . 54
6.9 Dual and adjoint operators for Hilbert spaces . . . 55
6.10 The Banach-Alaoglu Theorem . . . 55
6.11 The dual of $\boldsymbol{B}'$ with the w^*-topology . . . 57

7 Standard Material which can be done anytime **60**
7.1 Quotient Spaces . . . 60
7.2 Finite Products of Banach spaces . . . 61

8 Hilbert Spaces and Orthonormal Bases **63**
8.1 Scalar Products, Orthonormal Systems 63
8.2 Compact sets in concrete Banach spaces 66
8.3 Compact Operators 67

9 Spectral Theory for Compact Operators **72**

10 Comparing Linear Algebra with Functional Analysis **74**

11 Frames and Riesz projection bases **74**

12 Convolution algebras **77**
12.1 Locally compact Abelian Groups 77

13 Various Facts **77**

14 Further Literature **78**

2 Summary, Material to Digest

2.1 Core Material, Ideas, Motivation

TEST of notations and new symbols... The goal of the *functional analysis course* to provide a good general background of the theory of normae spaces and in particular *Banach spaces* and *Banach algebras*, bounded linear operators between such spaces, and in particular those into the field $\mathbb{C}$ resp. $\mathbb{R}$, i.e. the *linear functionals*[2], which constitute the *dual space* $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$. Therefore checking whether a given space is complete with respect to a suitably chosen norm, realizing certain continuous embeddings, establishing the boundedness of some operator or determine the condition number of an isomorphism of two Banach spaces are among the things to be understood and verified in a number of concrete examples.

In addition that course should raise awareness about the differences between finite dimensional and infinite dimensional situations (see subsection in this direction, towards the end of the script).

2.2 Beyond the Scope of this Course

What this course will not provide is e.g. a detailed background concerning topological vector spaces, or the theory of distributions (generalized functions).

Siehe u.a. auch das Buch von Dirk Werner [19]

```
http://de.wikipedia.org/wiki/Dualraum#
Der_starke_Dualraum_eines_lokalkonvexen_Raums

http://de.wikipedia.org/wiki/Schwach-*-Topologie

WICHTIG:
http://de.wikipedia.org/wiki/Satz_von_Banach-Alaoglu
Details on this topic publically available in:
http://www.mathematik.uni-muenchen.de/~lerdos/WS06/FA/alaoglu.pdf
```

[2] This is were the name of the theory, "functional analysis" comes from!

3 Basic Definitions, Banach Spaces

We take the concept of *vector spaces* (typically denoted by $\mathbf{V}, \mathbf{W}$ etc.) over a *field* ("Körper in German), e.g. $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ as granted.

A vector space (or a linear space) allows to form finite linear combinations of its elements (viewed as "arrows" or just as points, with the usual rules of computation). Therefore in any such vector space the concept of *linear independence* is well defined, as well as the concept of "generating systems".

A set $M \subset \mathbf{V}$ is called *linear independent* if every finite linear subset is linear independent in the classical set. As we will see later this is a bit in contrast with the consideration of infinite sums that we will promoted as a better concept during the rest of the course.

Recall that the core material of *Linear Algebra* concerns *finite dimensional vector spaces* and *linear mappings* between them. Due to the existence of finite bases their theory is equivalent to matrix analysis, the composition (or inversion) of linear mappings is equivalent to the corresponding action on matrices.

However, for many applications there are spaces which are too large to be finite dimensional. So clearly one thinks of *infinite-dimensional spaces*. But this should not be mis-interpreted as space having an infinite basis (!). In fact, there are Banach space which do not have a basis, the first example was found by Per Enflo in 1973, [6]. In fact the discussion of bases in Banach spaces is a fairly complicated field, if carried out in full generality (see the books of I. Singer, [14]).

Therefore it is important to understand "infinite-dimensional" as the logical negation of "finite-dimensional": A space is not-finite dimensional if for every $k \in \mathbb{N}$ there exist more (in the sense of "at least") k linear independent vectors. A prototypical example is the space of polynomial functions $\mathscr{P}(\mathbb{R})$ of all polynomial functions on the real line, because the infinite set of monomials $\{1, t, t^2, ...\}$ is a linear independent set (for obvious reasons).

There is a number of important concepts that are at the starting point of *functional analysis*.

Definition 1. A function $x \mapsto \|x\|$ from $\mathbf{V} \to \mathbb{R}^+$ is called a *norm* on the vector space $\mathbf{V}$ if it has the following three properties

1. $\|x\| = 0 \Rightarrow x = 0 \in \mathbf{V}$;
2. $\|x + y\| \leq \|x\| + \|y\|$ (called the *triangular inequality*, symbol used: $\stackrel{\triangle}{=}$);
3. $\|\lambda x\| = |\lambda| \|x\| \ \forall x \in \mathbf{V}, \lambda \in \mathbb{K}$;

A vector space $\mathbf{V}$ equipped with a norm $\|\cdot\|$ is called a *normed vector space* $(\mathbf{V}, \|\cdot\|)$.

Definition 2. A set $M \subseteq \mathbf{V}$ is called *total* in $(\mathbf{V}, \|\cdot\|)$ if its closed linear span coincides with the whole normed space $\mathbf{V}$.

Rewritten explicitly: For any $\mathbf{v} \in \boldsymbol{V}$ and $\varepsilon > 0$ there exists a finite linear combination of elements from M, i.e. some vector $v_M = \sum_{k=1}^{K} c_k m_k$ with $\|v - v_M\|_{\boldsymbol{V}} \leq \varepsilon$.

Definition 3. Convergent sequences, Cauchy sequence (**CS**) **in** $(\mathbf{V}, \|\cdot\|)$**; usual facts.**

Definition 4. A normed space which is complete, i.e. with the property that every CS has a limit in $(\mathbf{V}, \|\cdot\|)$, is called a *Banach space*, hence we use (mostly!) from now on $\big(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}}\big)$ as a description of a Banach space.

Definition 5. For a given norm there is a natural *metric* on $\mathbf{V}$ defined by

$$d(\mathbf{v}, \mathbf{w}) := \|\mathbf{v} - \mathbf{w}\|, \quad \mathbf{v}, \mathbf{w} \in \mathbf{V}. \tag{1}$$

Proposition 1. *The mapping defined via equ. 1 is indeed a metric on* $\mathbf{V}$*, i.e. satisfies the usual three axioms:*

1. $d(\mathbf{v}, \mathbf{w}) = 0 \Rightarrow \mathbf{v} = \mathbf{w}$*;*
2. $d(\mathbf{v}, \mathbf{w}) = d(\mathbf{w}, \mathbf{v})$*, (symmetry);*
3. $d(\mathbf{v}, \mathbf{x}) \leq d(\mathbf{v}, \mathbf{w}) + d(\mathbf{w}, \mathbf{x}), \mathbf{x}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$*; (Triangle inequality,* Δ*).*

It is also translation invariant, i.e.

$$d(\mathbf{v} + \mathbf{x}, \mathbf{w} + \mathbf{x}) = d(\mathbf{v}, \mathbf{w}). \tag{2}$$

As an obvious consequence the open ball of radius r, denoted by $B_r(\mathbf{v})$ around $\mathbf{v} \in \boldsymbol{V}$ is the same as moving the corresponding ball $B_r(\mathbf{0})$, i.e. we have $B_r(\mathbf{v})$

3.1 Basic Facts concerning Banach spaces

Proposition 2. *Any closed subspace* $\boldsymbol{B}_0$ *of a Banach space* $\big(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}}\big)$*, endowed with the norm inherited from* $\big(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}}\big)$*, is a Banach space itself.*

Also the converse is true: Assume that a subspace $\boldsymbol{B}_0 \subseteq \boldsymbol{B}$ *is a Banach space with respect to the norm inherited from* $\big(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}}\big)$*, then it must be a closed subspace.*

Proof. Details of the proof have been given during the course.

The direct part is easier and more important in practice: Assume that $\boldsymbol{B}_0$ is a closed subspace of a Banach space $\big(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}}\big)$. In order to verify completeness of $(\boldsymbol{B}_0, \|\cdot\|_{\boldsymbol{B}})$ we have to consider and arbitrary CS within $(\boldsymbol{B}_0, \|\cdot\|_{\boldsymbol{B}})$, e.g. $(\mathbf{b}_k^0)_{k=1}^{\infty}$. Since it is also (!obviously, the norm is the same) a CS in the larger space $\big(\boldsymbol{B},\, \|\cdot\|_{\boldsymbol{B}}\big)$ it will be convergent there, with some limit $b \in \boldsymbol{B}$. Hence we are looking into the situation that we have a *convergent* (!) sequence of elements from $\boldsymbol{B}_0$, but because this is (by assumption) a *closed subset* the limit must also belong to $\boldsymbol{B}_0$. Altogether the given (arbitrary) CS has a limit $b \in \boldsymbol{B}_0$, hence $(\boldsymbol{B}_0, \|\cdot\|_{\boldsymbol{B}})$ is a Banach space.

The verification of the converse is left to the reader. □

As an application of the above principle let us establish basic facts concerning the space $\boldsymbol{C}_0(\mathbb{R}^d)$ (in fact, this spaces, denoted by $\big(\boldsymbol{C}_0(G),\, \|\cdot\|_{\infty}\big)$, could be defined over any locally compact group $\mathscr{G}$ instead of $\mathscr{G} = \mathbb{R}^d$).

Definition 6. .
$\boldsymbol{C}_c(\mathbb{R}^d) := \{k : \mathbb{R}^d \to \mathbb{C},\ k \,\text{continuous},\ \text{supp}(k)\ \text{compact}\}$
where $\text{supp}(k)$ is the closure of $\{x \,|\, k(x) \neq 0\}$, and
$\boldsymbol{C}_0(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C},\ f \,\text{continuous},\ lim_{|x|\to\infty}|f(x)| = 0\}$

Proposition 3. *The space* $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ *is the closure (resp. the closed linear span) of* $\boldsymbol{C}_c(\mathbb{R}^d)$ *in* $\big(\boldsymbol{C}_b(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$*. Hence* $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ *is a Banach space, containing* $\boldsymbol{C}_c(\mathbb{R}^d)$ *as a dense subspace.*

Proof. to be given soon (after the course next week). Try yourself first! □

4 Bounded Linear Operators and Functionals

While in the finite-dimensional situations *all* linear mappings between two vector spaces are considered, one has to exclude some "pathological cases" in the infinity dimensional situation. Alternatively one can view the situation also in this way: The "natural" objects of considerations are the continuous linear mappings. While linear mappings between finite dimensional spaces are *automatically (!)* continuous [3] this is not true anymore if the domain space is infinite dimensional (in the sense that for any $N \in \mathbb{N}$ there exist at least N linear independent vectors in the space).

For this reason we are studying the "good linear mappings", which also "respect the convergence structure".

Definition 7. A mapping T between metric spaces (e.g. normed spaces) is continuous if the usual $\varepsilon - \delta$-condition is satisfied, or equivalently (o.B.!) if T maps convergent sequences (in the domain) to convergent sequences in the range; or in formulas

$$x_n \to x_0 \text{ for } n \to \infty \quad \Rightarrow \quad T(x_n) \to T(x_0), \text{ for } n \to \infty. \tag{3}$$

Exercise 1. Show that the mapping $\mathbf{v} \to \|\mathbf{v}\|$ from $(\mathbf{V}, \|\cdot\|)$ to $(\mathbb{R}, |\cdot|)$ is continuous! [4]

The norm also allows to define *boundedness* and thus clearly there is interest in mappings between normed spaces "respecting boundedness". For this reason let us give the pertinent definitions:

Definition 8. A subset of a normed space $M \subseteq (\boldsymbol{V},\, \|\cdot\|_{\boldsymbol{V}})$ is called *bounded* (with respect to that specific norm) if

$$\sup_{\mathbf{m} \in M} \|\mathbf{m}\|_{\boldsymbol{V}} < \infty.$$

Definition 9. A linear mapping T from $(\boldsymbol{B}^1,\, \|\cdot\|^{(1)})$ to $(\boldsymbol{B}^2,\, \|\cdot\|^{(2)})$ is called *bounded* if there exists some constant $C = C(T) \geq 0$ such that

$$\|T\mathbf{x}\|^{(2)} \leq C\|\mathbf{x}\|^{(1)}, \quad \text{for} \quad \mathbf{x} \in \boldsymbol{B}^{(1)}. \tag{4}$$

The infimum over all such constants is called the *operator norm* of T, and we write $|||T|||$.

Exercise 2. It is a good exercise to verify that

$$|||T||| = \sup_{\|\mathbf{v}\|_{\boldsymbol{V}} \leq 1} \|T(\mathbf{v})\| = \sup_{\mathbf{v} \neq 0} \frac{\|T(\mathbf{v})\|_{\boldsymbol{W}}}{\|\mathbf{v}\|_{\boldsymbol{V}}}. \tag{5}$$

[3] o.B, i.e. "ohne Beweis", resp. without proof, for now.

[4] In fact this is quite easy: using the estimate $|\|\mathbf{u}\| - \|\mathbf{v}\|| \leq \|\mathbf{u} - \mathbf{u}\|$, which is an immediate consequence of the triangular inequality for the norm.

If it concerns operators between two different norm spaces we will indicate that in the subscripts.

Exercise 3. Show that a linear mapping T between two normed spaces is *bounded* if and only if it maps bounded subsets into bounded subsets, i.e. iff it preserves boundedness.

It is easy to verify the following basic facts:

Lemma 1. *(i) Given two bounded set M_1 and M_2 their complex sum, i.e. the set*

$$M := M_1 + M_2 := \{m \,|\, m = m_1 + m_2, m_1 \in M_1, m_2 \in M_2\}$$

is a bounded set as well. In particular M is bounded if and only if $x + M$ is bounded (for some/all $x \in \boldsymbol{V}$).

(ii) A set is M is bounded if and only if λM is bounded for some (resp. all!) non-zero scalars $\lambda \in \mathbb{K}$.

Due to the fact that any linear mapping T is compatible with scalar multiplication one easily finds (Ex.!) that T is bounded if and only if it maps the unit ball $B_1(\mathbf{0})$ into some bounded set. Consequently it is natural to define the "size" (i.e. a norm) on the set of bounded linear operators (generalizing the size of a matrix, in MATLAB: `norm(A)`, providing the maximal singular value!) as the size of $T(B_1(\mathbf{0}))$, or more precisely:

Definition 10.

$$|||T||| := \sup_{\|\mathbf{v}\|_{\boldsymbol{V}} \leq 1} \|T(\mathbf{v})\|_{\mathbf{W}}.$$

Sometimes it is appropriate to indicate that the operator norm concerns a mapping from $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$ to $(\boldsymbol{W}, \|\cdot\|_{\boldsymbol{W}})$ by using the symbol $|||T|||_{\mathbf{V}\to\mathbf{W}}$ or in a similar situation $|||T|||_{\boldsymbol{B}^1\to\boldsymbol{B}^2}$. If the target space is the same as the domain of course only one of the two symbols suffices, e.g. $|||T|||_{\boldsymbol{B}}$ or $|||T|||_{\mathbf{V}}$.

Definition 11. A bounded linear mapping T between normed spaces with *bounded inverse* is called an *isomorphism* of two normed spaces.

The *condition number* of T is defined by

$$\mathrm{cond}(T) := |||T|||_{\boldsymbol{B}^1\to\boldsymbol{B}^2} \cdot |||T^{-1}|||_{\boldsymbol{B}^2\to\boldsymbol{B}^1}, \tag{6}$$

or in short (of the norms are clear) $\kappa(T) := |||T||| \cdot |||T^{-1}|||$.

This quantity describes the quality of the isomorphism. More as an exercise on terminology we note the following lemma:

Lemma 2. *Let T be an isomorphism between two normed spaces $(\boldsymbol{V}^1, \|\cdot\|^{(1)})$ resp. $(\boldsymbol{V}^2, \|\cdot\|^{(2)})$. Then $(v_k)_{k=1}^{\infty}$ is a CS in $(\boldsymbol{V}^1, \|\cdot\|^{(1)})$ if and only if $(T(v_k))_{k=1}^{\infty}$ is a CS in $(\boldsymbol{V}^2, \|\cdot\|^{(2)})$. In particular isomorphic normed spaces are simultaneously either complete (i.e. Banach) or incomplete.*

It is then of course also true that isomorphic normed spaces will have isomorphic completions.[5]

[5]The converse is of course *not true* for the following reason: Given a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and a dense subspace with the same norm. Then their completion is isomorphic to $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ for both of them, although - simply for set-theoretical reasons - they are not isomorphic as vector spaces.

It is also easy to check the following claim:

Lemma 3. *The composition of isomorphisms of normed spaces (as well as inverse of isomorphism) are again isomorphisms (and the condition numbers multiply).*

In particular one has: The set of automorphisms (i.e. isomorphism of a $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$ *onto itself) forms a group.*

Remark 1. For the finite dimensional case this situation is of course well known. Since every finite dimensional vector space over the field $\mathbb{K}$ (e.g. $\mathbb{R}$ or $\mathbb{C}$) is isomorphic to $\mathbb{K}^n$ it is clear that in this case the linear isomorphisms (they are automatically continuous in the finite dimensional case!) are just the mappings, which are described by invertible $n \times n$-matrices (for a fixed basis). In fact, the discussion of equivalence of matrices (via conjugation: $\boldsymbol{A} \mapsto \boldsymbol{C} * \boldsymbol{A} * \boldsymbol{C}^{-1}$) corresponds simply to the description of linear mappings with respect to different bases. The resulting group of matrices is known in the literature as $GL(n, \mathbb{R})$.

A special class of bounded operators are the *isometric* ones:

Definition 12. A linear mapping between Banach spaces is called *isometric* if

$$\|T(\mathbf{x})\|_{\boldsymbol{B}^2} = \|\mathbf{x}\|_{\boldsymbol{B}^1}, \quad \forall \mathbf{x} \in \boldsymbol{B}^1.$$

T is an *isometric isomorphism* if it is isometric and also surjective[6].

The unilateral shift in $\boldsymbol{\ell}^2(\mathbb{N})$, which is even isometric, hence injective, but *not surjective* is an instructive example of a situation that *does not occur in finite dimensions*, because in the finite-dimensional context one can argue using bases.

If a mapping is isometric or just has only a trivial nullset, the image of a linear independent set will stay linear independent. So the image of a basis under such a mapping is a linear independent, and has of course the correct number of elements ($= \dim(\mathbf{V})$). Therefore any injective linear mapping on a finite dimensional mapping is also surjective, hence an isomorphism (recall the different standard criteria for the non-singularity of square matrices!).

Lemma 4. *For any isometric isomorphism* J *one has* $\operatorname{cond}(J) = 1$.

Furthermore, for any isomorphism of T *of* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *one has* $T \circ J$ *and* $J \circ T$ *are isomorphisms as well with the same condition number.*

The proof is left to the reader. Note that a special consequence of the above lemma concerns *unitary operators* on Hilbert spaces (cf. below, they are just the analogue of unitary, complex $n \times n$-matrices in linear algebra). In particular it is clear that unitary equivalent invertible linear mappings (like $U_1 \circ T \circ U_2$ and T) have the same condition number.

Remark 2. It is easy to check (Ex.) that a linear map is bounded if and only if the image of the unit ball is a bounded set. In fact the "size" of a bounded linear map is defined as the "size" of the unit ball under T, measured in the norm of the target space.

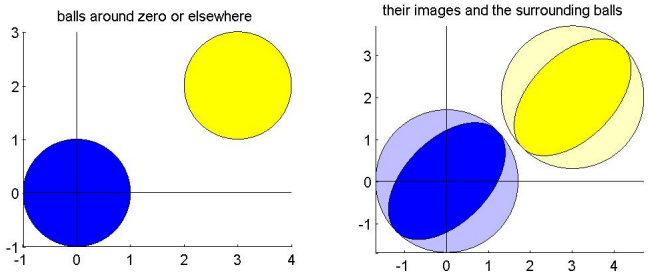


Abbildung 1: TESTBILD

Definition 13. For $\mathbf{b} \in \big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$, $\varepsilon > 0$ one defines the ball of radius ε around $\mathbf{b}$ as

$$B_\varepsilon(\mathbf{b}) := \{\mathbf{v} \in \boldsymbol{B}, \|\mathbf{v} - \mathbf{b}\| < \varepsilon\}. \tag{7}$$

Alternative typography using the macro "Ball" comes out equally: $B_\varepsilon(\mathbf{b})$

Theorem 1. *The operator norm is in fact a norm, and the space of bounded linear operators is a normed space. If the target space is complete, then so is the space of linear operators. In particular, the dual space, the space of linear operators (called functionals) into the underlying field ($\mathbb{R}$ or $\mathbb{C}$) is complete*[7]*.*

Proof. The only interesting part (to be described here) is the completeness claim. Assume the $(T_k)_{k=1}^\infty$ is a CS of operators in $\mathcal{L}(\boldsymbol{B})$, for some Banach space $\big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$ (the general case can be done by the reader in a similar way). Similarly to the proof of the completeness of $(\boldsymbol{C}(I), \|\cdot\|_\infty)$, where one goes from uniform convergence of functions to pointwise (ptw) convergence, we the same here. Fix any $\mathbf{v} \in \mathbf{V}$ and look at the sequence $(T_k(\mathbf{v}))_{k=1}^\infty$. Since

$$\|T_n(\mathbf{v}) - T_m(\mathbf{v})\|_{\mathbf{W}} \le |||T_n - T_m|||_{\mathbf{V}\to\mathbf{W}} \|\mathbf{v}\|_{\mathbf{V}},$$

this is in fact a CS in $\mathbf{W}$ and therefore has a limit. Let us call this limit (defined first for every $\mathbf{v} \in \mathbf{V}$ separately) $T(\mathbf{v})$, so at least we have a candidate for a limit operator. We have to verify that the mapping $\mathbf{v} \mapsto T(\mathbf{v})$ is in fact a linear and bounded operator, and finally that $|||T - T_k|||_{\mathbf{V}\to\mathbf{W}} \to 0$ for $k \to \infty$.

[6]... hence bijective, hence an isomorphism ...

[7]It is *not* necessary that the normed $(\mathbf{V}, \|\cdot\|)$ itself is complete, in order to have completeness of $\mathbf{V}^*$ with its natural norm!

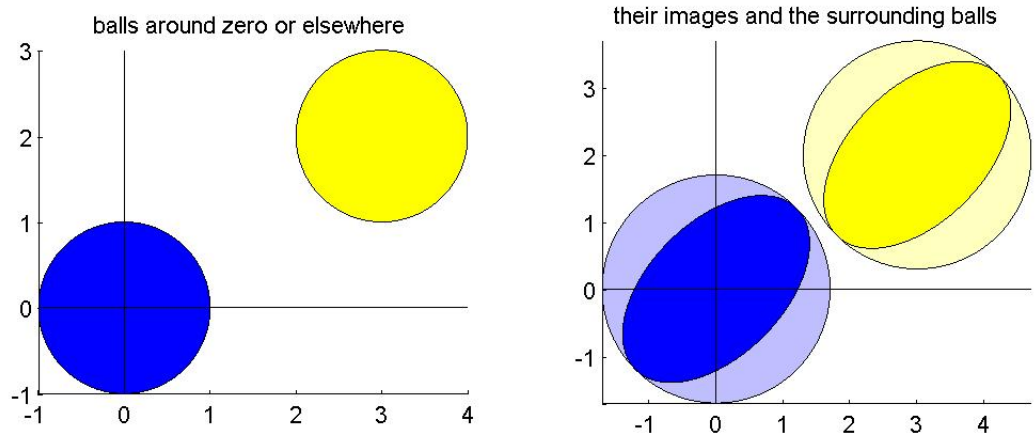


Abbildung 2: TESTBILD2

In fact, for every given $\varepsilon > 0$ we can find k_0 such that $m, n \geq k_0$ implies $|||T_n - T_m|||_{\mathbf{V}\to\mathbf{W}} < \varepsilon > 0$. For $\mathbf{v} \in \mathbf{V}$ with $\|v\|_{\mathbf{V}} \leq 1$ this implies that

$$\|T_n(\mathbf{v}) - T_m(\mathbf{v})\|_{\mathbf{W}} \leq \varepsilon > 0,$$

and hence, by taking the limit

$$\|T_n(\mathbf{v}) - T(\mathbf{v})\|_{\mathbf{W}} \leq \varepsilon > 0, \quad \text{for} \quad n \geq k_0.$$

□

Lemma 5. *A linear mapping T between two normed spaces is continuous if and only if it is bounded, i.e. maps bounded sets into bounded sets. In fact, if T is continuous at any point it is also continuous at zero and overall uniformly continuous.*

Proof. The statement is in fact a statement showing that a homomorphism between topological groups is continuous at zero of and only if it is (in fact even uniformly) continuous. We are here in a metric context, and use only the fact that T is compatible with the additive structure of the involved vector spaces.

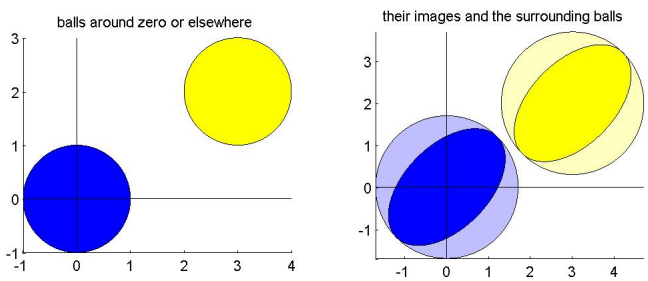


Abbildung 3: TESTBILD3

So assume that $T : \mathbf{V} \to \mathbf{W}$ is a linear mapping continuous at some fixed element $\mathbf{v}_0 \in \mathbf{V}$. This means that for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$T\left(B_\delta(\mathbf{v}_0)\right) \subseteq B_\varepsilon\left(T(\mathbf{v}_0)\right).$$

But since (!check it out yourself, using the properties of the norm)

$$B_\delta(\mathbf{v}_0) - \mathbf{v}_0 = B_\delta(0) \subset \mathbf{V} \quad \text{and} \quad B_\varepsilon(T(\mathbf{v}_0)) = T(\mathbf{v}_0) + B_\varepsilon(0) \subset \mathbf{W}$$

this implies

$$T(B_\delta(0)) = T((B_\delta(\mathbf{v}_0)) - T(\mathbf{v}_0) = T\left(B_\delta(\mathbf{v}_0)\right) \subseteq B_\varepsilon\left(T(\mathbf{v}_0)\right) - T(\mathbf{v}_0) = B_\varepsilon(0).$$

Multiplying both sides in the last containment relation by $1/\delta$ we obtain $T(B_1(0)) \subseteq B_{\varepsilon/\delta}(0)$ in the target space, or boundedness of the linear mapping, with a constant $C \leq \varepsilon/\delta$.

The proof that a *bounded linear mapping* is *uniformly* continuous, with $\varepsilon(\delta) \leq \delta \, |||T|||$ is an easy exercise, using almost the same arguments. $\square$

The following simple result is quite useful in many application situations.

Lemma 6. *Assume that T is a bounded linear mapping from a dense subspace D in $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ into a Banach space $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$. Then there is a uniquely determined extension of T (we keep the name) to all of $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$.*

Proof. Since $\boldsymbol{D}$ is a dense subspace of $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ for any given $\mathbf{x} \in \boldsymbol{B}^1$ there exists a sequence $(\mathbf{d}_k)_{k=1}^\infty$ such that $\mathbf{x} = lim_{k\to\infty}\mathbf{d}_n$, hence $(\mathbf{d}_k)_{k=1}^\infty$ is a CS in $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$, which is mapped onto a CS in $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ due to the boundedness of T:

$$\|T(\mathbf{d}_n) - T(\mathbf{d}_m)\|_{\boldsymbol{B}^2} \leq |||T|||_{\boldsymbol{B}^1 \to \boldsymbol{B}^2} \|\mathbf{d}_n - \mathbf{d}_m\|_{\boldsymbol{B}^1}.$$

According to the completeness of $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ this CS has a limit, which is in fact independent *independent* from the CS chosen (! mix two CS sequence to verify this uniqueness claim!) the only possible definition for $T(\mathbf{x})$ is via the limit relation

$$T(\mathbf{x}) := \lim_{k\to\infty} T(\mathbf{d}_n).$$

□

Remark 3. In many cases the *technical realization* of this extension is not necessarily by the same method.

Application: Proof of Plancherel's theorem. $\boldsymbol{L}^1 \cap \boldsymbol{L}^2$ is a dense subspace of $\boldsymbol{L}^2(\mathbb{R}^d)$, and the Fourier transform can be shown to be an isometry on $\boldsymbol{L}^1 \cap \boldsymbol{L}^2$. The range of $\mathcal{F}$ is *dense* in $\boldsymbol{L}^2(\mathbb{R}^d)$. Thus the above arguments are crucial for the proof of the following theorem:

Theorem 2. *PLANCHEREL's THEOREM (TEX-block 485?)*

The Fourier transform $f \mapsto \mathcal{F}(f)$ is an isometric mapping from $(\boldsymbol{L}^1 \cap \boldsymbol{L}^2)(\mathbb{R}^d), \|\cdot\|_2)$ into $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$, with dense range. Therefore it can be extended from this dense subspace of $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$ to the whole space. It is convenient to call this unique extension the Fourier-Plancherel transform[8]*, which is then establishing an isometric automorphism on $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$ (later we view it as a unitary mapping, because it also preserves scalar products).*

4.1 The Big Theorems Concerning Operators

Theorem 3. *Assume that T is a bijective and bounded linear operator from a Banach space $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to another Banach space $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ Then the two spaces are isomorphic, i.e. the linear mapping T has a bounded inverse, resp. T is an isomorphism between the two spaces.*

Equivalently, elements of the unit sphere (i.e. with norm one) have images well separated from zero, or formulated directly, by claiming that there exists some $C_0 > 0$ such that

$$\|T(\mathbf{x})\|_{\boldsymbol{B}^2} \geq C_0 \|\mathbf{x}\|_{\boldsymbol{B}^1} \quad \textit{for} \quad \mathbf{x} \in \boldsymbol{B}^1. \tag{8}$$

Proof. First of all it is clear that the identity mapping is a continuous mapping from $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ by assumption. If the norms are equivalent on $\boldsymbol{B}^1$ then it is also a bounded linear mapping from $(\boldsymbol{B}^1, \|\cdot\|^{(2)}$ to $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$. □

Proof. This is one of the basic theorems about Banach spaces and has a non-trivial proof, involving Baire's category theorem. It indicates that invertibility as a mapping plus good mapping properties in the forward direction automatically imply good properties of the inverse mapping (linearity is obvious in the same way as in the linear algebra courses). Boundedness is the issue. □

[8] Just at the beginning, to distinguish it from the classical FT defined using and integral transform, because it is not true anymore that for any $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ the Plancherel transform can be obtained pointwise, not even almost everywhere (a.e.) using integrals, even if one makes use of the best possible, namely the Lebesgue integral.

There are many applications of this "strong" result Let us provide two of them:

Corollary 1. *Given a vector space* $\mathbf{V}$*, endowed with two comparable norms, i.e. a weaker and a stronger one, or more precisely with the property, that for some* $C > 0$ *one has (without loss of generality)*

$$\|\mathbf{v}\|_{\boldsymbol{B}^2} \leq C \, \|\mathbf{v}\|_{\boldsymbol{B}^1} \quad \textit{for} \quad \mathbf{v} \in \boldsymbol{V}. \tag{9}$$

If $\mathbf{V}$ *is a Banach space with respect to both norms, then these norms are equivalent.*

Proof. One just has to apply Banach's theorem to the (obviously bijective) identity mapping from $\boldsymbol{V}$ to $\boldsymbol{V}$. The assumption (9) then just expresses the assumption concerning the boundedness of $Id_{\boldsymbol{V}}$ as a mapping from $(\boldsymbol{V}, \|\cdot\|^{(1)}$ to $(\boldsymbol{V}, \|\cdot\|^{(2)}$. □

Corollary 2. *Assume that one has one Banach space* $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ *continuously embedded into another (larger) Banach space* $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$*, and dense in* $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$*.*

Then the two spaces coincide if and only if the corresponding norms are equivalent. Or in other words: if (and only if) the two norms are not equivalent, then the inclusion must be a proper one.

Proof. TO BE DONE! □

4.2 Closed Graph Theorem

Closed Graph Theorem

The closed graph theorem provides one of the most useful methods to check whether a linear mapping between Banach spaces is bounded resp. continuous. It requires to make use of the Graph of a mapping T from $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$, namely

$$G(T) := \{(\mathbf{x}, T\mathbf{x}) \in \boldsymbol{B}^1 \times \boldsymbol{B}^2\}.$$

Theorem 4. *A linear mapping* T *between two Banach spaces is bounded resp. continuous if and only if it has closed graph, i.e. if and only if the set* $G(T)$*, considered as a linear subspace of* $(\boldsymbol{B}^1, \|\cdot\|^{(1)}) \times (\boldsymbol{B}^2, \|\cdot\|^{(2)})$ *is a closed subspace.*

In practice it suffices to show the following: Assume the $(\mathbf{x}_n)_{n=1}^{\infty}$ *is a sequence in* $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ *with* $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{0}$ *and furthermore that the image sequence satisfies* $\mathbf{z} = \lim_{n\to\infty} T\mathbf{x}_n$ *for some* $\mathbf{z} \in \boldsymbol{B}^2$*, then one has to identify (only) that* $\mathbf{z} = \mathbf{0}$*.*

Proof. The necessity is obvious, i.e. the fact that any bounded, hence continuous linear mapping T has closed graph.

So the interesting part of the argument (which will rely on Banach's theorem) is the verification that the closedness of the graph $G(T)$ already implies the continuity of the mapping T.

In order to do so one only has to recall two facts:

- Obviously the mapping $\mathbf{x} \mapsto (\mathbf{x}, T\mathbf{x})$ is a bijective mapping from a Banach space to a subspace of the product space $(\boldsymbol{B}^1, \|\cdot\|^{(1)}) \times (\boldsymbol{B}^2, \|\cdot\|^{(2)})$;

- If $G(T)$ is closed, it is a Banach space with respect to the norm in the product space;
- It is also clear that the inverse mapping, $(\mathbf{x}, T\mathbf{x}) \mapsto \mathbf{x}$ is a continuous and surjective mapping;
- hence by Banach's theorem also the inverse of this mapping, which is obviously $\mathbf{x} \mapsto (\mathbf{x}, T\mathbf{x})$ must be continuous as well, hence in particular the mapping $\mathbf{x} \to T\mathbf{x}$ from $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$.

□

4.3 The Uniform Boundedness Principle

The UBP (also called PUB by J. Conway, [4]) concerns the behavior of pointwise convergent sequences of operators. It will be discussed after some considerations concerning alternative forms of convergence (aside from the usual norm convergence).

For the subsequent discussion we need one more of the big theorems of functional analysis, the **PUB** (Principle of Uniform Boundedness).

See p.95 in [4] (Conway's book):

Theorem 5. *Let $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ be a Banach space and $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ be a normed space. If a set $M \subset \mathcal{L}(\boldsymbol{B}^1, \boldsymbol{B}^2)$ is such that for every $\mathbf{x} \in \boldsymbol{B}^1$ on has*

$$\sup\{\|T\mathbf{x}\|_{\boldsymbol{B}^2}, T \in M\} < \infty$$

(we may say that the family M of operators is pointwise bounded) then the set is also bounded in the operator norm sense, i.e.

$$\sup_{T \in M} |||T|||_{\boldsymbol{B}^1 \to \boldsymbol{B}^2} < \infty.$$

WE DO NOT PROVIDE A PROOF HERE FOR REASONS OF TIME

Lemma 7. *Let $\boldsymbol{B}$ be a Banach space and $M \subset \boldsymbol{B}'$. Then M is bounded (in $\boldsymbol{B}'$) if and only if it is pointwise bounded, i.e. if $\sup_{\mathbf{b}' \in M} \mathbf{b}'(\mathbf{b}) < \infty$ for any $\mathbf{b} \in \boldsymbol{B}$.*

There is also the Banach-Steinhaus Theorem.

Theorem 6. *Let $(T_n)_{n=1}^{\infty}$ be a* **sequence** *of bounded linear operators between two Banach spaces, i.e. in $\mathcal{L}(\boldsymbol{B}^1, \boldsymbol{B}^2)$, which is pointwise convergent, i.e. $T_n(\mathbf{x})$ is convergent for any $\mathbf{x} \in \boldsymbol{B}^1$. Then there exists some bounded linear operator $T_0 \in \mathcal{L}(\boldsymbol{B}^1, \boldsymbol{B}^2)$ such that T_n converges to T_0 in the strong operator topology and that the sequence (T_n) is (automatically) bounded in the operator norm sense:*

$$\sup_{n \geq 0} |||T|||_{\boldsymbol{B}^1 \to \boldsymbol{B}^2} \cdot$$

Proof. The main part of this theorem (cf. exercises) to show that the PUB implies

$$\sup_{n \geq 1} |||T|||_{\boldsymbol{B}^1 \to \boldsymbol{B}^2},$$

and hence this property is easily pushed to the limit, i.e. allows to include the limiting index $n = 0$! □

The theorem is also known as the "gliding hump" theorem.

A variant of the UBP gives:

Lemma 8. *A set $M \subset \boldsymbol{B}$ is bounded if and only if it is "weakly bounded", i.e. for every $\mathbf{b}' \in \boldsymbol{B}'$ one has*

$$\sup\{|\mathbf{b}'(\mathbf{b})|, \mathbf{b} \in M\} < \infty.$$

[see [4], p.96]

Since boundedness can be expressed in a weak sense it may now not be not *too* surprising to see that one has:

Proposition 4. *A linear mapping T between two Banach spaces is bounded resp. continuous if and only if it is weakly-to-weakly continuous, i.e. if for every weakly convergent net $(x_\alpha)_{\alpha \in I}$ in the domain $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ also the images $(T(x_\alpha))_{\alpha \in I}$ is weakly convergent in $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$.*

Note: somehow one can think of the situation as follows: continuity turns out to be equivalent to boundedness due to the linearity of T (and the compatibility of the considered topologies with the vector space structure). Since we have already seen that boundedness (in the norm senses) and boundedness in the weak sense is equivalent it is not surprising that the corresponding continuity concepts turn out to be equivalent.

We have to check whether HB is used already so far! most likely the material of this last sub-section has to be moved down

4.4 The Hahn-Banach Theorem

However there is a number of interesting consequences!

Corollary 3. *Let $\boldsymbol{V}$ be a normed space, and $M \subset \boldsymbol{V}$. Then M is bounded if and only if for each $\mathbf{v}' \in \mathbf{V}'$ one has*

$$\sup_{\mathbf{v} \in M} \mathbf{v}'(\mathbf{v}) < \infty.$$

Proof. The trick is to consider the space $\boldsymbol{V}$ (just a normed space) as a subspace of $\mathbf{V}''$ (via HB it is isometrically embedded). And as such it is bounded within $\mathcal{L}(\boldsymbol{V}', \mathbb{C})$. This also "explains" why one does not need the completeness in this theorem! □

The Hahn-Banach Theorem[9] guarantees the existence of sufficiently many linear functionals. A good summary is found in [11], section 9.

The linear algebra situation is quite simple: Assume a linear functional is given on subspace $\mathbf{W}$ of a vector space $\mathbf{V}$. Then one can easily extend it to the full vector space by putting it to zero "on the rest". But how can one really do it? The natural way (in the finite dimensional setting) would be to first take any basis for $\mathbf{W}$ and then extend it by adding more elements to a basis for the full space $\mathbf{V}$. In this way one can in principle expand every element of $\mathbf{v}$ in the full basis and *project* them into $\mathbf{W}$ by ignoring the additional basis vectors (putting them to zero, of you want so). This is certainly a linear procedure and also idempotent, i.e. applied to elements in the

[9] Hans Hahn was a professor at the University Vienna, look at the "Ehrentafel" at the entry hall of our building, OMP1. See also WIKIPEDIA: Hans Hahn Mathematiker

range of this linear mapping nothing changes anymore. Therefore, applying $\mathbf{w}^* \in \mathbf{W}^*$ to the projected version of $\mathbf{v}$ provides us with a linear functional $\mathbf{v}^*$ which is clearly an extension of original functional. But what can be said about the norm of this extended linear functional. It would be nice to know the norm of the projected elements, but in the generality just described nothing can be said about their length. In particular, it *not at all clear* whether the length of the projected vector should be smaller than the original one?

Here is a short MATLOW experiment:

```
>> A = rand(5), B = pinv(A'),
>> xx = rand(5,1);  xx = xx/norm(xx);
>> for kk = 1 : 5; norm( A(:,kk) * B(kk,:)), end
>> for kk = 1 : 5; norm( A(:,kk) * B(kk,:)*xx), end
```

It shows that most of the time at least in one direction the projection is longer than the full vector!!

Of course one could (and should) an orthonormal basis for $\mathbf{W}$ and extend that to an ONB of $\mathbf{V}$, because then Pythagoras could demonstrate that the (now orthogonal) projection onto $\mathbf{W}$ is non-expansive, hence the above procedure (putting the linear functional zero on the orthogonal complement) works.

But what about infinite dimensions? Is there still any extension of a functional on $\mathbf{W}$ to a functional on all of $\mathbf{V}$ (even if this is a huge Banach space?). Secondly - even in the situation just described - it is not at all clear why/whether it is possible to keep the norm of the linear functional unchanged during this extension process.

Both questions have a positive answer in the HB-Theorem, which comes in many different versions. It concerns Banach spaces which are vector spaces over the real or complex numbers. Let us first consider the crucial version, which concerns the *real case*:

Theorem 7. *Assume that $\mathbf{V}_0$ is a linear subspace of a real vector space $\mathbf{V}$. We assume that there exists a linear map $f_0 : \mathbf{V}_0 \to \mathbb{R}$, which is controlled by some sublinear function $p : \mathbf{V} \to \mathbb{R}$* [10] *which means that*

$$-p(\mathbf{y}) \leq f_0(\mathbf{y}) \leq p(\mathbf{y}) \ , \forall \mathbf{y} \in \mathbf{V}_0$$

.

Then there exists a linear form extending f defined on all of $\mathbf{V}$, i.e. with $f(\mathbf{y}) = f_0(\mathbf{y})$, $\forall \mathbf{y} \in \mathbf{V}_0$, such that f is now also controlled by p in the sense of

$$-p(-\mathbf{v}) \leq f(\mathbf{v}) \leq p(\mathbf{v}) \quad \forall \mathbf{v} \in \boldsymbol{V}.$$

Since $p(\mathbf{y}) = \|\mathbf{y}^*\|_{\mathbf{Y}^*}\|\mathbf{y}\|_{\boldsymbol{B}}$ is the most natural choice for such a sublinear functional, if $\mathbf{y}^*$ is a bounded linear functional on $\mathbf{V}_0$ one finds among others also some extension $\mathbf{b}^*$ with $\|\mathbf{b}^*\|_{\boldsymbol{B}^*} = \|\mathbf{y}^*\|_{\mathbf{Y}^*}$.

NEEDS CORRECTION: MIX OF SPACE $\boldsymbol{V}, \boldsymbol{B}$ ABOVE! MAYBE SEPARATE COROLLARY, SEPARATION VERSIONS?

[10] This means that p is defined on all of $\boldsymbol{V}$ and satisfies $p(\alpha\mathbf{x}) = \alpha p(\mathbf{x})$ for $\alpha > 0$ and $p(\mathbf{x}_1 + \mathbf{x}_2) \leq p(\mathbf{x}_1) + p(\mathbf{x}_2)$, $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{V}$.

Here comes the revised version!

We describe first (following [11]) a *real version* of the theorem, which in fact makes use of the total order of the real line! Since many important functionals on function space (often called spaces of test functions) are generalized functions or measures, we denote the functionals by the letters σ (rather than $\mathbf{v}^*$ or $\mathbf{y}*$). In other words the theorem involves a normed vector space $(\mathbf{V}, \|\cdot\|)$, a closed subspace $\boldsymbol{V}_0$ and the dual spaces $\boldsymbol{V}_0^*$ and VP respectively.

Theorem 8. *Assume that $\boldsymbol{V}_0$ is a linear subspace of a real vector space $\boldsymbol{V}$. We assume that there exists a linear map $\sigma_0 : \boldsymbol{V}_0 \to \mathbb{R}$, which is controlled by some sublinear function $p : \mathbf{V} \to \mathbb{R}$ [11] such that p controls σ_0, i.e. $\sigma_0(\mathbf{v}_0) \leq p(\mathbf{v}_0)$, $\forall \mathbf{v}_0 \in \mathbf{V}_0$.*

Then there exists a linear form $\sigma \in \boldsymbol{V}^*$ extending σ_0, i.e. which is defined on all of $\mathbf{V}$, i.e. with $\sigma(\mathbf{v}_0) = \sigma_0(\mathbf{v}_0)$, $\forall \mathbf{v}_0 \in \mathbf{V}_0$, such that σ is now also controlled by p in the sense of

$$-p(-\mathbf{v}) \leq \sigma(\mathbf{v}) \leq p(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}.$$

Since $p(\mathbf{y}) = \|\sigma_0\|_{\boldsymbol{V}_0^*} \|\mathbf{y}\|_{\boldsymbol{V}}$ is the most natural choice for such a sublinear functional for any $\sigma_0 \in \boldsymbol{V}_0^*$, one finds among others also some extension σ with $\|\sigma\|_{\boldsymbol{V}^*} = \|\sigma_0\|_{\boldsymbol{V}_0^*}$, hence with $\|\sigma\|_{\boldsymbol{V}^*} = \|\sigma_0\|_{\boldsymbol{V}_0^*}$.

WARNING: SOME INCONSISTENCIES ABOVE

4.5 Existence of projections onto closed subspaces

NOTE: The simple relation

$$P^2 = P \Rightarrow (Id - P)^2 = Id^2 - 2P + P^2 = Id - P \tag{10}$$

shows that the existence of a projection means that the space can be *split* into a direct sum! Indeed as we see from (10) above also $Id - P$ is a projection, and furthermore it is trivial to observe that $Id = P + (Id - P)$, hence $x = Px + (Id - P)x$, or $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, uniquely (!!, please check) with $\mathbf{x}_1$ in the range of P, and $\mathbf{x}_2$ in the range of the projection $(Id - P)$.

The existence of such projections *cannot be answered positively* in the general case (it is almost trivial in Hilbert spaces, because any subspace has an orthonormal bases, and hence the projection can be written in the most natural way, almost as in $\mathbb{R}^n$).

See [11], Satz 9.18, p.177:

Theorem 9. *Let $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$ be any normed space and let $\mathbf{W}$ be a finite-dimensional subspace ($\dim(\mathbf{W}) = n$). Then there exists a continuous projection P from $\boldsymbol{V}$ onto $\mathbf{W}$, i.e. (by definition) there exists a continuous linear map P from $(\mathbf{V}, \|\cdot\|)$ into $(\mathbf{W}, \|\cdot\|_{\mathbf{V}})$, with $P^2 = P$.*

Proof. Given a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ for $\mathbf{W}$ one can find a uniquely determined dual basis (! cf. linear algebra course) $\phi_1, \ldots, \phi_n$, i.e. a family of linear functionals on $\mathbf{W}$ with the property

$$\phi_k(\mathbf{v}_l) = \delta_{k,l}, \quad 1 \leq k, l \leq n.$$

[11] This means that p is defined on all of $\boldsymbol{B}$ and satisfies $p(\alpha\mathbf{x}) = \alpha p(\mathbf{x})$ for $\alpha > 0$ and $p(\mathbf{x}_1 + \mathbf{x}_2) \leq p(\mathbf{x}_1) + p(\mathbf{x}_2)$, $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{V}$. The use of the letter p in this context is the traditional one.

According to Hahn-Banach there exists bounded linear extensions $\mathbf{v}_k^*, 1 \leq k \leq n$, resp. $\mathbf{v}_k^*(\mathbf{w}) = \phi_k(\mathbf{w}), \forall \mathbf{w} \in \mathbf{W}$. The possible extensions can then be given by

$$P(\mathbf{v}) := \sum_{j=1}^{n} \mathbf{v}_j^*(\mathbf{v})\mathbf{v}_j, \quad \mathbf{v} \in \boldsymbol{V}. \tag{11}$$

□

However it is *not true at all* that one can find for a general closed subspace $\mathbf{W}$ in some normed or even Banach space *any* continuous projection having $\mathbf{W}$ as its range!! The spaces with this property (they are called *complemented* subspaces) can be characterized as follows ([11], Satz 9.17, p.176):

Proposition 5. *A Banach space $\boldsymbol{B}$ is the direct topological sum of two subspaces $\boldsymbol{B}_1 \oplus \boldsymbol{B}_2$, i.e. a direct sum with continuous projects on onto the individual subspaces if and only if both spaces are closed.*

Proof. The defining property of a projection, i.e. $P^2 = P$ implies that the "complementary projection" defined by $I - P$. This is obviously also a continuous linear mapping, and one has $(I - P)^2 = I - 2P + P^2 = I - P$. Clearly $I = P + (I - P)$, hence the following information about the range of the two operators: $\mathrm{Ran}(P) + \mathrm{Ran}(I - P) = \boldsymbol{B}$ and $\mathrm{Null}(P) \cap \mathrm{Null}(I - P) = \mathbf{0}$ (check it out!).

Next we show that $\mathrm{Ran}(P) = \mathrm{Null}(I - P)$ resp. (changing roles) $\mathrm{Ran}(I - P) = \mathrm{Null}(P)$. We just have to verify that the range of a projection operator is exactly the set of all invariant elements. In fact, $\mathbf{y} \in \mathrm{Ran}(P)$ if and only if $\mathbf{y} = P(\mathbf{b})$ for some $\mathbf{b} \in \boldsymbol{B}$. But then $P(\mathbf{y}) = P^2(\mathbf{v}) = P(\mathbf{v}) = \mathbf{y}$. Conversely it is trivial that $\mathbf{y} = P(\mathbf{y})$ implies that $\mathbf{y} \in \mathrm{Ran}(P)$. Since null-spaces are closed also the range spaces are closed. So altogether we have a splitting of $\boldsymbol{B}$ into the direct sum of the two closed subspaces, namely

$$\boldsymbol{B} = \mathrm{Ran}(P) \oplus \mathrm{Null}(P). \tag{12}$$

Conversely assume that we have a Banach space which is the direct sum of two closed subspaces $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ (hence both of them are also Banach spaces with the induced norm $\|\cdot\|_{\boldsymbol{B}}$). Then the product space is a Banach space and the linear mapping $T : \boldsymbol{B}_1 \times \boldsymbol{B}_2 \to \boldsymbol{B}$, defined by $T(\mathbf{b}_1, \mathbf{b}_2) = \mathbf{b}_1 + \mathbf{b}_2$ is clearly linear, bounded and surjective, in fact bijective (since it is assumed to be a *direct!* sum). By Banach's (open mapping) theorem T is an isomorphism, and the proof is complete. □

In the situation just described one calls $\boldsymbol{B}$ a topological direct sum of the two subspaces $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, and one may write

$$\boldsymbol{B} = \boldsymbol{B}_1 \oplus_t \boldsymbol{B}_2$$

where the subscript t indicates "'topologically"[12].

Another immediate consequence of the Hahn-Banach theorem reads as follows:

[12] For non-closed subspaces one might have only a representation respecting the algebraic properties, but not a continuous projection onto the direct factors. Hence we do not consider this case in the context of functional analysis!

Proposition 6. *If* $\mathbf{W}$ *is a closed subspace of* $(\mathbf{V}, \|\cdot\|)$ *and* $\mathbf{v}_1 \in \mathbf{V} \setminus \mathbf{W}$*. Then there exists exists* $\mathbf{v}^* \in \mathbf{V}^*$ *with* $\mathbf{v}^*(\mathbf{w}) = 0\ \forall \mathbf{w} \in \mathbf{W}$*, but* $\mathbf{v}^*(\mathbf{v}_1) \neq 0$*.*

Proof. We look at the quotient mapping $\pi : \mathbf{V} \to \mathbf{V}/\mathbf{W}$. Since $\pi(\mathbf{v}_1) \neq 0$ (!) there exists a bounded linear form ϕ on $\mathbf{V}/\mathbf{W}$ with $\phi(\pi\mathbf{v}_1) \neq 0$. Obviously $\mathbf{v}^* := \phi \circ \pi$ is doing the job. It is a bounded an linear mapping on all of $\mathbf{V}$ with the required property. □

By an indirect argument one derives from this result

Corollary 4. *Assume that* $\mathbf{W}$ *is a subspace of a normed space* $(\mathbf{V}, \|\cdot\|)$*, with the property, that every linear functional on* $(\mathbf{V}, \|\cdot\|)$ *which vanishes on* $\mathbf{W}$ *must be the zero-functional. Then* $\mathbf{W}$ *must be a dense subspace of* $(\mathbf{V}, \|\cdot\|)$*.*

Corollary 5. *For every* $\mathbf{b} \neq \mathbf{0} \in \big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$ *there exists some* $\mathbf{b}^* \in \boldsymbol{B}^*$*, with* $\|\mathbf{b}^*\|_{\boldsymbol{B}^*} = 1$ *such that* $\mathbf{b}^*(\mathbf{b}) = \|\mathbf{b}\|_{\boldsymbol{B}}$*.*

Proof. This is just a consequence of the simple one-dimensional situation and the HB-Theorem.

Reasonable?: It is convenient to reduce the discussion to the case that $\|\mathbf{b}\|_{\boldsymbol{B}} = 1$ **(by suitable rescaling of b, because the general case is easily reduced to this special case).**

First choose the one-dimensional subspace $\boldsymbol{B}_0 := [\mathbf{b}]$ of scalar multiples of $\mathbf{b}$. This is a closed linear subspace, and we can define $\phi_0(\lambda\mathbf{b}) := \lambda\|\mathbf{b}\|_{\boldsymbol{B}}$. This functional clearly satisfies $\phi_0(\mathbf{b}) = \|\mathbf{b}\|_{\boldsymbol{B}}$ and is of norm 1 on $\boldsymbol{B}_0$ (check!).[13] By the HB-Theorem this functional can be extended to all of $\boldsymbol{B}$ with the same norm restriction, as it was claimed. □

Corollary 5 above gives **one of the most important consequences** of the Hahn-Banach Theorem:

Theorem 10. *For every normed space there is a natural embedding of* $\boldsymbol{B}$ *into the double dual space* $\boldsymbol{B}''$*, given by*

$$i_{\boldsymbol{B}}(\mathbf{b})(\mathbf{b}') = \mathbf{b}'(\mathbf{b}), \quad \mathbf{b} \in \boldsymbol{B}, \mathbf{b}' \in \boldsymbol{B}'. \tag{13}$$

which is isometric, hence injective. In particular, $i_{\boldsymbol{B}}(\boldsymbol{B})$ *is a closed subspace of* $\boldsymbol{B}''$ *(with its natural norm).*

There is an immediate question coming up in this context: For which Banach spaces is the natural embedding $i_{\boldsymbol{B}}$ surjective, i.e. a bijection: $i_{\boldsymbol{B}}(\boldsymbol{B}) = \boldsymbol{B}''$? This situation is important and deserves separate terminology:

Definition 14. A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called *reflexive* if the natural embedding $i_{\boldsymbol{B}} : \boldsymbol{B} \to \boldsymbol{B}''$ is an (automatically isometric) bijection between $\boldsymbol{B}$ and $\boldsymbol{B}''$.

Remark 4. Often one finds the statement, that a Banach space of reflexive if and only if $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is isomorphic to $\boldsymbol{B}''$ (as a naturally normed space). There are however "exotic" examples of Banach spaces where such an isomorphism is possible even in the

[13] Since $\mathbf{b}$ is normalized one has $\|\lambda\mathbf{b}\| \leq 1$ if and only if $|\lambda| \leq 1$!

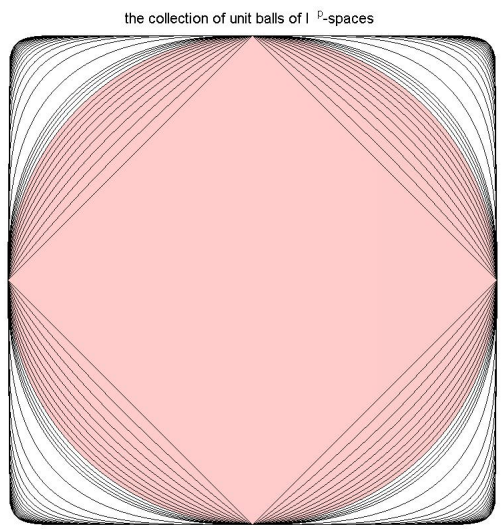
case that the Banach space is itself is not reflexive in the sense of the above definition (see [10]).

For the family of $\boldsymbol{L}^p$-spaces one finds, that there is a natural identification of the dual to $\big(\boldsymbol{L}^p, \|\cdot\|_p\big)$ with $\big(\boldsymbol{L}^q, \|\cdot\|_q\big)$, for the case that $1 \leq p < \infty$, with the pairing $1/p + 1/q = 1$ resp. $1/q = 1 - 1/p$. Note that sometimes people use the symbol p' instead of q for the *conjugate index*. The pairing is based on Hölder's inequality, which implies that under these circumstances on has

$$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q, \quad f \in \boldsymbol{L}^p, g \in \boldsymbol{L}^q. \tag{14}$$

There is of course the convention that $1/\infty$ has to be interpreted as zero.

It is interesting to look at the unit balls of $\mathbb{R}^2$, endowed with the discrete $\boldsymbol{\ell}^p$-norm, with p ranging from $p = 1$ (most internal) to p large forming the outer boundary.



For the sake of completeness let us describe the situation in the form of a proposition:

Proposition 7. *For any $p \in [1, \infty)$ the dual space to $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) = \big(\boldsymbol{L}^p, \|\cdot\|_p\big)$ can be isometrically identified with $\big(\boldsymbol{L}^q, \|\cdot\|_q\big)$, in the following sense:*

Given $g \in \boldsymbol{L}^q$ the functional $\sigma = \sigma_g \in \boldsymbol{L}^{p'}$ is given by

$$\sigma_g(f) := \int f(x)g(x)dx, \quad f \in \boldsymbol{L}^p, 1 \leq p \leq \infty$$

and $\|\sigma_g\|_{\boldsymbol{L}^{p'}} = \|g\|_q$. Conversely, for $p < \infty$ every bounded linear functional on $\big(\boldsymbol{L}^p, \|\cdot\|_p\big)$ is of this form.

For $p = \infty$ the dual space of $\big(\boldsymbol{L}^\infty, \|\cdot\|_\infty\big)$ is strictly larger than $\big(\boldsymbol{L}^1, \|\cdot\|_1\big)$.

4.6 Dual Operators and Annihilators

We first define the concept of dual operators. The mapping assigning to each bounded linear operator a so-called *dual operator*, acting on the corresponding dual spaces, in opposite order, corresponds in terms of linear algebra over $\mathbb{R}$ (using $\mathbf{V} = \mathbb{R}^n$) to the mapping $\boldsymbol{A} \mapsto \boldsymbol{A}^t$ (transposition of matrices). In fact:

Any $m \times n$-matrix $\boldsymbol{A}$ maps $\mathbb{R}^n$ into $\mathbb{R}^m$. The corresponding dual spaces are obviously the linear mappings from $\mathbb{R}^m$ resp. $\mathbb{R}^n$ into $\mathbb{R}$, which are given by $1 \times m$ (resp. n) matrices, i.e. *row-vectors* of corresponding length. On those *row vectors* the transpose matrix is acting (from the right) by matrix multiplication: T' is thus the mapping [14] $\mathbb{R}^m \to \mathbb{R}^n$ (viewed as row vectors), with $\mathbf{y} \mapsto \mathbf{y} * \boldsymbol{A}^t$.

In the reality of normed resp. Banach spaces one may thus expect that any bounded linear mapping is inducing a "corresponding" dual mapping (going between dual spaces, but in the opposite direction).

Definition 15. Given $T \in \mathcal{L}(\mathbf{V},\mathbf{W})$ one defines the *dual operator* $T' \in \mathcal{L}(\boldsymbol{W}^*,\boldsymbol{V}^*)$ via

$$T'(\mathbf{w}^*)(\mathbf{v}) = \mathbf{w}^*(T(\mathbf{v})), \quad \text{resp.} \quad T'(\mathbf{w}^*) = \mathbf{w}^* \circ T. \tag{15}$$

The same definition, given in a slightly different symbolic notation makes things look a bit more natural and we will therefore continue to use the dash-conventions in most of our discussions about dual operators:

Definition 16. Given $T \in \mathcal{L}(\mathbf{V},\mathbf{W})$ one defines the *dual operator* $T' \in \mathcal{L}(\boldsymbol{W}',\boldsymbol{V}')$ via

$$T'(\mathbf{w}')(\mathbf{v}) = \mathbf{w}'(T(\mathbf{v})), \quad \text{resp.} \quad T'(\mathbf{w}') = \mathbf{w}' \circ T. \tag{16}$$

The following lemma, also based on the HB-theorem, shows that the mapping $T \mapsto T'$ is isometric.

Lemma 9. *The mapping $T \mapsto T'$ is an isometric mapping from $\mathcal{L}(\mathbf{V},\mathbf{W})$ to $\mathcal{L}(\boldsymbol{W}',\boldsymbol{V}')$.*

Proof. First of all we show that $|||T'||| \leq |||T|||$, both norms taken in the respective operator norm. By definition we have estimate that action of T' on some element $\mathbf{w}' \in \boldsymbol{W}'$ with $\|\mathbf{w}'\|_{\boldsymbol{W}'} \leq 1$. For this we have then for any $\mathbf{v} \in \boldsymbol{V}$ with $\|\mathbf{v}\|_{\boldsymbol{V}} \leq$:

$$|T'(\mathbf{w}')(\mathbf{v})| = |\mathbf{w}'(T\mathbf{v})| \leq \|\mathbf{w}'\|_{\boldsymbol{W}'} |||T||| \, \|v\|_{\boldsymbol{V}} \leq |||T||| \, .$$

On the other hand we have (by the definition of the operator norm)

$$|||T'||| = \sup_{\|\mathbf{w}'\|_{\boldsymbol{W}'} \leq 1} \|T'(\mathbf{w}')\|_{\boldsymbol{V}'}.$$

which equals (written out in detail) the expression

$$\sup_{\|\mathbf{w}'\|_{\boldsymbol{W}'} \leq 1} \sup_{\|\mathbf{v}\|_{\boldsymbol{V}} \leq 1} |\mathbf{w}'(T\mathbf{v})|$$

[14] Unfortunately there is a conflict of symbols between MATLAB, where $\boldsymbol{A}'$ denotes the transpose conjugate matrix, i.e. $\overline{\boldsymbol{A}}^t$, while the transpose matrix is obtained but the inline command $\boldsymbol{A}.'$

but be changing the order of the sup one has

$$\sup_{\|\mathbf{v}\|_{\boldsymbol{V}}\leq 1}\ \sup_{\|\mathbf{w}'\|_{\boldsymbol{W}'}\leq 1} |\mathbf{w}'(T\mathbf{v})|.$$

By the HB-Theorem the inner sup in this order equals $\|T\mathbf{v}\|_{\boldsymbol{W}}$ and hence we find that

$$|||T'||| = \sup_{\|\mathbf{v}\|_{\boldsymbol{V}}\leq 1} \|T(\mathbf{v})\|_{\boldsymbol{W}} = |||T|||\,.$$

□

Lemma 10. *If T is an isometric isomorphism in $\mathcal{L}(\mathbf{V},\mathbf{W})$ then T' is an isometric*[15] *mapping in $\mathcal{L}(\boldsymbol{W}',\boldsymbol{V}')$.*

Proof. Given $\mathbf{w}' \in \mathbf{W}'$ with $\|\mathbf{w}'\|_{\mathbf{W}'} = 1$ we have to show that $\|T'\mathbf{w}'\|_{\mathbf{W}'} = 1$. So we take a look at

$$\|T'\mathbf{w}'\|_{\mathbf{V}'} = \sup_{\|\mathbf{v}\|_{\mathbf{V}}\leq 1} |T'\mathbf{w}'(\mathbf{v})| = sup_{\|\mathbf{v}\|_{\mathbf{V}}\leq 1}|\mathbf{w}'(T'\mathbf{v})| = sup_{\|T\mathbf{v}\|_{\mathbf{V}}\leq 1}|\mathbf{w}'(T\mathbf{v})| = sup_{\|\mathbf{u}\|_{\mathbf{V}}\leq 1}|\mathbf{w}'(\mathbf{u})|.$$

The last step is justified by the surjectivity assumption on T. Finally the last term equals $\|\mathbf{w}'\|_{\mathbf{W}'} = 1$, as claimed. □

I have not checked whether in each case on can derive that T' is an isometric isomorphism as well. Probably it is!

Next we are going to verify what the adjoint resp. dual operator of some linear operator is, resp. when a given linear operator on a Hilbert space is self-adjoint (i.e. $T^* = T$).

CONVENTION Since the mapping $(\mathbf{x}', \mathbf{x}) \to \mathbf{x}'(\mathbf{x})$ from $\mathbf{V}' \times \mathbf{V}$ to the field $\mathbb{R}$ (or $\mathbb{C}$) is obviously a bilinear mapping, like the ordinary scalar product on $\mathbb{R}^n$, which maps $(\overrightarrow{x}, \overrightarrow{y})$ to the real number $\langle \overrightarrow{x}, \overrightarrow{y}\rangle := \sum_{k=1}^n x_k y_k$ it is customary in functional analysis to also use the following alternative symbolic description for the duality of spaces

$$\langle \mathbf{v}, \mathbf{v}'\rangle_{\mathbf{V}\times\mathbf{V}'} := \langle \mathbf{v}, \mathbf{v}'\rangle := \mathbf{v}'(\mathbf{v}).$$

Given a nonempty set $M \subseteq \mathbf{V}$ resp. a non-empty set $N \subseteq \mathbf{V}'$ we define their *annihilators* as follows

For a normed space $(\mathbf{V}, \|\cdot\|)$ with dual space $\mathbf{V}'$

$$M^\perp := \{\mathbf{v}' \in \mathbf{V}' \,|\, \langle \mathbf{v}, \mathbf{v}'\rangle = 0\ \forall\, \mathbf{v} \in M\} \tag{17}$$

$$^\perp N := \{\mathbf{v} \in \mathbf{V} \,|\, \langle \mathbf{v}, \mathbf{v}'\rangle = 0\ \forall\, \mathbf{v}' \in N\} \tag{18}$$

Theorem 11.

- *The two sets $M^\perp$ and $^\perp N$ defined above are closed spaces;*
- *for $M \neq \emptyset$ one has $^\perp(M^\perp)$ or $^\perp(M^\perp) = \overline{[M]}$, the closed linear span of M in $(\mathbf{V}, \|\cdot\|)$. In particular $^\perp(M^\perp) = M$ for closed subspaces of $\mathbf{V}$;*
- *For $T \in \mathcal{L}(\mathbf{V},\mathbf{W})$*

  $$R(T)^\perp = N(T') \quad \textit{and} \quad \overline{R(T)} =^\perp N(T').$$

  if $R(T)$ is closed, then $R(T) =^\perp N(T')$.

[15] Obviously isometric operators have norm 1, but the converse is not true, therefore the claim of this lemma is *not* trivial!

4.7 Examples of dual and adjoint operators

Remark 5. Given a multiplication operator with some measurable function h, acting on $\mathcal{H} = \boldsymbol{L}^2(\mathbb{R}^d)$, we write $M_h : f \mapsto h \cdot f$. Then $T^* = M_{\bar{h}}$. Consequently we find: M_h is self-adjoint if (and only if!) h is a real-valued function.

Another nice example are convolution operators. For $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ (it is enough to consider $g \in \boldsymbol{C}_c(\mathbb{R}^d)$) one has: $f \mapsto g * f$ defines a bounded linear operator on $\big(\boldsymbol{L}^2(\mathbb{R}^d),\, \|\cdot\|_2\big)$, and in fact

$$|||T|||_2 = \sup_{z\in\mathbb{R}^b} \hat{|}h(z)| \leq \|g\|_1.$$

I this case it is easy to find out that T^* is another convolution operator, namely using the (convolution) kernel

$$g^*(t) = \overline{g}(-t), t \in \mathbb{R}^d$$

Proof. Using the standard scalar product for $\boldsymbol{L}^2(\mathbb{R}^d)$ one has

$$\langle M_h f, g\rangle = \int_{\mathbb{R}^d} h(t)f(t)\overline{g(t)}dt \int_{\mathbb{R}^d} f(t)\overline{g(t)\cdot\overline{h(t)}}dt = \langle f, M_{\overline{h}}g\rangle.$$

□

In order to illustrate this abstract concept let us give a couple of concrete examples: First we define a family of isometric dilation operators on $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$:

Definition 17. Dilation operators on $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ are defined by

$$D_\rho f(z) = f(\rho\cdot z),\ \rho \neq 0,\ z \in \mathbb{R}^d. \tag{19}$$

It is easy to verify that

$$\|D_\rho f\|_\infty = \|f\|_\infty \quad \text{and} \quad D_\rho(f\cdot g) = D_\rho(f)\cdot D_\rho(g) \tag{20}$$

The compatibility of dilation (as defined above) with pointwise multiplication can be expressed by saying that $(D_\rho)_{\rho>0}$ is a *group of isometric automorphisms* of the (pointwise) Banach algebra $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$, with the following composition rule

$$D_{\rho_1}\circ D_{\rho_2} = D_{\rho_1\cdot\rho_2} = D_{\rho_2}\circ D_{\rho_1} \quad \text{and consequently} \quad \mathrm{D}_\rho{}^{-1} = \mathrm{D}_{1/\rho}\,. \tag{21}$$

We (re)call the dual space to the Banach space the space of bounded measures (this is justified by another version of the Riesz representation theorem): $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}) := \big(\boldsymbol{C}_0'(\mathbb{R}^d),\, \|\cdot\|_{\boldsymbol{C}_0'}\big)$.

The motivation for this definition is the fact, that any bounded (and strictly speaking regular Borel) measure μ in the sense of measure theory (cf. real analysis courses) is defining a linear functional $\sigma \in \boldsymbol{M}_b(\mathbb{R}^d)$ via:

$$\sigma(h) := \int_{\mathbb{R}^d} h(t)d\mu(t), \quad h \in \boldsymbol{C}_0(\mathbb{R}^d); \tag{22}$$

in conjunction with a classical theorem that any bounded and linear functional on $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ is given in such a way. By turning this observation into a definition

we have a simple approach to *integration theory* avoiding the cumbersome technicalities of measure theory (but using general functional analytic arguments instead!).

Next we identify the dual operator for a given operator D_ρ, now obviously acting on $\boldsymbol{M}_b(\mathbb{R}^d)$. We will call it the *mass preserving* dilation operator (stretching[16]).

Definition 18.

$$\mathrm{St}_\rho := \mathrm{D}_\rho{}', \quad \text{i.e. } [\mathrm{St}_\rho(\mu)](f) := \mu(\mathrm{D}_\rho f), \quad \rho \neq 0,\ f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

In order to see what this new (dual) dilation is doing on measures we check the action on point measures. Recall that the so-called *Dirac Deltas* $\delta_x : x \mapsto f(x)$ is clearly among the bounded linear functionals on $\boldsymbol{C}_0(\mathbb{R}^d)$, with $\|\delta_x\|_{\boldsymbol{M}_b} = 1, \forall x \in \mathbb{R}^d$.

Lemma 11.

$$\mathrm{St}_\rho \delta_x = \delta_{\rho x}, \quad \rho \neq 0, x \in \mathbb{R}^d.$$

Proof.

$$(\mathrm{St}_\rho \delta_x)(f) := \delta_x(\mathrm{D}_\rho f) = \mathrm{D}_\rho f(x) = f(\rho x) = \delta_{\rho x}(f).$$

□

Obviously the single "point mass" concentrated at x is just moved to the dilated position ρx, without changing the amplitude. On can show (no details given here), that even for general measures (and with a suitable definition of the support of a measure ($\mathrm{supp}(\mu)$) one has

$$\mathrm{supp}(\mathrm{St}_\rho \mu) = \rho \cdot \mathrm{supp}(\mu), \quad \rho \neq 0, \mu \in \boldsymbol{M}_b(\mathbb{R}^d). \tag{23}$$

DUPLICATE RESULT NEXT:?

Lemma 12. *Assume that a bounded net (T_α) which is strongly convergent to T_0 on $(\mathbf{V}, \|\cdot\|)$. Then $(T'_\alpha(\mathbf{w}'))_{\alpha \in I}$ is w^*-convergent to $T_0(\mathbf{w}')$ for every $\mathbf{w}' \in \mathbf{W}'$.*

Proof.

$$T'_\alpha(\mathbf{w}')(\mathbf{v}) = \mathbf{w}'(T_\alpha \mathbf{v}) \ \to\ \mathbf{w}'(T_0 \mathbf{v}) = T'_0(\mathbf{w}')(\mathbf{v}),$$

the convergence following from the assumption about the net of operators (strong convergence) plus the continuity of the linear functional $\mathbf{w}'$ on $(\boldsymbol{W}, \|\cdot\|_{\boldsymbol{W}})$. □

The example of translation operators shows that one cannot expect norm convergence of $T_\alpha(\mathbf{w}')$ to $T_0(\mathbf{w}')$.

Another question arises naturally: Can it happen that the w^*- and the weak topology coincide for non-reflexive (dual) Banach spaces, but this is excluded, since their duals can be determined THIS STATEMENT SHOULD GO ELSEWHERE.

So there was a natural question: Can one show for certain sequences or nets that they are w^*-convergent, but *not weakly convergent*. Obviously one has to look at this question in some dual space of a non-reflexive Banach space. The most natural choice to think of is $\boldsymbol{\ell}^\infty = \big(\boldsymbol{\ell}^1, \|\cdot\|_1\big)'$. Clearly $\mathbf{1}$ is not in the norm closure of the linear span of the unit vectors (in fact it is clear that their closure is just $\big(\boldsymbol{c}_0, \|\cdot\|_\infty\big)$). Since

[16] In German: Stauch und Streckungsoperator

w^*-convergence in $\boldsymbol{\ell}^\infty$ is just coordinate-wise convergence it is clear that the sequence $\mathbf{y}_n := [1, 1, \ldots 1, 0, 0 \ldots] = \sum_{k=1}^n \mathbf{e}_k$ is w^*-convergent to $\mathbf{1}$. In fact, one can make the same statement for the family $\mathbf{y}_F = \sum_{k \in F} \mathbf{e}_k$ where F is running through the (directed) set of finite subsets of the index set $\mathbb{N}$ (this setting applies to $\boldsymbol{\ell}^\infty(I)$, for general index sets in the same way!).

Now we want to explain that this sequence (or the corresponding net) is *not weakly convergent*. Since the w^*-limit is already uniquely determined we do not have to discuss the potential logical possibility that it might be convergent but to a different limit. In order to fulfill the claim it is enough to find *at least one* linear functional $\sigma \in \left(\ell^\infty, \|\cdot\|_\infty\right)'$ such that $\sigma(\mathbf{y}_n)$ does not converge to $\sigma(\mathbf{1})$.

Such a functional can be obtained via Hahn-Banach. We take the zero-functional on $\boldsymbol{c}_0$, and add the one dimensional subspace generated by $\mathbf{1}$. The resulting subspace of $\left(\ell^\infty, \|\cdot\|_\infty\right)$ is then the set of all convergent sequences, with $\lim_{n\to\infty} = \alpha \in \mathbb{C}$, because they are exactly of the form $\mathbf{y} = \alpha\mathbf{1} + \mathbf{z}$ with $\mathbf{z} \in \boldsymbol{c}_0$. On this space obviously $\mathbf{y} \mapsto \alpha$ is a bounded linear functional of (functional) norm 1, which by Hahn Banach can be extended to all of $\boldsymbol{\ell}^\infty$. Since σ vanishes (i.e. is zero) on $\boldsymbol{c}_0$ we have $\sigma(\mathbf{y}_n) = 0$ for all $n \geq 1$ while in contrast obviously $\sigma(\mathbf{1}) = 1 \neq 0$. This completes the argument.

4.8 Consequences of the Big Theorems

The theorems above have a number of important consequences. The Open Mapping Theorem and Banach's Theorem (establishing the homomorphism theorem) establish the principle: If a mapping is bijective (i.e. invertible as a mapping from sets to sets) and bounded, linear (i.e. a *morphism* in the category of Banach spaces, preserving linearity and convergence) then it is also invertible in this category (the inverse is automatically linear - as we know from linear algebra courses - and continuous, as these theorems tell us.

The Closed Graph Theorem typically applies in a somewhat different situation. One has a linear mapping, and wants to establish continuity. If there is some more coarse topology on some ambient, i.e. larger topological vector space (still with the Hausdorff property) then one can show that the identification task necessary to establish the closed graph property can be carried out successfully. Hence, assuming that the mapping is between Banach spaces one gets continuity for free.

Typical applications where such a situation occurs are BK-spaces. Think e.g. of a pointwise multiplier from one sequence space (say $\ell^r(\mathbb{N})$ to another one, say $\ell^s(\mathbb{N})$), mapping one of the spaces into the other (for fixed values $s, r \geq 1$).

It is trivial, that each of the norms has the property that convergence in the norm of the Banach space of sequences implies pointwise convergence (for any $p \geq 1$ one has $|x_k| \leq \|\mathbf{x}\|_p$!) and also clearly any pointwise multiplication operator $\mathbf{x} \mapsto \mathbf{z}$, with $z_k = x_k m_k$ (for some multiplier sequence $(m_k)_{k\in\mathbb{N}}$ is continuous with respect to coordinate convergence (if $|x_k^n - u_k| \to 0$ for $n \to \infty$ for each $k \in \mathbb{N}$ obviously also $|(x_k^n - u_k)m_k| \to 0$ for $n \to \infty$!). Hence it maps spaces (more generally so-called BK-spaces) into each other if and only if the multiplication operator defines a bounded linear operator between the two Banach spaces.

As a trivial special case one has (choosing the sequence $m_k \equiv 1$): it Any embedding of BK-spaces is automatically continuous, and consequently the norm on such a space

is uniquely determined, up to equivalence.

5 Banach Algebras and Banach Modules

Many of the Banach spaces which we meet have an addition structure. Let us therefore , first of all, introduce the concept of a Banach algebra.

Many of the Banach spaces we encounter in functional analysis also have various additional structures. The most important one will be *Banach algebras* (with some kind of internal multiplication) and *Banach modules*, something on which such a Banach algebra can act, very much like scalars act on a general vector space.

Definition 19. A Banach space $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ which is endowed with an additional internal "multiplication", i.e. some *bilinear* mapping $(a, b) \to a \bullet b$, is called a *Banach algebra* if one has the norm estimate

$$\|a \bullet b\|_{\boldsymbol{A}} \leq \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{A}}, \ \forall a, b \in \boldsymbol{A}. \tag{24}$$

A Banach algebra will be called *commutative* if

$$\mathbf{a}_1 \bullet \mathbf{a}_2 = \mathbf{a}_2 \bullet \mathbf{a}_1, \ \forall \mathbf{a}_1, \mathbf{a}_2 \in \boldsymbol{A}.$$

Of course mappings between two Banach algebras respecting this additional structure are the appropriate homomorphisms:

Definition 20. A bounded linear operator from $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ is called a *Banach algebra homomorphism* if products in $\boldsymbol{B}^1$ are mapped into product in $\boldsymbol{B}^2$, resp.

$$T(\mathbf{a} \bullet \mathbf{b}) = T(\mathbf{a}) \star T(\mathbf{b}), \ \forall \mathbf{a}, \mathbf{b} \in \boldsymbol{B}^1. \tag{25}$$

It is called a *multiplicative linear functional* if $\boldsymbol{B}^2 = \mathbb{K}$. It is called a Banach algebra *isomorphism* if it is a bijective BA-homomorphism[17].

Definition 21. A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a *Banach module* over the Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ if there is a *bilinear* mapping $(a, b) \to a \circ b$ if the two multiplications are compatible, i.e. if the following associativity result is valid

$$(\mathbf{a}_1 \bullet \mathbf{a}_2) \circ \mathbf{b} = \mathbf{a}_1 \circ (\mathbf{a}_2 \circ \mathbf{b}), \ \forall \mathbf{a}_1, \mathbf{a}_2 \in \boldsymbol{A}, \mathbf{b} \in \boldsymbol{B}. \tag{26}$$

and furthermore the norm estimate is valid:

$$\|\mathbf{a} \circ \mathbf{b}\|_{\boldsymbol{B}} \leq \|\mathbf{a}\|_{\boldsymbol{A}} \|\mathbf{b}\|_{\boldsymbol{B}}, \ \forall a \in \boldsymbol{A}, \mathbf{b} \in \boldsymbol{B}. \tag{27}$$

More precisely, a Banach module satisfying property 26 above is called a *left* Banach module over $\boldsymbol{A}$ while it is called a *right* Banach module over $\boldsymbol{A}$ if instead one has the following variant of the associativity law ("as if multiplication took place from the right" !)

A simple finite-dimensional example of this situation is the following one (just to give an idea). Let $\boldsymbol{B}$ be the set of all complex $m \times n$-matrices, and $\boldsymbol{A}$ the set of all $m \times m$-matrices, with the module action being defined by $\boldsymbol{A} \bullet \boldsymbol{B} := \boldsymbol{A}^t * \boldsymbol{B}$ (matrix

[17] This is justified because in that case also the inverse mapping is automatically a Banach algebra homomorphism!

multiplication with the transpose matrix from the left). Since $(\boldsymbol{A}_1 * \boldsymbol{A}_2)^t = \boldsymbol{A}_1^t * \boldsymbol{A}_2^t$ this is a (!) right module action, although matrix multiplication is done from the left (otherwise it would not make sense). So it is the form of the associative law which decides, not the position of the multiplication symbol![18]

$$(\mathbf{a}_1 \bullet \mathbf{a}_2) \circ \mathbf{b} = \mathbf{a}_2 \circ (\mathbf{a}_1 \circ \mathbf{b}),\ \forall \mathbf{a}_1, \mathbf{a}_2 \in \boldsymbol{A}, \mathbf{b} \in \boldsymbol{B}. \tag{28}$$

Of course there is only a difference if the Banach algebra is non-commutative.

The symbols used in the context should not matter, e.g. $\circ, \bullet, \cdot, \star, *$ or similar, and most of the time the assumption refbanmoddef02 implies that it is *convenient* to use the same symbol for both the internal and external action. Examples are pointwise multiplication (denoted by $\cdot$, convolution denoted by $*$, or composition of operators with $\circ$ as "multiplication").

The concepts of *Banach module homomorphisms* and of *Banach module isomorphism* is defined in a completely analogous way (cf. above).

We have immediately a couple of such Banach algebras.

We have already a number of examples:

Theorem 12. *$\big(\boldsymbol{C}_b(\mathbb{R}^d),\ \|\cdot\|_\infty\big)$ or $\big(\boldsymbol{C}_0(\mathbb{R}^d),\ \|\cdot\|_\infty\big)$ are Banach algebras with respect to pointwise multiplication.*

Proof. Of course formally the pointwise product is given by the standard convention about pointwise multiplication. The pointwise product $f \cdot g$ is given pointwise via $(f \cdot g)(x) := f(x)g(x), x \in \mathbb{R}^d$. It is clear (from analysis) that $f \cdot g$ is continuous. For any $x \in \mathbb{R}^d$ one has

$$|(f \cdot g)(x)| = |f(x)||g(x)| \leq \|f\|_\infty \|g\|_\infty,$$

hence the smallest upper bound $\|f \cdot g\|_\infty$ will satisfy the required estimate

$$\|f \cdot g\|_\infty \leq \|f\|_\infty \|g\|_\infty,\ \forall f, g \in \boldsymbol{C}_b(\mathbb{R}^d). \tag{29}$$

□

As an exercise we suggest to investigate the question, whether $Lip\,(\alpha)$ is a Banach algebra.

The other prototypical example of non-commutative Banach algebras are Banach algebras of bounded linear operators on a fixed Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. As a special case we have of course the algebra of $n \times n$-matrices with respect to matrix multiplication. Just as an exercise on terminology let us formulate the following

Lemma 13. *Let $\mathbf{V}$ be an n-dimensional normed vector space over $\mathbb{C}$, and let $\mathcal{L}(\boldsymbol{V})$ be the algebra of bounded linear operators from $\mathbf{V}$ into itself, with composition as multiplication and the operator norm*

$$|||T||| := \sup_{\|\mathbf{v}\|_{\mathbf{V}} \leq 1} \|T(\mathbf{v})\|_{\mathbf{V}}. \tag{30}$$

is isomorphic to the algebra of $n \times n$-matrices, with the operator norm over $\mathbb{C}^n$, which can be computed as the maximal singular value of the corresponding matrix $\boldsymbol{A}$, resp. $\sigma_1 = \sqrt{max(eig(\boldsymbol{A}' * \boldsymbol{A}))}$.

[18] One could also multiply with the adjoint matrix (transpose conjugate) from the left, but this is a bit in conflict with the notations for dual operators and therefore we avoid this as an example here.

Theorem 13. *For any Banach space $\boldsymbol{B}$ the space $\mathcal{L}(\boldsymbol{B})$ of all bounded linear operators on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, endowed with the operator norm and ordinary composition is a Banach algebra. In fact, $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a Banach module over $(\mathcal{L}(\boldsymbol{B}), |||\cdot|||)$.*

Proof. It is obvious that $\mathcal{L}(\boldsymbol{B})$ is a linear space with the usual addition and scalar multiplication, and it is easy to verify that $T \mapsto |||T|||$ is actually a norm on $\mathcal{L}(\boldsymbol{B})$. It remains to check that any CS (T_n) in $(\mathcal{L}(\boldsymbol{B}), |||\cdot|||)$ is also convergent to some $T_0 \in \mathcal{L}(\boldsymbol{B})$. So we note that for any given $\varepsilon > 0$ there exists n_0 such that $m, n \geq n_0$ implies $|||T_n - T_m||| \leq \varepsilon > 0$.

For this purpose consider the sequence pointwise, i.e. fix any $\mathbf{b} \in \boldsymbol{B}$ and look at the sequence $(T_n\mathbf{b})$, which is obviously a CS in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, because

$$\|T_n\mathbf{b} - T_m\mathbf{b}\|_{\boldsymbol{B}} \leq |||T_n - T_m|||\, \|\mathbf{b}\|_{\boldsymbol{B}}.$$

For fixed $n \geq n_0$ (e.g. $n = n_0$) we find that the completeness of the space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ implies that there exists some limit $\mathbf{b}' \in \boldsymbol{B}$ of the sequence $(T_n\mathbf{b})_{n\geq 1}$. Because the limit is uniquely determined by $\mathbf{b}$ and the given sequence of operators we can define a new mapping $T_0 : \mathbf{b} \mapsto \mathbf{b}'$, or in other words, introduce the notation $T_0(\mathbf{b}) := \mathbf{b}'$ (as described above). It remains to be shown (Exercise) that T_0 is in fact a linear mapping. Since we have $\|T_m(\mathbf{b}) - T_n(\mathbf{b})\| \leq \varepsilon > 0$ for all $m \geq n_0$ it follows that $\mathbf{b}' = \lim_{m\to\infty} T_m(\mathbf{b})$ satisfies

$$\|\mathbf{b}' - T_n(\mathbf{b})\|_{\boldsymbol{B}} \leq \varepsilon,$$

hence both

$$\|T_0(\mathbf{b})\| \leq |||T_n|||\, \|\mathbf{b}\|_{\boldsymbol{B}} + \varepsilon,$$

i.e. boundedness of T_0 with $|||T_0||| \leq \sup_{n\geq n_0} |||T_n|||$ and finally convergence of (T_n) to T_0 in the operator norm, since for every $\mathbf{b} \in \boldsymbol{B}$ with $\|\mathbf{b}\|_{\boldsymbol{B}} \leq 1$ on has

$$\|T_0(\mathbf{b}) - T_m(\mathbf{b})\|_{\boldsymbol{B}} \leq \varepsilon > 0,\ \forall m \geq n_0,$$

or equivalently

$$|||T_0 - T_m||| \leq \varepsilon > 0,\ \forall m \geq n_0.$$

□

Remark 6. Observe that we have used the fact that a Cauchy-Sequence in a metric space, here the CS (T_n) in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is bounded, hence $\sup_{n\geq 1} |||T_n||| < \infty$.

5.1 Inversion in Banach Algebras

Many Banach algebras (such as $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$) do not contain a unit element, but only an *approximate unity (approximate identity)*

Definition 22. Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra (typically without unit element). A sequence, or in fact a directed family (! *net*!) $(e_\alpha)_{\alpha\in I}$ is a (left) approximate unit for $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ if for all $\mathbf{a} \in \boldsymbol{A}$ one has

$$\lim_{\alpha\to\infty} \|e_\alpha \bullet \mathbf{a} - \mathbf{a}\|_{\boldsymbol{A}} = 0. \tag{31}$$

The Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ has a *bounded approx. unit* if such a family can be chosen to be bounded in the $\boldsymbol{A}$-norm.

Whenever there is a bounded family which is a candidate for being an approximate unit it is enough to test it on a dense subspace, in fact, it is enough to test the action on a total subset of $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$.

Lemma 14. *Assume that a bounded net $(\mathbf{e}_\alpha)_{\alpha\in I}$ satisfies the relation*

$$\lim_\alpha \mathbf{e}_\alpha \bullet \mathbf{x} = \mathbf{x}, \quad \textit{for} \quad \mathbf{x} \in M,$$

where M is a total subset, then it is an approximate unit, i.e. equation (31) is valid.

Proof. First we argue that the relation $\lim_\alpha \mathbf{e}_\alpha \bullet \mathbf{x} = \mathbf{x}$ is valid for every $\mathbf{x}$ in the linear span of M. This is checked easily (!exercise) using induction on the length of the linear combination.

So it remains to proof the transition to the closure of the linear span (which by assumption is all of $\boldsymbol{A}$). Because we need the uniform bound on the family let us write $C := \sup_\alpha \|\mathbf{e}_\alpha\|_{\boldsymbol{A}}$. Obviously we may assume without loss of generality that $C \geq 1$ (why? find the easy argument for this claim).

Let now $\varepsilon > 0$ be given and $\mathbf{a} \in \boldsymbol{A}$ be given. Then there exists $\mathbf{x} \in \operatorname{span}(M)$ with $\|\mathbf{a} - \mathbf{x}\|_{\boldsymbol{A}} \leq \varepsilon/3C$. Choosing now α_0 such that $\|\mathbf{e}_\alpha \bullet \mathbf{x} - \mathbf{x}\|_{\boldsymbol{A}} \leq \varepsilon/3$ for $\alpha \succeq \alpha_0$, we end up with the estimate

$$\|\mathbf{e}_\alpha \bullet \mathbf{a} - \mathbf{a}\|_{\boldsymbol{A}} \leq \|\mathbf{e}_\alpha \bullet (\mathbf{a} - \mathbf{x})\|_{\boldsymbol{A}} + \|\mathbf{e}_\alpha \bullet \mathbf{x} - \mathbf{x}\|_{\boldsymbol{A}} + \|\mathbf{x} - \mathbf{a}\|_{\boldsymbol{A}}. \tag{32}$$

Obviously the first term can be estimated, for any $\alpha \succeq \alpha_0$ via

$$\|\mathbf{e}_\alpha \bullet (\mathbf{a} - \mathbf{x})\|_{\boldsymbol{A}} \leq \|\mathbf{e}_\alpha\|_{\boldsymbol{A}} \cdot \|\mathbf{a} - \mathbf{x}\|_{\boldsymbol{A}} \leq C \cdot \varepsilon/3C = \varepsilon/3.$$

while the second term is getting $\leq \varepsilon/3$ for $\alpha \succeq \alpha_1$. The last term is even smaller than the first one.

Choosing now some index α_3 such that $\alpha_3 \succeq \alpha_0$ and $\alpha_2 \succeq \alpha_1$ it is clear that the required estimate is valid for this general $\mathbf{a} \in \boldsymbol{A}$ as long as $\alpha \succeq \alpha_3$.

□

Sometimes it is also possible to "add" a unit element, i.e. to formally *adjoin* a unit element $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, which typically consists in adding a copy of the unit element (and its scalar multiples), or in other words form the direct sum of $\boldsymbol{A}$ and $\mathbb{C}$, with suitable multiplication and norm, very much similar to the construction of $\mathbb{C}$ from $\mathbb{R}$, using pairs of real numbers and define multiplication appropriately. We do not go into details here (it is one of the exercises for the course).

However, Banach algebras having a unit element $\mathbf{e}$ with

$$\mathbf{e} \bullet \mathbf{a} = \mathbf{a} = \mathbf{a} \bullet \mathbf{e}$$

are of special interest. Within such Banach algebras *invertibility* can be discussed, and in fact, quite general facts about invertibility can be derived using the idea of the so-called *Neumann series*[19].

[19] Carl Neumann, not the famous John von Neumann, who was involved in the invention of computers

Definition 23. Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be Banach algebra with (multiplicative) unit element $\mathbf{e}$, with $\|\mathbf{e}\|_{\boldsymbol{A}} = 1$ (does not follow from the axioms). Then every element $\mathbf{a}$ with $\|\mathbf{a}-\mathbf{e}\|_{\boldsymbol{A}} := q < 1$ is an invertible element in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$, and moreover

$$\|\mathbf{a}^{-1}\|_{\boldsymbol{A}} \leq \frac{1}{1-q}.$$

Proof. Set $\mathbf{y} := \mathbf{e} - \mathbf{a}$. Then it is clear that the Neumann series

$$\mathbf{b} := \sum_{k=0}^{\infty} \mathbf{y}^{\bullet k}$$

is *absolutely convergent*. Here we us the compound exponent $\bullet k$ to express the power k with respect to the algebra multiplication and make use of the obvious estimate

$$\|\mathbf{y}^{\bullet k}\|_{\boldsymbol{A}} \leq \|\mathbf{y}\|_{\boldsymbol{A}}^{k}, \quad \forall\, k \in \mathbb{N}. \tag{33}$$

By assumption $\|\mathbf{y}\|_{\boldsymbol{A}} = q < 1$ hence one obtains absolute convergence of the series in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ and $\|\mathbf{b}\|_{\boldsymbol{A}} \leq \sum_{k=0}^{\infty} q^k = \frac{1}{q}$. It is then easy to check that

$$\mathbf{a} \bullet \mathbf{b} = (\mathbf{e} - \mathbf{y}) \bullet \sum_{k=0}^{\infty} \mathbf{y}^{\bullet k} = \sum_{k=0}^{\infty} \mathbf{y}^{\bullet k} - \sum_{k=1}^{\infty} \mathbf{y}^{\bullet k} = \mathbf{e},$$

i.e. that $\mathbf{b} = \mathbf{a}^{-1}$ in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$. □

The proof also indicates how well the partial sums approximate the limit. Just consider

$$\mathbf{b}_n := \sum_{k=0}^{n} \mathbf{y}^{\bullet k},$$

then

$$\|\mathbf{b} - \mathbf{b}_n\|_{\boldsymbol{A}} \leq \sum_{k=n+1}^{\infty} \|\mathbf{y}^{\bullet k}\|_{\boldsymbol{A}} \leq \sum_{k=n+1}^{\infty} \|\mathbf{y}\|_{\boldsymbol{A}}^{k} \leq \frac{q^{n+1}}{1-q},$$

indicating at least "linear convergence" (look at a semi-logarithmic description of the decay rate!).

There is also an easy iterative way of computing $\mathbf{b}_{n+1}$ from $\mathbf{b}_n$:

$$\mathbf{b}_0 = \mathbf{e}\ , \quad \mathbf{b}_{n+1} := \mathbf{e} + \mathbf{y}\mathbf{b}_n.$$

(please check!) **Idea:** instead of adding a new term $\mathbf{y}^{k+1}$ the existing partial sum is multiplied by $\mathbf{y}$, giving $\sum_{k=1}^{n+1} \mathbf{y}^{\bullet k}$ which gives back $\mathbf{b}_{n+1}$ by adding the (now missing) term $\mathbf{e}$ (corresponding to the $k = 0$-term of the sum).

For the convergence of the series above in fact the strong condition $\|\mathbf{y}\|_{\boldsymbol{A}}$ can be replaced by the much weaker condition

$$r(\mathbf{y}) := \limsup \sqrt[k]{\|\mathbf{y}^k\|_{\boldsymbol{A}}} < 1 \tag{34}$$

The number $r(\mathbf{y})$ is called the *spectral radius* of $\mathbf{y} \in (\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$. It is in fact a limit! ([11], p.61).

The substructures in a Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ are either simple *subalgebras* (i.e. linear subspaces, which are closed under the multiplication inherited by the big algebra) and so-called *two-sided ideals*, which are closed subspaces $\mathbf{I} \subseteq \boldsymbol{A}$ with the property that $\boldsymbol{A} \bullet \mathbf{I} \bullet \boldsymbol{A} \subseteq \mathbf{I}$.

The natural quotient can then inherits a unique multiplicative structure, via the definition if a multiplication of classed, denoted by $\bullet_q$:

$$[\mathbf{a}_1]_{\mathbf{I}} \bullet_q [\mathbf{a}_2]_{\mathbf{I}} := [\mathbf{a}_1 \bullet \mathbf{a}_1]_{\mathbf{I}}. \tag{35}$$

Of course the point is to verify (using the two-sided ideal property) to show that this definition makes sense (i.e. that $\bullet_q$ is well-defined) and that the quotient norm is sub-multiplicative with respect to this norm! (could be an exercise).

A nice application of this (simple but important) principle in matrix analysis is the following one:

Lemma 15. *Assume that $\boldsymbol{A}$ is a diagonal dominant $n \times n$-matrix (real or complex), i.e. a matrix satisfying*

$$\rho := min_{1 \leq j \leq n} \left[|a_{j,j}| - \sum_{k \neq j} |a_{j,k}| \right] > 0.$$

Then $\boldsymbol{A}$ is invertible.

Proof. In fact, one has to consider the matrix as an operator from $(\mathbb{R}^n, \|)\|_\infty$ into itself. Then the corresponding operatornorm of $T : \mathbf{x} \to \boldsymbol{A} * \mathbf{x}$ is give by the expression (exercise)

$$|||T||| = \max_{1 \leq j \leq n} \sum_{k=1}^{n} |a_{j,k}|.$$

Since a diagonal dominant matrix has evidently an invertible diagonal part $\mathbf{D}$ we can look at the matrix $\boldsymbol{B} := \boldsymbol{A} * \mathbf{D}^{-1}$, which means that we are multiplying all the rows of $\boldsymbol{A}$ by the numbers $d_j^{-1} = 1/a_{j,j}$. Hence $\boldsymbol{B}$ has a diagonal consisting of ones, and thus the operator norm of $Id_n - \boldsymbol{B}$ on $(\mathbb{R}^n, \|)\|_\infty$ can be estimated by

$$\sum_{k \neq j} \sum_{k=1}^{n} |a_{j,k}|/|a_{j,j}| < 1 \quad \text{for} \quad 1 \leq j \leq n$$

and hence $|||\boldsymbol{B}||| < 1$, which allows to apply the Neumann series argument explained above. □

In (numerical) linear algebra courses one can learn that for such matrices Gaussian elimination without pivoting always works well, because the diagonal dominance can be shown to be preserved during the Gauss procedure.

5.2 Examples of bounded linear operators

There will be many more of those operators in the future. First a selection of operators which are isometric (up to some constant), i.e. with the property that T/λ is an isometry, for a suitably chosen value λ which may depend on the space under consideration.

- translation operators $T_z f(x) := f(x - z)$ on $\big(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\big)$;
- dilation operators $\mathrm{St}_\rho f(x) = \rho^{-d} f(x/\rho)$, for $f \in \boldsymbol{L}^1(\mathbb{R}^d)$.

5.3 Banach algebras with convolution

Among the most interesting Banach algebras are the *convolution algebras* (for many reasons, to be explained later). Typically one starts with the introduction of the Lebesgue space $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$ of Lebesgue integrable functions, or more precisely with the Banach space of equivalence classes of measurable functions, where two functions which coincide almost everywhere, we write $f(x) = g(x) a.e.$ (almost everywhere) or in other words where the set where the set $\{x \,|\, f(x) \neq g(x)\}$ is a nullset in the sense of the Lebesgue measure. Unfortunately this approach requires to study measure theory and in particular the Lebesgue integral, at the end to understand that $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$ is a Banach space with respect to the (well-defined) norm $\|f\|_1 := \int_{\mathbb{R}^d} |f(x)| dx$, containing $\boldsymbol{C}_c(\mathbb{R}^d)$ as a dense subspace. Also, making use of Fubini's theorem one finds that $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$ is even a commutative Banach algebra, with the multiplication being the so-called *convolution*.

Of course, technically speaking one has to show that this is defining a new equivalence class (which does not depend on the representative of f resp. g), that the procedure is associative etc. etc..

The classical definition of the convolution of two sequences is modeled after the so-called *Cauchy product* of sequences, which arises in a natural way through the pointwise multiplication of *polynomial functions* (and if one wants of *Laurent series* which allow a finite range of negative powers, of course only for non-zero arguments).

In standard mathematical definition we have for a sequence $(a_k)_{k=0}^K$ of coefficients of length $K+1$ (the *order of the polynomial* is the length of this vector) resp. of *degree* K:

$$p(z) = p_a(z) = \sum_{k=0}^{K} a_k z^k$$

and the pointwise product of $p_a(z)$ with $q_b(z)$ equals $r(z) = r_c(z)$, with

$$c_n = \sum_{k+j=n} a_k b_j = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{j=0}^{n} a_{n-j} b_j; \tag{36}$$

Clear enough the product polynomial is of degree $N = K + J$ (sum of the degrees of the factors), as the highest monomial from each term is meeting only the highest term from the second factor to give the (single) coefficient of $z^N = z^K \cdot z^J$.

It is a well known and now obvious fact that the set of polynomials as a commutative algebra under pointwise multiplication, hence the set of "finite sequences" (meaning sequences on $\mathbb{N}_0$ (natural numbers including zero) with only finitely many non-zero entries (K describes the position of the highest order term) is closed under the Cauchy product. We will call this a *discrete convolution*.

Now it is interesting for us to even put a norm on this algebra, namely the $\boldsymbol{\ell}^1$-norm:

$$\|\mathbf{a}\|_{\boldsymbol{\ell}^1} = \|\mathbf{a}\|_1 := \sum_{k=1}^{\infty} |a_k|,$$

where of course the sum is finite. The following estimate establishes submultiplicativity of this $\boldsymbol{\ell}^1$-norm:

Short proof of the statement in class:

$$c_k = \sum_{k=0}^{\infty} a_s b_{k-s}$$

$$\begin{aligned}
\|c\|_1 = \sum_{k=0}^{\infty} |c_k| = \sum_{k=0}^{\infty} \left| \sum_{s=0}^{k} a_s b_{k-s} \right| &\leq \text{ by } \Delta \\
&\leq \sum_{k=0}^{\infty} \sum_{s=0}^{k} |a_s||b_{k-s}| = \text{! important!} \qquad (37) \\
&= \sum_{s=0}^{\infty} \sum_{k=s}^{\infty} |a_s||b_{k-s}| = \text{set } j := k-s \\
&= \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} |a_s||b_j| = \qquad (38) \\
&= \sum_{s=0}^{\infty} |a_s| \sum_{j=0}^{\infty} |b_j| = \|a\|_1 \|b\|_1
\end{aligned}$$

From (1) to (2): In order to obtain $|a_p||b_q|$ in (2) (for some fixed $p, q \in \mathbb{N}$), take $k = p + q, s = p$ in (1).

Similarly one can write the Cauchy product for Laurent sequences, but even for sequences non-zero at any coordinate k, as long as the sequence $(a_k)_{k\in\mathbb{Z}}$ is absolutely summable, i.e. $a \in \boldsymbol{\ell}^1$ or (by definition) $\|\mathbf{a}\|_{\boldsymbol{\ell}^1} := \sum_{k\in\mathbb{Z}} |a_k| < \infty$.

Since this implies both the convergence of the part with positive and negative powers for $|z| \leq 1$ and $|z| \geq 1$ respectively, the overall sum is certainly convergent for any $z \in \mathbb{U} := \{z \,|\, |z| = 1\}$, the unit circle or 1D-torus.

Coming to the domain $\mathbb{R}$ or $\mathbb{R}^d$ it is therefore natural to ask that (first only) $f, g \in \boldsymbol{L}^1(\mathbb{R}^d)$ should have a convolution, but it is also possible to convolve two $\boldsymbol{L}^2(\mathbb{R}^d)$-functions, according to the following rule, featuring first the convolution defined in a pointwise sense [20].

Definition 24.

$$f*g(x) := \int_{\mathbb{R}^d} g(x-y)f(y)dy = \int_{\mathbb{R}^d} T_y g(x) f(y) dy = \left(\int_{\mathbb{R}^d} T_y g f(y) dy \right)(x), \quad f, g \in \boldsymbol{C}_c(\mathbb{R}^d). \tag{39}$$

FOR NOW take the first expression as the definition, later on we will see that also the second and third expression (vector-valued integral with values in a Banach space) are meaningful, as limits of Riemannian Sums (which form a Cauchy net!).

It is easy to verify the following properties of *convolution* (Exercises): Convolution is *commutative* (i.e. $f*g(x) = g*f(x)$) and bilinear (for example $f*(g_1+g_2) = f*g_1+f*g_2$ as functions!) etc., and even *associative*, i.e.

$$f * (g * h) = (f * g) * h,$$

[20] We will learn elsewhere about the *relevance* of the concept of convolution, e.g. for the description of translation invariant linear systems!

at least in the sense of classes. We do not claim that the iterated convolution product exists for a given point x in one case if it works for the other, but this is a minor concern. Due to the associativity it is justified to just write $f * g * h$ and use the "$*$" symbol as a kind of multiplication.

There are different arguments that can be used to verify that *convolution makes sense*. First of all let us recall the so-called Cauchy-Schwartz inequality which will be used to ensure the existence of the convolution in the case that both factors are $\boldsymbol{L}^2$-functions (in fact classes):

The Cauchy-Schwartz Inequality

$$|\langle f, g\rangle| \leq \int_{\mathbb{R}^d} |f(x)||g(x)|dx \leq \|f\|_{\boldsymbol{L}^2}\|g\|_{\boldsymbol{L}^2}, \quad \forall f, g, \in \boldsymbol{L}^2(\mathbb{R}^d). \tag{40}$$

From this equality it is easy to derive that

$$\|f * g\|_\infty \leq \|f\|_2\|g\|_2 \quad \forall f, g \in \boldsymbol{L}^2(\mathbb{R}^d). \tag{41}$$

Proof. Let us first define the FLIP-operator: $h^\checkmark(z) := h(-z)$. [21] □

Let us observe that the convolution product can be recast as a scalar product in the case that the factors are both in $\boldsymbol{L}^2$:

$$f * g(x) = \langle f, T_x(g^\checkmark)\rangle^{22} \tag{42}$$

It clearly implies via CS (40):

$$|f * g(x)| \leq \|f\|_2\|T_x g^\checkmark\|_2 = \|f\|_2\|g\|_2, \tag{43}$$

because obviously $g \to g^\checkmark$ and $g \to T_x g$ are isometric (in fact unitary) operators on $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$.

WORK in progress: It is also possible to show the submultiplicativity of the $\boldsymbol{L}^1(\mathbb{R}^d)$-norm:

$$\|f * g\|_1 \leq \|f\|_1\|g\|_1 \quad \forall f, g \in \boldsymbol{L}^1(\mathbb{R}^d). \tag{44}$$

[21] There is not much uniqueness in the literature in the choice of symbols for this simple and quite important operation. Think of plotting the graph of a function on a transparency and look at it from the backside! We follow [18].

[22] Later on we just write $T_x g^\checkmark$.

6 Different kinds of convergence

It is clear that in a space which is endowed with a *metric* one has a natural concept of *convergence* and of *Cauchy sequence*, hence of *density*, *totalness*, *separability*, etc.., but for a given space there are important other possible ways of convergence, in particular for spaces of operators or functionals. For them (as for general functions over general sets) the concept of *pointwise convergence* is of great importance. Since in this case the topology may perhaps not be described by just looking at *sequences* (and even if it was like that, in many situations naturally more general forms of convergence appear) we have to talk about the more general concept of *Moore-Smith sequences* or (better actually) *nets*, which may be convergent or *Cauchy nets* etc. and allow to describe topological facts more properly.

As a kind of training let us recall that important topological concepts can be characterized in many cases making use of sequences only. Thus we have in a metric space (hence in any normed space), correct statements of the form:

- A point $\mathbf{v}_0 \in \mathbf{V}$ belongs to the closure of a set $M \subseteq \mathbf{V}$ if and only if there is a sequence $(\mathbf{v}_k)_{k=1}^{\infty}$ **in** $\mathbf{V}$ such that $\mathbf{v}_0 = lim_{k\to\infty}\mathbf{v}_k$.
- A mapping between metric spaces is continuous if and only if convergent sequences are mapped into convergent sequences (with the correct/natural limit), i.e. if and only if
$$x_k \to x_0 \quad \Rightarrow \quad f(x_k) \to f(x_0) \text{ for } k \to \infty.$$
- A normed space is *complete* if every Cauchy-sequence is convergent.
- Any Cauchy-sequence is bounded.

Where in analysis have we seen already "generalized concepts of convergence"? Let us quickly recall a few situations:

- $x_0 = lim_{k\to\infty} x_k$;
- $lim_{x\to 0+} \; f(x) = a$;
- $lim_{x\to\infty} \; e^x = +\infty$.
- $\sum_{(k,n)\in\mathbb{Z}\times\mathbb{Z}} c_{n,k} = b$;
- The Riemannian sums converge to $\int_a^b f(x)dx$ for every $f \in \boldsymbol{C}([a,b])$;

There are various "general principles" connected with the use of NETS versus SEQUENCES. On the one hand topology (in its most general setting) *enforces* the use of nets, simply because it is NOT sufficient to consider sequences in order to reach a point in the closure of a set M (the intersection of all closed subsets containing M, resp. the smallest closed subset of the ambient topological space X containing M). This is in particular the case of the topology does *not* allow for a countable basis of the neighborhood-system (of the zero-element in a vector space). In this case it is at least plausible that a *countable* family (any sequence contains only countably many elements)

may not suffice to "reach the limit", while nets (typically with a much larger, but less sorted, and certainly not totally ordered, index set) will do the job.

For this we need the definition of an *directed* resp. *oriented* set is required. At first sight it looks similar to that of a semi-ordering, but there are differences, as we see:

Think about this: What should be the concept of a *subnet* of a given net $(x_\alpha)_{\alpha\in I}$. Of course this concept should a natural generalization of a *subsequence*. In other words, if we consider an ordinary sequence $(x_k)_{k=1}^\infty$ as a net (which we can do!), then the usual subsequences $(x_{k_n})_{n=1}^\infty$ should be special cases of the (still to be defined) concept of a *subnet* (see later/below).

The answer can be found in Kelley's classical book on topology ([12]), p.70.

Exercise: Let I be any countable index set (e.g. $\mathbb{Z}^n, n \geq 1$), and assume that the net $\sum_{i\in F} \mathbf{x}_i$ is convergent in a normed space $(\mathbf{V}, \|\cdot\|)$, with limit $\mathbf{s} \in \mathbf{V}$. Then for any enumeration $\pi : \mathbb{N} \to I$ the series $\sum_{k=1}^\infty \mathbf{x}_{\pi(k)}$ is (unconditionally) convergent.

Definition 25. A *subnet* of a given net $(x_\alpha)_{\alpha\in I}$ is obtained by means of a cofinal mapping from another index set J. By this we mean the following situation. We assume that

- I is directed by means of some *orientation* $\succeq$;
- The index set J is endowed with another orientation, let us write $\sqsupseteq$ (for the sake of distinction);
- a mapping ϕ from J to I is called *cofinal* [23] if for every $\alpha_0 \in I$ there exists some $\beta_0 \in J$ such that
$$\beta \sqsupseteq \beta_0 \quad \Rightarrow \quad \phi(\beta) \succeq \alpha_0. \tag{45}$$
In such a situation the mapping $\phi : J \to X$, given by $\beta \to x_{\phi(\beta)}$ is called a *subnet* of the original net (x_α). Instead of writing $(x_{\phi(\beta)})_{\beta\in J}$ we simple write $(x_\beta)_{\beta\in J}$.

There are various alternative ways of looking at the concept of subnets. One of them (compatible with the view on subsequences) is the idea that one has a subnet, if the *labels* (the indices β attached to elements) are attached to the elements of the *!set!* which is generated by the net (x_α), i.e. a mapping from the directed set J into the set $\{x_\alpha, \alpha \in I\} \subset X$ (which is of course different from *indexed family* (x_α), e.g. because in $(x_\alpha)_{\alpha\in I}$ repetitions are possible and such nets are not just subsets of X). So the only requirement is that the indexing using the $\beta's$ should reach "arbitrary strong" values within the set.

I was doing a plot with blue (= old, original) labels from I and red (= new, J-labels).

In this context we have the following interesting situation, which does not have any analogue in the case of a subsequence (except a trivial one): It is possible to have a situation that two nets $(x_\alpha)_{\alpha\in I}$ and $(x_\beta)_{\beta\in I}$ are subnets of each other without being equal (in a trivial sense).

Note that in many cases the mapping ϕ used to generate a subnet is preserving "orientation", i.e. has a kind of *monotonicity* property, as we are used to have it in the case of *subsequences*. However we mention explicitly that this is not required here (in order to obtain stronger statements!). If such a monotonicity is available it is of course enough to assume that for every α_0 there exists some β_0 such that $\phi(\beta_0$

[23] Warning, this terminology is not at all standard, but hopefully intuitive!

6.0.1 Examples of subnets

The following cases are (hopefully) obvious:

1. Any subsequence of a given sequence is also a subnet (of both sequences are considered as nets, mapping $(N, \geq)$ to the given set X; Both I, J are the set $\mathbb{N}$ of natural numbers, with the natural (!total) ordering. The mapping ϕ is what is usually described by sub-indexing, i.e. the mapping $n \mapsto k_n$, describing the subsequence. In fact it is clear (by induction) that $k_n \geq n$ because by the general convention about $k_{n_1} > k_{n_2}$, or expressed differently: ϕ is assumed to be a strictly increasing (monotonous) function from $\mathbb{N}$ to $\mathbb{N}$.

2. The $x_1, x_4, x_3, x_5, x_6, \ldots$ is a subnet, but NOT a subsequence (because monotonicity is violated (but it is not requested for subnets);

3. The sequence $x_1, x_2, x_2, x_3, x_3, x_3, x_4 \ldots$ is again not a subsequence, but it is a subnet. In other words, *r*epetitions are allowed in the context of subnets!

4. $x_1, x_2, x_1, x_3, x_1, x_4, x_1, x_5 \ldots$ is *not!* a subnet of the sequence $(x_k)_{k=1}^{\infty}$ because it is not cofinal (over and over again there are elements with the fixed index 1 are occurring!

5. any permutation of a sequence, i.e. a new sequence generated from a given sequence $(x_k)_{k=1}^{\infty}$ via some bijection $\pi : \mathbb{N} \to \mathbb{N}$, leading to a new sequence (hence net) $(x_{\pi(n)})$ can be considered as a subnet. In particular one can argue (mostly for the sake of training of ideas) that the stability of convergence behavior of a sequence under permutation of the index set is a consequence of the fact that subnets of convergent nets are convergent and have the same limit. (see Exercises).

Lemma 16. *Let $(\mathbf{v}_\beta)_{\beta \in J} := (\mathbf{v}_{\phi(\beta)})_{\beta \in J}$ be a subnet of a Cauchy-net $(\mathbf{v}_\alpha)_{\alpha \in I}$. Then the following statements are valid:*

- *Any tail, i.e. the subfamily of elements with indices $\alpha \succeq \alpha_0$ is a subnet;*
- *$(\mathbf{v}_\beta)_{\beta \in J}$ is also a Cauchy-net;*
- *if $(\mathbf{v}_\beta)_{\beta \in J}$ is convergent, then so is $(\mathbf{v}_\alpha)_{\alpha \in I}$.*

Proof. For the first claim check the definition. One just has to take $J := \{\alpha \in I, \alpha \succeq \alpha_0\}$ and $\phi(\alpha) = \alpha$. The main conditions for subnets can be verified (try it yourself, it was done in the course) by making use of what has been called the *majorization property* for nets.

Next we show that the convergence of a subnet of a CN (Cauchy net) implies the convergence of the net itself. This results isolates nicely the logical basis for such an argument (the corresponding condition for CS is done in the exercises) and also exhibits that the very weak request made in the definition of a subnet (without any monotonicity requirement) is still strong enough to verify (technically) this result:

Given $\varepsilon > 0$ we can find α_0 such that

$$\|\mathbf{v}_\alpha - \mathbf{v}_{\alpha'}\|_{\boldsymbol{A}} < \varepsilon \quad \text{for} \quad \alpha, \alpha' \succeq \alpha_0.$$

Choosing now (according to the definition of subnet, using the cofinality condition) β_0 such that for $\beta \sqsupseteq \beta_0$ the elements $\mathbf{v}_\beta := \mathbf{v}_{\phi(\beta)}$ are such that $\phi(\beta) \succeq \alpha_0$ it is clear that the "tail" of the subnet (the elements having labels stronger than β_0 in the subnet) are part of the a tail $\square$

Let us formalize the term "tail" (non-standard, no good reference, again: hopefully intuitive):

Definition 26. Given a net $(\mathbf{v}_\alpha)_{\alpha \in I}$ in some set $\mathbf{V}$ we call the family $(\mathbf{v}_\alpha)_{\alpha \succeq \alpha_0}$ the *tail of the net*, starting at $\alpha_0 \in I$.

Remark 7. In many cases (e.g. subsequences) the mapping ϕ is *monotonous*

Similarities and Differences: Sequences and Nets

In many cases the use of nets allows to consider just more general convergence structures (such as Riemannian sums with general subdivision). It is remarkable that completeness with respect to sequences (the usual CS-criterion) is also implying convergence of Cauchy-nets (due to the fact that the ε-balls with radius $1/n$ or 2^{-n} (for example) form a *countable* basis of the neighborhood system of each element $\mathbf{v} \in (\mathbf{V}, \|\cdot\|)$.

6.0.2 Similarities

- all Cauchy nets are convergent if and only if the space is complete;
- a subnet is convergent if only the net itself is convergent;
- for Cauchy-nets the converse is true;
- continuity/boundedness of linear operators can be characterized by the preservation of convergence of nets;

6.0.3 Dissimilarities

- a Cauchy-net, even a convergent net, does not have to be bounded ;
- two nets can be mutually subnets of each other without being equal;
- a subnet can be obtained from a given sequence by arbitrary permutation of the element and finite repetition of each of the elements;

Lemma 17. *A sequence (y_n) is a subnet of a given sequence $(x_k)_{k=1}^\infty$ if and only if each of the elements from the original sequence $(x_k)_{k=1}^\infty$ is chosen (if at all) at most finitely many times.*

Proof. Just the starting point. A sequence is obviously just a net with a specific index set $\mathbb{N}$, oriented towards ∞, i.e. with the natural order (satisfying the properties of an oriented set, with the *majorization* being just the usual game: Given $n_1, n_2 \in \mathbb{N}$ it is clear that $n_0 = \max(n_1, n_2)$ satisfies $n_0 \geq n_1$ and $n_0 \geq n_2$.

Now one has to verify that the request coming from the subnet definition are (logically/practically) equivalent to the situation described in the lemma.

For now this is given as an exercise. $\square$

6.1 Cauchy nets in Banach Spaces II

THIS PART WAS TYPED PREVIOUSLY, USING DIFFERENT CONVENTIONS

Definition 27. A net $\{f_\alpha\}_{\alpha\in A}$ in a Banach space $\mathscr{X}$ is said to be a Cauchy net if for every $\epsilon > 0$, there is a α_0 in A such that $\alpha_1, \alpha_2 \succeq \alpha_0$ implies $||f_{\alpha_1} - f_{\alpha_2}|| < \epsilon$.

Proposition 8. *In a Banach space each Cauchy net is convergent. Consequently a normed space is complete if and only if every Cauchy net is convergent*[24]

Proof. Let $\{f_\alpha\}_{\alpha\in A}$ be a Cauchy net in the Banach space $\mathscr{X}$. Choose α_1 such that $\alpha \geq \alpha_1$ implies $||f_\alpha - f_{\alpha_1}|| < 1$. Having chosen $\{\alpha_k\}_{k=1}^n$ in A, choose $\alpha_{n+1} \geq \alpha_n$ such that $\alpha \geq \alpha_{n+1}$ implies

$$||f_\alpha - f_{\alpha_{n+1}}|| < \frac{1}{n+1}.$$

The sequence $\{f_{\alpha_n}\}_{n=1}^\infty$ is clearly Cauchy and, since $\mathscr{H}$ is complete, there exists f in $\mathscr{H}$ such that $\lim_{n\to\infty} f_{\alpha_n} = f$.
It remains to prove that $\lim_{\alpha\in A} f_\alpha = f$. Given $\epsilon > 0$, choose n such that $\frac{1}{n} < \frac{\epsilon}{2}$ and $\|f_{\alpha_n} - f\| < \frac{\epsilon}{2}$. Then for $\alpha \geq \alpha_n$ we have

$$||f_\alpha - f|| \leq ||f_\alpha - f_{\alpha_n}|| + ||f_{\alpha_n} - f|| < \frac{1}{n} + \frac{\epsilon}{2} < \epsilon.$$

A similar argument can be used (by choosing $\varepsilon = 2^n$) can be used to prove that a normed space with the property that every absolutely convergent series is also (norm) convergent is enough to show that the Cauchy-sequences exist. (this is the way how the proof was given in the course on Thursday, 24th of October).

□

Definition 28. Let $\{f_\alpha\}_{\alpha\in A}$ be a set of vectors in the Banach space $\mathscr{X}$. Let $\mathscr{F} = \{F \subset A : F \text{ finite}\}$. If we define $F_1 \leq F_2$ for $F_1 \subset F_2$, then $\mathscr{F}$ is a directed set. For each F in $\mathscr{F}$, let $g_F = \sum\limits_{\alpha\in F} f_\alpha$. If the net $\{g_F\}_{F\in\mathscr{F}}$ converges to some g in $\mathscr{H}$, then the series $\sum\limits_{\alpha\in A} f_\alpha$ is said to be *(unconditionally) convergent* and we write $g = \sum\limits_{\alpha\in A} f_\alpha$.

Proposition 9. *If $\{f_\alpha\}_{\alpha\in A}$ is a set of vectors in the Banach space $\mathscr{X}$ such that $\sum\limits_{\alpha\in A} ||f_\alpha||$ converges in the real line $\mathbb{R}$, then $\sum\limits_{\alpha\in A} f_\alpha$ converges in $\mathscr{X}$.*

Proof. It suffices to show, in the notation of Definition 28, that the net $\{g_F\}_{F\in\mathscr{F}}$ is Cauchy. Since $\sum\limits_{\alpha\in A} ||f_\alpha||$ converges, for $\epsilon > 0$, there exists F_0 in $\mathscr{F}$ such that $F \geq F_0$ implies

$$\sum_{\alpha\in F} ||f_\alpha|| - \sum_{\alpha\in F_0} ||f_\alpha|| < \epsilon.$$

[24]For the practical use of Banach spaces this is the more convenient characterization of completeness, while for the verification of completeness it is of course easier to just verify the convergence of Cauchy nets or even better, the convergence of absolutely convergent series.

Thus for $F_1, F_2 \geq F_0$ we have

$$\begin{aligned} ||g_{F_1} - g_{F_2}|| &= \left\| \sum_{\alpha \in F_1} f_\alpha - \sum_{\alpha \in F_2} f_\alpha \right\| \\ &= \left\| \sum_{\alpha \in F_1 \setminus F_2} f_\alpha - \sum_{\alpha \in F_2 \setminus F_1} f_\alpha \right\| \\ &\leq \sum_{\alpha \in F_1 \setminus F_2} ||f_\alpha|| + \sum_{\alpha \in F_2 \setminus F_1} ||f_\alpha|| \\ &\leq \sum_{\alpha \in F_1 \cup F_2} ||f_\alpha|| - \sum_{\alpha \in F_0} ||f_\alpha|| < \epsilon. \end{aligned}$$

Therefore, $\{g_F\}_{F \in \mathscr{F}}$ is Cauchy and $\sum\limits_{\alpha \in A} f_\alpha$ converges by definition. □

Corollary 6. *A normed linear space $\mathscr{X}$ is a Banach space if and only if for every sequence $\{f_n\}_{n=1}^\infty$ of vectors in $\mathscr{X}$ the condition $\sum\limits_{n=1}^\infty ||f_n|| < \infty$ implies the convergence of $\sum\limits_{n=1}^\infty f_n$.*

Proof. If $\mathscr{X}$ is a Banach space, then the conclusion follows from the preceding proposition. Therefore, assume that $\{g_n\}_{n=1}^\infty$ is a Cauchy sequence in a normed linear space $\mathscr{X}$ in which the series hypothesis is valid. Then we may choose a subsequence $\{g_{n_k}\}_{k=1}^\infty$ such that $\sum\limits_{k=1}^\infty ||g_{n_{k+1}} - g_{n_k}|| < \infty$ as follows: Choose n_1 such that for $i, j \geq n_1$ we have $||g_i - g_j|| < 1$; having chosen $\{n_k\}_{k=1}^N$ choose $n_{N+1} > n_N$ such that $i, j > n_{N+1}$ implies $||g_i - g_j|| < 2^{-N}$. If we set $f_k = g_{n_k} - g_{n_{k-1}}$ for $k > 1$ and $f_1 = g_{n_1}$, then $\sum\limits_{k=1}^\infty ||f_k|| < \infty$, and the hypothesis implies that the series $\sum\limits_{k=1}^\infty f_k$ converges. It follows from the definition of convergence that the sequence $\{g_{n_k}\}_{k=1}^\infty$ converges in $\mathscr{X}$ and hence so also does $\{g_n\}_{n=1}^\infty$. Thus $\mathscr{X}$ is complete and hence a Banach space. □

Theorem 14. *Let $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ be a Banach space. Then every Cauchy net (x_α) in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is convergent, i.e., there exists $x_0 \in \boldsymbol{B}$ such that for every $\varepsilon > 0$ there exists some index α_0 such that $\alpha \succeq \alpha_0$ implies $\|x_\alpha - x_0\|_{\boldsymbol{B}} < \varepsilon$.*

Proof. First we select a Cauchy (subnet, resp.) sequence (x_{α_n}), in the following way: For every $n \in \mathbb{N}$ there exists α'_n such that for α, α' with $\alpha \succeq \alpha'_n$ and $\alpha' \succeq \alpha'_n$ implies

$$\|x_\alpha - x_{\alpha'}\|_{\boldsymbol{B}} < 1/n.$$

We may choose $\alpha_1 = \alpha'_1$, $\alpha_2 \succeq \alpha_1, \alpha'_2$, $\alpha_3 \succeq \alpha_1, \alpha_2, \alpha'_3$, inductively.

It is clear that we have thus

$$\|x_{\alpha_k} - x_{\alpha_m}\|_{\boldsymbol{B}} < \frac{1}{n} \quad \text{for} \quad k, m \geq n.$$

By the completeness of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ this sequence will have a limit, let us call it x_0. Although we do *not claim* that the Cauchy sequence chosen in this way can be considered as a subnet [25] we can obtained the required estimates using it.

In order to verify convergence of the Cauchy-net itself let $\varepsilon > 0$ be given. Choosing n such that $\varepsilon > 1/n$ we can use α_n etc.

$$\|\mathbf{x}_\alpha - x_0\|_{\boldsymbol{B}} \leq \|\mathbf{x}_\alpha - x_{\alpha_n}\|_{\boldsymbol{B}} + \|x_{\alpha_n} - x_0\|_{\boldsymbol{B}} < \varepsilon, \quad \forall \alpha \succeq \alpha_l.$$

□

HINT: In Heine's book [9] one finds a general argument, why for pseudo-metric spaces the completeness assumption (involving sequences only) is equivalent to the convergence of Cauchy-nets (Satz 3.1-2).

6.2 Banach's fixed point theorem

Valid in general metric spaces one has:

Certainly relevant for many applications:

Theorem 15. *For every contractive mapping $S : \big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big) \to \big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$ which is contractive, i.e. satisfies for some $\gamma < 1$:*

$$\|S(\mathbf{b})\|_{\boldsymbol{B}} \leq \gamma \|\mathbf{b}\|_{\boldsymbol{B}}, \quad \forall \mathbf{b} \in \boldsymbol{B}, \tag{46}$$

then there exists a unique fix point $\widetilde{\mathbf{b}} \in \boldsymbol{B}$, i.e. with $S(\widetilde{\mathbf{b}}) = \widetilde{\mathbf{b}}$

Proof. It is easy to verify that $S^n(\mathbf{b}_0)$ is a Cauchy sequence in $\big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$ for any given (starting point) $\mathbf{b}_0 \in \boldsymbol{B}$. Consequently it is convergent. In addition it is easy (!the reader is still encouraged to check it!) to verify that the limit $\widetilde{\mathbf{b}} := \lim_{n\to\infty} S^n(\mathbf{b}_0$ is in fact a fix point for the mapping S and also the uniqueness of the fixed point. In particular the limit $\widetilde{\mathbf{b}}$ does *not depend* on the choice of the initial value $\mathbf{b}_0 \in \boldsymbol{B}$. □

6.3 Strong Operator Convergence

The typical case of *pointwise convergence* is in the context linear functionals or operators.

A net $(T_\alpha)_{\alpha\in I}$ is convergent in the pointwise sense if $\lim_\alpha T_\alpha x$ exists for any $x \in X$. If each of the operators T_α is a linear one the limit depends linearly on the argument, hence $T_0 x := \lim_\alpha T_\alpha x$ defines the limit (in the *strong operator sense*). (verifying linearity is a good exercise).

Lemma 18. *The net T_α is strongly convergent (to the operator T_0) if and only if for every finite set $F \subset X$ and $\varepsilon > 0$ there exists some index α_0 such that*

$$\|T_\alpha x - T_0 x\|_{\boldsymbol{B}} < \varepsilon \ \ \forall \alpha \succeq \alpha_0.$$

[25] The notion of a subnet requires that it is "cofinal", which seems to be difficult to prove, if not false for the general case, but we will not need any argument of the form: " if a subnet is convergent, then the Cauchy-net is convergent itself", instead we can argue directly:

Proof. The formulated condition implies of course the strong convergence, because one can choose $F = \{x\}$, an arbitrary one-point set.

Conversely, strong convergence plus the axioms concerning directed sets imply the condition, through a proof by induction. For sets F of cardinality 1 this is just the assumption. Assume that it is valid for sets F of cardinality n, we have to show that it is true for sets with $n+1$ elements.

We may assume that F with elements $x_1, \ldots, x_n$ is OK and x_{n+1} is an additional element. Given $\varepsilon > 0$ there exist two indices α_1 for the finite set F and α_2 for x. Choosing then $\alpha_3 \succeq \alpha_1$ and $\alpha_3 \succeq \alpha_2$ we have for $\alpha \succeq \alpha_3$

$$\|T_\alpha x_k - T_0 x_k\|_{\boldsymbol{B}} < \varepsilon \ \ \forall \alpha \succeq \alpha_3 \ \ k = 1, \ldots, n+1.$$

□

A typical "application" (making future proofs easier) is based on the following

Definition 29. A *! bounded !* directed family (a net or sequence) $(h_\alpha)_{\alpha \in I}$ in a Banach algebra $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called a **BAI** (= *bounded approximate identity* or "*approximate unit*" for $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$) if

$$\lim_\alpha \|h_\alpha \cdot h - h\|_{\boldsymbol{B}} = 0 \quad \forall h \in \boldsymbol{B}.$$

The terminology of (!one approximate unit, meaning one family) is mostly justified by the following lemma:

Lemma 19. *A Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ has an approximate identity if and only if there exists a bounded net $\mathbf{e}_\alpha$ with*

$$lim_\alpha \mathbf{e}_\alpha \bullet \mathbf{a} = \mathbf{a}, \quad \textit{for} \quad \mathbf{a} \in \boldsymbol{A}. \tag{47}$$

Proof. The validity of **??** implies of course the existence of elements satisfying the above definition.

Conversely, one has to form such an indexed family $(\mathbf{e}_\alpha)_{\alpha \in I}$, given the validity of the definition. For this purpose one generates as index set all pairs (F, ε), where F is a finite subset of $\boldsymbol{A}$ and $\varepsilon > 0$.

The main step (more or less allowing to give an inductive argument) is the following: Given $\mathbf{a}_1, \mathbf{a}_2$, which (for simplicity of the argument) may be assumed to be normalized, i.e. satisfy $\|\mathbf{a}_i\|_{\boldsymbol{A}} = 1, i = 1, 2$..

If we assume there exists $C < \infty$ (the uniform a priori bound) such that for every $\mathbf{a} \in \boldsymbol{A}$ and $\varepsilon > 0$ one can find $h \in \boldsymbol{A}$ with[26]

$$\|\mathbf{h}\|_{\boldsymbol{A}} \leq C \quad \text{and} \quad \|\mathbf{h} \bullet \mathbf{a} - \mathbf{a}\|_{\boldsymbol{A}} < \varepsilon. \tag{48}$$

So we have $\|\mathbf{h}_i \bullet \mathbf{a}_i - \mathbf{a}_i\|_{\boldsymbol{A}}$ for $i = 1, 2$.

As an index set one can choose which consists of pairs of the form (F, ε), where $F \subset \boldsymbol{A}$ is any finite set of elements from the algebra $\boldsymbol{A}$ and $\varepsilon > 0$. As usual the natural order on these pairs in term of the size of F and the smallness of ε.

□

For details of the proof one probably can look up the Lecture Notes of H. Reiter [13].

[26] One way of looking at this assumption is to say that there is no price to be paid, in terms of an increased norm of the element $\mathbf{h}$ serving as replacement of the identity element, for better and better approximation. Instead, the costs are known to be controlled independent of the smallness of $\varepsilon > 0$.

6.4 Iterated limits of nets and interchange of order

The following result should be compared with a theorem on *iterated limits* provided in Kelley's book ([12], p.69).

Lemma 20. *Assume that $(T_\alpha)_{\alpha\in I}$ and $(S_\beta)_{\beta\in J}$ are two bounded nets of operators in $\mathcal{L}(\boldsymbol{V})$, which are strongly convergent to limits T_0, and S_0 resp. i.e.*

$$T_o(\mathbf{v}) = \lim_\alpha T_\alpha(\mathbf{v}) \quad \forall\, \mathbf{v} \in \boldsymbol{V} \quad \textit{and} \quad S_0(\mathbf{w}) = \lim_\beta S_\beta(\mathbf{w}) \quad \forall\, \mathbf{w} \in \boldsymbol{V}.$$

Then the net $(T_\alpha \circ S_\beta)_{(\alpha,\beta)}$ (with index set $I \times J$ and natural order[27]) is also strongly convergence, with limit $T_0 \circ S_0$, i.e. for each $\mathbf{v} \in \mathbf{V}$ one has:

$$T_o[S_0(\mathbf{v})] = [T_0 \circ S_0]\,(\mathbf{v}) = \lim_{\alpha,\beta} [T_\alpha \circ S_\beta]\,(\mathbf{v}). \tag{49}$$

In detail: For any $\mathbf{v} \in \boldsymbol{V}$ and $\varepsilon > 0$ there exists a pair of indices $(\alpha_0, \beta_0) \in I \times J$ such that for every $\alpha \succeq \alpha_0$ in I and $\beta \succeq \beta_0$ in J implies

$$\|T_0(S_0(\mathbf{v})) - T_\alpha(S_\beta(\mathbf{v}))\| \le \varepsilon. \tag{50}$$

In particular we have in the sense of strong limits:

$$T_0 \circ S_0 = \lim_\alpha \lim_\beta T_\alpha \circ S_\beta = \lim_\beta \lim_\alpha T_\alpha \circ S_\beta \tag{51}$$

Proof. Note that $|||T_\alpha||| \le C < \infty$. The statement depends on the following estimate

$$\|T_\alpha[S_\beta(\mathbf{v})] - T_0[S_0(\mathbf{v})]\| \le \|T_\alpha[S_\beta(\mathbf{v})] - T_\alpha[S_0(\mathbf{v})]\| + \|T_\alpha[S_0(\mathbf{v})] - T_0[S_0(\mathbf{v})]\| \tag{52}$$

The first expression can be estimated as follows:

$$\|T_\alpha[S_\beta(\mathbf{v})] - T_\alpha[S_0(\mathbf{v})]\| \le |||T_\alpha|||\, \|S_\beta(\mathbf{v}) - S_0(\mathbf{v})\| \le C\|S_\beta(\mathbf{v}) - S_0(\mathbf{v})\|, \tag{53}$$

which gets $< \varepsilon/2$ for $\beta \succeq \beta_0$ (chosen for ε/C), while the second term can be estimated by

$$\|T_\alpha[S_0(\mathbf{v})] - T_0[S_0(\mathbf{v})]\| \le \varepsilon/2, \tag{54}$$

for any α with $\alpha \succeq \alpha_0$ (choosing $\mathbf{w} = S_0(\mathbf{v})$).

Finally we have to check that the validity of (54) implies that also the iterated limits exist (of course with the same limit). We elaborate on the first iterated limit (namely $\lim_\alpha \lim_\beta$) because the other one works in the same way[28]. So for the situation in (50) we can first fix any α and look at the net $(T_\alpha(S_\beta(\mathbf{v})))_{\beta\in J}$. By assumption the limit $S_0(\mathbf{v}) = lim_\beta S_\beta(\mathbf{v})$ exists. Since T_α (for fixed α first) is a bounded linear operator also the following limits exist and can be estimated by

$$\|T_0(S_0(\mathbf{v})) - T_\alpha(\lim_\beta S_\beta(\mathbf{v}))\| \le \varepsilon, \quad \text{if only} \quad \alpha \succeq \alpha_0.$$

Setting $\mathbf{w}_0 := S_0(\mathbf{v})$ we see that by assumption also $(T_\alpha(\mathbf{w}_0))$ is convergent, and of course the last estimate remains true in the limit (due to the continuity of the norm), hence

$$\|T_0[S_0(\mathbf{v})] - \lim_\alpha \lim_\beta T_\alpha[S_\beta(\mathbf{v})]\| \le \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, equation (51) is valid. □

[27] Of course I and J may have completely different order!

[28] Observe however that the *order* in which the operators are applied does NOT change!!!

Remark 8. Note that only the boundedness of the net (T_α) is only required, not that of (S_β). Moreover, in the case of sequences (instead of nets) the *uniform boundedness principle* can be applied, which tells us that the strong convergence of operators implies the norm convergence of the corresponding sequence. This situation could be formulated as a corollary then, i.e. strong convergence of (T_n) and (S_k) implies

$$\lim_n \lim_k T_n[S_k(\mathbf{v})] = T_0[S_0(\mathbf{v})] = \lim_k \lim_n T_n[S_k(\mathbf{v})] \tag{55}$$

A very special case of this situation occurs in the context of (double) Banach modules. Assume that (h_β) is bounded approximate unit in $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ and (g_α) an approximate unit in $\big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$.

Lemma 21. *For $f \in \big(\boldsymbol{L}^p(\mathbb{R}^d),\, \|\cdot\|_p\big), 1 \le p < \infty$, one has norm convergence:*

$$\lim_\alpha \lim_\beta g_\alpha * (h_\beta \cdot f) = \lim_\beta \lim_\alpha g_\alpha * (h_\beta \cdot f) = f = \lim_\alpha \lim_\beta h_\beta \cdot (g_\alpha * f) = \lim_\beta \lim_\alpha h_\beta \cdot (g_\alpha * f) \tag{56}$$

Remark 9. The same relations are valid for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, just assuming that (h_β) is a bounded approximate unit in $\big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$. Typically it could be the dilation of the form $(\mathrm{D}_\rho h(x))_{\rho>0}$, for some $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ with $h(0) = 1$, where $\rho \succeq \rho_0$ if $\rho \le \rho_0$.

Corollary 7. *Assume that $(\mathbf{e}_\alpha)_{\alpha\in I}$ and $(\mathbf{e}'_\beta)$ are two bounded approximate identities in a Banach algebra $(\boldsymbol{A},\, \|\cdot\|_{\boldsymbol{A}})$. Then $(\mathbf{e}_\alpha \bullet \mathbf{e}'_\beta)_{(\alpha,\beta)}$ is a BAI in $(\boldsymbol{A},\, \|\cdot\|_{\boldsymbol{A}})$ as well, with the natural "orientation" on the index set $I \times J$.*

Applying a simple induction argument one obtains from this the following fact:

Corollary 8. *Assume that $(\mathbf{e}_\alpha)_{\alpha\in I}$ is a bounded approximate identity in a Banach algebra $(\boldsymbol{A},\, \|\cdot\|_{\boldsymbol{A}})$. Then $(\mathbf{e}_\alpha^{(\bullet k)})$ is a BAI in $(\boldsymbol{A},\, \|\cdot\|_{\boldsymbol{A}})$ as well, for every $k \in \mathbb{N}$.*

There is a completely similar argument applicable (hence left to the reader) in the following situation:

Proposition 10. *Given three normed spaces $\mathbf{V}, \mathbf{W}$ and $\boldsymbol{X}$ we consider a a bounded, bilinear mapping B from $\mathbf{V} \times \mathbf{W}$ into $\boldsymbol{X}$, i.e. a mapping B which is linear in each variable and satisfies for some $C > 0$:*

$$\|B(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{X}} \le C\|\mathbf{v}\|_{\mathbf{V}}\|\mathbf{w}\|_{\mathbf{W}}, \quad \forall \mathbf{v} \in \mathbf{V}, \mathbf{w} \in \mathbf{W}.$$

Then for any two convergent nets $(\mathbf{v}_\alpha)_{\alpha\in I}$ and $(\mathbf{w}_\beta)_{\beta\in J}$ one has

$$B(\mathbf{v}_0, \mathbf{w}_0) = \lim_{(\alpha,\beta)} B(\mathbf{v}_\alpha, \mathbf{w}_\beta), \tag{57}$$

and in particular (as a consequence of (57))

$$B(\mathbf{v}_0, \mathbf{w}_0) = \lim_\alpha \lim_\beta B(\mathbf{v}_\alpha, \mathbf{w}_\beta) = \lim_\beta \lim_\alpha B(\mathbf{v}_\alpha, \mathbf{w}_\beta). \tag{58}$$

6.5 Strong operator convergence and dual operators

Convergence of operators (in the operator norm) implies (and is in fact equivalent) to the convergence of the adjoint operators, since clearly $T \mapsto T'$ is isometric (see above) and compatible with additivity (> Exercise). However often one has only strong operator convergence, and then a separate consideration is needed.

Recall the following definition:

Definition 30. A bounded net $(T_\alpha)_{\alpha\in I}$ of operators in $\mathcal{L}(\mathbf{V},\mathbf{W})$ is *strongly convergent to* $T_0 \in \mathcal{L}(\mathbf{V},\mathbf{W})$ (equivalently one says: the convergence takes place in the strong operator topology) if one has "pointwise" norm convergence in $(\boldsymbol{W}, \|\cdot\|_{\boldsymbol{W}})$:

$$\lim_\alpha T_\alpha(v) = T_0(v) \quad \forall \mathbf{v} \in \boldsymbol{V}. \tag{59}$$

Note furthermore that due to the boundedness and linearity of the operators it is enough to test for the validity of (30) on a *total subset* for $\mathbf{V}$ resp. for $\mathbf{v}$ from a dense subspace of of $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$.

Our simple observation is the following one. If the net (or sequence) of operators, applied to an arbitrary element $\mathbf{v} \in \boldsymbol{V}$ shows norm convergence one (only resp. still has) w^*-convergence[29], if one applies the corresponding net of dual operators to any given $\mathbf{w}' \in \mathbf{W}'$:

Lemma 22. *Assume that the bounded net $(T_\alpha)_{\alpha\in I}$ is strongly convergent T_0, then the dual operators, applied to any $\mathbf{w}' \in \mathbf{W}'$, i.e. the net $(T'_\alpha \mathbf{w}')_{\alpha\in I}$ is (at least resp. still) w^*-convergent, i.e. one has*

$$\forall \mathbf{w}' \in \mathbf{W}' : \quad \lim_\alpha [T'_\alpha \mathbf{w}'](\mathbf{v}) = [T'_0 \mathbf{w}'](\mathbf{v}), \ \forall \mathbf{v} \in \mathbf{V}. \tag{60}$$

Proof. The proof is quite straightforward, following the definition. Since the net is bounded the same is true for the net of dual operators, hence the net $(T'_\alpha \mathbf{w}')_{\alpha\in I}$ is bounded for each $\mathbf{w}' \in \mathbf{W}'$. According to the definition of the dual operator one has

$$[T'_\alpha \mathbf{w}'](\mathbf{v}) = \mathbf{w}'(T_\alpha \mathbf{v}) \to \mathbf{w}'(T_0 \mathbf{v}) = [T'_0 \mathbf{w}'](\mathbf{v}). \tag{61}$$

□

A variant of this proof would allow to replace the fixed $\mathbf{w}' \in \mathbf{W}'$ by a net $(\mathbf{w}'_\beta)_{\beta\in J}$ with limit (in the w^*-sense!) $\mathbf{w}'_0$ and still have convergence (of the iterated limit)

$$[T'_\alpha \mathbf{w}'_\beta](\mathbf{v}) = \mathbf{w}'_\beta(T_\alpha \mathbf{v}) \to \mathbf{w}'_0(T_0 \mathbf{v}) = [T'_0 \mathbf{w}'_0](\mathbf{v}). \tag{62}$$

There are other things to be observed, since one has now *three*! topologies (resp. modes of convergence of nets) we should take care a little bit about the hierarchy of those types of convergence (and also the corresponding notions of continuity of linear mappings T).

For example we have seen (check once more) that a linear operator between dual spaces, e.g. the dual operator $T' \in \mathcal{L}(\mathbf{W}', \mathbf{V}')$ for some $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$, i.e. maps convergent nets in the domain (with norm convergence) into equally norm convergent nets in the range space.

[29] To be discussed later on!

Now we have e.g. $w^* - w^*$- continuous operators T, which means that $\mathbf{v}'_0 = w^*\text{-}\lim_\alpha \mathbf{v}'_\alpha$ in $\mathbf{V}'$ implies $T(\mathbf{v}'_0) = w^*\text{-}\lim_\alpha T(\mathbf{v}'_\alpha)$ in $\mathbf{W}'$.

In principle one can also mix topologies and ask about operators which convert one of the type of convergence of nets into another:

Lemma 23. *The convergence in norm implies weak convergence in a normed vector space. If the vector space is the dual of some other normed space (e.g. $\mathbf{V} = \mathbf{W}'$), we also have the w^*-convergence, which follows in turn from weak convergence*[30].

Proof. That norm convergence implies weak convergence is an easy exercise. Now assume that you have weak convergence of a net $(\mathbf{w}'_\alpha)$ in $\mathbf{V} = \mathbf{W}'$ with limit $\mathbf{w}'_0$, and you want to verify that this net is also w^*-convergent to the same limit.

The main tool is the Hahn-Banach theorem, resp. its corollary showing that the natural embedding $i_{\mathbf{W}} : \mathbf{w} \mapsto \imath_{\mathbf{W}}(\mathbf{w})$, given by

$$\imath_{\mathbf{W}}(\mathbf{w}') := \mathbf{w}'(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{W}$$

defines an isometric (!) embedding from $\mathbf{W}$ into $\mathbf{W}''$.

Now given $\mathbf{w}$ we have to check whether

$$\lim_\alpha \mathbf{w}'_\alpha(\mathbf{w}) = i_{\mathbf{W}}(\mathbf{w})(\mathbf{w}'_\alpha)$$

is convergent, but obviously $i_{\mathbf{W}}(\mathbf{w})$ defines a linear functional in $(\mathbf{W}')'$, and hence by assumption (of weak convergence) we have

$$i_{\mathbf{W}}(\mathbf{w})(\mathbf{w}'_\alpha) \to \imath_{\mathbf{W}}(\mathbf{w})(\mathbf{w}'_0).$$

□

Knowing that different topologies correspond to different types of convergence we may have in fact different set of "continuous" linear functionals. For example, a linear functional σ (a linear mapping with values from the vector space into the field $\mathbb{C}$ or $\mathbb{R}$) is continuous with respect to the w^*-topology if and only any w^*-convergent net $(\mathbf{v}'_\alpha)$ is mapped into a convergent net of (complex) numbers, i.e. $\sigma(\mathbf{v}'_\alpha) \to \sigma(\mathbf{v}'_0)$ in $\mathbb{C}$. Clearly any (ordinary) such continuous linear functional also respecting the ordinary convergence (which is a stronger assumption), so can search for those (new) linear w^*-continuous functionals among the elements of the ordinary dual space $\mathbf{V}'$.

Therefore the following result (important, but for now without proof) about dual space (i.e. the description of all continuous linear functionals):

Theorem 16.
- *For any normed space $\mathbf{V}$ the dual space of $\mathbf{V}$ with the weak topology is just $\mathbf{V}'$;*
- *For any normed space $\mathbf{V}$ the dual space of $\mathbf{V}'$ with the w^*-topology is just $\mathbf{V}$.*
- *In particular, if $\big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$ is a reflexive Banach space, then the dual space of $\boldsymbol{B}'$ with the weak topology is just $\boldsymbol{B}$.*

[30] One can show that w^*-convergence is equivalent to weak convergence of nets, i.e. the converse below is true if and only if the underlying Banach space $\mathbf{W}$ is reflexive!

Proof. Just comments first: The first statement says: Although the convergence in the weak sense is typically *strictly* weaker than the concept of norm convergence (i.e. there are almost all the time weakly convergent nets which are *not* norm convergent. Still there is the same dual space, i.e. one has to verify that any linear functional which respects norm convergence is automatically respecting weakly convergent nets.

The other statement says, that if only w^*-convergence is used, one really does not have the same (full) dual space, but just those functionals within $\mathbf{W}''$ which are coming from the original space $\mathbf{W}$ via the natural embedding! (once more: this is not a proof but just an explanation of the view!)

Finally it is clear that for reflexive space there is *no difference* between the w^* and the weak topology, and thus we are again in the situation of the first case. □

MAYBE PLACED AT A WRONG POSITION

6.6 A number of simple applications in HA

- For $x_0 \in \mathscr{G}$ we may consider the net $(T_x)_{x\in\mathscr{G}}$ as a net, directed by "closeness" to x_0. It is easier to formulate the condition here for metric groups $\mathscr{G}$, where one can simply define $y \succeq x$ if $d(y, x_0) \leq d(x, x_0)$.
- Similar one can take the operators (D_ρ) oriented towards ρ_0 (e.g. $\rho_0 = 1$) with $\rho_1 \succeq \rho_1$ if $d(\rho_1, \rho_0) \leq d(\rho_2, \rho_0)$.
- Again assume $\mathscr{G}$ to be a metric group (for simplicity), such as $\mathbb{R}^d$. Consider the family (Sp_Ψ), where Ψ is running through the family of all BUPUs, with $\Psi_1 \succeq \Psi_2$ if $|\Psi_1| \leq |\Psi_2|$, where $|\Psi|$ is the maximal support size of $\Psi = (\psi_i)_{i\in I}$, i.e. the infimum over all $\delta > 0$ such that

  $$\forall i \in I : \mathrm{supp}(\psi_i) \subseteq B_\delta(x_i),$$

  for suitable families $(x_i)_{i\in I}$.

It is then interesting to look at the corresponding dual operators. Of course, the given operators can be viewed as a bounded family of operators on different spaces, and therefore the dual operators, defined on the corresponding dual spaces, may appear as quite different.

First let us consider the case $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}}) = \big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, because this is a rather concrete and simple space. The dual space is (by convention, chosen here) just the space of *bounded measures* on $\mathbb{R}^d$, i.e. $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}) := \big(\boldsymbol{C}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{C}_0'}\big)$, by definition.

Remark 10. It is not difficult to check that the mapping, which assigns to each operator $T \in \mathcal{L}(\boldsymbol{V})$ the corresponding dual operator $T' \in \mathcal{L}(()\boldsymbol{V}')$ is an isometric anti-isomorphism of Banach algebras, since $(T_1 \circ T_2)' = T_2' \circ T_1'$.

Bounded families of operators on $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ then correspond to bounded families of operators on the dual spaces. In other words, we consider the mapping $T \mapsto T'$ as an isometric mapping from $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ to $\mathcal{L}(\boldsymbol{M}_b(\mathbb{R}^d))$.

Remark 11. Just for the sake of completeness let us recall that the dual operators of the operators defined above are easily identified, e.g. by their action on typical representatives of the dual space.

Lemma 24.

$$T_x'(\delta_z) = \delta_{z+x}$$

Proof.

$$[T_x'(\delta_z)](f) = \delta_z(T_x f) = T_x f(z) = f(z-(-x)) = [\delta_{z+x}](f), \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

□

This example also shows that *one cannot expect* to have for all $\mathbf{w}' \in \boldsymbol{C}_0'(\mathbb{R}^d)$: $\|T_x'(\mathbf{w}') - T_0'(\mathbf{w}')\|_{\boldsymbol{C}_0'} \to 0 \quad \text{for } x \to 0.$

Next we look at the dilation operator:

Lemma 25.

$$\mathrm{D}_\rho{}'(\delta_z) = \delta_{\rho z}, \quad \rho \neq 0, z \in \mathbb{R}^d.$$

Proof.

$$[\mathrm{D}_\rho{}'(\delta_z)](f)\delta_z(\mathrm{D}_\rho f) = f(\rho z) = \delta_{\rho z}](f).$$

□

Many of the linear functionals on $\big(\boldsymbol{C}_0(\mathbb{R}^d),\, \|\cdot\|_\infty\big)$ are in fact given by ordinary function, which create bounded linear functionals via integration: For $g \in \big(\boldsymbol{L}^1(\mathbb{R}^d),\, \|\cdot\|_1\big)$ one finds that

$$\mu_g(f) := \int_{\mathbb{R}^d} f(x)\, g(x) dx, \quad f \in \boldsymbol{C}_0(\mathbb{R}^d). \tag{63}$$

Applying the adjoint dilation operator to such a function shows that for "ordinary functions" the dual of the (value preserving) dilation operator is an integral (hence $\boldsymbol{L}^1$-norm) preserving dilation operator (going in the opposite direction).

Lemma 26. *For any $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ one has:*

$$[\mathrm{D}_\rho{}'\, \mu_g] = \mu_h, \quad \textit{for } h = \mathrm{St}_{1/rho}\, g,$$

where we may define St_γ *via*

$$\mathrm{St}_\gamma\, g(x) = \gamma^{-d} g(x/\gamma).$$

Finally let us compute the action of the dual operator to Sp_Ψ on $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$:

Lemma 27. *Given any partition of unity $\Psi = (\psi_i)_{i\in I}$ one has*

$$\mathrm{Sp}_\Psi{}'(\mu) = D_\Psi(\mu) := \sum_{i\in I} \mu(\psi)\delta_{x_i}.$$

Proof. One can in fact show that this sum is absolutely convergent and that $\mu = \sum_{i\in I} \psi_i \mu$ is also absolutely convergent, with $\sum_{i\in I} \|\psi_i \mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)} = \|\mu\|_{\boldsymbol{M}_b(\mathbb{R}^d)}$. □

There are several concrete situations in analysis where in fact w^*-convergent nets are considered. The most popular occurs in the description of the Riemann integral on $(\boldsymbol{C}(I), \|\cdot\|_\infty)$.

Proposition 11. *For every pair of Banach space* $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}}), (\boldsymbol{W}, \|\cdot\|_{\boldsymbol{W}})$ *and any* $T \in \mathcal{L}(\mathbf{V},\mathbf{W})$ *the operator* T'' *is an extension of* T *to an operator from* $\boldsymbol{W}''$ *to* $\boldsymbol{V}''$*, i.e.* [31]

$$T(i_{\boldsymbol{V}}(\mathbf{v})) = i_{\boldsymbol{W}}(T(\mathbf{v}), \quad \mathbf{v} \in \boldsymbol{V}.$$

In particular, for the case $\boldsymbol{W} = \boldsymbol{V}$ *one has: For any* $T \in \mathcal{L}(\boldsymbol{V})$ *the operator* $T'' : \boldsymbol{W}'' \to \boldsymbol{W}''$ *leaves the closed subspace* $i_{\boldsymbol{V}}$ *invariant and can be "naturally identified" with* T *on this subspace.*

The situation can be well described by the following diagram:

$$\begin{array}{ccc} \boldsymbol{V}'' & \xrightarrow{T''} & \boldsymbol{W}'' \\ \cup\!\mid & & \cup\!\mid \\ \boldsymbol{V} & \xrightarrow{T} & \boldsymbol{W} \end{array}$$

Proof. Of course we will make heavy use of one of the important consequences of the *Hahn-Banach Theorem*, namely the fact, that due to the existence of plenty of linear functionals the *natural embedding* $i_{\boldsymbol{B}}$ (for any normed space $\boldsymbol{B}$), given by

$$i_{\boldsymbol{B}}(\mathbf{b})(\mathbf{b}') = \mathbf{b}'(\mathbf{b}), \quad \mathbf{b} \in \boldsymbol{B}, \mathbf{b}' \in \boldsymbol{B}'. \tag{64}$$

So we have to check on the action of T'' on $i_{\boldsymbol{V}}(\mathbf{v})$.

$$T''(i_{\boldsymbol{V}}(\mathbf{v}))(\mathbf{w}') = i_{\boldsymbol{V}}(\mathbf{v})(T'(\mathbf{w}')) = [T'(\mathbf{w}')](\mathbf{v}) = \mathbf{w}'(T(\mathbf{v})) = i_{\boldsymbol{W}}(T(\mathbf{v}))(\mathbf{w}').$$

□

An easy example (where of course a more direct argument could be applied) is given by the following situation. Consider $\boldsymbol{W} = \boldsymbol{V} = \text{cosp}$, endowed with the sup-norm. Then it is not difficult to verify that $\boldsymbol{V}' = \left(\boldsymbol{\ell}^1, \|\cdot\|_1\right)$ and furthermore $\boldsymbol{V}'' = \boldsymbol{\ell}^\infty$, again with the sup-norm (and the natural embedding is just the usual embedding from $\boldsymbol{c}_0$ into $\boldsymbol{\ell}^\infty$.

Assume that one has a multiplication operator, which acts [boundedly] on $\boldsymbol{c}_0$: $T := M_{\mathbf{y}} : \mathbf{x} \to \mathbf{z} := \mathbf{x}. * \mathbf{y}$ (in the sense of pointwise multiplication: $z_n = x_n \cdot y_n, n \in \mathbb{N}$). Clearly such an operator has closed graph and therefore has (!) to a bounded operator, which in turn implies that $\mathbf{x}$ (the multiplier sequence) has to be a bounded sequence, i.e. $\mathbf{z} \in \boldsymbol{\ell}^\infty$. Obviously T' is just the same operator, but now considered on $\boldsymbol{\ell}^1$, while T'' is now the same multiplication operator, viewed as operator on $\boldsymbol{\ell}^\infty$.

6.7 Weak versus w^*-convergence

There are various types of convergence which are *weaker than the usual norm convergence*. In fact all of them correspond to *topologies* which turn a given vector space in to *topological vector spaces* (which means, that addition and scalar multiplication are continuous operations with respect to the given topology). We do not want to go deeply into abstract set-theoretic topology, but it is convenient to note that one simply has to replace the collection of ε-balls around zero by other systems of neighborhoods of the

[31] Here we denote by $i_{\boldsymbol{B}}$ the natural embedding of $\boldsymbol{B}$ into the double dual space $\boldsymbol{B}''$.

origin (the zero-element $\mathbf{0} \in \boldsymbol{V}$), which can then be "moved around" (any neighborhood of $\mathbf{v} \in \boldsymbol{V}$ is just a neighborhood $\mathbf{U}$ of $\mathbf{0}$, shifted to the position $\mathbf{v}$, i.e. of the form $\mathbf{v} + \mathbf{U}$.

The two most important are the so-called *weak* convergence in an arbitrary normed space, and in the case of a dual space one has (in addition to the weak convergence) also the w^*-convergence.

Definition 31. For a normed space $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$ the concept of *weak convergence* is described as follows:
A net $(\mathbf{v}_\alpha)_{\alpha \in I}$ is weakly convergent with limit $\mathbf{v}_0 \in \boldsymbol{V}$ if one has

$$\lim_\alpha \sigma(\mathbf{v}_\alpha) = \sigma(\mathbf{v}_0), \quad \forall \sigma \in \boldsymbol{V}^*.$$

Note that, although the norm is not playing an explicit role in the definition of weak convergence it still is important in this definition. Since a stronger norm (think of a more sensitive measurement) has fewer convergent nets, hence more continuous linear functionals the reservoir of linear functional varies (potentially considerably) with the norm.

In fact the following observation is a good exercise about weak convergence:

Lemma 28. *Assume that a normed space $\boldsymbol{V}^1$ is continuously embedded into another normed space $\boldsymbol{V}^2$ as a dense subspace. Then weak convergence of a net $(\mathbf{v}_\alpha)_{\alpha \in I}$ within $\boldsymbol{V}^1$ implies weak convergence of this net as a net within $\boldsymbol{V}^2$.*

Proof. Since $\boldsymbol{V}^1$ is continuously embedded into $\boldsymbol{V}^2$ every continuous linear functional on $\boldsymbol{V}^2$ is also a continuous linear functional on $\boldsymbol{V}^1$. Alternatively, one can argue, that one has

$$\|\mathbf{v}\|^{(2)} \leq C \cdot \|\mathbf{v}\|^{(1)}, \forall \mathbf{v} \in \boldsymbol{V}^1,$$

hence for any σ in the dual space of $\boldsymbol{V}^2$ one has

$$|\sigma(\mathbf{v})| \leq \|\sigma\|'_{\boldsymbol{V}^1} \|\mathbf{v}\|^{(2)} \leq C \cdot \|\sigma\|'_{\boldsymbol{V}^1} \|\mathbf{v}\|^{(1)}$$

This implies that every bounded linear functional with respect to the coarse norm defines also a bounded linear functional in the stronger sense. On the other hand the density assumption implies that the mapping $\sigma \mapsto \sigma|_{\boldsymbol{V}^2}$ is in fact an embedding, i.e. is injective (and not only bounded).

Since there are (potentially) more bounded linear functionals on the smaller space (with the more sensitive norms) the weak convergence induced from the strong/sensitive norm implies the weak convergence with respect to the less sensitive norm[32]. □

Remark 12. It is also clear that in the case of a basis for the dual space it may not be necessary to test for weak-convergence for an arbitrary element $\sigma \in \boldsymbol{V}^*$, but rather for the elements of a basis of this space. If the net is a priori bounded in the norm it is even sufficient to check the convergence for a generating system! (Exercise).

For example: What does it mean to have weak convergence in $\boldsymbol{\ell}^p$, for $1 < p < \infty$? We claim that this is just coordinatewise convergence (at least for bounded nets this is an easy exercise). To be discussed (hopefully) later on.

[32] In other words: the hierarchy of convergencies which is valid for the norms is - given the density of embedding - also covariantly, meaning in the same direction, going over to the weak convergence.

First of all it is clear that $\mathbf{x} \to x_k$ is a bounded linear functional on $\left(\boldsymbol{\ell}^p, \|\cdot\|_p\right)$ (due to the trivial estimate $|x_k| \leq \|\mathbf{x}\|_p$!) and consequently $\delta_k : \mathbf{x} \mapsto x_k$ is a bounded linear functional. Thus the definition of weak convergence for $(\mathbf{x}_\alpha)_{\alpha\in I}$ to some $\mathbf{y} \in \boldsymbol{\ell}^p$ immediately implies the convergence in each coordinate, i.e. $(x_{\alpha,k} \to y_k$ for each $k \in \mathbb{N}$: (once more: each δ_k is one of the possible set of all bd. linear functionals that have to respect convergence).

Conversely, assume that we have a bounded net $(\mathbf{x}_\alpha)_{\alpha\in I}$ with $C_1 := \sup_\alpha \|\mathbf{x}_\alpha\|$, and ask, whether for a given bd. linear functional $\sigma \in \boldsymbol{\ell}^{p\prime}$ one can guarantee convergence of $\sigma(\mathbf{x}_\alpha)$ (in $\mathbb{C}$), assuming only coordinatewise convergence. By taking differences ($\mathbf{x}_\alpha$ minus the limit) one may assume without loss of generality that the limit is just $\mathbf{0} \in \boldsymbol{\ell}^p$.

Knowing that the dual space of $\left(\boldsymbol{\ell}^p, \|\cdot\|_p\right)$ is just $\left(\boldsymbol{\ell}^q, \|\cdot\|_q\right)$, we know that there exists a (uniquely determined) sequence $\mathbf{z} \in \boldsymbol{\ell}^q$ such that $\sigma(\mathbf{x}) = \sum_{k\in\mathbb{N}} z_k x_k$.. Using next the fact that $q < \infty$ (because $p > 1$ was assumed!) we can find for every $\varepsilon > 0$ some $K > 0$ such that $(\sum_{k>K} |z_k|^q)^{1/q} \leq \varepsilon$. Defining $\mathbf{y} \in \boldsymbol{\ell}^q$ by $y_k = z_k$ for $k \leq K$ and zero for $k > K$ we have thus $\|\mathbf{y} - \mathbf{z}\|_q \leq \varepsilon$, and consequently (using Hoelder's inequality)

$$|\sigma(\mathbf{x}_\alpha) - \sigma_{\mathbf{y}}(\mathbf{x}_\alpha)| = |\sigma_{\mathbf{z}-\mathbf{y}}(\mathbf{x}_\alpha)| \leq \|\mathbf{y} - \mathbf{z}\|_q \|\mathbf{x}_\alpha\|_p, \leq \varepsilon \cdot C_1$$

for all $\alpha\in I$. Now given $\varepsilon > 0$ and having chosen K as above one can find an index (using the rules of the convergence of nets, plus the observation that there is only a finite!! set of indices $k = 1, 2, \ldots, K$ which has to be observed) in order to find out that there exists some α_0 such that for $\alpha \succeq \alpha_0$ one can be assured (as a consequence of the coordinatewise convergence) that one has (just a finite sum!)

$$|\sigma_{\mathbf{y}}(\mathbf{x}_\alpha) - \sigma_{\mathbf{y}}(\mathbf{0})| \leq \sum_{k=1}^{K} |z_k| \sigma_{\mathbf{e}_k}(\mathbf{x}_\alpha)| \leq \sum k = 1^K |z_k||\delta_k(\mathbf{x}_\alpha)| \leq \varepsilon$$

e.g. by making every term smaller than $\varepsilon/(K\|\mathbf{z}\|_q)$.

Definition 32. For a dual $\boldsymbol{V}^*$ of a normed space $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$ the concept of w^*-*convergence* is described as follows: A net $(\mathbf{v}'_\alpha)_{\alpha\in I}$ is w^*-convergent in $\left(\boldsymbol{V}^*, \|\cdot\|_{\boldsymbol{V}^*}\right)$ with limit $\mathbf{v}'_0 \in \boldsymbol{V}^*$ if one has

$$\lim_\alpha \mathbf{v}'_\alpha(\mathbf{v}) = \mathbf{v}'_0(\mathbf{v}), \quad \forall \mathbf{v} \in \boldsymbol{V}.$$

Lemma 29. *Weak convergence within* $\left(\boldsymbol{V}^*, \|\cdot\|_{\boldsymbol{V}^*}\right)$ *implies* w^**-convergence. Obviously the two concepts coincide if* [33] *the Banach space* $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$ *is reflexive.*

Proof. It suffices to observe that the w^*-convergence can be considered as *weakening* of the concept of weak convergence. In fact, it restricts to necessity to have convergence of $\sigma(\mathbf{v}'_\alpha)$ to those elements which are in the image of $\boldsymbol{V}$ within $\boldsymbol{V}''$, since for them we have obviously

$$i_{\boldsymbol{V}}(\mathbf{v})(\mathbf{v}'_\alpha) = \mathbf{v}'_\alpha(\mathbf{v}), \mathbf{v} \in \boldsymbol{V}.$$

For reflexive Banach spaces there are simply "no more functionals than those". The comment (see footnote) is less obvious (but still true). □

[33].. and in fact only if..

Remark 13. Just a hint: since (as a consequence of the Riesz representation theorem, which allows to identify the dual space of a Hilbert space $\mathcal{H}$ with the space itself (in a "natural way") one can show that "of course" every Hilbert space is reflexive. Hence we have at least for Hilbert space the coincidence of weak and w^*-convergence.

Remark 14. Historical remarks concerning w^*-convergence, in connection with the central limit theorem: there one may understand where the word "weak" comes in (the star simply refers to the fact that one is having a net or sequence in a dual space, hence $\boldsymbol{V}^*$).

6.8 Characterizing weak convergence in Banach spaces

We want to demonstrate now that Thm. (9) (which was an immediate consequence of the Hahn-Banach theorem).

Theorem 17. *A net $(\mathbf{v}_\alpha)_{\alpha\in I}$ in a normed space $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$ is weakly convergent to $\mathbf{v}_0 \in \boldsymbol{V}$ if and only if for any finite-dimensional subspace $\boldsymbol{V}_0$ and any projection P onto such a space the net $(P(\mathbf{v}_\alpha))_{\alpha\in I}$ is convergent (to $P(\mathbf{v}_0)$) in $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$.*

Proof. Assume that one has a net which is weakly convergent to $\mathbf{v}_0$. Then the proof of Theorem 9 implies that the projections onto the finite dimensional space can be written (!! any such projection can be described using a basis, not necessarily the biorthogonal basis,... to be explained a little bit better!) using continuous linear functionals. Hence (using the finiteness of the sum) one finds that also the projections onto arbitrary finite-dimensional spaces will converge properly (and of course towards $P(\mathbf{v}_0)$).

Conversely, assume that any finite-dimensional projection applied to the net $(\mathbf{v}_\alpha)$ produces a norm convergent family. We still have to show that then weak convergence follows. Hence one should be able (!open now) to show: For any given $\mathbf{v}' \in \boldsymbol{V}^*$ we have to make the connection between functionals and finite dimensional subspaces of $\boldsymbol{V}$. Of course, the null-space N of $\mathbf{v}'$, i.e. $\boldsymbol{N}_{\mathbf{v}'} := \{\mathbf{n}, \mathbf{v}'(\mathbf{n}) = 0\}$ is a *hyperplane*, i.e. *codimension* one. Expressed differently, using the quotient theorem 21 implies that $\boldsymbol{V}/\boldsymbol{N}$ is isomorphic to $\mathbb{C}$ (resp. $\mathbb{K}$). In fact, for any $\mathbf{v}_1 \notin \boldsymbol{N}$ one has $\mathbf{v}'(\mathbf{v}_1) \neq 0$ (otherwise $\mathbf{v}'$ would vanish on all the elements of $\mathbf{V}$, because $\mathbf{V}$ decomposes into a direct sum of $\boldsymbol{N}$ and the (one-dimensional) linear span of $\mathbf{v}_1$ in $\boldsymbol{V}$. The projection onto the one-dimensional generated by $\mathbf{v}_1$ assigns to each $\lambda\mathbf{v}_1$ the value $\gamma \cdot \lambda$ (for some fixed $\gamma \neq 0$, depending on the choice of $\mathbf{v}_1$). But the restriction of $\mathbf{v}'$ onto this one-dimensional space if also non-trivial, hence another (non-trivial) multiple of λ, hence knowing convergence of $P(\mathbf{v}_\alpha)$ onto the one-dim. subspace generated by $\mathbf{v}_1$ is just the same as knowing the convergence of $\mathbf{v}'(\mathbf{v}_\alpha)$ to (obviously) $\mathbf{v}'(\mathbf{v}_0)$. □

WE ALSO HAVE TO DISCUSS THE UNIQUENESS OF THE WEAK LIMIT, ANOTHER CONSEQUENCE OF THE HB THEOREM!

There is also the concern, how to describe a basis of the neighborhood of $\mathbf{0} \in \boldsymbol{V}$ for the weak topology. As such one can choose some finite subset $F \subset \boldsymbol{V}^*$ and $\varepsilon > 0$ and

define

$$U(F,\varepsilon) := \{\mathbf{v} \in \boldsymbol{V} \,|\, |\mathbf{v}'(\mathbf{v})| < \varepsilon \ \forall \mathbf{v}' \in F\}. \tag{65}$$

The neighborhoods for $\mathbf{0} \in \boldsymbol{V}^*$ for the w^*-topology looks quite similar: Given any finite set $E \subset \boldsymbol{V}$ and $\varepsilon > 0$ we define

$$V(E,\varepsilon) := \{\mathbf{v}' \in \boldsymbol{V}^* \,|\, |\mathbf{v}'(\mathbf{v})| < \varepsilon \ \forall \mathbf{v} \in E\}. \tag{66}$$

The comparison of the two topologies, when applied to $\boldsymbol{V}^*$ is again based on the fact that $i_{\boldsymbol{V}}(E) \subset (\boldsymbol{V}^*)'$.

From these descriptions it is also clear why it is necessary to go to nets (and not to restrict the attention to sequences) in order to describe continuity of convergence.

6.9 Dual and adjoint operators for Hilbert spaces

We are discussing in this section that the adjoint operator is more or less the same as the adjoint operator (which is only defined via the scalar product in the Hilbert space, which in turn allows to identify the dual space of the Hilbert space with the Hilbert space itself, using the so-called Riesz representation system.

The idea is relatively simple: Given an operator $T \in \mathcal{L}(\mathcal{H})$ there exists the dual operator $T' \in \mathcal{L}(()\mathcal{H}')$. Since (via Riesz Repr. Theorem) $\mathcal{H} = \mathcal{H}'$ (well, it is not an isometric isomorphism of Banach space, because it is anti-linear, i.e. does not completely respect scalar multiplication) this means that it also can be viewed as an operator from $\mathcal{H}$ to $\mathcal{H}$. Given $\mathbf{y} \in \mathcal{H}$ we view it as a linear functional $\sigma_{\mathbf{y}} \in \mathcal{H}'$, with $\sigma_{\mathbf{y}}(\mathbf{x}) = \langle \mathbf{y}, \mathbf{x}\rangle_{\mathcal{H}}, \forall \mathbf{y} \in \mathcal{H}$, and by the definition of the dual operator $T'(\sigma_{\mathbf{y}})$ is a well defined linear functional, which can be represented by a uniquely determined vector (let us call it) $\mathbf{z} \in \mathcal{H}$, i.e. $\langle \mathbf{x}, \mathbf{z}\rangle_{\mathcal{H}} = T'(\sigma_{\mathbf{y}})(\mathbf{x}), \forall \mathbf{x} \in \mathcal{H}$. By the definition of the dual operator we ave

$$T'(\sigma_{\mathbf{y}})(\mathbf{x}) := \sigma_{\mathbf{y}}(T(\mathbf{x})) = \langle T(\mathbf{x}), \mathbf{y}\rangle_{\mathcal{H}}$$

and using the definition of the adjoint operator T^* we see that

$$\langle \mathbf{x}, \mathbf{z}\rangle_{\mathcal{H}} = T'(\sigma_{\mathbf{y}})(\mathbf{x}) = \langle \mathbf{x}, T^*(\mathbf{z})\rangle_{\mathcal{H}},$$

or in other words, $T^*(\mathbf{y})$ is just the element which corresponds to the image of $T'(\sigma_{\mathbf{y}})$. So all together we have

$$\sigma_{T^*(\mathbf{y})} = T'(\sigma_{\mathbf{y}}), \forall \mathbf{y} \in \mathcal{H}. \tag{67}$$

6.10 The Banach-Alaoglu Theorem

One of the most important *existence theorems* used in many applications to obtain (e.g. in the context of PDF) from a bounded sequence of *approximate solutions* a *"true solution"* to a given problem, or an element, which has *"some ideal property"* (as the limit of elements which have this property more and more) is the Banach-Alaoglu theorem.

It states, that the unit-ball $B_1(\mathbf{0}) \subset \boldsymbol{B}'$ (consequently any w^**-closed, bounded subset) in a dual space is compact in the w^*-topology.

Theorem 18. *The unit ball $B_1(\mathbf{0}) = \{\mathbf{b}' \mid \|\mathbf{b}'\|_{\boldsymbol{B}'} \leq 1\}$ is compact with respect to the w^*-topology, or in other words, any net $(\mathbf{b}'_\alpha)_{\alpha\in I}$ in $B_1(\mathbf{0})$ there exists a subnet $(\mathbf{b}'_\beta)_{\beta\in J}$ which has a w^*-limit $\mathbf{b}'_0$ within $B_1(\mathbf{0})$.*

Consequently $B_1(\mathbf{0})$ is weakly compact in $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$ if $\big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$ is a reflexive Banach space, but also the converse is true.

Note that it is part of the claim that the unit ball of the dual space is not only closed with respect to the very strong form of norm convergence, but also (as a consequence of the PUB!) closed with respect to the much coarser w^*-topology. Since the increased coarseness of the topology allows for many more convergent sequences and indeed nets the closedness property is in fact a stronger one! (to be discussed also in the exercises). This is one of the *important* theorems[34] in functional analysis. See also

`http://de.wikipedia.org/wiki/`
`Satz_von_Banach-Alaoglu#F.C3.BCr_Banachr.C3.A4ume`

For the proof of this theorem one makes use of one of the strong results from general topology: the product of an arbitrary family of *compact* topological spaces, endowed with the correct, i.e. the *product topology* (the topology of pointwise convergence, if we view the product space as a space of functions over the index set) is as again a *compact space*. The strength of this result is partially due to the fact that *arbitrary index sets* are allowed, which in turn makes it necessary to use the *axiom of choice* resp. *Hausdorff's maximality principle*, see

`http://de.wikipedia.org/wiki/Hausdorffs_Maximalkettensatz`

We do not give a proof here, but rather point out the two main steps that are used in the argument (aside from the power of Tychonoff's product result):

- First of all the elements of $B_1(\mathbf{0})$ are considered not as functions on the full Banach space $\big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$, but rather again only on the unit ball $B_1(\mathbf{0}) \subset \boldsymbol{B}$. Clearly, if linear functionals from $\boldsymbol{B}'$ are known on $B_1(\mathbf{0})$ it is trivial to know their (using scalar multiplication for dilation) on all of $\boldsymbol{B}$ (simply because for $\lambda = 1/\|\mathbf{b}\|_{\boldsymbol{B}}$ one has $\mathbf{b}'(\mathbf{b}) = \mathbf{b}'(\lambda\mathbf{b})/\lambda$). In this way the range of $\mathbf{b}'$ on $B_1(\mathbf{0}) \subset \boldsymbol{B}$ is compact (!) within $\mathbb{C}$!

- Secondly one needs to show that the linear functionals are a w^*-closed subset within this collection. But this is easy, because the linearity can be characterized by pointwise relationships, namely $\mathbf{b}'(\mathbf{b}_1+\mathbf{b}_2) = \mathbf{b}'(\mathbf{b}_1)+\mathbf{b}'(\mathbf{b}_2)$ and $\mathbf{b}'(\gamma\mathbf{b}) = \gamma\mathbf{b}'(\mathbf{b})$ and if a net of (linear) functionals $(\mathbf{b}'_\alpha)_{\alpha\in I}$ fulfills these conditions and is pointwise convergent also the limit is satisfying this condition. Indeed

$$\mathbf{b}'_0(\mathbf{b}_1+\mathbf{b}_2) = \lim_\alpha \mathbf{b}'_\alpha(\mathbf{b}_1+\mathbf{b}_2) = \lim_\alpha \mathbf{b}'_\alpha(\mathbf{b}_1) + \mathbf{b}'_\alpha(\mathbf{b}_2) =$$

[34] The validity of this theorem is also one of the reasons why the w^*-convergence has to be discussed in a FA course! The final/additional claim characterizing reflexivity via the weak compactness of $B_1(\mathbf{0})$ is usually not part of the B-A theorem, but has been added for convenience of the reader here. Maybe it should be given as a separate statement.

$$= \lim_{\alpha} \mathbf{b}'_\alpha(\mathbf{b}_1) + \lim_{\alpha} \mathbf{b}'_\alpha(\mathbf{b}_2) = \mathbf{b}'_0(\mathbf{b}_1) + \mathbf{b}'_0(\mathbf{b}_2)$$

and by a similar argument $\mathbf{b}'_0$ is compatible with scalar multiplication.

This results is given in most functional analysis courses and books (e.g. [4, 20]).

6.11 The dual of $\boldsymbol{B}'$ with the w^*-topology

Motivation: We know from linear algebra, that the set of solutions to the homogeneous system of linear equations described in matrix form, i.e. the matrix equation $\boldsymbol{A} * \mathbf{x} = \mathbf{0}$ is just the kernel of $\boldsymbol{A}$ or $\mathrm{Null}(\boldsymbol{A})$, and is known to be the orthogonal complement of the rowspace $\mathrm{Row}(\boldsymbol{A})$ of $\mathbb{C}^n$ (for a typical $m \times n$-matrix), actually it should be seen as the column space of the adjoint operator (matrix) $\boldsymbol{A}' = \overline{(\boldsymbol{A}^t)}$.

In fact, each line in the system of equations describes a hyperplane, where the corresponding row vector is used as a normal vector (to describe the hyperplane). In this way the system of equations expresses that a solution to the *homogeneous system of equations* has to be a vector *perpendicular* to each of the row vectors, but any such has to be orthogonal to any linear combination of row vectors (whether they are linear independent or now, they are by definition a spanning set for the row space), i.e. $\mathbf{x} \in \mathrm{Null}(\boldsymbol{A}) \Rightarrow \mathbf{x} \perp \mathrm{Row}(\boldsymbol{A})$.

A technical lemma in this (linear algebra) spirit needed for the subsequent proof is the following one:

Lemma 30. *Given a finite set of linear functionals $F = (\kappa_j)_{k=1}^K$ in the dual space of some normed space $(\mathbf{V}, \|\cdot\|)$, and one additional linear functional $\kappa_0 \in \mathbf{V}'$ with the property that*

$$\cap_{k=1}^K \mathrm{Null}(\kappa_k) \subseteq \mathrm{Null}(\kappa_0). \tag{68}$$

Then κ_0 is just a finite linear combination of the given functionals, i.e. there are coefficients $(c_k)_{k=1}^K$ in $\mathbb{C}$ such that $\kappa_0 = \sum_{k=1}^K c_k \kappa_k$.

This is Prop. 1.4. of Appendix A in [4].

Proof. One can reduce the problem to the situation that for each $j \in 1, \ldots, K$

$$\cap_{k \neq j} \mathrm{Null}(\kappa_j) \subset \cap_{k=1}^K \mathrm{Null}(\kappa_k), \tag{69}$$

i.e. $\mathrm{Null}(\kappa_j)$ actually contributes to the intersection of null-spaces[35]. In fact, after eliminating possible unnecessary functionals from F one still has the same intersection, but a "minimal set" of linear functionals with the same intersection of null-spaces.[36]

[35] One may think that otherwise it is already a linear combination of the other ones and therefore one does not have to make use of it.

[36] This is, slightly in disguise, the assumption that one may - without loss of generality - assume that the finite family F is linear independent in $\boldsymbol{B}'$, resp. that one should select from a given set a linear independent set of generators and work from that.

Assuming the validity of (69) we can find for each $j \in \{1, \ldots, K\}$ some $\mathbf{v}_j \in \mathbf{V}$ with $\mathbf{v}_j \in \mathbf{V}$ such that $\kappa_j(\mathbf{v}_j) \neq 0$ but $\kappa_j(\mathbf{v}_k) = 0$ for $k \neq j$. Of course one can assume (after rescaling of the elements $\mathbf{v}_j$ found in this way that

$$\kappa_j(\mathbf{v}_k) = \delta_{j,k} \quad \text{(Kronecker Delta function)} \tag{70}$$

In other words, the vectors $(\mathbf{v}_j)_{k=1}^K$ are a set of linear independent vectors such that $(\kappa_j)_{k=1}^K$ (the given [linear independent!] sequence of linear functionals) are *biorthogonal* to those chosen vectors.[37]

Similar to the way in which one expands a given functional on a space into a linear combination of *dual basis* vector elements we can now obtain the given functional κ_0:

Once the sequence $(\mathbf{v}_j)_{j=1}^K$ is given one can define a mapping $\mathbf{v} \mapsto \widetilde{\mathbf{v}}$ on $\boldsymbol{V}$ as follows:

$$\widetilde{\mathbf{v}} = \sum_{j=1}^K \kappa_j(\mathbf{v})\mathbf{v}_j,$$

and consequently one has for $1 \leq k \leq K$, due to the biorthogonality property (70).

$$\kappa_k(\widetilde{\mathbf{v}}) = \sum_{j=1}^K \kappa_j(\mathbf{v})\kappa_k(\mathbf{v}_j) = \kappa_k(\mathbf{v}),$$

or all together

$$\widetilde{\mathbf{v}} - \mathbf{v} \in \cap_{j=1}^K \operatorname{Null}(\kappa_j).$$

We can now consider any κ_0 satisfying the assumption (68) that

$$\kappa_0(\mathbf{v}) = \kappa_0(\widetilde{\mathbf{v}}) = \sum_{j=1}^K \kappa_j(\mathbf{v})\kappa_0(\mathbf{v}_j) = [\sum_{j=1}^K \kappa_0(\mathbf{v}_j)\kappa_j](\mathbf{v}) \tag{71}$$

which is exactly the claimed statement, with $c_j = \kappa_0(\mathbf{v}_j)$, for $1 \leq j \leq K$. □

****

Remark 15. In spirit also quite close to the well known fact that one has in the finite dimensional setting, for any finite set $F \subset \mathbf{V}$:

$$\mathbf{v}_0 \in (F^\perp)^\perp \quad \Rightarrow \quad \mathbf{v}_0 \in \operatorname{span}(F).$$

****

We are now going to show that any w^*-continuous linear functional κ_0 on a dual space $\boldsymbol{B}'$ is the linear combination of a finite set of elements of the form $i_{\boldsymbol{B}}(b_j), 1 \leq j \leq K$. So in fact we will derive directly from the w^*-continuity that there is such a finite set. So our functional κ_0 will be a w^*-continuous functional on $\mathbf{V} = \boldsymbol{B}'$.

[37] The situation is in fact very much the same that one has by showing that on the vector space $\mathscr{P}_3(\mathbb{R})$ of cubic polynomials the sequence of Dirac measures $\delta_{x_1}, \delta_{x_2}, \delta_{x_3}$ and δ_{x_4}, defined via $\delta_y(p(t)) = p(y)$ (evaluation mapping) are linear independent (and hence a basis for the 4-dim. dual space of the four-dimensional space $\mathscr{P}_3(\mathbb{R})$). The basis (alternative to the standard basis of monomials for $\mathscr{P}_3(\mathbb{R})$) is then the family of Lagrange-Interpolation-Polynomials. Just look at $\prod_{k \neq j}(x - x_k)$ for $j \in \{1, 2, 3, 4\}$ and one will find exactly the situation described for the elements $(\mathbf{v}_j)$ above, in the abstract setting!

Since it is w^*-continuous it is also norm-continuous, i.e. $\kappa_0 \in \mathbf{V}' = \boldsymbol{B}''$, but we have to search whether κ_0 has extra properties. The assumption certainly implies that one can find for any $\varepsilon > 0$ some w^*-neighborhood U of $\mathbf{0}$ such that $\mathbf{b}' \in U$ implies $|\kappa_0(\mathbf{b}')| < \varepsilon$. Since a basis for the neighborhood systems of the form $U(F, \varepsilon')$ is known we can find a finite subset $F \subset \boldsymbol{B}$, say $\{\mathbf{b}_1, \ldots, \mathbf{b}_K\}$ and $\varepsilon' > 0$ such that

$$|\mathbf{b}'(\mathbf{b}_k)| < \varepsilon' \quad \text{for } 1 \leq k \leq K \quad \Rightarrow \quad |\kappa_0(\mathbf{b}')| < \varepsilon. \tag{72}$$

Now we just view for each such k $\mathbf{b}_k$ $\mathbf{b}' \to \mathbf{b}'(\mathbf{b}_k) = i_{\boldsymbol{B}}(\mathbf{b}_l)(\mathbf{b}')$ as a linear functional on $\boldsymbol{B}'$. Note that certainly $\mathbf{b}' \in \mathrm{Null}(i_{\boldsymbol{B}}(\mathbf{b}_k))$ implies that (72) can be applied and implies $|\kappa_0(\mathbf{b}')| < \varepsilon$ according to (72). But also $\lambda \mathbf{b}'$ is in the same nullspace for any $\lambda \in \mathbb{C}$, so we conclude that

$$|\lambda||\kappa_0(\mathbf{b}')| = |\kappa_0(\lambda \mathbf{b}')| < \varepsilon \quad \forall \lambda \in \mathbb{C}.$$

but his obviously implies that $\kappa_0(\mathbf{b}') = 0 \in \mathbb{C}$, or $\mathbf{b}' \in \mathrm{Null}(\kappa_0)$.

The above argument shows that the w^*-continuity of $\kappa_0 \in \boldsymbol{B}''$ puts ourselves into the situation described in the above lemma (with $\kappa_k = i_{\boldsymbol{B}}(\mathbf{b}_k), 1 \leq k \leq K$) and so finally we have verified that

$$(\boldsymbol{B}', w^*)' = \boldsymbol{B}.$$

This completes the proof.

As a consequence one can say: If one has an operator which is mapping $(\boldsymbol{B}', w^*)$ continuously into itself, then it has a dual operator which maps the dual spaces, which are in fact the dual, which are in fact (cf. above) the *pre-dual* spaces (in the general case of two dual spaces, both with their w^*-topology). In this sense we have

Theorem 19. *Any bounded linear operator T from $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$ into itself which is also wwst-continuous is the dual operator of some (uniquely determined) operator $S \in \mathcal{L}(\boldsymbol{B})$. In fact, the mapping $S \mapsto T = S'$ is a surjective and isometric mapping from $\mathcal{L}(\boldsymbol{B})$ into the subspace of $\mathcal{L}(\boldsymbol{B}')$ which consists of all w^*-w^*--continuous linear operators on $\boldsymbol{B}'$.*

Proof. It is a good exercise the verify that any dual operator $T := S'$ is not just norm-continuous on $\boldsymbol{B}'$ (with $|||T|||_{\boldsymbol{B}'} = |||S|||_{\boldsymbol{B}}$) but also w^*-w^*--continuous.

The converse follows from the general considerations about the w^*-dual spaces and dual operators (well: now for the topological vector space $(\boldsymbol{B}, w^*)$. □

7 Standard Material which can be done anytime

7.1 Quotient Spaces

In the same way as within groups the "sub-objects" are subgroups (and not just subsets), and linear algebra *linear subspaces* are natural objects one has to deal with *closed linear subspaces* in the case of functional analysis. The natural "morphisms" (structure preserving) applications are of course the bounded linear mappings resp. boundedly invertible bijective linear mappings (called *isomorphisms of normed spaces*). Using these to terms one can come up with the natural identification of the range space of a linear map with the quotient of the domain of the linear map, divided by the null-space of the linear mapping.

Recall the following situation in linear algebra (perfectly expressed by the SVD, the singular value decomposition). There is even an orthonormal basis $\mathbf{v}_1, \cdots \mathbf{v}_r$, in the row space of $\boldsymbol{A}$ (better: the column space of $\boldsymbol{A}'$) with $r = rank(T)$, such that $T : \mathbf{x} \to \boldsymbol{A} * \mathbf{x}$, maps this system onto an orthogonal basis of the column space of $\boldsymbol{A}$.

Definition 33. For any subspace $\mathbf{W} \subset \mathbf{V}$ the quotient $\mathbf{V}/\mathbf{W}$ consists of *equivalence classes* of elements, given by the equivalence relation $\mathbf{x} \sim_{\mathbf{W}} \mathbf{y}$ if $\mathbf{x} + \mathbf{W} = \mathbf{y} + \mathbf{W}$. In fact, $\mathbf{x} + \mathbf{W}$ is the equivalence class generated by $\mathbf{x}$ and $\mathbf{x} \sim \mathbf{y}$ if and only if $x - y \in \mathbf{W}$. Sometimes we write also $[\mathbf{x}]_{\mathbf{W}}$ for such an equivalence class.

Lemma 31. *The quotient* $\mathbf{V}/\mathbf{W}$*, endowed with the natural addition* [38]

$$(\mathbf{x}_1 + \mathbf{W}) +_Q (\mathbf{x}_2 + \mathbf{W}) := (\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{W}$$

i.e. we define

$$[\mathbf{x}]_{\mathbf{W}} +_Q [\mathbf{x}]_{\mathbf{W}} := [\mathbf{x}_1 + \mathbf{x}_2]_{\mathbf{W}}$$

and the natural scalar product are (obviously) again a vector space.

Theorem 20. *The mapping* $[\mathbf{x}]_{\mathbf{W}} \to \|[\mathbf{x}]_{\mathbf{W}}\|$*, defined by*

$$\|[\mathbf{x}]_{\mathbf{W}}\| := \inf_{\mathbf{w} \in \mathbf{W}} \{\|\mathbf{x} + \mathbf{w}\|_{\boldsymbol{V}}\} \tag{73}$$

is a norm on the quotient space if (and only if) $\mathbf{W}$ *is a closed subspace of* $\boldsymbol{V}$*.*

Moreover, if $\mathbf{V}$ *is complete, then also the quotient space* $\mathbf{V}/\mathbf{W}$ *is a Banach space.*

Proof. The first part, including the "if and only if" statement, is left to the reader. It is interesting to see how the closedness comes into the game here.

For the completeness let us use the criterion involving absolutely convergent series. Assume we have completeness of $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$ and a sequence $(\mathbf{x}_n)$ is given, with

$$\sum_{n=1}^{\infty} \|[\mathbf{x}_n]_{\mathbf{W}}\| = C < \infty.$$

Since for fixed $n \in \mathbb{N}$ the quotient norm of the element is given as an infimum it is always possible to find some $\mathbf{w}_n \in \mathbf{W}$ such that for any fixed and given $\varepsilon > 0$ one has:

$$\|[\mathbf{x}_n]_{\mathbf{W}}\| \geq \|\mathbf{x}_n + \mathbf{w}_n\|_{\boldsymbol{V}} + 2^{-n}\varepsilon.$$

[38] Just here we write $+_Q$ for the quotient addition, because it is a new addition.

Then it is clear that also the series $\mathbf{v}_0 := \sum_{n=1}^{\infty} \mathbf{x}_n + \mathbf{w}_n$ is absolutely convergent, in fact $\mathbf{v}_0 = \mathbf{x}_0 + \mathbf{w}_0$ with

$$\|\mathbf{x}_0\|_{\boldsymbol{V}} \leq \sum_{n=1}^{\infty} \|\mathbf{x}_n\|_{\boldsymbol{V}}$$

and

$$\|\mathbf{w}_0\|_{\boldsymbol{V}} \leq \sum_{n=1}^{\infty} \|\mathbf{w}_n\|_{\boldsymbol{V}} \leq \varepsilon.$$

Hence the class $[\mathbf{x}_0]_{\boldsymbol{W}}$ has a representative with

$$\|\mathbf{x}_0 + \mathbf{w}_0\| \leq \sum_{n=1}^{\infty} \|[\mathbf{x}_n]_{\mathbf{W}}\| + \varepsilon = C + \varepsilon,$$

but this is valid for any $\varepsilon > 0$, and consequently one has

$$\|[\mathbf{x}_o]_{\mathbf{W}}\| \leq C.$$

The convergence of the partial sums follows now easily by repeating the same argument to the tails of the series.

Given $\delta > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{n=n_1+1}^{\infty} \|[\mathbf{x}_n]_{\mathbf{W}}\| \leq \delta,$$

which implies that as a consequence this series is convergent as well, and the limit is obviously the class $[\mathbf{x}_0]_{\mathbf{W}} + \sum_{n=1}^{n_1} [\mathbf{x}_n]_{\mathbf{W}}$. The above argument (replacing C above by δ now) implies that

$$\|[\mathbf{x}_0]_{\mathbf{W}} - \sum_{n=1}^{N} [\mathbf{x}_n]_{\mathbf{W}}\| \leq \delta,$$

as long as $N > n_1$, and thus the proof is complete.

□

7.2 Finite Products of Banach spaces

<u>Product</u>: The (set-theoretical) product of two normed spaces can be endowed naturally with various different, but *equivalent* norms, for example $\|(x,y)\|_\infty = max(\|x\|, \|y\|)$, or $\|(x,y)\|_1 = \|x\| + \|y\|)$ or $\|(x,y)\|_2 = \sqrt{(\|x\|^2 + \|y\|^2)}$.

For any finite Produkt the with respect to any of these norms is in fact equivalent to coordinatewise convergence in the product space, i.e. norm convergence in the appropriate norm of the component.

siehe Übungen

There is also a quite natural *isomorphism theorem*, which generalizes the fact that in linear algebra matrix multiplication by some matrix $\boldsymbol{A}$ establishes an isomorphism between the range space of the matrix, i.e. the column space of $\boldsymbol{A}$, and the row-space of $\boldsymbol{A}$ (hence row-rank equals column-rank), which can/should be identified with the

quotient space of the domain (usually $\mathbb{R}^n$) divided by the Null-space ($\boldsymbol{A}$) of the mapping $\mathbf{x} \mapsto \boldsymbol{A} * \mathbf{x}$. In fact, it is well known that $\mathbb{R}^n = \mathrm{Null}(\boldsymbol{A}) \bigoplus \mathrm{Row}(\boldsymbol{A})$, as an orthogonal direct sum (resp. $\mathrm{Row}(\boldsymbol{A})^\perp = \mathrm{Null}(\boldsymbol{A})$).

For linear operators between Banach spaces we have the following result:

Theorem 21. *Given two Banach spaces* $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ *and* $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ *and a surjective, bounded linear operator* $T \in \mathcal{L}(\boldsymbol{B}^1, \boldsymbol{B}^2)$ *we have the following natural isomorphism, induced by the mapping* $T_1 := T \in \pi$*, where* π *is the canonical projection from* $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ *onto* $\boldsymbol{B}^1/\mathrm{Null}(T)$*.*

$$T_1 : \boldsymbol{B}^1/\mathrm{Null}(T) \to \boldsymbol{B}^2 \tag{74}$$

defines an isomorphism of Banach spaces.

Proof. Recall that $N := \mathrm{Null}(T)$ is a closed subspace of $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ and hence the quotient space if a well-defined Banach space (with the natural quotient norm). It is clear that the mapping $T_1 : \mathbf{b} + N \mapsto T(\mathbf{b})$ is well defined, because any two elements from $\mathbf{b} + N$ have the same image in $\boldsymbol{B}^2$ under T. It is also clear that T_1 is surjective, since it has the same range as T. Finally we have to verify that it is continuous. In fact it has the same operator norm as T itself!

Note that we have to estimate $T_1([\mathbf{b}]_N)$ under the assumption that the norm of this class in $\boldsymbol{B}^1/N$ has norm ≤ 1 in the quotient norm. Hence for $\varepsilon > 0$ one can find some $\mathbf{b}_1 \in \mathbf{b} + N$ such that $\|\mathbf{b}_1\|_{\boldsymbol{B}^1} \leq 1 + \varepsilon$, and consequently

$$\|T_1(\mathbf{b} + N)\|_{\boldsymbol{B}^2} = \|T(\mathbf{b}_1)\|_{\boldsymbol{B}^2} \leq |||T|||\, \|\mathbf{b}_1\|_{\boldsymbol{B}^1} \leq |||T|||\, (1 + \varepsilon).$$

Since such an estimate is valid for any $\varepsilon > 0$ we have

$$|||T_1|||_{\boldsymbol{B}^1/N \to \boldsymbol{B}^2} \leq |||T|||_{\boldsymbol{B}^1 \to \boldsymbol{B}^2}.$$

The converse estimate is left to the reader for individual consideration!

In particular it is now clear that T_1 is surjective, continuous and obviously injective (since the nullspace of T has been "factored out"). Hence we can apply Banach's theorem and obtain that T_1 defines an isomorphism. $\square$

8 Hilbert Spaces and Orthonormal Bases

8.1 Scalar Products, Orthonormal Systems

Definition of a scalar product etc.

Lemma 32. *For any sesquilinear form on a complex vector space one has the following estimate which is called the Cauchy-Schwartz inequality:*

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_{\mathcal{H}} \|\mathbf{y}\|_{\mathcal{H}} \quad \textit{for} \quad \mathbf{x}, \mathbf{y} \in \mathcal{H}. \tag{75}$$

STILL MISSING: RIESZ REPRESENTATION THEOREM, GRAM-SCHMIDT ORTHONORMALIZATION

For the classical Riesz Representation Theorem see

In this section the use of *inner products* (i.e. positive definite sesquilinear forms) will be discussed.

Definition 34. A mapping from $\boldsymbol{V} \times \boldsymbol{V} \to \mathbb{K}$, usually described as $(\mathbf{v}_1, \mathbf{v}_2) \to \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$, is called a *sesqui-linear form* if ... cf. linear algebra

It is called *positive definite* of $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \Rightarrow \mathbf{v} = \mathbf{0}$.

For any pos. def. sesqui-linear form there is an associated norm, defined by the expression $\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. A vector space $\boldsymbol{V}$ endowed with such a sequi-linear form (viewed as a normed space) is called a *pre-Hilbert space*.

If the space $\boldsymbol{V}$, endowed with this canonical norm is a Banach space, then it is called a *Hilbert space*, and usually then the symbol $\mathcal{H}$ is used, resp. $(\mathcal{H}, \langle \cdot \rangle)$.

Standard concepts are just like in linear algebra:

Definition 35. A family $(x_i)_{i \in I}$ in $(\mathcal{H}, \langle \cdot \rangle)$ is called an *orthogonal system* if $\langle x_i, x_j \rangle = 0$ for $i \neq j$. It is called an *orthonormal family*, if $\langle x_i, x_j \rangle = \delta_{i,j}$ (Kronecker delta).

Ideas, goals: Every separable Hilbert space $(\mathcal{H}, \langle \cdot \rangle)$ has a countable orthonormal basis, thus establishing an isometric (in fact unitary) isomorphism between $\mathcal{H}$ and $\boldsymbol{\ell}^2(\mathbb{N})$. This is essentially another application of the idea of the Gram-Schmidt method in the context of the Hilbert spaces.

Alternative symbol used: $\mathcal{H} = \mathscr{H}$.

Definition 36. A bounded linear mapping T from $(\mathcal{H}, \langle \cdot \rangle)$ to $(\mathcal{H}, \langle \cdot \rangle)$ is *unitary* if it is surjective and satisfies

$$\langle T\mathbf{x}, T\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle, \quad , \forall \mathbf{x}, \mathbf{y} \in \mathcal{H}. \tag{76}$$

Evidently this implies that T is isometric, i.e. $\|T\mathbf{x}\|_{\mathcal{H}} = \|\mathbf{x}\|_{\mathcal{H}}$ for all $\mathbf{x} \in \mathcal{H}$, hence injective, thus consequently a bijective linear mapping.

Remark 16. The preservation of scalar products (or equivalently, by the polarization identity, at least for Hilbert spaces over $\mathbb{C}$), equivalent to the preservation of length (and also the preservation of angles) by itself is only equivalent with the identity $T^*T = Id_{\mathcal{H}}$, but T might not be invertible in such a case, because the range can be a proper subspace! (if and only if the Hilbert space is infinite dimensional).

The right shift T_1 on $\boldsymbol{\ell}^2(\mathbb{N})$, which maps $[x_1, x_2, \dots]$ into $[0, x_1, x_2, \dots]$ is a classical example. Obviously it is *not surjective* but isometric. Check why the left shift (dropping the first coordinate!!!) is just T_{-1} and consequently the reverse composition, namely $T_1 \circ T_{-1}[x_1, x_2, \dots..] = [0, x_2, x_3, \dots]$ and is *not* the identity!!

For the case $\mathbb{K} = \mathbb{C}$ also the converse is true, due to the *polarization identity*:

Make a connection to the so-called *polarization identity* which hold true for arbitrary complex Hilbert spaces $\mathscr{H}$. For any $x, y \in \mathscr{H}$ the scalar product $\langle x, y\rangle_{\mathscr{H}}$ can be expressed as a sum of norms:

$$\langle x, y\rangle_{\mathscr{H}} = \frac{1}{4}\sum_{k=0}^{3} i^k \langle (x + i^k y), (x + i^k y)\rangle_{\mathscr{H}} = \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2_{\mathscr{H}}, \tag{77}$$

where i denotes to complex unit in $\mathbb{C}$.

There are a few important basic facts concerning unitary linear mappings on a Hilbert space:

Lemma 33.
- *The family of unitary operators on a Hilbert space $(\mathcal{H}, \langle\,\cdot\,\rangle)$ forms a (usually non-commutative) group of operators; in particular the composition of two unitary operators as well as the inverse of a unitary operator is again unitary.*
- *A bounded linear operator on a Hilbert space if unitary if and only if it maps any complete orthonormal system into a complete orthonormal system;*
- *A bounded linear mapping is unitary if and only if it has an infinite matrix representation with respect to some fixed complete orthonormal system $(h_i)_{i\in I}$, with coefficients which form a complete orthonormal system for $\boldsymbol{\ell}^2(I)$ (where I is the index set of that orthonormal basis).*

One could even claim that the *unitary group* acts *transitively* on the set of all complete orthonormal systems in a given Hilbert space, i.e. for any two such systems there exists a (uniquely determined) unitary transformation of the underlying Hilbert space mapping one onto the other.

The usage of orthonormal systems (and in particular orthonormal bases for separable Hilbert spaces) is very much like that use of *unitary* matrices for vectors of length n, and matrices describing the general linear mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$.

As a first lemma (good Exercise!) we state:

Lemma 34. *Assume that $(\mathbf{x}_i)_{i\in I}$ is any orthonormal system in a Hilbert space $(\mathcal{H}, \langle\,\cdot\,\rangle)$. Then the length of any finite linear combination (only a finite set $F \subset I$ of coefficients is non-zero) of the form $\mathbf{x} = \sum_{i\in F} c_i \mathbf{x}_i$ is just the Euclidean length of the coefficient vector, or more explicitly*

$$\left\| \sum_{i\in F} c_i \mathbf{x}_i \right\|_{\mathcal{H}} = \sqrt{\sum_{i\in F} |c_i|^2}. \tag{78}$$

As a consequence (taking limits) one has: the infinite sum is (in fact then automatically unconditionally, but not necessarily absolutely, convergent to $\mathbf{x}_0 = \sum_{i\in I} c_i \mathbf{x}_i$ if and only if the coefficient sequence $(c_i) \in \boldsymbol{\ell}^2(I)$. In fact, if it is a Cauchy sequence, the

corresponding sequence of coefficients (more precisely: the collection of coefficients, with all but finitely many non-zero coefficients put to zero, indexed by the collection of finite subsets $F \subset I$) forms a Cauchy sequence in $\boldsymbol{\ell}^2(I)$ (and vice versa).

Another nice/easy consequence is the following: Whenever $J \subseteq I$ is any subset, be it finite of infinite, it is guaranteed that for $(c_i) \in \boldsymbol{\ell}^2(I)$ the sum $\sum_{i \in J} c_i \mathbf{x}_i$ is well defined.

Another easy exercise would be to establish the following fact:

Lemma 35. *For the one-dimensional subspace $\mathbf{V}_{\mathbf{y}}$ generated by a vector $\mathbf{y} \in \mathcal{H}$ (we may assume for convenience that $\|\mathbf{y}\|_{\mathcal{H}} = 1$) the orthogonal projection onto $\mathbf{V}_{\mathbf{y}}$ is given by the mapping*

$$\mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{y}, \quad \mathbf{x} \in \mathcal{H}. \tag{79}$$

We need one more observation:

Theorem 22. *For any given (finite or infinite) set $J \subseteq I$ we consider the closed linear span of the subfamily $(\mathbf{x}_i)_{i \in J}$ of an orthonormal family $(\mathbf{x}_i)_{i \in I}$, which we denote by $\mathbf{V}_J$.*

Then for any $\mathbf{x} \in \mathcal{H}$ we can define the mapping

$$P_J : \mathbf{x} \to \sum_{i \in J} \langle \mathbf{x}, \mathbf{x}_i \rangle \mathbf{x}_i, \quad \mathbf{x} \in \mathcal{H}, \tag{80}$$

which describes the orthogonal projection from $\mathcal{H}$ onto $\mathbf{V}_J$.

In particular we have:

$$\|\mathbf{x} - P_J(\mathbf{x})\|_{\mathcal{H}} \leq \|\mathbf{x} - \mathbf{v}_J\|, \quad \forall \mathbf{v}_J \in \mathbf{V}_J. \tag{81}$$

It is part of the claim above that for any subset and any family J the sum

$$\sum_{i \in J} \langle \mathbf{x}, \mathbf{x}_i \rangle \mathbf{x}_i$$

is convergent in $\mathcal{H}$, for any $\mathbf{x} \in \mathcal{H}$ (because it is obviously a CS, thanks to Pythagoras!). The sum is just the projection onto the (automatically closed!) linear span of the elements $(\mathbf{x}_j)_{j \in J}$.

It is not even necessary that those projections are one-dimensional, but it could be general projections, as long as they are pairwise orthogonal. So assume that one has a sequence of pairwise orthogonal projections $(P_l)_{l \in I}$, with $P_l \circ P_{l'} = 0$ for $l \neq l'$, then $\sum_{l \in I} P_l(\mathbf{x})$ is convergent (and the limit is the orthogonal projection onto the closed linear span of all the spaces involved).

The above result can also be described as a description of the projection of P_J as diagonal matrices with entries 1 for $i \in J$ and 0 for $i \in I \setminus J$. There are other operators obtained in this way (we are preparing the stage for the diagonalization results for compact self-adjoint linear operators on Hilbert spaces).

The topic of "matrix representations" is an interesting in itself.

The notion of compactness is the same as in $\mathbb{R}^d$ and we thus just recall the definition:

Definition 37. A subset $M \subset (\mathbf{V}, \|\cdot\|)$ is compact if for any $\varepsilon > 0$ every covering by ε-balls there exists a finite (sub-)covering of M.

8.2 Compact sets in concrete Banach spaces

It is of course interesting to study (relatively) compact set in concrete Banach spaces. One of the classical examples in the literature (see e.g. [4], p.175) is the so-called Arzela-Ascoli Theorem, a prototype result characterizing relative compactness in $(\boldsymbol{C}(X), \|\cdot\|_\infty)$ over compact domains.

Theorem 23. *[Arzela-Ascoli Theorem] A bounded and closed subset of $M \subset (\boldsymbol{C}(X), \|\cdot\|_\infty)$, where X is a compact (hence completely regular) topological space, is compact if and only if it is equicontinuous, which means: For every $\varepsilon > 0$ and $x_0 \in X$ there exists some neighborhood U_0 of x_0 such that*

$$|f(z) - f(x_0)| \leq \varepsilon, \quad \forall f \in M.$$

Remark 17. Since it is also true that a continuous functions on a compact, metric space (to take the easy case of a [**?**]) one even could say that a set M of continuous functions on a compact, metric space X (e.g. any compact subset of $\mathbb{R}^d$) is *uniformly equicontinuous* on X if

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \quad |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall f \in M. \tag{82}$$

In other words, the $\delta > 0$ d depends only on ε but not on the position x nor on the function f chosen, as long as $f \in M$.

The characterization of relatively compact subsets of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$ is given as Ex. 17 (p.177) in [4]:

Theorem 24. *A bounded and closed subset $M \in \boldsymbol{C}_0(\mathbb{R}^d)$ (with the sup-norm) is relatively compact resp. totally bounded if and only it is (uniformly) equicontinuous and tight, i.e. as the property*

$$\forall \varepsilon > 0 \; \exists R > 0 \text{ such that: } |f(x)| \leq \varepsilon \text{ if only } |x| \geq R, \forall f \in M. \tag{83}$$

There are certain manipulations which one can carry out with compact sets. The best way to verify these results is to use their characterization using the concept of *total boundedness* (for every $\varepsilon > 0$ there exists a finite covering [chosen by the user!] with balls of radius at most ε). >> Ex.!

Lemma 36. *For every totally bounded set M set the closure is totally bounded as well. In principle we even have much more: Assume that a set M_0 is - with respect to the Hausdorff distance - the limit of a sequence of totally bounded sets M_n, then M_0 is itself a totally bounded set.*

The continuity of addition gives:

Lemma 37. *Given two compact sets M_1 and M_2 in some normed space $(\mathbf{V}, \|\cdot\|)$, then also the complex sum $M_1 + M_2$ is a compact subset of $(\mathbf{V}, \|\cdot\|)$.*

Certainly slightly more surprising, although not too difficult to prove, using the ideas of the above two lemmas, is the following **Theorem of Mazur**:

Proposition 12. *([4],p.180): Assume that M is a totally bounded set, then the closed convex hull of M, i.e. the closure of the set of all convex linear combinations of the form*

$$\mathbf{y} = \sum_{k=1}^{K} c_k m_k, \quad \textit{with} \sum_{k=1}^{K} c_k = 1, c_k \geq 0,$$

is totally bounded in $(\mathbf{V}, \|\cdot\|)$ as well.

8.3 Compact Operators

Definition 38. A linear operator T from a Banach space $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ is a *compact operator* if the image of any bounded set $M \subset \boldsymbol{B}^1$ within $\boldsymbol{B}^2$ is relatively compact, i.e. has compact closure.

We will write $\mathcal{B}_0(\boldsymbol{B}^1, \boldsymbol{B}^2)$ for the family of all such compact operators.

A nice result which we will *not prove here*, see [11], Satz 11.3, is due to Schauder:

Theorem 25. *A continuous operator $T \in \mathcal{L}(\boldsymbol{B}^1, \boldsymbol{B}^2)$ is compact if and only if the dual operator $T' \in \mathcal{L}(\boldsymbol{B}^{2'}, \boldsymbol{B}^{1'})$ is compact.*

Proof. In principle only one direction has to be shown, namely that the compactness of T implies the compactness of T', because then this argument can be used to prove that the compactness of T' implies the compactness of T'', but the restriction of T'' to $i_{\boldsymbol{B}^1}(\boldsymbol{B}^1) \subset \boldsymbol{B}^{1''}$ is then compact as well. This is just one of the arguments, the rest is left to the reference. □

One of the standard results (mentioned here without proof) is the following one:

Lemma 38. *For any Hilbert space $\mathcal{H}$ the closure of the subspace $\mathcal{B}_{0,0}(\mathcal{H})$ of all finite rank operators within $\mathcal{L}(\mathcal{H})$ is exactly the closed subspace, in fact closed ideal of all compact operators, i.e. $\mathcal{B}_0(\mathcal{H})$.*

The Arzela-Ascoli Thm. is also at the basis of a proof that for certain Banach spaces (such as $(\boldsymbol{C}(X), \|\cdot\|_\infty)$) a similar statement is valid, but it is not true for general Banach spaces:

Lemma 39. *If X is compact, then the finite rank operators are dense in the compact operators (of course with respect to the operator algebra norm).*

Proof. Just a hint: on is using finite dimensional approximations using kind of *quasi-interpolation operators*, which in the most simple case would be piecewise linear interpolation operators. Details are in [4], p.176. □

It is a good exercise to check that the compact operators form a closed subspace within bounded linear mapping from the Banach space $\boldsymbol{B}^1$ to $\boldsymbol{B}^2$, i.e. within the operator space $\mathcal{B}(\boldsymbol{B}^1, \boldsymbol{B}^2)$.

It is an important observation that due to the *general properties* of a *complete metric space* (hence a Banach space), that a set is relatively compact if and only if it is *precompact*, which means (by definition): For every $\varepsilon > 0$ there exists a finite covering

by balls of radius ε. In fact, from a purely logical point of view it is trivial that relative compactness implies precompactness, so only the converse if of some interest (proof not given here), because obviously for every set M it is true that its closure is contained in the union of all balls of the form $B_\varepsilon(m)$, where m runs through the set M. If this (trivial) covering has a finite subcovering, we have the situation of prae-compactness.

There is also the notion (just another word) of total boundedness of a set:

Definition 39. A set M in a metric space is *totally bounded* if for every $\varepsilon > 0$ there exists some finite covering of M of the form

$$M \subseteq \bigcup_{k=1}^{K} B_\varepsilon(x_i)$$

It is of course equivalent to say: there exists a finite subset $x_1, \ldots, x_K$ of points (Exercise: those points my be choosen from the ambient space X or even from M itself, since the request is made for every $\varepsilon > 0$! [check!]) such that each point $m \in M$ is at most at distance ε to one of these points, i.e.

$$\sup_{m \in M} \left[\inf_{1 \le k \le K} d(x_k, m) \right] \le \varepsilon.$$

In the discussion of compact operators we also need another class of operators.

Definition 40. Given two Banach spaces $\boldsymbol{B}^1, \boldsymbol{B}^2$ an operator is said to be *completely continuous* if for any sequence which is weakly convergent in $\boldsymbol{B}^1$, the image is norm convergent (in $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$).

See the section on the UBP or PUB above!

Remark 18. As a preparatory step of the next result we wont to point out that the weak topology is a Hausdorff topology in particular, weak limits are unique, or more concrete: *If a net* $(\mathbf{b}_\gamma)$ *is convergent to a limit element* $\mathbf{b}0$ *and another limit element* $\mathbf{c}_0$ *then* $\mathbf{b}_0 = \mathbf{c}_0 \in \boldsymbol{B}$. This is in fact clear because $\mathbf{b}_0 - \mathbf{c}_0 \neq \mathbf{0}$ would imply the existence of some bounded linear functional $\mathbf{b}' \in \boldsymbol{B}'$ of norm one with $\mathbf{b}'(\mathbf{b}_0 - \mathbf{c}_0)$ or equivalently $\mathbf{b}'(\mathbf{b}_0) \neq \mathbf{b}'(\mathbf{c}_0)$, which is inconsistent with the fact that $\mathbf{b}'(\mathbf{b}_\gamma)$ is convergent to both $\mathbf{b}'(\mathbf{b}_0)$ and $\mathbf{b}'(\mathbf{c}_0)$ by assumption.

The following result is just a transcription of Satz 11.4 in [11].

Proposition 13.
1. *A compact operator between Banach spaces is always completely continuous;*
2. *if* $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ *is a reflexive Banach space (e.g. a Hilbert space) then the converse is true as well.*

Proof. (a) First we have to show that a compact operator T maps a weakly sequence or net, say $(\mathbf{b}_\alpha)$ into a norm convergent net $(T(\mathbf{b}_\alpha))_{\alpha \in I}$. What is clear (from the boundedness of T is the fact that this new sequence/net is bounded.

So let us assume the contrary, i.e. that there is a weakly convergent sequence $(\mathbf{b}_n)$ with limit $\mathbf{b}_0$ such that $T(\mathbf{b}_n)$ (which is known to be weakly convergent to $T(\mathbf{b}_0)$) would

not have a convergent subsequence with limit $T(\mathbf{b}_0)$, or equivalently there is some $\varepsilon_0 > 0$ such for some subsequence $(\mathbf{b}_{n_j})$ one has

$$\|T(\mathbf{b}_0) - T(\mathbf{b}_{n_j})\|_{\boldsymbol{B}} \geq \varepsilon_0 \quad \text{for} j \in \mathbb{N}.$$

Since $(T(\mathbf{b}_{n_j}))_{j\geq 1}$ is the image of a bounded sequence (recall: weak convergence implies weak boundedness which is the same as boundedness, due to the PUB!) it is relatively compact (in a metric space) hence totally bounded and from this (Heine-Borel-type) argument we obtain that it has a convergent (further) subsequence $(T(\mathbf{b}_{n_{j_k}}))_{k\geq 1}$. But this sequence cannot have as a (strong or weak) limit the "expected" element $T(\mathbf{b}_0)$, because all its elements are at a fixed minimal distance away from it (by the construction of the index sequence $(n_j)_{j\geq 1}$.

(b) For the second part just observe that a weakly convergent sequence in $\big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$ must be bounded (by Banach-Steinhaus, see ??) and therefore (by Banach-Alaoglu, combined with the reflexivity of the space) must have a weakly convergent subsequence $(\mathbf{b}_{n_j})_{j\geq 1}$.[39].

Of this implies norm convergence of $T(\mathbf{b}_{n_j})_{j\geq 1}$ we have established the existence of a norm convergent subsequence of $(T(\mathbf{b}_n))_{n\geq 1}$, hence compactness.

□

COMMENT: TO BE PUT ELSEWHERE PROBABLY,EARLIER!

Remark 19. See [4], p.173. The proof makes use of the PUB (Principle of Uniform Boundedness). If a sequence (x_n) is weakly convergent, i.e. $x'(x_n)$ is convergent in $\mathbb{C}$ for any $x' \in \boldsymbol{B}^{1'}$, then it has to be (norm!) bounded.

The converse is not discussed here, but we will make use of the equivalence of those concepts below!

`http://en.wikipedia.org/wiki/Compact_operator_on_Hilbert_space`

Definition 41. An operator T from a Banach space $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ is a *finite rank operator* if the range of T, i.e. $T(\boldsymbol{B}^1)$ is a finite dimensional subspace of $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$.

It is not too difficult (but also not trivial) to show that

Lemma 40. *An operator T from a Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$ is compact if and only if it can be approximated in the operator norm by finite rank operators.*

Proof. One direction is relatively obvious. Let M the image of $B_1(0)$ under T, and let $B_\varepsilon(\mathbf{y}_j)$ be a covering of M.

.....

Assume that for every $\varepsilon > 0$ there exists some finite-rank operator T_1 such that $|||T - T_1||| \leq \varepsilon/2$

□

[39]For separable Banach spaces $\big(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\big)$ this follows from general consideration, allowing to replace nets by sequences, describing topological properties, for the general case a small extra argument has to be used, which is explained in Satz 10.14, p.199, of [11]

We also need information about *self-adjoint operators* on Hilbert spaces, i.e. operators which are "highly compatible" with the scalar product on the given Hilbert spaces.

Definition 42. A bounded linear operator T on a Hilbert space $(\mathcal{H}, \langle\,\cdot\,\rangle)$ is called *self-adjoint* if one has

$$\langle T\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, T\mathbf{y}\rangle \quad \text{for} \quad \mathbf{x}, \mathbf{y} \in \mathcal{H}. \tag{84}$$

The *spectral theorem* for compact self-adjoint operators is the analogue of the spectral theorem for symmetric matrices and reads as follows:

Theorem 26. *Let T be a self-adjoint and compact operator on a Hilbert space $(\mathcal{H}, \langle\,\cdot\,\rangle)$. Then there exists an orthonormal system of vectors $(\mathbf{h}_n)_{n=1}^{\infty}$ and a null- sequence of real eigenvalues, i.e. $(c_n)_{n=1}^{\infty} \in \boldsymbol{c}_0$ such that*

$$T\mathbf{x} = \sum_{n=1}^{\infty} c_n \langle \mathbf{x}, \mathbf{h}_n\rangle \mathbf{h}_n \quad \textit{for} \quad \mathbf{x} \in \mathcal{H}. \tag{85}$$

One of the important steps towards the proof of this theorem is to find that there is a "realization" of the maximal eigenvalue, i.e. a vector whose length is multiplied under the action of T by exactly the operator norm $|||T|||$, or in other words, there exists a unit vector $\mathbf{x} \in \mathcal{H}$ with $\|\mathbf{x}\|_{\mathcal{H}} = 1$ such that $\|T\mathbf{x}\|_{\mathcal{H}} = |||T|||$.

MATERIAL of Jan. 16th, 2014

Theorem 27. *a) For any self-adjoint linear operator $T = T^*$ in $\mathcal{L}(\mathcal{H})$ on has*

$$|||T||| = \sup\{|\langle T\mathbf{x}, \mathbf{x}\rangle, \|\mathbf{x}\|_{\boldsymbol{B}} \leq 1\}.$$

b) For normal operators T (i.e. with $T^*T = TT^*$) the operator T^*T is self-adjoint and

$$|||T^*T||| = |||T|||^{\,2}.$$

Proof. Let us set $q(T) := \sup\{|\langle T\mathbf{y}, \mathbf{y}\rangle \|\mathbf{y}\|_{\boldsymbol{B}} \leq 1\}$.

Then it is clear that

$$|\langle T\mathbf{z}, \mathbf{z}\rangle| \leq q(T)\|\mathbf{z}\|_{\mathcal{H}}^2, \quad \mathbf{z} \in \mathcal{H}. \tag{86}$$

MORE (page 136 of [11], Satz 7.10). Also making use of the parallelogram rule: Another important fact which can be verified by (lengthly but simple) direct computation is the so-called parallelogram rule:

$$\|\xi + \eta\|^2 + \|\xi - \eta\|^2 = 2(\|\xi\|^2 + \|\eta\|^2), \quad \text{for } \xi, \eta \in \mathcal{H}. \tag{87}$$

Since the quadratic form $Q_T(z) := \langle T\mathbf{z}, \mathbf{z}\rangle$ on $\mathcal{H}$ takes only real values the parallelogram rule can be applied and gives us

$$4Re\langle T\mathbf{x}, \mathbf{y}\rangle = Q_T(\mathbf{x} + \mathbf{y}) - Q_T(\mathbf{x} - \mathbf{y}).$$

For given $\mathbf{x}, \mathbf{y}$ with $\|\mathbf{x}\|_{\mathcal{H}} \leq 1$, $\|\mathbf{y}\|_{\mathcal{H}} \leq 1$ there exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that

$$|\langle T\mathbf{x}, \mathbf{y}\rangle| = \langle T\alpha\mathbf{x}, \mathbf{y}\rangle = \frac{1}{4}(Q_T(\alpha\mathbf{x} + \mathbf{y}) - Q_T(\alpha\mathbf{x} - \mathbf{y})) \leq$$

$$\leq \frac{1}{4}q(T)\left(\|\alpha\mathbf{x}+\mathbf{y}\|_{\mathcal{H}}{}^2+\|\alpha\mathbf{x}-\mathbf{y}\|_{\mathcal{H}}{}^2\right)\leq \frac{1}{2}q(T)\left(\|\alpha\mathbf{x}\|\|_{\mathcal{H}}{}^2+\|\mathbf{y}\|\|_{\mathcal{H}}{}^2\right)=q(T).$$

Taking now the sup over all $\mathbf{y}\in B_1(\mathbf{0})$

□

Lemma 41. *Let T be a self-adjoint and compact operator on a Hilbert space $(\mathcal{H},\langle\cdot\rangle)$. Then there exists a unit vector in $\mathcal{H}$, i.e. $\mathbf{x}\in\mathcal{H}$ with $\|\mathbf{x}\|_{\mathcal{H}}=1$, such that $\|T\mathbf{x}\|_{\mathcal{H}}=|||T|||$.*

Proof. For $\varepsilon=1/n$ there exists some $\mathbf{x}_n\in\mathcal{H}$ with $\|\mathbf{x}_n\|_{\mathcal{H}}=1$ such that $\|T\mathbf{x}_n\|_{\mathcal{H}}\geq |||T|||-1/n$. Since the image of the unit ball in $\mathcal{H}$ under T is a relatively compact set there is a subsequence $(\mathbf{x}_{n_k})$ such that $T\mathbf{x}_{n_k}$ is convergent, to some limit, say $\mathbf{x}_0\in\mathcal{H}$. Since the norm is continuous on $(\mathcal{H},\langle\cdot\rangle)$ (as on any Banach space) we have $\|\mathbf{x}_0\|_{\mathcal{H}}=\lim_{k\to\infty}\|Tx_{n_k}\|=|||T|||$. For convenience of notation let us relabel this sequence simply to $(\mathbf{x}_k)_{k\geq 1}$.

Let us now look at the scalar products $\langle T\mathbf{x}_k,\mathbf{x}_k\rangle$. Since $T=T^*$ it follows that this are real numbers. In fact:

$$\langle T\mathbf{x},\mathbf{x}\rangle=\langle T^*\mathbf{x},\mathbf{x}\rangle=\langle\mathbf{x},T\mathbf{x}\rangle=\overline{\langle T\mathbf{x},\mathbf{x}\rangle}.$$

REST open! [11], p.233 □

The following theorem is just a nice application of the closed graph theorem and will not be presented during the course!

An interesting consequence of the Closed Graph Theorem is the Theorem of Hellinger-Toeplitz: (see [11], p.153, for the proof).

Proposition 14. *Let T be a linear and self-adjoint mapping on $(\mathcal{H},\langle\cdot\rangle)$. Then T is continuous.*

Proof. Assume there is a convergent sequence $(\mathbf{x}_n)$ in $\mathcal{H}$ with $\mathbf{x}_0=\lim_{n\to\infty}\mathbf{x}_n$ and $\mathbf{y}=\lim_{n\to\infty}T(\mathbf{x}_n)$. Then one has for $\mathbf{z}\in\mathcal{H}$:

$$\langle T\mathbf{x}_0,\mathbf{z}\rangle=\langle\mathbf{x}_0,T\mathbf{z}\rangle=\lim_{n\to\infty}\langle\mathbf{x}_n,T\mathbf{z}\rangle=\lim_{n\to\infty}\langle T\mathbf{x}_n,\mathbf{z}\rangle=\langle\mathbf{y},\mathbf{z}\rangle,$$

and hence $\mathbf{y}=T\mathbf{x}_0$, i.e. T has a closed graph, and therefore it is a bounded linear mapping on $\mathcal{H}$. □

END of the interlude

Theorem 28. *Let $(x_i)_{i\in I}$ be an orthormal family in a Hilbert space $(\mathcal{H}, \langle\cdot\rangle)$. For any bounded sequence $(c_i)_{i\in I}$ the operator defined by the (in fact unconditionally convergent) series*

$$T : \mathbf{x} \mapsto \sum_{i\in I} c_i \langle \mathbf{x}, \mathbf{x}_i\rangle \mathbf{x}_i, \quad \mathbf{x} \in \mathcal{H}, \tag{88}$$

defines a bounded operator, with $|||T||| = \|\mathbf{c}\|_\infty$.

The operator is compact if and only if $(c_i)_{i\in I}$ tends to zero, i.e. for every $\varepsilon > 0$ there exists a finite subset $J_0 \subset I$ such that one has $|c_j| \le \varepsilon > 0$ for all $i \in I \setminus J_o$.

Remark 20. For the case of the system of pure frequencies

$$\chi_n(t) := e^{2\pi int}, t \in [0,1], n \in \mathbb{Z},$$

the corresponding operators are called *Fourier multipliers* because they are operators which are just multiplication operators "on the Fourier transform side". Originally they arose in the process of inverting the Fourier transform, i.e. the question of recovering the function from its Fourier coefficients. In fact, the naive approach (taking simply more and more Fourier coefficients, resp. summing up from $-K$ to K was giving some problems, known as *Gibb's phenomenon*. In other words, the simple view-point of $\boldsymbol{L}^2$-analysis (and best approximation in the quadratic mean) was somewhat in conflict with pointwise considerations, at least as soon as functions having discontinuities come into the picture.

9 Spectral Theory for Compact Operators

Without doubt the existence of a complete orthonormal basis of *eigenvectors* which allows the diagonalization of an arbitrary given symmetric (resp. self-adjoint) matrix is one of the high-lights in linear algebra. It is also the basis for many other important applications, such as the SVD (Singular Value Decomposition), the polar-decomposition of matrices, minimal norm least square solutions to linear equations, i.e. the determination of the pseudo-inverse matrix and other questions.

It is also clear, that a basis of eigenvectors is ideally suited to understand the operator (in terms of the corresponding diagonal matrix). Invertibility (only non-zero diagonal elements), inversion, pseudo-inversion can also easily determined in this basis.

So in a functional analytic setting the question is: How can the so-called *spectral theorem* be generalized. What could be the right setting? Is there a natural analogue for the finite-dimenionsal setting?

At first sight it is most natural to concentrate on the setting of Hilbert spaces, because only there the concept of self-adjointness makes sense. Recall, that a bd. linear operator T on a Hilbert space $(\mathcal{H}, \langle\cdot\rangle)$ is *self-adjoint* if and only if

$$\langle T\mathbf{x}, \mathbf{y}\rangle_{\mathcal{H}} = \langle \mathbf{x}, T\mathbf{y}\rangle_{\mathcal{H}} \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{H}. \tag{89}$$

Note that this definition depends on the particular choice of the scalar product on the Hilbert space (in fact, one can have many different scalar products which define equivalent norms, but with a very different set of self-adjoint operators!).

Hence one may ask for eigenvectors for general self-adjoint operators. Of course we have the usual definition:

Definition 43. A *non-zero* vector $\mathbf{h} \in \mathcal{H}$ is called an *eigen-vector* for the linear operator $T \in \mathcal{L}(\mathcal{H})$ if there exists some $\lambda \in \mathbb{K}$ such that $T(\mathbf{h}) = \lambda \mathbf{h}$. Obviously λ is called an *eigenvalue* for T, and $\mathbf{h}$ is called an eigenvector corresponding to the eigenvalue λ.

One of the new questions is the following one: Does every self-adjoint linear operator on a given Hilbert space have some eigen-vector?

Note that there is a problem for quite simple and natural operators, say multiplication operators $M_h : f \mapsto h \cdot f$ on $\mathcal{H} = \boldsymbol{L}^2(\mathbb{R}^d)$. One may think of $h(x) = \exp -\pi|x|^2$ (the normalized Gaussian function) or $h(x) = 1/(1+|x|^2)^s$ (for some $s > 0$). These are prototypical examples of self-adjoint and injective (!check it out) linear operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ (think of $d = 1$), however without having $\lambda = 0$ as eigen-value! In fact, for any possible value $\lambda \in \mathbb{R}$ the level-set of h is a set of measure zero (two points in the case of $d = 1$) and consequently every possible $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ will not just be multiplied by a given number! (details are left to the reader).

This also raises the question, of what the "spectrum" of such an operator should be. Since obviously the set of eigenvalues is only a proper subset of the set of all numbers in $\mathbb{C}$ (selfadjoint operators can have only real eigenvalues, however, as in the linear algebra case!) of the set of all values where $T - \lambda Id_{\mathcal{H}}$ is not invertible one has to see how to describe the spectrum of T: We will put ourselves in a more general context, i.e. that of the Banach algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators with the operator norm, so we write $(\mathcal{L}(\mathcal{H}), |||\cdot|||\,)$ from now on.

Definition 44. Given any Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ with unit element $\mathbf{e}$ the *spectrum* of $\mathbf{a} \in \boldsymbol{A}$ is defined as the set of all (complex) numbers $\lambda \in \mathbb{C}$ such that $\mathbf{a} - \lambda\mathbf{e}$ is not invertible:

$$\operatorname{spec}(\mathbf{a}) := \{\lambda \,|\, \mathbf{a} - \lambda\mathbf{e} \text{ is not invertible in } \boldsymbol{A}\}$$

Just to repeat, invertibility of $T \in \mathcal{L}(\mathcal{H})$ means that there exists another operator S such that $S \circ T = \mathrm{Id}_{\mathcal{H}} = T \circ S$.

Fortunately the situation is quite different for *compact* self-adjoint operator. Such operators will have at least ONE non-trivial eigenvector, and by a reduction argument one can find the other, i.e. remaining eigenvectors, corresponding to a sequence of eigenvalues which tends to zero!

Remark 21. The spectral theory for self-adjoint operators (resp. *normal* operators) generalized of course the familiar theory which is part of linear algebra courses. On the other hand it is itself a special instance for the theory of commutative C^*-algebras $\boldsymbol{A}$, where the *Gelfand-theory* shows (using the maximal ideals in such algebras, resp. the *multiplicative, bounded linear functionals* on such Banach algebras (with the defining property that $\|\mathbf{a}\mathbf{a}^*\|_{\boldsymbol{A}} = \|\mathbf{a}\|_{\boldsymbol{A}}{}^2$) to identify them in a *canonical* way with some space $(\boldsymbol{C}(X), \|)\|_\infty$, where X is a suitable topological (compact) space, whenever $\boldsymbol{A}$ has a unit element (otherwise such an element has to be *adjoined*, by turning the direct sum $\boldsymbol{A} \bigoplus \mathbb{C}$ into an algebra, containing a copy of $\boldsymbol{A}$ (via $\mathbf{a} \approx (\mathbf{a}, 0)$,) with "natural" multiplication rules of the form

$$(\mathbf{a}, \lambda)(\mathbf{b}, \mu) \approx (\mathbf{a} + \lambda Id)(\mathbf{b} + \mu Id) = (\mathbf{ab} + \lambda\mathbf{b} + \mu\mathbf{a}) + \lambda\mu Id \approx (\mathbf{ab} + \lambda\mathbf{b} + \mu\mathbf{a}, \lambda\mu).$$

10 Comparing Linear Algebra with Functional Analysis

Linear Algebra	Functional Analysis
matrix	bounded linear operator
generating system	Frame (stable)
generating system	total subset
linear independent system	Riesz basic sequence

11 Frames and Riesz projection bases

While it is easy to explain how the concept of orthonormal bases can be extended to the situation of Hilbert spaces, it turns out that the concepts of "generating system" and "linear independence" **appear to have natural generalizations to the non-finite dimensional situation**, but this is a bit deceptive!

Let us recall the situation. The definition of a total set (see definition 2 appears as a quite natural generalization of the concept of generating system in a finite dimensional vector spaces) is useful and wide spread, but has to be distinguished from the idea that the elements of a set (a candidate for a basis) allows to write every element as a series. Well, one has to be quite careful in the infinite dimensional setting, because there are conditionally convergent series (the order of elements matters!) and unconditionally convergent series (any permutation of the index set is allowed without changing the fact that the corresponding series is convergent and in addition all those possible limits are the same!).

This is something which is NOT required in the case of a total set. Given $\mathbf{v}$ and $\varepsilon > 0$ we just claim that $\|\mathbf{v} - \sum_{k=1}^{K} c_k m_K\|_{\boldsymbol{V}} \leq \varepsilon$ for a suitable choice of elements $m_1, \ldots, m_k \in M$ and appropriate coefficients, but if a better approximation is required a completely different and new set of elements and coefficients might do the job, and in addition the norm of the (finite) sequence in $\mathbb{C}^K$ need not be controlled, in fact, it may tend to ∞ for $\varepsilon \to 0$.

Hence there is a much better concept relevant, which guarantees that every element can be written as an infinite and unconditionally convergent sum with $\boldsymbol{\ell}^2$-coefficients.

Definition 45. A family $(f_i)_{i \in I}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if there exist

constants $A, B > 0$ such that for all $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2 \tag{90}$$

Alternative Symbol for $\mathcal{H}$!

It is well known that condition (90) is satisfied if and only if the so-called *frame operator*

Definition 46.

$$S(f) := \sum_{i \in I} \langle f, f_i \rangle f_i, \quad \text{for} \quad f \in \mathcal{H}$$

is invertible. The obvious fact $S \circ S^{-1} = Id = S^{-1} \circ S$ implies that the (canonical) *dual frame* $(\tilde{f}_i)_{i \in I}$, defined by $\tilde{f}_i := S^{-1}(f_i)$ has the property that one has for $f \in \mathcal{H}$:

Definition 47.

$$f = \sum_{i \in I} \langle f, \tilde{f}_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle \tilde{f}_i \tag{91}$$

Since S is *positive definite* in this case we can also get to a more symmetric expression by defining $h_i = S^{-1/2} g_i$. In this case one has

$$f = \sum_{i \in I} \langle f, h_i \rangle h_i \quad \forall f \in \mathcal{H}. \tag{92}$$

The family $(h_i)_{i \in I}$ defined in this way is called the *canonical tight frame* associated to the given family $(g_i)_{i \in I}$.

Remark 22. Note that frames are "indexed families" of vectors, and not just sets. It is quite possible that a given vector appears finitely many times within a frame (if the length is fixed) or even infinitely many times, if their norms behave properly. On the other hand it is not just a sequence, because the *order* of elements does *not play any role*, and of course the frame operator of a permuted sequence is just the same (and correspondingly the dual frame or tight frame are permuted in the same way!).

LITERATURE: Book by Ole Christensen: [3]

If the frame is also linear independent in a suitable way, the family deserves the name *Riesz basis*.

Overall (this has to be describe in more detail later on) one can say: Using the concepts of frames, Riesz bases and Riesz basic sequences (as replacement for total, basis, or linear independence) in the Hilbert space setting allows to have a setting which is very similar to the finite dimensional setting, including also solutions to the minimal norm least square problem.

Definition 48. A sequence (h_k) in a separable Hilbert space $\mathcal{H}$ is a *Riesz basis* for its closed linear span (sometimes also called a *Riesz basic sequence*) if for two constants $0 < D_1 \leq D_2 < \infty$,

$$D_1 \|c\|_{\ell^2}^2 \leq \Big\| \sum_k c_k h_k \Big\|_{\mathcal{H}}^2 \leq D_2 \|c\|_{\ell^2}^2, \qquad \forall c \in \ell^2 \tag{93}$$

A detail descripton of the concept of *Riesz basis* can be found in ([21]) where the more general concept of Riesz projection bases is explained.

In particular a sequence (h_k) is a *Riesz basis* if and only if the corresponding *Gram matrix*, whose entries are the scalar products $(< h_k, h'_k >)_{k,k'}$ is invertible on the corresponding $\boldsymbol{\ell}^2$-space.

[1]: Book entitled: "Studies in Functional Analysis".

Or in GERMAN:

Lineare Algebra	Funktionalanalysis
Matrix	(beschränkter) linearer Operator
Erzeugendensystem	Frame (stabil)
Erzeugendensystem	totale Teimenge
linear unabhängige Menge	Riesz Basis

FOR people reading GERMAN material I can recommend to take a look at my lecture notes on LINEAR ALGEBRA, especially the last section (from p. 86 onwards):

`http://www.univie.ac.at/nuhag-php/login/skripten/data/MainLinAlg12.pdf`

where many results (and views) of the linear algebra background are explained in more detail.

12 Convolution algebras

Material concerning convolution is also found in Conway's book [4].

12.1 Locally compact Abelian Groups

LCA:= Locally Compact Abelian Group

Conway, p.189 (Example): convolution defined in the measure algebra, convolution with δ_e (e = neutral element in $\mathscr{G}$) is the identity element in $\boldsymbol{M}(\mathscr{G})$. Moreover one has (end of page 189:

$$\delta_x * \delta_y = \delta_{xy}, \quad \forall x, y \in \mathscr{G}.$$

The *group algebra* $\boldsymbol{M}(\mathscr{G})$ is commutative (Abelian) if and only if the group $\mathscr{G}$ is Abelian. Furthermore, $\big(\boldsymbol{L}^1(G),\, \|\cdot\|_1\big)$ is a closed idealR (using the Radon-Nikodym Theorem and the concept of absolutely continuous functions over groups!).

Continuation of the material then in section VII.9, page 223. Continuous shift property in $\big(\boldsymbol{L}^p(G),\, \|\cdot\|_p\big)$, for $1 \leq p < \infty$, based on the density of $\boldsymbol{C}_c(\mathscr{G})$ in these Banach spaces.

The very last Appendix C in [4] shows how dual of $\boldsymbol{C}_0(X)$ can be identified with the space of regular Borel measures on X (but what are the exact assumptions: X locally compact!? ...). At appears to be built on the **Hahn-Jordan Decomposition** for signed measures and the **Radon-Nikodym Theorem** characterizing absolutely continuous measures! The correct reference is Thm.C18 (the **Riesz Representation Theorem**!), showing that there is an *isometric isomorphism* between $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ and $\big(\boldsymbol{C}_0'(G),\, \|\cdot\|_{\boldsymbol{C}_0'}\big)$ (for "general" X, not just locally compact groups, whatever this means!).

13 Various Facts

One can use the Krein-Milman Theorem to verify that cosp is NOT a dual space. This would also follow from the fact that it a solid BF space (Banach function space in the terminology of Luxemburg and Zaanen) but fails to have to Fatou property.

14 Further Literature

Most of the books mentioned below are to be found in the NuHAG library at 5.131, OMP1, next to my office.

1. x [Heil, Christopher] A Basis Theory Primer. Expanded ed. [8]
2. x [Christensen, Ole] An Introduction to Frames and Riesz Bases. [3]
3. x [Gröchenig, Karlheinz] Foundations of Time-Frequency Analysis [7]
4. x [Daubechies, Ingrid] Ten lectures on wavelets [5]

Concerning function spaces one has to mention a whole long list of books, including several recent ones, by Hans Triebel, see e.g. [15–17].

Literatur

[1] R. G. Bartle, editor. *Studies in Functional Analysis*, volume 21 of *MAA Studies in Mathematics*. Mathematical Association of America, Washington, D.C., 1980. 76

[2] J. Cerda. *Linear Functional Analysis*. Graduate Studies in Mathematics 116. Providence, RI: American Mathematical Society (AMS) and Madrid: Real Sociedad Matem'atica Espa nola. xiii, 330 p., 2010. 1

[3] O. Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, 2003. 75, 78

[4] J. B. Conway. *A Course in Functional Analysis. 2nd ed.* Springer, New York, 1990. 15, 16, 57, 66, 67, 69, 77

[5] I. Daubechies. *Ten Lectures on Wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. SIAM, Philadelphia, PA, 1992. 78

[6] P. Enflo. A counterexample to the approximation problem in Banach spaces. *Acta Math.*, 130:309–317, 1973. 5

[7] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001. 78

[8] C. Heil. *A Basis Theory Primer. Expanded ed.* Applied and Numerical Harmonic Analysis. Basel: Birkhäuser, 2011. 78

[9] J. Heine. *Topologie und Funktionalanalysis, Grundlagen der abstrakten Analysis mit Anwendungen.* Oldenbourg, München, Oldenbourg, 2002. 43

[10] R. James. Bases and reflexivity of Banach spaces. 52(3):518–527, 1950. 21

[11] W. Kaballo. *Grundkurs Funktionalanalysis*. Spektrum Akademischer Verlag, 2011. 16, 18, 19, 32, 67, 68, 69, 70, 71

[12] J. L. Kelley. *General Topology. 2nd ed.* Springer, Berlin, Heidelberg, New York, 1975. 38, 45

[13] H. Reiter. *L^1-algebras and Segal Algebras.* Springer, Berlin, Heidelberg, New York, 1971. 44

[14] I. Singer. *Bases in Banach Spaces. I.* Number 154 in Grundlehren Math. Wiss. Springer, Berlin, 1970. 5

[15] H. Triebel. *Local Function Spaces, Heat and Navier-Stokes Equations.* Zürich: European Mathematical Society (EMS), 2013. 78

[16] H. Triebel. *Hybrid Function Spaces, Heat and Navier-Stokes Equations.* Zürich: European Mathematical Society (EMS), 2015. 78

[17] H. Triebel. *Tempered Homogeneous Function Spaces.* Zürich: European Mathematical Society (EMS), 2016. 78

[18] J. von Neumann. *Invariant Measures.* Providence, RI: American Mathematical Society (AMS). xiv, 1999. 36

[19] H. Wendland. Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree. *Adv. Comput. Math.*, 4(4):389–396, 1995. 4

[20] D. Werner. *Functional Analysis. (Funktionalanalysis) 5., Erweiterte Aufl.* Springer-Verlag, Berlin, 2002. 57

[21] G. Zimmermann. *Projective Multiresolution Analysis and Generalized Sampling.* PhD thesis, University of Maryland, College Park, December 1994. 76