Convolution theory in a space of generalized functions

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Communicated by Prof. N. G. de Bruijn at the meeting of September 30, 1978

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MATHEMATICS

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INTRODUCTION

This paper presents a convolution theory for the test function space S of smooth functions and the space S^* of generalized functions as introduced by De Bruijn (the terminology and notation is the one used in [B], where these spaces are defined). The space S can be regarded as an example of a test function space of the type studied in [GS], Ch. IV (actually, our space S can be identified with the space $S_{\frac{1}{2}}^{\frac{1}{2}}$ of [GS], Ch. IV, § 2.3). Since the spaces S and S^* are adapted to the needs of Fourier analysis (cf. [B], section 8 and 9, and [GS], Ch. IV, § 6), it was to be expected that it is possible to develop a satisfactory convolution theory for these spaces; it seems however that no such theory has been published thus far.

Let us summarize the contents of this paper. Section 1 gives the main definitions and theorems about the spaces S and S^*, and some results about continuous linear transformations in these spaces are mentioned. This section is mainly included here for ease of reference.

Section 2 serves as a preparation. The convolution operators introduced here involve smooth functions only, and they are defined as follows. If $g \in S$, then the convolution operator T_g of S is defined by

(1) $$(T_g f)(x) = \int_{-\infty}^{\infty} f(x-t)\overline{g(t)}dt \quad (x \in \mathbb{C})$$

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for $f \in S$. Instead of the integral at the right hand side in (1) we can also write $(T_x f, g_-)$, where T_x is the shift operator over distance x, and g_- is the smooth function with values $g_-(t) = g(-t)$ for $t \in \mathbb{C}$.

In section **3** we generalize the notion of convolution operator. The g in (1) is replaced by a generalized function: if $G \in S^*$, then the convolution operator T_G of S is defined by

$$(2) \qquad (T_G f)(x) = (T_x f, G_-) \quad (x \in \mathbb{C})$$

for $f \in S$. Here G_- bears a similar relation to G as g_- does to g in the previous paragraph. Special attention is paid to the case that T_G maps S into S, and the class of all $G \in S^*$ with this property is called the convolution class $\mathscr{C}$. For $G \in \mathscr{C}$ we prove that T_G has an adjoint, and that T_G can be extended in a natural way to a continuous linear operator of S^*. We also discuss some alternative descriptions of the class $\mathscr{C}$.

Section **4** presents a link between convolution theory and Fourier analysis. This involves what we call multiplication operators of S and S^*. If $g \in S$, then the multiplication operator M_g of S is defined by $M_g f = g \cdot f$ for $f \in S$, where the dot denotes pointwise multiplication; this multiplication operator can be extended in a natural way to a continuous linear operator of S^*. We obtain a useful characterization of the class $\mathscr{C}$ in terms of the Fourier transforms of its elements. Furthermore the convolution theorem is generalized in section 4, and a version of Titchmarsh's theorem is proved. Finally we mention some results about the solutions $F \in S^*$ of equations of type $T_G F = 0$, where G is a fixed element of $\mathscr{C}$.

Section **5** contains some additional material. There we prove that the class of generalized functions of the form $M_g G$ with $g \in S$, $G \in S^*$ is a proper subset of $\mathscr{C}$. We make some remarks about convergence in $\mathscr{C}$, and finally we pay some attention to convolution theory for the spaces of smooth and generalized functions of several variables.

NOTATION

We use Church's lambda calculus notation, but instead of his λ we have the symbol $\curlyvee$, as suggested by Freudenthal: If S is a set, then putting $\curlyvee_{x \in S}$ in front of an expression (usually containing x) means to indicate the function with domain S and with the function values given by the expression. For example, if $g \in S$ then $T_g = \curlyvee_{f \in S} T_g f$. In case it is clear from the context which set S is meant, we write $\curlyvee_x$ instead of $\curlyvee_{x \in S}$.

1. THE SPACES S AND S^*

1.1. We give a survey of the fundamental notions and theorems of De Bruijn's theory of generalized functions (as far as relevant for this paper). Also, the main theorems of [J], appendix 1 about continuous linear operators of S and S^* are given. More details can be found in [B] and [J].

The class S (of *smooth* functions) is the set of all analytic functions f of one complex variable that satisfy inequalities

$$|f(t)| \leq M \exp(-\pi A (\mathrm{Re}\, t)^2 + \pi B (\mathrm{Im}\, t)^2) \quad (t \in \mathbb{C}),$$

where $M > 0$, $A > 0$, $B > 0$ depend on f. In S we take the usual inner product, denoted by $(\, , \,)$. Cf. [B], 2.1.

1.2. We consider a semigroup $(N_\alpha)_{\alpha > 0}$ of linear operators of S (the *smoothing* operators); they satisfy $N_{\alpha+\beta} = N_\alpha N_\beta$ $(\alpha > 0, \beta > 0)$. These operators are integral operators (integration over $\mathbb{R}$); the kernels K_α $(\alpha > 0)$ are given by

$$K_\alpha(z, t) = (\sinh \alpha)^{\frac{1}{2}} \exp\left(\frac{-\pi}{\sinh \alpha}((z^2 + t^2)\cosh \alpha - 2zt)\right) \quad (z \in \mathbb{C},\ t \in \mathbb{C}).$$

Cf. [B], section 4, 5 and 6. The operators N_α $(\alpha > 0)$ can be defined on the larger space S^+ consisting of all mappings $f: \mathbb{R} \to \mathbb{C}$ such that

$$\curlyvee_{t \in \mathbb{R}} f(t) \exp(-\pi \varepsilon t^2) \in \mathscr{L}_1(\mathbb{R}) \text{ for every } \varepsilon > 0.$$

We have $N_\alpha f \in S$ for $f \in S^+$, $\alpha > 0$ (compare [B], section 20, where an equivalent definition of S^+ is used). Note that $\mathscr{L}_2(\mathbb{R}) \subset S^+$.

1.3. We summarize some properties of N_α $(\alpha > 0)$.

(i) $(N_\alpha f, g) = (f, N_\alpha g)$ for $\alpha > 0$, $f \in S$, $g \in S$ (cf. [B], 6.5).

(ii) If $f \in S$, $\alpha > 0$, then there is at most one $g \in S$ with $f = N_\alpha g$. Also, if $f \in S$, then there exists an $\alpha > 0$, $g \in S$ with $f = N_\alpha g$. And if $f \in S$, and the numbers $M > 0$, $A > 0$, $B > 0$ are such that

$$|f(t)| \leq M \exp(-\pi A (\mathrm{Re}\, t)^2 + \pi B (\mathrm{Im}\, t)^2) \quad (t \in \mathbb{C}),$$

then we can find an $\alpha > 0$, $M' > 0$, $A' > 0$, $B' > 0$, only depending on A and B, such that the inequalities

$$|g(t)| \leq M M' \exp(-\pi A' (\mathrm{Re}\, t)^2 + \pi B' (\mathrm{Im}\, t)^2) \quad (t \in \mathbb{C})$$

hold for the unique $g \in S$ with $f = N_\alpha g$ (cf. [B], 10.1).

1.4. We list some other linear operators of S (cf. [B], section 8 and 11).

(i) The Fourier transform $\mathscr{F}$ and its inverse $\mathscr{F}^*$:

$$\mathscr{F} f = \curlyvee_{z \in \mathbb{C}} \int_{-\infty}^{\infty} e^{-2\pi i z t} f(t)\, dt, \quad \mathscr{F}^* f = \curlyvee_{z \in \mathbb{C}} (\mathscr{F} f)(-z) \quad (f \in S).$$

(ii) The shift operators T_a $(a \in \mathbb{C})$ and R_b $(b \in \mathbb{C})$:

$$T_a f = \curlyvee_{z \in \mathbb{C}} f(z + a), \quad R_b f = \curlyvee_{z \in \mathbb{C}} e^{-2\pi i b z} f(z) \quad (f \in S).$$

(iii) The operators P and Q:

$$P f = \curlyvee_{z \in \mathbb{C}} \frac{f'(z)}{2\pi i}, \quad Q f = \curlyvee_{z \in \mathbb{C}} z f(z) \quad (f \in S).$$

1.5. A *generalized function* F is a mapping $\alpha \in (0, \infty) \to F_\alpha \in S$ such that $N_\alpha F_\beta = F_{\alpha+\beta}$ $(\alpha > 0, \beta > 0)$. We also write $F(\alpha)$ or $N_\alpha F$ instead of F_α.

It follows from 1.3(ii) that $F=0$ in case $F_\alpha = 0$ for some $\alpha > 0$ $(F \in S^*)$.

If $F \in S^*$, $g \in S$, then the inner product (F, g) is defined as follows: write $g = N_\alpha h$ with some $\alpha > 0$, $h \in S$ (cf. 1.3(ii)), and put $(F, g) := (F_\alpha, h)$ (this number depends only on F and g; cf. [B], section 17 and 18). We have $(N_\alpha F, g) = (F, N_\alpha g)$ for $\alpha > 0$, $F \in S^*$, $g \in S$.

We further define $(g, F) := \overline{(F, g)}$ for $F \in S^*$, $g \in S$.

1.6. We give some examples of generalized functions.

(i) If $f \in S^+$, then the embedding of f (notation: emb (f)) is defined by

$$\text{emb}\,(f) := \Upsilon_{\alpha>0}\, N_\alpha f.$$

Cf. [B], section 20. We have for $f \in S^+$, $g \in S$

$$(\text{emb}\,(f), g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\,dt.$$

It may be proved that $f = 0$ (a.e.) if and only if emb $(f) = 0$.

(ii) For $b \in \mathbb{C}$, the "delta function at b" is defined by

$$\delta_b := \Upsilon_{\alpha>0}\, \Upsilon_{t\in\mathbb{C}} K_\alpha(t, b).$$

Now $(g, \delta_b) = g(\bar{b})$ for $g \in S$ (cf. [B], 17.3 and 27.18).

1.7. We next define convergence in S. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in S, and let $f \in S$. We write $f_n \xrightarrow{S} 0$ if there are positive numbers A and B such that $f_n(t) \exp(\pi A(\mathrm{Re}\, t)^2 - \pi B(\mathrm{Im}\, t)^2) \to 0$ uniformly in $t \in \mathbb{C}$; we write $f_n \xrightarrow{S} f$ if $f_n - f \xrightarrow{S} 0$. Similarly we define $f^{(\alpha)} \xrightarrow{S} 0$ $(\alpha \downarrow 0)$ and $f^{(\alpha)} \xrightarrow{S} f$ $(\alpha \downarrow 0)$ if $f^{(\alpha)} \in S$ $(\alpha > 0)$, $f \in S$. Cf. [B], section 23.

1.8. The following theorem on S-convergence is useful.

THEOREM. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in S. The three following statements are equivalent.

(i) $f_n \xrightarrow{S} 0$.

(ii) There exist $\alpha > 0$ and $g_n \in S$ $(n \in \Omega)$ such that $f_n = N_\alpha g_n$, $g_n \xrightarrow{S} 0$.

(iii) There exists an $M > 0$, $A > 0$, $B > 0$ such that

$$|f_n(t)| \leq M \exp(-\pi A(\mathrm{Re}\, t)^2 + \pi B(\mathrm{Im}\, t)^2) \quad (t \in \mathbb{C}),$$

and $f_n \to 0$ pointwise.

PROOF. Equivalence of (i) and (ii) follows from [B], 23.1, and equivalence of (i) and (iii) follows from [J], appendix 2, theorem 1. □

1.9. We proceed by defining convergence in S^*. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in S^*, and let $F \in S^*$. We write $F_n \xrightarrow{S^*} 0$ if $N_\alpha F_n \xrightarrow{S} 0$ for every

$\alpha > 0$; we write $F_n \xrightarrow{S^*} F$ if $F_n - F \xrightarrow{S^*} 0$. Similarly we define $F^{(\beta)} \xrightarrow{S^*} 0$ $(\beta \downarrow 0)$ and $F^{(\beta)} \xrightarrow{S^*} F$ $(\beta \downarrow 0)$ if $F^{(\beta)} \in S^*$ $(\beta > 0)$, $F \in S^*$.

[B], 24.2 states: a sequence $(F_n)_{n\in\mathbb{N}}$ in S^* is S^*-convergent if and only if $\lim_{n\to\infty} (F_n, g)$ exists for every $g \in S$.

It is not hard to prove from 1.8 that $(F_n, f_n) \to (F, f)$ if $F_n \xrightarrow{S^*} F$, $f_n \xrightarrow{S} f$, where $(F_n)_{n\in\mathbb{N}}$ is a sequence in S^* and $(f_n)_{n\in\mathbb{N}}$ is a sequence in S.

1.10. We are going to study continuous linear transformations of S and S^*.

DEFINITION. A linear functional L of S is called *continuous* if $Lf_n \to 0$ for every sequence $(f_n)_{n\in\mathbb{N}}$ in S with $f_n \xrightarrow{S} 0$. A linear operator T of S is called *continuous* if $Tf_n \xrightarrow{S} 0$ for every sequence $(f_n)_{n\in\mathbb{N}}$ in S with $f_n \xrightarrow{S} 0$. The definitions of continuous linear functionals and operators of S^* are similar. (We use the word continuous instead of quasi-bounded, cf. [B], 22.2, and [J], appendix 1, 2.2.)

1.11. DEFINITION. A linear operator T of S is said to *have an adjoint* if for every $g \in S$ there is a $g^* \in S$ such that $(Tf, g) = (f, g^*)$ for every $f \in S$. Such a g^* is unique, and g^* depends linearly on $g \in S$. If we define $T^*g := g^*$ for $g \in S$, then T^* is a linear operator of S, called the *adjoint* of T.

Note that if T has an adjoint, then so has T^*, and $(T^*)^* = T$.

1.12. EXAMPLE. We introduce some notation. If $g \in S$, then we define

$$\bar{g} := \Upsilon_{z\in\mathbb{C}}\overline{g(\bar{z})},\quad g_- := \Upsilon_{z\in\mathbb{C}}\, g(-z),\quad \bar{g}_- := \Upsilon_{z\in\mathbb{C}}\overline{g(-\bar{z})}.$$

Note that $\bar{g} \in S$, $g_- \in S$, $\bar{g}_- \in S$, and that $(\bar{g})_- = (\overline{g_-}) = \bar{g}_-$. If $F \in S^*$, then we define $\bar{F} := \Upsilon_{\alpha>0}\bar{F}_\alpha$, $F_- := \Upsilon_{\alpha>0}(F_\alpha)_-$, $\bar{F}_- := \Upsilon_{\alpha>0}(\overline{F_\alpha})_-$. Note that (by symmetry of the K_α's; cf. 1.2) $\bar{F} \in S^*$, $F_- \in S^*$, $\bar{F}_- \in S^*$, and that $(\bar{F})_- = (\overline{F_-}) = \bar{F}_-$. We have $(\bar{F}, g) = \overline{(F, \bar{g})}$, $(F_-, g) = (F, g_-)$, $(\bar{F}_-, g) = \overline{(F, \bar{g}_-)}$ for $F \in S^*$, $g \in S$.

If T is a continuous linear operator of S, then we define

$$\bar{T} := \Upsilon_{g\in S}\overline{T\bar{g}},\quad T_- := \Upsilon_{g\in S}(Tg_-)_-,\quad \bar{T}_- := \Upsilon_{g\in S}(\overline{T\bar{g}_-})_-.$$

Now $\bar{T}$, T_- and $\bar{T}_-$ are continuous linear operators of S with $(\bar{T})_- = (\overline{T_-}) = \bar{T}_-$, and if T has an adjoint, then so have $\bar{T}$, T_- and $\bar{T}_-$: $(\bar{T})^* = (\overline{T^*})$, $(T_-)^* = (T^*)_-$, $(\bar{T}_-)^* = (\overline{T^*})_-$.

If T is a continuous linear operator of S^*, then we define

$$\bar{T} := \Upsilon_{F\in S}\,\overline{T\bar{F}},\quad T_- := \Upsilon_{F\in S^*}(TF_-)_-,\quad \bar{T}_- := \Upsilon_{F\in S^*}(\overline{T\bar{F}_-})_-.$$

Now $\bar{T}$, T_- and $\bar{T}_-$ are continuous linear operators of S^* and $(\bar{T})_- = (\overline{T_-}) = \bar{T}_-$.

1.13. THEOREM. L is a continuous linear functional of S if and only if there exists an $F \in S^*$ such that $Lf = (f, F)$ $(f \in S)$. Such an F is unique.

PROOF. Follows easily from [B], 22.2. □

1.14. THEOREM. Let T be a linear operator of S. The four following statements are equivalent.

(i) T is continuous.

(ii) $\Upsilon_{f\in S}(Tf)(x)$ is a continuous linear functional of S for every $x \in \mathbb{C}$.

(iii) TN_α has an adjoint for every $\alpha>0$.

(iv) For every $\alpha>0$ there is a $\beta>0$ and a bounded linear operator T_1 of S (bounded with respect to inner product norm) such that $TN_\alpha = N_\beta T_1$.

PROOF. This is proved in [J], appendix 1, 2.2 through 2.10. □

REMARK. A useful alternative formulation of (iv) is: for every $M>0$, $A>0$, $B>0$ there exists $M_0>0$, $A_0>0$, $B_0>0$ such that

$$|(Tf)(t)| \leqslant M_0 \exp(-\pi A_0(\operatorname{Re} t)^2+\pi B_0(\operatorname{Im} t)^2) \quad (t \in \mathbb{C})$$

whenever $f \in S$ and

$$|f(t)| \leqslant M \exp(-\pi A(\operatorname{Re} t)^2+\pi B(\operatorname{Im} t)^2) \quad (t \in \mathbb{C}).$$

Equivalence of both conditions easily follows from the equivalence of (i) and (iv), and from [B], 6.3.

The linear operators of 1.4 are continuous.

1.15. THEOREM. If T is a linear operator of S with an adjoint, then it is possible to extend T to a continuous linear operator $\tilde{T}$ of S^* such that $\tilde{T}(\operatorname{emb}(f)) = \operatorname{emb}(Tf) (f \in S)$, $(\tilde{T}F, f) = (F, T^*f) (F \in S^*,\ f \in S)$. Here emb (f) for $f \in S$ is to be read as emb (f_0), where f_0 is the restriction of f to $\mathbb{R}$ (cf. 1.2).

PROOF. This is [J], appendix 1, theorem 3.2. □

We denote the extended operator again by T. For examples, see 1.4.

1.16. We finally devote some attention to (generalized) functions of several variables. The previous definitions and theorems can be given and proved (with the proper modifications) without any restriction for the more dimensional case. For instance, the class S^n (where $n \in \mathbb{N}$) is defined as the set of all complex-valued functions f of n complex variables that are analytic in all variables, for which there exist positive numbers M, A and B such that

$$|f(t_1, \ldots, t_n)| \leqslant M \exp(\pi \sum_{k=1}^{n} (-A(\operatorname{Re} t_k)^2+B(\operatorname{Im} t_k)^2))$$

for $(t_1, \ldots, t_n) \in \mathbb{C}^n$.

As an example of a smooth function of n variables we have

$$f_1 \otimes \ldots \otimes f_n := \Upsilon_{(t_1, \ldots, t_n)\in \mathbb{C}^n} f_1(t_1) \cdot \ldots \cdot f_n(t_n),$$

where $f_1 \in S, \ldots, f_n \in S$.

The classes S^{n+} and S^{n*} (of embeddable and generalized functions respectively) are introduced in a similar way (the smoothing operators $N_{\alpha,n}$ are defined as the n-fold tensor products of N_α $(\alpha>0)$). Cf. [B], section 7 and 21.

As an example of a generalized function of n variables we have

$$F_1 \otimes \ldots \otimes F_n := \Upsilon_{\alpha>0} N_\alpha F_1 \otimes \ldots \otimes N_\alpha F_n,$$

where $F_1 \in S^*, \ldots, F_n \in S^*$.

The notions of convergence and continuity are adapted correspondingly, and theorems 1.13, 1.14, 1.15 hold for the present case.

1.17. The following theorem is important (we state it only for the case $n=2$).

THEOREM. If T_i $(i=1, 2)$ are continuous linear operators of S, then the mapping $T_1 \otimes T_2$, defined by

$$(T_1 \otimes T_2)f := \Upsilon_{(z_1, z_2)} T_1(\Upsilon_{t_1}(T_2(\Upsilon_{t_2} f(t_1, t_2))(z_2)))(z_1)$$

for $f \in S^2$, is a continuous linear operator of S^2. If T_i $(i=1, 2)$ have adjoints, then so has $T_1 \otimes T_2$ (with respect to the inner product in S^2), and $(T_1 \otimes T_2)^* = T_1^* \otimes T_2^*$. If furthermore T_1, T_2 and $T_1 \otimes T_2$ are extended to linear operators of S^*, S^* and S^{2*} (according to 1.15), then we have $(T_1 \otimes T_2)(F_1 \otimes F_2) = T_1F_1 \otimes T_2F_2$ for $F_1 \in S^*$, $F_2 \in S^*$.

PROOF. This follows from [J], appendix 1, 2.13 and 3.12. □

1.18. An example of an operator of S^2 (not of the type discussed in 1.17) that can be extended to a continuous linear operator of S^{2*} is the following one. Define

$$Z_U f := \Upsilon_{(t_1, t_2)\in \mathbb{C}^2} f\left(\frac{t_1+t_2}{\sqrt{2}}, \frac{t_1-t_2}{\sqrt{2}}\right) \quad (f \in S).$$

It is not hard to see that Z_U is a continuous linear operator of S^2 that satisfies $Z_U^* = Z_U$.

2. PREPARATION

2.1. We introduce in this section convolution operators defined on S in which only smooth functions appear. Some simple results are derived.

2.2. DEFINITION. For $g \in S$ the convolution operator T_g is defined by

$$T_g f := \curlyvee_{x \in \mathbb{C}} \int_{-\infty}^{\infty} f(x-t)\overline{g(t)}dt \quad (f \in S).$$

Note that $T_g f$ is the ordinary convolution of f and $\curlyvee_t \overline{g(t)}$ (it will have some notational convenience in the subsequent sections to take $\curlyvee_t \overline{g(t)}$ instead of g).

To avoid confusion with the translation operators T_a $(a \in \mathbb{C})$ of 1.4(ii), we shall always denote convolution operators by

$$T_f, T_g, T_h, \ldots, T_F, T_G, T_H, \ldots,$$

where

$$f \in S,\ g \in S,\ h \in S, \ldots, F \in S^*,\ G \in S^*,\ H \in S^*, \ldots,$$

whereas translation operators are denoted by $T_a, T_b, T_c, \ldots, T_x, T_y, T_z, \ldots$ with $a \in \mathbb{C}$, $b \in \mathbb{C}$, $c \in \mathbb{C}, \ldots$, $x \in \mathbb{C}$, $y \in \mathbb{C}$, $z \in \mathbb{C}, \ldots$.

2.3. THEOREM. If $g \in S$, then we have
(i) T_g maps S linearly and continuously into S.
(ii) T_g has an adjoint, viz. $T_g^* = T_{\bar{g}_-}$, and $\overline{T}_g = T_{\bar{g}}$, $(T_g)_- = T_{g_-}$ (cf. 1.12).
(iii) If $h \in S$, then $T_g T_h = T_h T_g$ and $T_{\bar{g}} h = T_{\bar{h}} g$.
(iv) If $h \in S$, then $\mathscr{F}(T_{\bar{g}} h) = \mathscr{F}g \cdot \mathscr{F}h$ (pointwise multiplication).

PROOF. If $f \in S$, then it is easily seen that $T_g f$ is an analytic function, and we therefore concentrate on the estimation. Let $M_1, A_1, B_1, M_2, A_2, B_2$ be positive numbers such that

$$|f(x+iy)| \leqslant M_1 \exp(-\pi A_1 x^2 + \pi B_1 y^2) \quad (x \in \mathbb{R},\ y \in \mathbb{R}),$$

$$|g(x+iy)| \leqslant M_2 \exp(-\pi A_2 x^2 + \pi B_2 y^2) \quad (x \in \mathbb{R},\ y \in \mathbb{R}).$$

Using the optimal shift technique as displayed in the proof of [B], theorem 8.1, we obtain

$$|(T_g f)(x+iy)| \leqslant \frac{M_1 M_2}{\sqrt{A_1 + A_2}} \exp\left(-\pi \frac{A_1 A_2}{A_1 + A_2} x^2 + \pi \frac{B_1 B_2}{B_1 + B_2} y^2\right) \quad (x \in \mathbb{R},\ y \in \mathbb{R}).$$

This proves smoothness of $T_g f$, and it also shows continuity of T_g. It is trivial that T_g is linear.

Assertions (ii) and (iii) follow from elementary calculations which we shall omit, and (iv) is the well known convolution theorem for S. □

3. CONVOLUTION OPERATORS AND GENERALIZED FUNCTIONS

3.1. In this section we define convolution operators T_F with $F \in S^*$. We pay special attention to operators T_F that map S into S, and it shall be proved that such operators have an adjoint (so that we can extend

them to linear operators of S^* according to 1.15). Furthermore we shall derive a number of useful properties of these convolution operators.

3.2. DEFINITION. We define for $F \in S^*$ the mapping T_F (cf. 1.4 and 1.12) by

$$T_F f := \curlyvee_{x \in \mathbb{C}} (T_x f, F_-) \quad (f \in S).$$

We shall write T_f instead of $T_{\mathrm{emb}(f)}$ in case $f \in S^+$ (cf. 1.2). Note that in case $f \in S$ definition 2.2 and the present one yield the same operator T_f.

3.3. THEOREM. If $F \in S^*$ and $f \in S$, then $T_F f$ is an analytic function.

PROOF. It is sufficient to prove analyticity of the function $\curlyvee_{x \in \mathbb{C}}(T_x f, F_-)$ in a point $x_0 \in \mathbb{C}$. It is easy to prove (by using Cauchy's theorem and a continuous version of 1.8) that

$$\curlyvee_{x \in \mathbb{C}} \frac{f(x_0+x+h) - f(x_0+x)}{h} \xrightarrow{S} \curlyvee_{x \in \mathbb{C}} f'(x_0+x) \quad (h \to 0).$$

Hence, by 1.9,

$$\lim_{h \to 0} \left(\frac{T_{x_0+h} f - T_{x_0} f}{h},\ F_- \right) = (\curlyvee_{x \in \mathbb{C}} f'(x_0+x), F_-),$$

and this shows analyticity of $\curlyvee_{x \in \mathbb{C}}(T_x f, F_-)$ in x_0. □

REMARK. We shall prove in 5.4 that $\curlyvee_{z \in \mathbb{C}} \exp(-\pi\varepsilon z^2)(T_F f)(z) \in S$ for every $\varepsilon > 0$, $F \in S^*$, $f \in S$.

3.4. DEFINITION. The class $\mathscr{C}$ is defined as the set of all generalized functions F for which $T_F(S) \subset S$.

REMARK. This definition is somewhat uneasy to handle, but we shall give alternative descriptions of the class $\mathscr{C}$ later on.

3.5. EXAMPLES.
(i) If $f \in S$, then $\mathrm{emb}(f) \in \mathscr{C}$.
(ii) If $a \in \mathbb{C}$, $F = \delta_a$, then $T_F = T_{-\bar{a}}$ (cf. 1.4(ii)), so $\delta_a \in \mathscr{C}$.
(iii) If $F = P\delta_0$, then $T_F = P$, so $P\delta_0 \in \mathscr{C}$ (cf. 1.4(iii)).
(iv) If f is an integrable function defined on $\mathbb{R}$ with a compact support, then $\mathrm{emb}(f) \in \mathscr{C}$.
(v) If P is a measure on $\mathbb{R}$, and if there is an $\varepsilon > 0$ such that

$$\int_{|t| \geqslant x} dP(t) = 0(\exp(-\pi\varepsilon x^2)) \quad (x \geqslant 0),$$

then F, defined by

$$(F, f) := \int_{-\infty}^{\infty} \overline{f(t)} dP(t) \quad (f \in S)$$

(cf. 1.13), belongs to $\mathscr{C}$.

3.6. The following lemma turns out to be very useful in this section.

LEMMA. If $F \in \mathscr{C}$, $f \in S$, then $T_{F(\alpha)} f \xrightarrow{S} T_F f$ $(\alpha \downarrow 0)$.

PROOF. We first note that T_F is a continuous linear operator: if $x \in \mathbb{C}$, then $\forall_{f \in S} (T_F f)(x)$ is a continuous linear functional of S (cf. 1.10 and 1.14).

For $\alpha > 0$, $a \in \mathbb{C}$ we have $(T_{F(\alpha)} f)(a) = (N_\alpha T_a f, F_-)$ and using the formula $N_\alpha T_a = \exp(-\pi a^2 \cosh \alpha \cdot \sinh \alpha) T_{a \cosh \alpha} R_{ia \sinh \alpha} N_\alpha$ (that easily follows from [B], (11.11)), we obtain

$$\begin{aligned}(T_{F(\alpha)} f)(a) &= \exp(-\pi a^2 \cosh \alpha \cdot \sinh \alpha)(T_{a \cosh \alpha} R_{ia \sinh \alpha} N_\alpha f, F_-)\\ &= \exp(-\pi a^2 \cosh \alpha \cdot \sinh \alpha)(T_F(R_{ia \sinh \alpha} N_\alpha f))(a \cosh \alpha).\end{aligned}$$

We are going to estimate $R_{ia \sinh \alpha} N_\alpha f$ for $a \in \mathbb{C}$, $\alpha > 0$. It is not hard to prove from smoothness of f that there is an $M > 0$, $A > 0$, $B > 0$ such that

$$|(N_\alpha f)(t)| \leq M \exp(-\pi A(\operatorname{Re} t)^2 + \pi B(\operatorname{Im} t)^2) \quad (t \in \mathbb{C},\ \alpha > 0).$$

(This may be proved by using the optimal shift technique of the proof of [B], 8.1.) Hence, using the inequality

$$|2\,at| \leq |a|^2 + (\operatorname{Re} t)^2 + (\operatorname{Im} t)^2 \quad (a \in \mathbb{C},\ t \in \mathbb{C}),$$

$$|(R_{ia \sinh \alpha} N_\alpha f)(t)| = |\exp(2\pi a t \sinh \alpha)(N_\alpha f)(t)| \leq$$
$$\leq M \exp(-\pi(A - \sinh \alpha)(\operatorname{Re} t)^2 + \pi(B + \sinh \alpha)(\operatorname{Im} t)^2) \exp(\pi |a|^2 \sinh \alpha)$$

for every $a \in \mathbb{C}$, $t \in \mathbb{C}$, $\alpha > 0$. This shows that for sufficiently small $\alpha > 0$

$$|(R_{ia \sinh \alpha} N_\alpha f)(t)| \leq M \exp(-\tfrac{\pi}{2} A(\operatorname{Re} t)^2 + 2\pi B(\operatorname{Im} t)^2) \exp(\pi |a|^2 \sinh \alpha)$$

for every $a \in \mathbb{C}$, $t \in \mathbb{C}$.

Now we use continuity of T_F. It follows from 1.14, remark that there are numbers $M_0 > 0$, $A_0 > 0$, $B_0 > 0$ such that

$$|(T_F(R_{ia \sinh \alpha} N_\alpha f))(a \cosh \alpha)| \leq$$
$$\leq M_0 \exp(-\pi A_0 (\operatorname{Re} a)^2 \cosh^2 \alpha + \pi B_0 (\operatorname{Im} a)^2 \cosh^2 \alpha) \exp(\pi |a|^2 \sinh \alpha)$$

for sufficiently small $\alpha > 0$ and all $a \in \mathbb{C}$. Hence

$$|(T_{F(\alpha)} f)(a)| \leq$$
$$\leq M_0 \exp(-\pi A_0 (\operatorname{Re} a)^2 \cosh^2 \alpha + \pi B_0 (\operatorname{Im} a)^2 \cosh^2 \alpha) \times$$
$$\times \exp(2\pi |a|^2 \cosh \alpha\ \sinh \alpha)$$

for sufficiently small $\alpha > 0$ and all $a \in \mathbb{C}$.

It is easy to see now that there exists $\alpha_1 > 0$, $M_1 > 0$, $A_1 > 0$, $B_1 > 0$ such that

$$0 < \alpha < \alpha_1 \Rightarrow |(T_{F(\alpha)} f)(a)| \leq M_1 \exp(-\pi A_1 (\operatorname{Re} a)^2 + \pi B_1 (\operatorname{Im} a)^2) \quad (a \in \mathbb{C}).$$

Since $T_{F(\alpha)} f \to T_F f$ pointwise we easily conclude from a continuous version of 1.8 that $T_{F(\alpha)} f \xrightarrow{S} T_F f$. □

3.7. A lot of properties of the T_F's with $F \in \mathscr{C}$ easily follow now.

THEOREM. Let $F \in \mathscr{C}$. Then $\overline{F} \in \mathscr{C}$, $F_- \in \mathscr{C}$ and $\overline{F}_- \in \mathscr{C}$ (cf. 1.12), and furthermore T_F and $T_{\overline{F}_-}$ are adjoint operators.

PROOF. We shall only prove that $\overline{F}_- \in \mathscr{C}$ (the other cases can be treated similarly). Let $g \in S$. Using the relation $T_{\overline{h}_-} g = (\overline{T_h \overline{g}_-})_-$ that holds by 2.3(ii) for $h \in S$, we obtain by 3.6

$$T_{\overline{(F(\alpha))_-}}\, g = (\overline{T_{F(\alpha)} \overline{g}_-})_- \xrightarrow{S} (\overline{T_F \overline{g}_-})_-$$

if $\alpha \downarrow 0$. Furthermore $T_{\overline{(F(\alpha))}_-} g \to T_{\overline{F}_-} g$ pointwise. So $T_{\overline{F}_-} g = (\overline{T_F \overline{g}_-})_- \in S$, and hence $\overline{F}_- \in \mathscr{C}$.

We further have for $f \in S$, $g \in S$ by 2.3(ii) and 3.6

$$(T_F f, g) = \lim_{\alpha \downarrow 0} (T_{F(\alpha)} f, g) = \lim_{\alpha \downarrow 0} (f, T_{\overline{(F(\alpha))}_-}\, g) = (f, T_{\overline{F}_-} g),$$

so T_F and $T_{\overline{F}_-}$ are adjoint operators. □

REMARK 1. According to 1.15 we can extend T_F to a linear operator of S^* in case $F \in \mathscr{C}$. We denote this extended operator again by T_F. The following properties are satisfied

(i) $T_F(\operatorname{emb}(f)) = \operatorname{emb}(T_F f)$ $\quad (f \in S)$,

(ii) $G_n \in S^*$ $(n \in \mathbb{N})$, $G_n \xrightarrow{S^*} 0 \Rightarrow T_F G_n \xrightarrow{S^*} 0$,

(iii) $(T_F G, g) = (G, T_{\overline{F}_-} g)$ $\quad (G \in S^*,\ g \in S)$.

REMARK 2. For $F \in \mathscr{C}$ the following relations hold:

$$\overline{T}_F = T_{\overline{F}},\ (T_F)_- = T_{F_-},\ (\overline{T_F})_- = T_{\overline{F}_-}.$$

3.8. THEOREM. Let $F \in \mathscr{C}$, $G \in \mathscr{C}$. We have $T_F T_G = T_G T_F$.

PROOF. First assume that $F \in \mathscr{C}$, $G \in \operatorname{emb}(S)$, and let $f \in S$. We have

$$T_F T_G f = \lim_{\alpha \downarrow 0} T_{F(\alpha)} T_G f = \lim_{\alpha \downarrow 0} T_G T_{F(\alpha)} f = T_G T_F f$$

(the limits are in S-sense by 3.6). The general case is reduced to the above one by noting that

$$T_F T_G f = \lim_{\alpha \downarrow 0} T_F T_{G(\alpha)} f = \lim_{\alpha \downarrow 0} T_{G(\alpha)} T_F f = T_G T_F f$$

(the limits are again in S-sense). □

3.9. Another theorem of the above type is the following one.

THEOREM. Let $F \in \mathscr{C}$, $G \in \mathscr{C}$. We have $T_{\overline{F}} G = T_{\overline{G}} F$, $T_{T_{\overline{F}} G} = T_F T_G$.

PROOF. Let us first note that $T_{\overline{F}} G$ and $T_{\overline{G}} F$ are well defined by 3.7, remark 1.

To show $T_{\overline{F}}G = T_{\overline{G}}F$ first assume that $F \in \mathscr{C}$, $G \in \text{emb}\,(S)$. Then

$$T_{\overline{F}}G = \lim_{\alpha \downarrow 0} T_{\overline{F(\alpha)}}G = \lim_{\alpha \downarrow 0} T_{\overline{G}}(\text{emb}\,(F(\alpha))) = T_{\overline{G}}F$$

(the limits are in S^*-sense). Here we used 2.3(iii) and the fact that emb $(F(\alpha)) \xrightarrow{S^*} F (\alpha \downarrow 0)$. The general case can be reduced to this one by noting that

$$T_{\overline{F}}G = \lim_{\alpha \downarrow 0} T_{\overline{F}}(\text{emb}\,(G(\alpha))) = \lim_{\alpha \downarrow 0} T_{\overline{G(\alpha)}}F = T_{\overline{G}}F$$

(the limits are again in S^*-sense), where the latter equality follows from

$$(T_{\overline{G(\alpha)}}F, f) = (F, T_{(G(\alpha))\_}f) \to (F, T_{G\_}f) = (T_{\overline{G}}F, f) \quad (\alpha \downarrow 0)$$

holding for $f \in S$ (cf. 1.9).

Now we show $T_{T_{\overline{F}}G} = T_F T_G$, and we use therefore the relation

$$T_a T_K = T_K T_a \quad (a \in \mathbb{C},\ K \in \mathscr{C}),$$

that follows at once from the definition of T_K. We easily see from 3.7, remark 2 that $(T_{\overline{F}}G)_- = T_{\overline{F}\_}G_-$, so

$$(T_{T_{\overline{F}}G}f)(x) = (T_x f, (T_{\overline{F}}G)_-) = (T_x f, T_{\overline{F}\_}G_-) =$$
$$= (T_F T_x f, G_-) = (T_x T_F f, G_-) = (T_G(T_F f))(x)$$

for $f \in S$, $x \in \mathbb{C}$. Hence $T_{T_{\overline{F}}G} = T_G T_F$. □

REMARK 1. The preceding theorem states (among other things) that T_F maps $\mathscr{C}$ into $\mathscr{C}$ in case $F \in \mathscr{C}$. For if $G \in \mathscr{C}$ then $T_{T_F G} = T_{\overline{F}} T_G$, and $T_{\overline{F}} T_G$ maps S into S, hence $T_F G \in \mathscr{C}$.

REMARK 2. Let $F \in S^*$. We mention the possibility to extend the linear mapping T_F (which maps S into the class of all entire functions) to a linear mapping of the space $\mathscr{C}$ into S^*. This is done by putting $T_F f := T_{\overline{f}}\overline{F}$ for $f \in \mathscr{C}$. In case $F \in \mathscr{C}$ this definition coincides with the one given in 3.7, remark 1.

REMARK 3. In 3.7, remark 1 we have extended the operator T_F to a linear operator of S^* (in case $F \in \mathscr{C}$). If $F \in S$ however, there is a more direct way to define this operator on S^*, namely by putting $T_F G = T_{\overline{G}}\overline{F}$ for $G \in S^*$ (this $T_{\overline{G}}\overline{F}$ has been defined in 3.2, and is an entire function). It is not obvious yet that this alternative definition yields the same operator, i.e. that emb $(T_{\overline{G}}\overline{F})$ equals $T_F G$ (as it is defined in 3.7, remark 1) for $G \in S^*$. The proof is pretty hard, and will be postponed until 5.5.

REMARK 4. Let $F \in \mathscr{C}$, $G \in S^*$. We can define the *convolution* $F \star G$ of F and G by putting $F \star G = T_F G$. If we restrict ourselves to $F \in \mathscr{C}$, $G \in \mathscr{C}$, then this convolution product has the usual properties. We mention commutativity (follows from the preceding theorem), and associativity:

if $H \in S^*$, then $F \star (G \star H) = (F \star G) \star H$, for

$$F \star (G \star H) = T_{\overline{F}}(G \star H) = T_{\overline{F}} T_{\overline{G}} H =$$
$$= T_{T_{\overline{F}}\overline{G}} H = T_{\overline{F \star G}} H = (F \star G) \star H.$$

We mention furthermore Titchmarsh's theorem for $\mathscr{C}$: if $F \in \mathscr{C}$, $G \in \mathscr{C}$, then $F \star G = 0 \Rightarrow F = 0 \vee G = 0$. This will be proved in the next section.

3.10. We consider an alternative description of the class $\mathscr{C}$ which is related to Weyl correspondence (cf. [B], 26). If $F \in S^*$, then we can define for every $g \in S$ the continuous linear functional (cf. 1.10 and 1.18)

$$\forall_{f \in S}(f \otimes \bar{g}, Z_U(E \otimes \overline{F}))$$

of S (E is the function emb $(\forall_t 1)$). For every $g \in S$ we can find (by 1.13) exactly one $K_F g \in S^*$ such that

$$(f \otimes \bar{g}, Z_U(E \otimes \overline{F})) = (f, K_F g) \quad (f \in S).$$

This K_F is a linear mapping of S into S^*.

If $F \in \mathscr{C}$, $g \in S$, then we can prove that $K_F g \in \text{emb}\,(S)$. It suffices therefore to show that $K_F g = \text{emb}\,(T_G g)$, where G is the element of S^* that satisfies $(G, f) = (F, \forall_x \sqrt{2} f(x\sqrt{2}))$ for $f \in S$ (cf. 1.13; note that $G \in \mathscr{C}$). This equation holds in case $F \in \text{emb}\,(S)$, and the general case can be handled by using $T_{G(\alpha)}g \xrightarrow{S} T_G g$, $K_{F(\alpha)}g \xrightarrow{S^*} K_F g$ if $\alpha \downarrow 0$. The converse of the above statement is also true, i.e. if $K_F g \in \text{emb}\,(S)$ for every $g \in S$, then $F \in \mathscr{C}$. We shall prove this in 5.6.

3.11. One of the main features of the convolution operators is the fact that they commute with the time shifts T_a ($a \in \mathbb{C}$). That this fact actually characterizes the convolution operators is expressed in the following theorem.

THEOREM. Let T be a continuous linear operator of S that satisfies $TT_a = T_a T$ for every $a \in \mathbb{R}$. Then there is a $G \in \mathscr{C}$ such that $T = T_G$.

PROOF. We note that $\forall_{f \in S}(Tf)(0)$ is a continuous linear functional of S. This means that there exists an $H \in S^*$ such that

$$(Tf)(0) = (f, H) \quad (f \in S).$$

Now we have for every $f \in S$, $x \in \mathbb{C}$

$$(Tf)(x) = (T_x Tf)(0) = (TT_x f)(0) = (T_x f, H).$$

This proves the theorem with $G = H_-$. □

COROLLARY. Let T be a continuous linear operator of S that commutes with all time shifts T_a ($a \in \mathbb{R}$) and all frequency shifts R_b ($b \in \mathbb{C}$). Then there is a $c \in \mathbb{C}$ such that $T = cI$. This is proved as follows. We infer from

3.11 that there is an $H \in \mathscr{C}$ such that $(Tf)(x)=(T_x f, H)$ $(f \in S,\ x \in \mathbb{C})$; we have $H=T^*\delta_0$. Since T, and hence T^* commutes with all frequency shifts we have $R_b T^* \delta_0 = T^* R_b \delta_0 = T^*\delta_0$ $(b \in \mathbb{R})$. So, by a theorem that will be proved in 4.11, remark, $T^*\delta_0$ is a multiple of δ_0. Hence T is a multiple of I.

REMARK. Let T be a continuous linear operator of S satisfying $TP=PT$ (cf. 1.4(iii)). It may be proved from 3.11 that there is a $G \in \mathscr{C}$ with $T=T_G$.

4. FOURIER TRANSFORM AND GENERALIZED CONVOLUTION OPERATORS

4.1. This section is devoted to the Fourier transform in its relation to convolution theory. We shall generalize the convolution theorem, and we shall give a characterization of the class $\mathscr{C}$ in terms of Fourier transforms. Some remarks are made on the equation $T_f F=0$ with $f \in \mathscr{C}$, $F \in S^*$.

4.2. DEFINITION. Let h be a mapping of $\mathbb{C}$ into $\mathbb{C}$. We define the *multiplication operator* M_h by

$$M_h f := \curlyvee_{z\in\mathbb{C}} h(z) f(z) \quad (f \in S).$$

We also write $h \cdot f$ instead of $M_h f$.

4.3. LEMMA. If $h:\mathbb{C}\to\mathbb{C}$ satisfies $V_{\varepsilon>0}[\curlyvee_{z\in\mathbb{C}} h(z) \exp(-\pi\varepsilon z^2) \in S]$, then M_h is a continuous linear operator of S with an adjoint, viz. $M_{\bar h}$.

PROOF. Almost trivial. □

REMARK. The M_h of the above lemma can be extended in the familiar way to a continuous linear operator of S^*, which is again denoted by M_h. We shall also write $h \cdot F$ instead of $M_h F$ if $F \in S^*$.

4.4. DEFINITION. Let $\mathscr{M}$ be the class of all generalized functions F for which there exists an analytic g satisfying $V_{\varepsilon>0}[\curlyvee_{z\in\mathbb{C}} g(z) \exp(-\pi\varepsilon z^2) \in S]$ such that $F=\operatorname{emb}(g)$ (cf. 1.6(i)). On $\mathscr{M}$ we define the mapping emb^{-1} by putting $\operatorname{emb}^{-1}(F)=g$ if $F \in \mathscr{M}$, $F=\operatorname{emb}(g)$, where g satisfies the above description (note that such a g is unique, hence the mapping emb^{-1} is well defined on $\mathscr{M}$).

4.5. The following characterization of $\mathscr{C}$ is very useful.

THEOREM. $F \in \mathscr{C} \Leftrightarrow \mathscr{F}F \in \mathscr{M}$.

PROOF. Let $F \in \mathscr{C}$. We have for every $f \in S$

$$(\mathscr{F}F)(\alpha)\cdot\mathscr{F}f = (\mathscr{F}F(\alpha))\cdot\mathscr{F}f = \mathscr{F}(T_{\overline{F(\alpha)}}f) \xrightarrow{S} \mathscr{F}(T_{\overline{F}}f)$$

if $\alpha \downarrow 0$ by [B], Theorem 9.1, 2.3(iv) and lemma 3.6. It easily follows that $g := \curlyvee_{z\in\mathbb{C}} \lim_{\alpha\downarrow 0} ((\mathscr{F}F)(\alpha))(z)$ is an analytic function that satisfies

$$\curlyvee_{z\in\mathbb{C}} g(z) \exp(-\pi\varepsilon z^2) \in S \text{ for } \varepsilon>0.$$

Furthermore $\mathscr{F}F=\operatorname{emb}(g)$ since we have for every $h \in S$

$$(\mathscr{F}F, h) = \lim_{\alpha\downarrow 0} ((\mathscr{F}F)(\alpha), h) =$$

$$= \lim_{\alpha\downarrow 0} \int_{-\infty}^{\infty} ((\mathscr{F}F)(\alpha))(t)\overline{h(t)}\,dt = \int_{-\infty}^{\infty} g(t)\overline{h(t)}\,dt.$$

Hence $\mathscr{F}F \in \mathscr{M}$.

Now assume that $\mathscr{F}F \in \mathscr{M}$. It suffices to show that for every $f \in S$ the function $\curlyvee_{z\in\mathbb{C}} (T_z f, \overline{F}_-) \in S$. Note therefore that for every $z \in \mathbb{C}$

$$\lim_{\alpha\downarrow 0} (T_z f, \overline{F_-(\alpha)}) = (T_z f, \overline{F}_-),$$

and that the proof will be complete if we can show that this limit is achieved in S-sense.

We have by 2.3(iv) for every $\alpha>0$

$$\mathscr{F}(T_{\overline{F(\alpha)}}f) = (\mathscr{F}F)(\alpha)\cdot\mathscr{F}f.$$

Now let $M>0$, $A>0$, $B>0$ be such that

$$|(\mathscr{F}f)(x+iy)| \leqslant M \exp(-\pi A x^2 + \pi B y^2)$$

for every $x \in \mathbb{R}$, $y \in \mathbb{R}$. Since $\mathscr{F}F \in \mathscr{M}$, we infer the existence of a g that satisfies $V_{\varepsilon>0}[\curlyvee_{z\in\mathbb{C}} g(z) \exp(-\pi\varepsilon z^2) \in S]$ such that $F=\operatorname{emb}(g)$. This means that there exists an $M_1>0$, $B_1>0$ such that

$$|g(x+iy)| \leqslant M_1 \exp(\tfrac{1}{2}\pi A x^2 + \pi B_1 y^2)$$

for every $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is not hard to show that there are numbers $M_2>0$, $B_2>0$, $\alpha_0>0$ such that

$$0<\alpha<\alpha_0 \Rightarrow |(N_\alpha g)(x+iy)| \leqslant M_2 \exp(\tfrac{1}{2}\pi A x^2 + \pi B_2 y^2)$$

for every $x \in \mathbb{R}$, $y \in \mathbb{R}$. Since $N_\alpha g = (\mathscr{F}F)(\alpha)$ for every $\alpha>0$, we see that $(\mathscr{F}F)(\alpha) f \xrightarrow{S} g\cdot f$ if $\alpha \downarrow 0$ (here we have used a continuous version of 1.8), so $\mathscr{F}(T_{\overline{F(\alpha)}}f) \xrightarrow{S} g\cdot\mathscr{F}f$, and hence $T_{\overline{F(\alpha)}}f \xrightarrow{S} \mathscr{F}^*(g\cdot\mathscr{F}f)$ if $\alpha \downarrow 0$ (cf. 1.4(i)). This shows that $T_{\overline{F}}f = \mathscr{F}^*(g\cdot\mathscr{F}f)$, and hence that $T_F f \in S$. □

COROLLARY. If $F \in \mathscr{C}$, $f \in S$, then $\mathscr{F}(T_{\overline{F}}f) = \operatorname{emb}^{-1}(\mathscr{F}F)\cdot\mathscr{F}f$. This follows easily from the second part of the proof of the above theorem.

4.6. EXAMPLES

(i) Let $F := \curlyvee_{z\in\mathbb{C}} e^{\pi i z^2}$. It is not hard to check that $\mathscr{F}F$ is the embedding of $\curlyvee_{z\in\mathbb{C}} (-i)^{-\frac{1}{2}} e^{-\pi i z^2}$ (principal root), and it follows that $F \in \mathscr{C}$.

(ii) Let h_τ be the function defined by $h_\tau := 1/2\tau\, \chi_{(-\tau,\tau)}$. Its Fourier transform is given by

$$\curlyvee_\lambda \frac{\sin 2\pi\lambda\tau}{2\pi\lambda\tau},$$

hence $\mathrm{emb}\,(h_\tau) \in \mathscr{C}$.

(iii) Let $(d_n)_{n\in\mathbb{Z}}$ be a complex sequence that satisfies $|d_n| = O(e^{-\varepsilon n^2})$ $(n \in \mathbb{Z})$ for some $\varepsilon > 0$, and let $F := \sum_{n=-\infty}^{\infty} d_n\delta_n$ (cf. [B], 27.24.2(ii)). Then $\mathscr{F}F$ is the embedding of the analytic function $\curlyvee_z \sum_{n=-\infty}^{\infty} d_n\, e^{-2\pi i n z}$, and it is not hard to show that $F \in \mathscr{C}$.

4.7. THEOREM. If $F \in \mathscr{C}$, $G \in S^*$, then $\mathscr{F}(T_{\bar{F}}G) = \mathrm{emb}^{-1}(\mathscr{F}F)\cdot\mathscr{F}G$.

PROOF. The case with $G \in \mathrm{emb}\,(S)$ follows from 4.5, corollary, and the general case is deduced from this one by noting that

$$\mathscr{F}(T_{\bar{F}}G) = \lim_{\alpha\downarrow 0} \mathscr{F}(T_{\bar{F}}G(\alpha)) =$$

$$= \lim_{\alpha\downarrow 0} \mathrm{emb}^{-1}(\mathscr{F}F)\cdot\mathrm{emb}((\mathscr{F}G)(\alpha)) = \mathrm{emb}^{-1}(\mathscr{F}F)\cdot\mathscr{F}G,$$

where the limit is achieved in S^*-sense. □

4.8. THEOREM. If $F \in S^*$, $\mathscr{F}F \in \mathrm{emb}\,(S^+)$, $G \in \mathscr{C}$, and if $T_{\bar{G}}F = 0$, then $F = 0$ or $G = 0$.

PROOF. By 4.7 we have $\mathrm{emb}^{-1}(\mathscr{F}G)\cdot\mathscr{F}F = 0$. Write $g = \mathrm{emb}^{-1}(\mathscr{F}G)$, and let $f \in S^+$ be such that $\mathrm{emb}\,(f) = \mathscr{F}F$. Then $g\cdot\mathscr{F}F = \mathrm{emb}\,(g\cdot f)$, for we have for $h \in S$

$$(g\cdot\mathscr{F}F, h) = (\mathscr{F}F, \bar{g}\cdot h) = (\mathrm{emb}\,(f), \bar{g}\cdot h) =$$

$$= \int_{-\infty}^{\infty} f(t)\overline{\overline{g(t)}h(t)}dt = \int_{-\infty}^{\infty} g(t)f(t)\overline{h(t)}dt = (\mathrm{emb}\,(g\cdot f), h).$$

It follows from 1.6(i) that $g\cdot f = 0$ (a.e.). We conclude by analyticity of g that $g = 0$ or that $f = 0$ (a.e.), and so $G = 0$ or $F = 0$. □

4.9. We enter somewhat further into questions of the type: if $f \in \mathscr{C}$, $F \in S^*$ and $T_fF = 0$, then what can we tell about F. As we see from 4.7 such questions can be translated into $(g = \mathscr{F}\bar{f},\ G = \mathscr{F}F)$: if $g \in \mathscr{M}$, $G \in S^*$, and if $\mathrm{emb}^{-1}(g)\cdot G = 0$, then what can we tell about G. In [S] these problems have been solved for the space K' (dual space of K), but we cannot use the techniques employed there, since we have (by lack of non-trivial elements of S of compact support) not the occasion to consider the elements of S^* locally, as it is done in [S] for the elements of K' (cf. [S], Ch. V, § 4).

We shall prove here some simple results in this direction, and we shall mention some further theorems without proof.

4.10. THEOREM. Let $g \in \mathscr{M}$, and assume that $\mathrm{emb}^{-1}(g)$ has no zeros. If $G \in S^*$ and $\mathrm{emb}^{-1}(g)\cdot G = 0$, then $G = 0$.

PROOF. Put $h := \mathrm{emb}^{-1}(g)$. We infer from the fact that

$$\curlyvee_z h(z) \exp\,(-\pi\varepsilon z^2) \in S$$

for every $\varepsilon > 0$, and the fact that h has no zeros, that there are complex numbers a_0, a_1 and a_2 with $\mathrm{Re}\,(a_2) \leqslant 0$ such that

$$h(z) = \exp\,(a_0 + a_1 z + a_2 z^2) \quad (z \in \mathbb{C}).$$

If $\alpha > 0$ is such that $\coth\alpha > |a_2|$, and if we denote for $t \in \mathbb{C}$

$$k_t := \curlyvee_{z\in\mathbb{C}} (\sinh\alpha)^{\frac{1}{2}} \exp\left(\frac{-\pi}{\sinh\alpha}((z^2+t^2)\cosh\alpha - 2zt)\right)\overline{h^{-1}(\bar{z})},$$

then we have $k_t \in S$, $\bar{h}\cdot k_t = \delta_\alpha(t)$ (cf. 1.6(ii)). Hence for $t \in \mathbb{C}$

$$0 = (h\cdot G, k_t) = (G, \bar{h}\cdot k_t) = (G, \delta_\alpha(t)) = G_\alpha(\bar{t})$$

(cf. [B], 27.18), so $G_\alpha = 0$, and therefore $G = 0$ by 1.5. □

4.11. THEOREM. If $G \in S^*$, $QG = 0$ (cf. 1.4(iii)), then there is a $c \in \mathbb{C}$ such that $G = c\delta_0$. This c is uniquely determined by G.

PROOF. Let $\alpha > 0$ be fixed. We infer from [B], (11.9) and (11.11) that

$$N_\alpha Q = \cosh\alpha\, QN_\alpha + i\sinh\alpha\, PN_\alpha,$$

so we find

$$\cosh\alpha\, QG_\alpha + i\sinh\alpha\, PG_\alpha = 0.$$

The solution of this differential equation is given by

$$G_\alpha = \curlyvee_{z\in\mathbb{C}} G_\alpha(0)\exp\,(-\pi z^2\coth\alpha) = G_\alpha(0)(\sinh\alpha)^{\frac{1}{2}}(\delta_0)_\alpha.$$

Hence, with $c := G_\alpha(0)(\sinh\alpha)^{\frac{1}{2}}$, $(G - c\delta_0)_\alpha = G_\alpha - c(\delta_0)_\alpha = 0$. It follows from 1.5 that $G = c\delta_0$.

Uniqueness of c is trivial. □

REMARK. It follows easily from the above theorem that an $F \in S^*$ that satisfies $T_aF = F$ $(a \in \mathbb{R})$ is the embedding of a constant function. For we have $PF = \lim_{h\to 0} 1/2\pi i h\,(T_hF - F) = 0$ (the limit is in S^*-sense), hence $Q\mathscr{F}F = \mathscr{F}PF = 0$. This means that $\mathscr{F}F = c\delta_0$ with some $c \in \mathbb{C}$, and hence $F = \mathrm{emb}\,(\curlyvee_{t\in\mathbb{C}} c)$. Also, if $F \in S^*$, $R_bF = F$ $(b \in \mathbb{R})$, then $F = c\delta_0$ for some $c \in \mathbb{C}$.

4.12. Theorem 4.11 can be generalized as follows. Let

$$n \in \mathbb{N},\ a_1 \in \mathbb{C},\ \ldots,\ a_n \in \mathbb{C},\ \nu_1 \in \mathbb{N},\ \ldots,\ \nu_n \in \mathbb{N},$$

and let $h := \curlyvee_{z\in\mathbb{C}} \prod_{k=1}^{n} (z-a_k)^{\nu_k}$. If $G \in S^*$ satisfies $h\cdot G=0$, then there are complex numbers d_{kl} $(l=0, \ldots, \nu_k-1;\ k=1, \ldots, n)$ such that

$$(\star)\qquad F = \sum_{k=1}^{n} \sum_{l=0}^{\nu_k-1} d_{kl}\, P^l \delta_{a_k}.$$

The numbers d_{kl} are uniquely determined by F.

We still can go further. Assume that $h \in \mathrm{emb}^{-1}(\mathscr{M})$, and let h have its zeros in $a_1, a_2, \ldots$ with multiplicities $\nu_1, \nu_2, \ldots$. Let V_n be the set of all elements $F \in S^*$ of the form $(\star)$, and let V be the union of all V_n's. We assume that $\sum_{k=1,|a_k|\neq 0}^{\infty} \nu_k/|a_k|^2 < \infty$. Then every $F \in S^*$ that satisfies $h\cdot F=0$ is S^*-limit of a sequence in V, and every $F \in S^*$ that is S^*-limit of elements of V satisfies $h\cdot F=0$.

We note that every element h of $\mathrm{emb}^{-1}(\mathscr{M})$ has order $\leqslant 2$, and this means that the limit exponent of h does not exceed 2 (cf. [Bi], Ch. VI, § 4): if $a_1, a_2, \ldots$ and $\nu_1, \nu_2, \ldots$ are as in the above, then $\sum_{k=1,|a_k|\neq 0}^{\infty} \nu_k/|a_k|^{2+\varepsilon} < \infty$ for every $\varepsilon>0$. In case that the order of h is less than 2, we have $\sum_{k=1,|a_k|\neq 0}^{\infty} \nu_k/|a_k|^2 < \infty$, so the above theorem applies to h. We do not know how to handle the general case in which functions h like $\curlyvee_{z\in\mathbb{C}}(\exp(\pi i z^2)-1)$ occur (here $\sum_{k=1,|a_k|\neq 0}^{\infty} \nu_k/|a_k|^2=\infty$).

5. SOME FURTHER REMARKS ON CONVOLUTION THEORY

5.1. In this section we give some further theorems and definitions about the class $\mathscr{C}$. We shall also pay attention to convergence in $\mathscr{C}$, and to convolution operators in S^n $(n \in \mathbb{N})$.

5.2. We are going to show that $g\cdot G \in \mathscr{C}$ if $g \in S$, $G \in S^*$. This means that $T_{\bar{f}}F \in \mathscr{M}$ if $f \in S$, $F \in S^*$ (cf. 4.5 and 4.7), and it will turn out that $T_{\bar{f}}F = \mathrm{emb}\,(T_{\bar{F}}f)$ (in particular $T_{\bar{F}}f \in S^+$). The following lemma is useful in the proofs of the above statements.

LEMMA. Let $f \in S$, $g \in \mathrm{emb}^{-1}(\mathscr{M})$, and let $M_1>0$, $A_1>0$, $B_1>0$, $M_2>0$, $A_2 \geqslant 0$, $B_2>0$ be such that

$$|f(z)| \leqslant M_1 \exp(-\pi A_1(\mathrm{Re}\, z)^2 + \pi B_1(\mathrm{Im}\, z)^2),$$
$$|g(z)| \leqslant M_2 \exp(-\pi A_2(\mathrm{Re}\, z)^2 + \pi B_2(\mathrm{Im}\, z)^2)$$

for every $z \in \mathbb{C}$. To every ε with $0<\varepsilon<A_1+A_2$ there exists a $C>0$ and a $\beta>0$ (only depending on B_1, B_2 and ε) such that for every $F \in S^*$, $y \in \mathbb{C}$ ($\|\ \|$ denotes inner product norm in S)

$$|(g\cdot T_y f, F)| \leqslant M_1 M_2 C \|F(\beta)\| \exp\Big(-\pi A_1 \frac{A_2-\varepsilon}{A_1+A_2-\varepsilon} (\mathrm{Re}\, y)^2 + 2\pi B_1 (\mathrm{Im}\, y)^2\Big).$$

PROOF. We have for every $y \in \mathbb{C}$, $z \in \mathbb{C}$

$$|g(z)(T_y f)(z)| \leqslant M_1 M_2 \exp(\pi P(z, y)),$$

where P is defined by

$$P(z, y) = -(A_1+A_2)(\mathrm{Re}\, z)^2 + (B_1+B_2)(\mathrm{Im}\, z)^2 + 2A_1|\mathrm{Re}\, z\ \mathrm{Re}\, y|$$
$$+ 2B_1|\mathrm{Im}\, z\ \mathrm{Im}\, y| - A_1(\mathrm{Re}\, y)^2 + B_1(\mathrm{Im}\, y)^2 \quad (z \in \mathbb{C},\ y \in \mathbb{C}).$$

Let ε satisfy $0<\varepsilon<A_1+A_2$. Applying the inequality $2|ab| \leqslant \gamma a^2 + \gamma^{-1} b^2$ (valid for $a \in \mathbb{R}$, $b \in \mathbb{R}$, $\gamma>0$) to $2|\mathrm{Re}\, z\ \mathrm{Re}\, y|$ and $2|\mathrm{Im}\, z\ \mathrm{Im}\, y|$ with $\gamma = 1+(A_2-\varepsilon)A_1^{-1}$ and $\gamma=1$ respectively, we obtain

$$P(z, y) \leqslant -\varepsilon(\mathrm{Re}\, z)^2 + (2B_1+B_2)(\mathrm{Im}\, z)^2 - A_1 \frac{A_2-\varepsilon}{A_1+A_2-\varepsilon} (\mathrm{Re}\, y)^2 + 2B_1(\mathrm{Im}\, y)^2$$

for $y \in \mathbb{C}$, $z \in \mathbb{C}$. Now put for every $y \in \mathbb{C}$

$$h_y := \curlyvee_{z\in\mathbb{C}}\, g(z)(T_y f)(z) \exp\Big(\pi A_1 \frac{A_2-\varepsilon}{A_1+A_2-\varepsilon} (\mathrm{Re}\, y)^2 - 2\pi B_1 (\mathrm{Im}\, y)^2\Big).$$

Then we have $h_y \in S$, and

$$|h_y(z)| \leqslant M_1 M_2 \exp(-\pi\varepsilon(\mathrm{Re}\, z)^2 + \pi(2B_1+B_2)(\mathrm{Im}\, z)^2) \quad (z \in \mathbb{C}).$$

It follows from 1.3(ii) that we can find a $C>0$ and a $\beta>0$ (only depending on ε and $2B_1+B_2$) such that for every $y \in \mathbb{C}$ there exists an $l_y \in S$ with $h_y = N_\beta l_y$, $\|l_y\| \leqslant M_1 M_2 C$. So we have for every $y \in \mathbb{C}$

$$|(g\cdot T_y f, F)| \leqslant M_1 M_2 C \|F(\beta)\| \exp\Big(-\pi A_1 \frac{A_2-\varepsilon}{A_1+A_2-\varepsilon} (\mathrm{Re}\, y)^2 + 2\pi B_1 (\mathrm{Im}\, y)^2\Big)$$

if $F \in S^*$ (here we apply

$$|(h_y, F)| = |(N_\beta l_y, F)| = |(l_y, F(\beta))| \leqslant M_1 M_2 C \|F(\beta)\|). \qquad \square$$

5.3. THEOREM. If $g \in S$, $G \in S^*$, then $g\cdot G \in \mathscr{C}$.

PROOF. Let $f \in S$. Then $T_{g\cdot G}f = \curlyvee_{y\in\mathbb{C}}(\bar{g}\cdot T_y f, G_-)$. Analyticity of $T_{g\cdot G}f$ follows from theorem 3.3, and it follows easily from lemma 5.2 that $T_{g\cdot G}f \in S$. $\square$

5.4. THEOREM. If $f \in S$, $F \in S^*$, then $\mathrm{emb}\,(T_{\bar{F}}f) \in \mathscr{M}$.

PROOF. Take $g = \curlyvee_{z\in\mathbb{C}} 1$ and $\bar{F}_-$ (instead of F) in 5.2 to conclude that $T_{\bar{F}}f \in S^+$. It further follows from lemma 5.2 and theorem 3.3 that $\curlyvee_{z\in\mathbb{C}} \exp(-\pi\varepsilon z^2)(T_{\bar{F}}f)(z) \in S$ for every $\varepsilon>0$. Hence $\mathrm{emb}\,(T_{\bar{F}}f) \in \mathscr{M}$. $\square$

5.5. We can prove now the statement in 3.9, remark 3.

THEOREM. If $F \in S^*$, $f \in S$ then $T_{\bar{f}}F = \mathrm{emb}\,(T_{\bar{F}}f)$.

PROOF. We first prove the formula with $F \in \mathscr{C}$. We have in that case by theorem 3.9, theorem 3.7 and 3.7, remark 1 for every $g \in S$

$$(T_{\bar{f}}F, g) = (T_{\bar{F}}(\mathrm{emb}\,(f)), g) = (\mathrm{emb}\,(f), T_{F\_}g) =$$
$$= (f, T_{F\_}g) = (T_{\bar{F}}f, g) = (\mathrm{emb}\,(T_{\bar{F}}f), g),$$

hence $T_{\bar{f}}F = \mathrm{emb}\,(T_{\bar{F}}f)$ by the uniqueness part of theorem 1.13.

The general case is reduced to the above one as follows. Denote $h_\delta := \curlyvee_{z\in\mathbb{C}} \exp(-\pi\delta z^2)$, and define $F_\delta := h_\delta \cdot F$ for $\delta > 0$. Now $F_\delta \in \mathscr{C}$ by 5.3, and we have $T_{\bar{f}}F_\delta = \mathrm{emb}\,(T_{\bar{F}_\delta}f)$ for $\delta > 0$. The proof will be complete if we can show that $T_{\bar{f}}F_\delta \overset{S^*}{\to} T_{\bar{f}}F$, $\mathrm{emb}\,(T_{\bar{F}_\delta}f) \overset{S^*}{\to} \mathrm{emb}\,(T_{\bar{F}}f)$ if $\delta \downarrow 0$. We note therefore that $F_\delta \overset{S^*}{\to} F$ if $\delta \downarrow 0$, so, by 3.7 remark 1, we have $T_{\bar{f}}F_\delta \overset{S^*}{\to} T_{\bar{f}}F$ if $\delta \downarrow 0$. Furthermore we have $(T_{\bar{F}_\delta}f)(y) \to (T_{\bar{F}}f)(y)$ if $\delta \downarrow 0$ for every $y \in \mathbb{C}$, and it is easily proved now with the aid of lemma 5.2 and 1.9 that $\mathrm{emb}\,(T_{\bar{F}_\delta}f) \overset{S^*}{\to} \mathrm{emb}\,(T_{F}f)$ if $\delta \downarrow 0$. □

REMARK. Note that not every element of $\mathscr{C}$ can be obtained as the product of a $g \in S$ and a $G \in S^*$ (cf. theorem 5.3), and so not every element of $\mathscr{M}$ can be obtained as $T_{\bar{f}}F$ with some $f \in S$, $F \in S^*$.

EXAMPLE. $k := \mathrm{emb}\,(\curlyvee_{z\in\mathbb{C}} e^{\pi i z^2}) \in \mathscr{C}$ cannot be of the form $g \cdot G$ with some $g \in S$, $G \in S^*$. For if so, then we consider the sequence $(f_n)_{n\in\mathbb{N}}$ defined by $f_n := \curlyvee_{z\in\mathbb{C}} e^{\pi i z^2 - \pi(z+n)^2}$ $(n \in \mathbb{N})$. Now we have $f_n \cdot \bar{g} \overset{S}{\to} 0$, but $(f_n \cdot \bar{g}, G) = (f_n, g \cdot G) = (f_n, k) = 1$ $(n \in \mathbb{N})$.

5.6. We are going to prove the statement at the end of 3.10. With the notation used there we have to show that $F \in \mathscr{C}$ in case $K_F g \in \mathrm{emb}\,(S)$ for every $g \in S$. Let $f \in S$, $g \in S$. We have $(K_F g, f) = (E \otimes \bar{F}, Z_U(f \otimes \bar{g}))$ by definition and 1.18, and it is not hard to see from [B], (21.4) that $(E \otimes \bar{F}, Z_U(f \otimes \bar{g})) = (\bar{G}, T_{g\_}f)$, where G is the generalized function that satisfies $(G, h) = (F, \curlyvee_x \sqrt{2}h(x\sqrt{2}))$ for $h \in S$ (cf. 1.13). Hence, by 3.7, remark 1 and 5.5,

$$(K_F g, f) = (\bar{G}, T_{g\_}f) = (T_{\bar{g}}\bar{G}, f) = (\mathrm{emb}\,(T_G g), f).$$

This means that $K_F g = \mathrm{emb}\,(T_G g)$, and by analyticity of $T_G g$ we conclude that $T_G g \in S$. Hence $G \in \mathscr{C}$, and so $F \in \mathscr{C}$.

5.7. We make some remarks on convergence of convolution operators.

DEFINITION. Let $f_n \in \mathscr{C}$ $(n \in \mathbb{N})$, $f \in \mathscr{C}$. We write $f_n \overset{C}{\to} 0$ if $T_{f_n}g \overset{S}{\to} 0$ for every $g \in S$; we write $f_n \overset{C}{\to} f$ if $f_n - f \overset{C}{\to} 0$.

If $f_n \in \mathscr{C}$ $(n \in \mathbb{N})$, then the following statements are equivalent.

(i) $f_n \overset{C}{\to} 0$,
(ii) $\bar{f}_n \overset{C}{\to} 0$,
(iii) $(f_n)_- \overset{C}{\to} 0$,
(iv) $\forall_{F\in S^*}[T_{f_n}F \overset{S^*}{\to} 0]$,

(v) $\forall_{g\in S}[\mathrm{emb}^{-1}(\mathscr{F}f_n) \cdot g \overset{S}{\to} 0]$,
(vi) $\forall_{F\in S^*}[\mathrm{emb}^{-1}(\mathscr{F}f_n) \cdot F \overset{S^*}{\to} 0]$.

The proofs are almost trivial.

5.8. THEOREM. Let $f_n \in \mathscr{C}$ $(n \in \mathbb{N})$. We have

$$\exists_{f\in C}[f_n \overset{C}{\to} f] \Leftrightarrow \forall_{g\in S}[(T_{f_n}g)_{n\in\mathbb{N}} \text{ is } S\text{-convergent}].$$

PROOF. If $f \in \mathscr{C}$ is such that $f_n \overset{C}{\to} f$, then we have $T_{f_n}g - T_f g = T_{f_n - f}g \overset{S}{\to} 0$ for every $g \in S$.

Now assume that $(T_{f_n}g)_{n\in\mathbb{N}}$ is S-convergent for every $g \in S$. Denote $Tg = \lim_{n\to\infty} T_{f_n}g$ for $g \in S$. It follows from [J], appendix 1, 2.12 that T is a continuous linear operator of S, and $TT_a = T_aT$ for every $a \in \mathbb{R}$. So by 3.11, there is an $f \in \mathscr{C}$ such that $T = T_f$. It follows at once that $f_n \overset{C}{\to} f$. □

5.9. THEOREM. Let $f \in \mathscr{C}$, $f_n \in \mathscr{C}$ $(n \in \mathbb{N})$, $g \in \mathscr{C}$, $g_n \in \mathscr{C}$ $(n \in \mathbb{N})$, and assume that $f_n \overset{C}{\to} f$, $g_n \overset{C}{\to} g$. Then $f_n * g_n \overset{C}{\to} f * g$ (cf. 3.9, remark 4).

PROOF. Let $u \in S$, and denote

$$h_n := \mathrm{emb}^{-1}(\mathscr{F}f_n) \quad (n \in \mathbb{N}), \quad k_n := \mathrm{emb}^{-1}(\mathscr{F}\bar{g}_n) \quad (n \in \mathbb{N}),$$
$$h := \mathrm{emb}^{-1}(\mathscr{F}f), \quad k := \mathrm{emb}^{-1}(\mathscr{F}\bar{g}), \quad v := \mathscr{F}u.$$

By 5.7(v) it suffices to show that $h_n \cdot k_n \cdot v \overset{S}{\to} h \cdot k \cdot v$. Therefore we note that $h_n \cdot k_n \cdot v \overset{S}{\to} h \cdot k \cdot v$ pointwise, and that there is an $M > 0$, $A > 0$, $B > 0$ such that

$$|h_n(z)k_n(z)v(z)| \leq M \exp(-\pi A(\mathrm{Re}\, z)^2 + \pi B(\mathrm{Im}\, z)^2) \quad (z \in \mathbb{C},\ n \in \mathbb{N})$$

as one easily sees from the fact that $k_n \cdot v \overset{S}{\to} k \cdot v$, $h_n \cdot w \overset{S}{\to} h \cdot w$ $(w \in S)$. So, by 1.8, $h_n \cdot k_n \cdot v \overset{S}{\to} h \cdot k \cdot v$. □

REMARK. The above theorem is a special case of the following one. If $(T_n)_{n\in\mathbb{N}}$, $(U_n)_{n\in\mathbb{N}}$ are sequences of continuous linear operators of S for which $(T_n g)_{n\in\mathbb{N}}$, $(U_n g)_{n\in\mathbb{N}}$ are S-convergent for every $g \in S$, then

$$T := \curlyvee_{g\in S} \lim_{n\to\infty} T_n g, \quad U := \curlyvee_{g\in S} \lim_{n\to\infty} U_n g$$

are continuous linear operators of S, and we have $T_n U_n g \overset{S}{\to} TUg$ for every $g \in S$.

5.10. EXAMPLES

(i) If $(f_n)_{n\in\mathbb{N}}$ is an S-convergent sequence in S, then $(\mathrm{emb}\,(f_n))_{n\in\mathbb{N}}$ is an $\mathscr{C}$-convergent sequence in $\mathscr{C}$. If $(g_n)_{n\in\mathbb{N}}$ is a $\mathscr{C}$-convergent sequence in $\mathscr{C}$, then $(g_n)_{n\in\mathbb{N}}$ is an S^*-convergent sequence in S^*.
(ii) If $f \in \mathscr{C}$, then $\mathrm{emb}\,(N_\alpha f) \overset{C}{\to} f$ if $\alpha \downarrow 0$ (we have of course a similar

definition of $\mathscr{C}$-convergence for this case as in 5.7). This is lemma 3.6.

(iii) If $(d_n)_{n\in\mathbf{Z}}$ is a complex sequence satisfying $d_n = 0(e^{-\varepsilon n^2})$ $(n \in \mathbf{Z})$ for some $\varepsilon > 0$, then $\sum_{n=-\infty}^{\infty} d_n \delta_n$ is a $\mathscr{C}$-convergent series in the sense that

$$\Upsilon_{x\in\mathbf{C}} \sum_{n=-N}^{M} d_n g(n+x) \overset{S}{\to} \Upsilon_{x\in\mathbf{C}} \sum_{n=-\infty}^{\infty} d_n g(n+x)$$

$(N \to \infty,\ M \to \infty)$ for every $g \in S$.

(iv) If $g_\gamma := \Upsilon_{z\in\mathbf{C}}\, \gamma^{\frac{1}{2}} \exp(-\pi\gamma z^2)$ $(\gamma > 0)$, then $\operatorname{emb}(g_\gamma) \overset{C}{\to} \delta_0$ $(\gamma \to \infty)$. If $h_\tau := 1/2\tau\, \chi_{(-\tau,\tau)}$ $(\tau > 0)$, then $\operatorname{emb}(h_\tau) \overset{C}{\to} \delta_0$ $(\tau \downarrow 0)$. More generally: if $h \in \mathscr{C}$, and $(\operatorname{emb}^{-1}(\mathscr{F}h))(0) = 1$, then $V_\lambda h \overset{C}{\to} \delta_0$ $(\lambda \to \infty)$, where $V_\lambda h$ is the generalized function that satisfies

$$(V_\lambda h, f) = (h, \Upsilon_{x\in\mathbf{C}} f(x/\lambda)) \quad (f \in S) \text{ for } \lambda > 0$$

(cf. 1.13). This may be proved by using $\mathscr{F} V_\lambda = \lambda^{-1} V_{\lambda^{-1}} \mathscr{F}$ $(\lambda > 0)$, the equivalence of 5.7(i) and 5.7(v), and 1.8.

(v) Let $g \in S$, $g_n \in S$ $(n \in \Omega)$, $G \in S^*$, $G_n \in S^*$ $(n \in \Omega)$, and assume that $g_n \overset{S}{\to} g$, $G_n \overset{S^*}{\to} G$. Then $g_n \cdot G_n \overset{C}{\to} g \cdot G$. For it easily follows from 1.9 that $g_n \cdot G_n \overset{S^*}{\to} g \cdot G$. So if $f \in S$, then we have $T_{g_n \cdot G_n} f \to T_{g \cdot G} f$ pointwise, and it may be proved from lemma 5.2 that $T_{g_n \cdot G_n} f \overset{S}{\to} T_{g \cdot G} f$.

We also have: if $g \in S$, $G \in S^*$, then $g \cdot G(\alpha) \overset{C}{\to} g \cdot G$ $(\alpha \downarrow 0)$.

5.11. We finally make some remarks about convolution theory for (generalized) functions of several variables. It is possible to develop the theory as it is presented here almost entirely for the more dimensional case (an exception should be made for the results of 4.12). We shall restrict ourselves here to the case of functions of two variables.

The definition of T_K with $K \in S^{2*}$ becomes

$$T_K f = \Upsilon_{(x,y)\in\mathbf{C}^2} (T_x \otimes T_y f, K_-) \quad (f \in S),$$

where K_- is the generalized function $\Upsilon_{\alpha>0} \Upsilon_{(z,w)\in\mathbf{C}^2} (N_{\alpha,2} K)\ (-z, -w)$ (cf. 1.16). In order to prove the two-dimensional version of theorem 3.3, we can use a theorem of Hartogs ([BT], Ch. III, § 4, Satz 15) about the analyticity of functions of several variables.

We introduce the set $\mathscr{C}^2$ as the class of all generalized functions K for which T_K maps S^2 into itself. The crucial lemma 3.6 still holds for the present case, and its proof differs only from that of lemma 3.6 in notational respect. This enables us to prove the two-dimensional versions of the theorems of section 3 and 4. We mention in particular theorems 3.5 and 3.7 (the definition of the class $\mathscr{M}^2$ is obvious).

An important example of an element of $\mathscr{C}^2$ is the tensor product of two elements of $\mathscr{C}$. Let $g_1 \in \mathscr{C}$, $g_2 \in \mathscr{C}$. We claim that $g_1 \otimes g_2 \in \mathscr{C}^2$, and that $T_{g_1 \otimes g_2} = T_{g_1} \otimes T_{g_2}$ (cf. 1.17). To prove this note that

$$(\mathscr{F} \otimes \mathscr{F})(g_1 \otimes g_2) = \mathscr{F} g_1 \otimes \mathscr{F} g_2 \in \mathscr{M}^2,$$

hence, by the two-dimensional version of theorem 3.5, $g_1 \otimes g_2 \in \mathscr{C}^2$. Furthermore we have

$$T_{g_1 \otimes g_2}(h_1 \otimes h_2) = T_{g_1} h_1 \otimes T_{g_2} h_2 \text{ for } h_1 \in S,\ h_2 \in S,$$

and the proof can be completed in the style of [J], appendix 1, 2.13.

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