ON THE LOCUS AND SPREAD OF PSEUDO-DENSITY FUNCTIONS IN THE TIME-FREQUENCY PLANE

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Abstract

This paper compares various time-frequency pseudo-density functions used in signal analysis with respect to spread. Among the members of Cohen's class of pseudo-density functions satisfying the finite support property as well as Moyal's formula, the Wigner distribution is the most well-behaved one in the sense that it has the least amount of global spread around its centre of gravity. The Wigner distribution does not perform significantly better globally than the real part of Rihaczek's function; it does, though, if the global criterion is replaced by a local one, especially for signals f of the form $f(t) = a(t)\exp(2\pi\, i\varphi(t))$ where φ is a smooth real-valued function and a is a slowly varying positive function. We formulate a general principle according to which the various pseudo-density functions of f should be concentrated around the curve $(t, \varphi'(t))$, and we present a more detailed qualitative analysis of the behaviour of the Wigner distribution of f around this curve.

1. Introduction and announcement of results

It has frequently been observed that a tool is needed for accurately describing signals in time and frequency simultaneously. This is for example the case for a signal whose spectral content varies rapidly in time. Then the usual methods of Fourier analysis fail since the time windows needed are either too long to follow the transient behaviour of the signal adequately or too short to give satisfactory resolution in frequency. In the past, several time-frequency transformations have been proposed; well-known in this respect are Rihaczek's function and its real part, and the Wigner distribution [20,23,25]). These time-frequency transformations are supposed to yield density functions indicating how the signal energy is distributed over time and frequency. The interpretation as density function is seriously hampered, though, by the fact that the time-frequency functions considered take in general negative, and sometimes even complex, values. This is the reason why we speak of time-frequency pseudo-density functions. We compare the various possibilities by verifying whether for certain test signals a particular time-frequency transformation yields density functions concentrated mainly in those regions in the time-frequency plane where, according to intuition or experience, the signal energy is expected to be located. As test signals we consider in this paper what we may call concentrated signals and signals of the type $\exp(2\pi\, i\varphi(t))$ with φ a smooth, real-valued function. The first type consists of signals that can be considered as having their signal energy concentrated around a point in the plane (e.g. $\exp(-\pi(t-a)^2 + 2\pi\, \mathrm{i} bt)$). The signal energy of $\exp(2\pi\, \mathrm{i}\varphi(t))$ is, according to intuition, concentrated around the curve $(t, \varphi'(t))$ of instantaneous frequencies. For the first kind of signals the relevant quantity to consider is a sort of spread of the various density functions around the appropriate point. For the second type one may consider the spread of the density functions in frequency direction at time t around the instantaneous frequency $\varphi'(t)$. This leads to what we call a global and a local notion of spread. Before we describe this in detail, though, we shall consider a general class of time-frequency pseudo-density functions.

In ref. 9, part III, the Wigner distribution is compared with several other time-frequency pseudo-density functions, in particular with those introduced and studied by Cohen and Margenau[11,21]). These density functions are defined as follows. Let Φ be a reasonably behaved complex-valued function of two real variables. Then for any reasonably behaved complex-valued function f of one real variable one defines $H_{f,f}^{(\Phi)}$ by°)

$$H_{f,f}^{(\Phi)}(t,\omega) = \int\int\int \exp(-2\pi\, \mathrm{i}\,(\theta t + \tau\omega - \theta u))\, \Phi(\theta,\tau) \times f(u + \tfrac{1}{2}\tau)\,\overline{f(u - \tfrac{1}{2}\tau)}\, \mathrm{d}\theta\, \mathrm{d}\tau\, \mathrm{d}u \qquad (1)$$

for $t \in \mathbb{R}$, $\omega \in \mathbb{R}$.

It can be checked that

$$H_{T_a f,\, T_a f}^{(\Phi)}(t,\omega) = H_{f,f}^{(\Phi)}(t+a,\omega),\quad H_{R_b f,\, R_b f}^{(\Phi)}(t,\omega) = H_{f,f}^{(\Phi)}(t,\omega+b) \qquad (2)$$

for all $a \in \mathbb{R}$, $b \in \mathbb{R}$ and all f.
Here $(T_a f)(t) = f(t+a)$, $(R_b f)(t) = \exp(-2\pi\, \mathrm{i} bt) f(t)$.

In order that an $H_{f,f}^{(\Phi)}$ deserves to be called a time-frequency density function, certain requirements must be made. In ref. 9, part III, sec. 3, one requires that for all f

$$\int H_{f,f}^{(\Phi)}(t,\omega)\, \mathrm{d}\omega = |f(t)|^2 \quad (t \in \mathbb{R}), \qquad (3)$$

$$\int H_{f,f}^{(\Phi)}(t,\omega)\, \mathrm{d}t = |(\mathscr{F}f)(\omega)|^2 \quad (\omega \in \mathbb{R}), \qquad (4)$$

$$\int \omega\, H_{f,f}^{(\Phi)}(t,\omega)\, \mathrm{d}\omega \Big/ \int H_{f,f}^{(\Phi)}(t,\omega)\, \mathrm{d}\omega = \frac{1}{2\pi} \operatorname{Im} \frac{\mathrm{d}}{\mathrm{d}t} \ln f(t) \quad (t \in \mathbb{R}), \qquad (5)$$

$$\int t\, H_{f,f}^{(\Phi)}(t,\omega)\, \mathrm{d}t \Big/ \int H_{f,f}^{(\Phi)}(t,\omega)\, \mathrm{d}t = -\frac{1}{2\pi} \operatorname{Im} \frac{\mathrm{d}}{\mathrm{d}\omega} \ln \mathscr{F}f(\omega) \quad (\omega \in \mathbb{R}), \qquad (6)$$

$$f(t) = 0 \quad (|t| > T) \Rightarrow H_{f,f}^{(\Phi)}(t,\omega) = 0 \quad (|t| > T,\ \omega \in \mathbb{R}), \qquad (7)$$

$$(\mathscr{F}f)(\omega) = 0 \quad (|\omega| > \Omega) \Rightarrow H_{f,f}^{(\Phi)}(t,\omega) = 0 \quad (t \in \mathbb{R},\ |\omega| > \Omega). \qquad (8)$$

°) $\int$ denotes integration over the whole real line; our normalizations differ from those used in ref. 9 in such a way that the Fourier inversion formula takes a completely symmetric form.

Here $\mathscr{F}f$ is the Fourier transform of f, and T and Ω are positive real numbers. Of course, proper assumptions on f and Φ should be made so that the formulas (3)-(8) make sense.

Conditions (3) and (4) are quite natural requirements; conditions (5) and (6) have to do with the expected instantaneous frequency of f and with the group delay of a linear time-invariant system with impulse-response f respectively; conditions (7) and (8) are called the finite support properties in ref. 9, part III.

It can be shown (ref. 9, part III) that

(3) holds for all f if and only if $\Phi(\theta,0) = 1$ for all θ,
(4) holds for all f if and only if $\Phi(0,\tau) = 1$ for all τ,
(5) holds for all f if and only if $\Phi(\theta,0) = 1$, $\Phi_\tau(\theta,0) = 0$ for all θ,
(6) holds for all f if and only if $\Phi(0,\tau) = 1$, $\Phi_\theta(0,\tau) = 0$ for all τ,
(7) holds for all f and all $T > 0$ if and only if, for any τ, $\Phi(\cdot,\tau)$ is entire with $\Phi(z,\tau) = 0(\exp(\pi(|\tau| + \varepsilon)|z|))$ for all $\varepsilon > 0$,
(8) holds for all f and all $\Omega > 0$ if and only if, for any θ, $\Phi(\theta,\cdot)$ is entire with $\Phi(\theta,z) = 0(\exp(\pi(|\theta| + \varepsilon)|z|))$ for all $\varepsilon > 0$.

A further restriction, which is quite convenient for this paper and natural in quantum mechanics (but perhaps not really necessary for signal analysis), is the validity of Moyal's formula (ref. 8, Theorem 14.2 and 27.15),

$$\int\int H_{f,f}^{(\Phi)}(t,\omega)\, \overline{H_{g,g}^{(\Phi)}(t,\omega)}\, \mathrm{d}t\, \mathrm{d}\omega = \left|\int f(t)\, \overline{g(t)}\, \mathrm{d}t\right|^2 \qquad (9)$$

for all f and all g. We shall show that (9) holds for all f and g if and only if $|\Phi(\theta,\tau)| = 1$ for all θ and all τ, and also that validity of (3), (4), (7), (8) and (9) for all f and g implies that Φ is of the form $\Phi(\theta,\tau) = \exp(2\pi\, \mathrm{i}\alpha\theta\tau)$ with $\alpha \in \mathbb{R}$, $|\alpha| \leqslant \frac{1}{2}$.

Requirements of a more restrictive type than (9) are (a) for all f and all g we have

$$H_{f*g,\, f*g}^{(\Phi)} = H_{f,f}^{(\Phi)} *^t H_{g,g}^{(\Phi)}$$

and (b) for all f and all g we have

$$H_{f\cdot g,\, f\cdot g}^{(\Phi)} = H_{f,f}^{(\Phi)} *^\omega H_{g,g}^{(\Phi)}.$$

Here $*$ and $\cdot$ denote ordinary convolution and multiplication, and $*^t$ and $*^\omega$ denote convolution in the time and frequency domain respectively. Although neither of these conditions are equivalent with validity of Moyal's formula for all f and all g, it can be shown that (3), (4), (7), (8) together with (a) or (b) also implies that Φ is of the form just given. The proofs follow the same lines as the one presented here for the case that (9) (instead of (a) or (b)) is assumed to hold. Observe that (3) and (a) imply that for all f and g

$$\int\int H_{f,f}^{(\Phi)}(t,\omega)\, H_{g,g}^{(\Phi)}(-t,\omega)\, \mathrm{d}t\, \mathrm{d}\omega = \left|\int f(t)\, g(-t)\, \mathrm{d}t\right|^2, \qquad (10)$$

and that (4) and (b) imply that for all f and g

$$\iint H_{f,f}^{(\Phi)}(t,\omega)\, H_{g,g}^{(\Phi)}(t,-\omega)\,\mathrm{d}t\,\mathrm{d}\omega = \left|\int f(t)\, g(t)\,\mathrm{d}t\right|^2. \tag{11}$$

The class of pseudo-density functions $H_{f,f}^{(\alpha)}$ obtained by taking

$$\Phi(\theta,\tau) = \exp(2\pi\,\mathrm{i}\alpha\,\theta\,\tau)$$

in (1) for all f contains e.g. the Wigner distribution ($\alpha = 0$) and Rihaczek's function[23] $\overline{f(t)}\,(\mathscr{F}f)(\omega)\exp(2\pi\,\mathrm{i}t\omega)$ ($\alpha = -\frac{1}{2}$), but not the real part of the latter which is considered sometimes (ref. 9, part III, (3.21)).

We demonstrate that, among the members of this restricted class, the Wigner distribution has the least amount of spread in the sense that for any f and any $(t_0,\omega_0)\in\mathbb{R}^2$ the global spread $\sigma_f^{(\alpha)}(t_0,\omega_0)$ of $|H_{f,f}^{(\alpha)}|^2$ around (t_0,ω_0), defined by

$$\sigma_f^{(\alpha)}(t_0,\omega_0) = \iint [(t-t_0)^2 - (\omega-\omega_0)^2]\,|H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega \tag{12}$$

is minimal for $\alpha = 0$. We consider (12) and not e.g.

$$\iint [(t-t_0)^2 + (\omega-\omega_0)^2]\, H_{f,f}^{(\alpha)}(t,\omega)\,\mathrm{d}t\,\mathrm{d}\omega \tag{13}$$

since $H_{f,f}^{(\alpha)}$ takes negative (even complex) values so that, by cancellations in the integral in (13), we may get a wrong impression of the amount of spread. Also, it can be shown that (13) does not depend on α; it is therefore of no use for comparing the $H_{f,f}^{(\alpha)}$'s.

The quantity of $\sigma_f^{(\alpha)}(t_0,\omega_0)$ is not particularly useful for e.g. signals f of the form $f(t) = a(t)\exp(2\pi\,\mathrm{i}\,\varphi(t))$, where φ is a real-valued, smooth phase function and a is a positive slowly varying amplitude function. Intuition tells us that a pseudo-density function for such an f should be concentrated around the curve $(t,\varphi'(t))$ (also cf. (5)), while $\sigma_f^{(\alpha)}(t_0,\omega_0)$ indicates how well $H_{f,f}^{(\alpha)}$ is concentrated around (t_0,ω_0). A more useful notion for such signals is the local spread $\sigma_{f,t_0}^{(\alpha)}(\omega_0)$ of $|H_{f,f}^{(\alpha)}|^2$ at time t_0 around frequency ω_0, defined by

$$\sigma_{f,t_0}^{(\alpha)}(\omega_0) = \int (\omega-\omega_0)^2\,|H_{f,f}^{(\alpha)}(t_0,\omega)|^2\,\mathrm{d}\omega, \tag{14}$$

in particular when ω_0 equals the instantaneous frequency $\varphi'(t_0)$.

Since the computations get rather involved for the general case, we shall mainly consider the cases $\alpha = 0$, $\alpha = -\frac{1}{2}$ and the case with $\mathrm{Re}\,H_{f,f}^{(-\frac{1}{2})}$ instead of $H_{f,f}^{(\alpha)}$ in (14). We shall show that the Wigner distribution performs more than marginally better than the other pseudo-density functions, especially when the curve $(t,\varphi'(t))$ can be approximated satisfactorily by a straight line through $(t_0,\varphi'(t_0))$.

Let us summarize the further content of this paper. In sec. 2 we consider the pseudo-density functions of Cohen in the context where they arose, viz. quantum mechanics. In sec. 3 we prove the claim made in connection with Moyal's formula (9) about the global spread. In sec. 4 we formulate a general principle for locating time-frequency pseudo-density functions of functions f that satisfy an equation $Tf = 0$ with T a self-adjoint operator. In sec. 5 we prove the assertions made above about the spread of the various pseudo-density functions, and in sec. 6 we present more details for the behaviour of Wigner distributions of functions f of the form $f(t) = \exp(2\pi\,\mathrm{i}\,\varphi(t))$ along the curve $(t,\varphi'(t))$. We have tried to organize the paper in such a way that the theoretical part (secs 2, 3 and 4) and the more practical part (secs 5 and 6) can be read more or less independently of each other. Finally a remark about rigor: we present things in a rather informal way, although much of the paper can be put into a rigorous form.

2. Time-frequency pseudo-density functions and quantum mechanics

The history of pseudo-density functions of two variables starts in 1932 with the paper by Wigner, see ref. 26. In this paper Wigner introduced what is now called the Wigner distribution as a device that allows one to express quantum mechanical expectation values in the same mathematical form as the averages of classical statistical mechanics. This line of development was taken up again in the late forties notably by Groenewold[13]) and Moyal[22]). Since then many papers have appeared on the role of the Wigner distribution and other pseudo-density functions in quantum mechanics; see e.g. the papers by Berry[5]), Cohen and Margenau[11,21]) and the references therein. For a good understanding it is necessary to indicate how pseudo-density functions arose in quantum mechanics.

In classical mechanics the motion of a particle on the real line is described by specifying at each moment t the position $q(t)$ and momentum $p(t)$ of the particle (to avoid confusion with the notion of time used in the introduction we note that it is more proper to identify the notions time and frequency of signal analysis with the notions position and momentum of quantum mechanics). The particle moves according to the Hamilton equations

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q},$$

where $H = H(q,p)$ is the Hamiltonian which we assume to be time-independent.

In quantum mechanics a particle is described by specifying at each moment t a so-called wave function, or state, $\psi_t(q)$ with $\int|\psi_t(q)|^2\,\mathrm{d}q = 1$. The probability that the particle lies, at time t, between a and b equals

$$\int_a^b |\psi_t(q)|^2 \, dq,$$

and the probability that the momentum of the particle, at time t, has a value between c and d equals

$$\int_c^d |(\mathscr{F}_h \psi_t)(p)|^2 \, dp.$$

Here $(\mathscr{F}_h f)(p) = h^{-\frac{1}{2}} \int \exp(-2\pi\, i q p/h) f(q) \, dq$, and h is Planck's constant. The motion of the particle is described now by Schrödinger's equation

$$\frac{\partial}{\partial t} \psi_t(q) = i(\mathscr{H}\psi_t)(q),$$

where $\mathscr{H}$ is an operator acting on functions of q and that replaces in some sense the Hamiltonian H (we give more details below).

In classical mechanics an observable, a, is any function of q and p. In quantum mechanics an observable is represented by an operator A; the role of the function values $a(q,p)$ is taken over by the values assumed by $(A\psi, \psi)$. Here (ψ, φ) denotes the usual inner product $\int \psi(q)\overline{\varphi(q)}\, dq$ of two square integrable functions ψ and φ. Thus, if $\psi = \psi_t$ and the observable a is represented by the operator A, $(A\psi, \psi)$ is the expectation value of a for the particle at time t. The problem is to assign properly quantum mechanical operators to the observables from classical mechanics. For observables depending on q or p only this is not so hard. For example, the quantum mechanical version of the observable "position" $a(q,p) = q$ is the operator Q defined by $(Q\psi)(q) = q\,\psi(q)$ (if ψ_t is concentrated in a small interval around a point q_t, then $(Q\psi_t, \psi_t) = \int q |\psi_t(q)|^2 dq$ will also lie in that interval), and for the observable "momentum" $a(q,p) = p$ we have the operator P defined by $(P\psi)(q) = h\psi'(q)/2\pi i$ (if $\mathscr{F}_h\psi_t$ is concentrated in a small interval around a point p_t, then $(P\psi_t, \psi_t) = \int (h\psi_t'(q)/2\pi i)\overline{\psi_t(q)}\, dq = \int p |(\mathscr{F}_h \psi_t)(p)|^2 dp$ will also lie in that interval). Thus the operator A corresponding to $a(q,p) = f(q)$ and $a(q,p) = g(p)$ are given by

$$(A\psi)(q) = f(q)\,\psi(q) \quad \text{and} \quad (A\psi)(q) = \mathscr{F}_h^{-1}(g \cdot \mathscr{F}_h \psi)(q)$$

respectively.

To handle more general observables a correspondence principle between functions of q and p and operators of $L^2(\mathbb{R})$ (= the set of square integrable functions defined on $\mathbb{R}$) needs to be formulated. The first step in setting up such a principle consists in defining for any $\psi \in L^2(\mathbb{R})$ a kind of density function $H_{\psi,\psi}(q,p)$ that indicates the location of a particle described by ψ as far

as position and momentum are concerned. A rather obvious condition is that $H_{\psi,\psi}$ gives the "correct" values for the moments of functions of q and p alone. More specifically, one requires that [24]),

$$\iint a(q)\, \overline{H_{\psi,\psi}(q,p)}\, dq\, dp = \int a(q) |\psi(q)|^2 \, dq, \tag{15}$$

$$\iint b(p)\, \overline{H_{\psi,\psi}(q,p)}\, dq\, dp = \int b(p) |\mathscr{F}_h \psi)(p)|^2 \, dp \tag{16}$$

for all sufficiently well-behaved functions a and b. Note that, apart from the overhead bar, (15) and (16) coincide with the conditions (3) and (4) for $H_{f,f}^{(\Phi)}$.

A fairly general way °) to get functions $H_{\psi,\psi}$ such that (15) and (16) hold for all ψ is by taking $\varphi = \psi$ in

$$H_{\psi,\varphi}^{(\Phi)}(q,p) = \iiint \exp(-2\pi\, i\,(\theta q + \tau p - \theta u))\, \Phi(\theta, \tau) \times \psi(u + \tfrac{1}{2}\tau h)\, \overline{\varphi(u - \tfrac{1}{2}\tau h)}\, d\theta\, d\tau\, du. \tag{17}$$

Here Φ is a reasonably behaved function of two variables with

$$\Phi(0, \tau) = \Phi(\theta, 0) = 1 \text{ [11,19]}).$$

The next step in setting up the correspondence principle is to associate with a function a of two variables a linear operator A by requiring that

$$(A\psi, \psi) = \iint a(q,p)\, \overline{H_{\psi,\psi}(q,p)}\, dq\, dp \tag{18}$$

for all $\psi \in L^2(\mathbb{R})$. That is, instead of substituting a particular value (q,p) in a, one integrates a against the conjugate $\overline{H}_{\psi,\psi}$ of the pseudo-density function.

The various choices of Φ in (17) give rise to a great number of correspondence rules. The best known rule of these is Weyl's rule where one takes $\Phi \equiv 1$. One also uses $\Phi(\theta,\tau) = \exp(-\pi\, i\, \theta\tau)$ or $\Phi(\theta,\tau) = \cos \pi\theta\tau$ (called standard rule in ref. 2 and symmetrization rule in ref. 11 respectively).

An important reason why Weyl's rule is popular is mathematical elegance. A further important advantage is (quoting Hörmander in ref. 15) that in the calculus based on Weyl's rule there is closer agreement between composition of linear operators and pointwise multiplication of the functions they correspond to than in the calculus based on the rule with $\Phi(\theta,\tau) = \exp(-\pi\, i\, \theta\tau)$. There is, furthermore, in Weyl's rule definitely a relation between the notions of positivity of operators and positivity of the functions they correspond to. We refer to refs 7 and 16 where some results in this direction can be found.

°) The claim in ref. 21 that (17) gives all $H_{\psi,\psi}$ satisfying (15) and (16) for all ψ is not correct; for this to be the case the function Φ should also depend on q and p.

3. Time-frequency pseudo-density functions and Moyal's formula

In this section we prove the claim made in connection with (9). Adopting the notation of the introduction again we assume that we have a reasonably behaved function Φ in (1) such that (3), (4), (7), (8) and (9) hold for all f and g for which these formulas makes sense. It is already known that $\Phi(\theta,0) = \Phi(0,\tau) = 1$, and that $\Phi(\theta,\cdot)$ and $\Phi(\cdot,\tau)$ are entire functions with certain restrictions on growth for all θ and all τ because of (7) and (8).

We shall first show that $|\Phi(\theta,\tau)| = 1$ for all θ and all τ. To that end we apply (9) to $f = zf_1 + wf_2$ and g. By using

$$H^{(\Phi)}_{zf_1+wf_2,\,zf_1+wf_2} = |z|^2 H^{(\Phi)}_{f_1,f_1} + z\bar{w}\, H^{(\Phi)}_{f_2,f_2} + + w z\, H^{(\Phi)}_{f_2,f_1} + |w|^2 H^{(\Phi)}_{f_2,f_2} \tag{19}$$

and equating coefficients we get

$$\iint H^{(\Phi)}_{f_1,f_2}(t,\omega)\,\overline{H^{(\Phi)}_{g,g}(t,\omega)}\,\mathrm{d}t\,\mathrm{d}\omega = (f_1,g)\overline{(f_2,g)} \tag{20}$$

for all f_1, f_2 and g. Applying (20) with f_1, f_2 and $g = zg_1 + wg_2$ we get in a similar way

$$\iint H^{(\Phi)}_{f_1,f_2}(t,\omega)\,\overline{H^{(\Phi)}_{g_1,g_2}(t,\omega)}\,\mathrm{d}t\,\mathrm{d}\omega = (f_1,g_1)\overline{(f_2,g_2)} \tag{21}$$

for all f_1, $g_1 f_2$ and g_2.

Define a mapping T: $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by

$$(TF)(\theta,\tau) = \int \exp(2\pi\,\mathrm{i}\,\theta\,u)\,F(u + \tfrac{1}{2}\tau, u - \tfrac{1}{2}\tau)\,\mathrm{d}u. \tag{22}$$

It is not hard to see that T maps $L^2(\mathbb{R}^2)$ onto $L^2(\mathbb{R}^2)$, that it preserves inner products in $L^2(\mathbb{R}^2)$, and that for any f_1, f_2

$$H^{(\Phi)}_{f_1,f_2}(t,\omega) = \iint \exp(-2\pi\,\mathrm{i}(\theta\,t + \tau\,\omega))\,(T(f_1 \otimes \bar{f}_2))(\theta,\tau)\,\Phi(\theta,\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau. \tag{23}$$

Here $(f_1 \otimes \bar{f}_2)(t,s) = f_1(t)\overline{f_2(s)}$. Now by Parseval's formula we can rewrite (21) as

$$\iint |\Phi(\theta,\tau)|^2\, T(f_1 \otimes \bar{f}_2)(\theta,\tau)\,\overline{T(g_1 \otimes \bar{g}_2)(\theta,\tau)}\,\mathrm{d}\theta\,\mathrm{d}\tau = (f_1,g_1)\,\overline{(f_2,g_2)}. \tag{24}$$

Since T preserves inner products, we may write (24) as

$$\iint (|\Phi(\theta,\tau)|^2 - 1)\,T(f_1 \otimes \bar{f}_2)(\theta,\tau)\,\overline{T(g_1 \otimes \bar{g}_2)(\theta,\tau)}\,\mathrm{d}\theta\,\mathrm{d}\tau = 0. \tag{25}$$

If (ψ_n) is a complete orthonormal system for $L^2(\mathbb{R})$, then the properties of T ensure that $(\varphi_{k,l})$ is a complete orthonormal system for $L^2(\mathbb{R}^2)$. Here $\varphi_{k,l} = T(\psi_k \otimes \bar{\psi}_l)$. Hence $\iint (|\Phi(\theta,\tau)|^2 - 1)\,\varphi_{k,l}(\theta,\tau)\,\overline{\varphi_{n,m}(\theta,\tau)}\,\mathrm{d}\theta\,\mathrm{d}\tau = 0$ for all k,l,n,m, and this implies that $|\Phi(\theta,\tau)|^2 - 1 = 0$, i.e. $|\Phi(\theta,\tau)| = 1$ for all θ and all τ.

We must show now that $\Phi(\theta,\tau) = \exp(2\pi\,\mathrm{i}\,\alpha\theta\tau)$ for some $\alpha \in \mathbb{R}$, $|\alpha| \leqslant \frac{1}{2}$. Fix $\theta \in \mathbb{R}$. We know that $\Phi(\theta,\cdot)$ is an entire function, and that $\Phi(\theta,z) = 0(\exp(\pi(|\theta| + \varepsilon)|z|))$ for all $\varepsilon > 0$. Hence $\Phi(\theta,\cdot)$ is regular in the upper half plane, continuous in the closed half plane, its zeros z_n in the upper half plane have no finite limit points,

$$\liminf_{r\to\infty} f\,(r^{-1} \log_{\max} |z| = r,\quad \operatorname{Im} z \geqslant 0,\ |\Phi(\theta,z)|) < \infty,$$

and we also have $\int [\log_+ |\Phi(\theta,z)|]/(1 + z^2)\,\mathrm{d}z = 0 < \infty$ since $|\Phi(\theta,z)| = 1$ for all real z ($\log_+ a = \max(0,\log a)$ for $a > 0$). The conditions of theorem 6.5.4 in ref. 6 are satisfied, whence we have the representation

$$\log |\Phi(\theta,z)| = \log \left| \Pi_n \frac{1 - z/z_n}{1 - z/\bar{z}_n} \right| + c\,\operatorname{Im} z \tag{26}$$

in the closed upper half plane for some $c \in \mathbb{R}$.

The function

$$g(z) = \exp(-\mathrm{i}\,c\,l)\,\Pi_n \frac{1 - z/z_n}{1 - z/\bar{z}_n}$$

is regular in the closed upper half plane, and we have $|\Phi(\theta,z)| = |g(z)|$ there. Hence, as $\Phi(\theta,0) = g(0) = 1$ we must have $\Phi(\theta,z) = g(z)$ in the closed half plane. But $\Phi(\theta,\cdot)$ is an entire function, therefore so is g. We conclude that g has no poles in the lower half plane, i.e. no zeros in the upper half plane. This means that $\Phi(\theta,z) = \exp(-\mathrm{i}\,c\,z)$.

We have shown now that for every $\theta \in \mathbb{R}$ there is a $c_1(\theta) \in \mathbb{R}$ with $\Phi(\theta,\tau) = \exp(-\mathrm{i}\,c_1(\theta)\,\tau)$. In addition, $|c_1(\theta)| \leqslant \pi|\theta|$. Similarly, for every $\tau \in \mathbb{R}$ there is a $c_2(\tau) \in \mathbb{R}$ with $|c_2(\tau)| \leqslant \pi|\tau|$ such that $\Phi(\theta,\tau) = \exp(-\mathrm{i}\,c_2(\tau)\,\theta)$. We conclude easily that $\Phi(\theta,\tau) = \exp(-\mathrm{i}\,c\theta\tau)$ some $c \in \mathbb{R}$ with $|c| \leqslant \pi$, and this completes the proof.

The choice $\Phi(\theta,\tau) = \exp(2\pi\,\mathrm{i}\,\alpha\theta\tau)$ with $\alpha \in \mathbb{R}$, $|\alpha| \leqslant \frac{1}{2}$ leads to the pseudo-density function $H^{(\alpha)}_{f,f}$ which can be written in the convenient form

$$H^{(\alpha)}_{f,f}(t,\omega) = \int \exp(-2\pi\,\mathrm{i}\,s\omega)\,f(t + s(\tfrac{1}{2} - \alpha))\,\overline{f(t - s(\tfrac{1}{2} + \alpha))}\,\mathrm{d}s. \tag{27}$$

It can easily be checked directly that Moyal's formula is satisfied. Hence, the converse of the theorem proved in this section also holds. For $\alpha = \frac{1}{2}$, $-\frac{1}{2}$ (27) simplifies to $\exp(-2\pi\,\mathrm{i}\,t\omega)\,f(t)\,\overline{(\mathscr{F}f)(\omega)}$ and $\exp(2\pi\,\mathrm{i}\,t\omega)\,\overline{f(t)}\,(\mathscr{F}f)(\omega)$ respectively, and the choice $\alpha = 0$ leads to the Wigner distribution.

We finally note that the pseudo-density functions of (27) also satisfy $H^{(\alpha)}_{f,f}(\gamma t, \gamma^{-1}\omega) = H^{(\alpha)}_{z_\gamma f,\, z_\gamma f}(t,\omega)$ where $(z_\gamma f)(t) = \gamma^{\frac{1}{2}} f(\gamma t)$ for $\gamma > 0$.

4. A general principle for locating time-frequency pseudo-density functions

We take in this section $\Phi(\theta,\tau) = \exp(2\pi\,\mathrm{i}\,\alpha\theta\tau)$ with $\alpha \in \mathbb{R}$, $|\alpha| \leq \frac{1}{2}$ so that we get the pseudo-density function $H_{f,f}^{(\alpha)}$ of (27). We shall formulate a principle according to which we can get a rough idea where the $H_{f,f}^{(\alpha)}$'s of an f satisfying $Tf = 0$, with T being a self-adjoint operator are located. This principle is based on the fact that under the rule (18) of assigning linear operators to functions of two variables there is agreement to a certain extent between composition of linear operators and pointwise multiplication of the functions they correspond to.

If we take $H_{f,f} = H_{f,f}^{(\alpha)}$, the rule (18) can be reversed so as to assign functions of two variables to linear operators. Let T be a linear operator. We define the α-symbol $P_T^{(\alpha)}$ of T by requiring that

$$(Tg, f) = \iint P_T^{(\alpha)}(t,\omega)\,\overline{H_{f,g}^{(\alpha)}(t,\omega)}\,\mathrm{d}t\,\mathrm{d}\omega \tag{28}$$

for all f and all g. It can be checked, by using the fact that Moyal's formula hold for $H_{f,g}$, that $P_T^{(\alpha)}$ is given by

$$P_T^{(\alpha)}(t,\omega) = \int \exp(-2\pi\,\mathrm{i}\,s\omega)\,h_T(t + (\tfrac{1}{2} - \alpha)s,\; t - (\tfrac{1}{2} + \alpha)s)\,\mathrm{d}s, \tag{29}$$

where $h_T(u,v) = (T\delta_v)(u)$ is the kernel of T (δ_v is the delta function at v). Of course, in a rigorous approach (29) should be interpreted as an identity between distributions (see ref. 12, where the case $\alpha = 0$ is treated rigorously).

If T represents a linear system, h_T is the impulse response function of the system, and $P_T^{(\alpha)}$ can be considered as a kind of time-dependent transmission function. Indeed, if we take $\alpha = \frac{1}{2}$ we get

$$P_T^{(\frac{1}{2})}(t,\omega) = \exp(-2\pi\,\mathrm{i}\,t\omega)(Te_\omega)(t) = \exp(-2\pi\,\mathrm{i}\,t\omega)(Te_\omega, \delta_t),$$

where e_ω is defined as $e_\omega(t) = \exp(2\pi\,\mathrm{i}\,\omega t)$. This is the transmission function considered in ref. 10. And if we take $\alpha = -\frac{1}{2}$ we find

$$P_T^{(-\frac{1}{2})}(t,\omega) = \exp(2\pi\,\mathrm{i}\,t\omega)(T\delta_t, e_\omega) = \exp(2\pi\,\mathrm{i}\,t\omega)(\delta_t, T^* e_\omega) = \overline{P_{T^*}^{(\frac{1}{2})}(t,\omega)},$$

where T^* is the adjoint operator, given by $(T^*f, g) = (f, Tg)$ for all f and g.

Let us give some examples. Let T be the modulator given by $(Tf)(t) = V(t)f(t)$, where V is a well-behaved function. It can be checked from (29) that $P_T^{(\alpha)}(t,\omega) = V(t)$ for all α. And if T represents a linear time-invariant system with transmission function $g(\omega)$, then $P_T^{(\alpha)}(t,\omega) = g(\omega)$ for all α.

The following example will be used later on. Define T by $Tf = (f,\varphi)\psi$ for all f (projection operator). Then

$$P_T^{(\alpha)}(t,\omega) = H_{\psi,\varphi}^{(\alpha)}(t,\omega) \tag{30}$$

for all α. This follows easily from Moyal's formula.

As already said, there is some agreement between composition of operators T and multiplication of the $P_T^{(\alpha)}$'s. This gives a method of finding approximations for $H_{Tf,Tf}^{(\alpha)}$ in terms of $H_{f,f}^{(\alpha)}$, $P_T^{(\alpha)}$ and $P_{T^*}^{(\alpha)}$. For, according to (30), $H_{Tf,Tf}^{(\alpha)}$ can be considered as the α-symbol of the operator $g \to (g, Tf)Tf = (T^*g, f)Tf$. This operator is the composition of the operators T^*, $g \to (g,f)f$ and T. Hence, an approximation for $H_{Tf,Tf}^{(\alpha)}$ is given by $P_{T^*}^{(\alpha)} \cdot H_{f,f}^{(\alpha)} \cdot P_T^{(\alpha)}$, the product of the three α-symbols.

The case $\alpha = 0$ is especially convenient. For we have $H_{f,g}^{(0)} = \overline{H_{g,f}^{(0)}}$ follows from (26)), so that $P_{T^*}^{(0)} = \overline{P_T^{(0)}}$ (follows from (28) and the fact that $(Tg, f) = (g, T^*f)$). Hence the approximation for $H_{Tf,Tf}^{(0)}$ found above can be written as $|P_T^{(0)}|^2 H_{f,f}^{(0)}$. This is a nice result considering the fact that $P_T^{(0)}$ has an interpretation as a time-dependent transmission function and $H_{f,f}^{(0)}$ is something like a time-frequency density function.

To give an idea how this approximation result should be looked upon, we give two examples where $H_{Tf,Tf}^{(0)} = |P_T^{(0)}|^2 H_{f,f}^{(0)}$.

(1) Let T be a modulator with modulation function $V(t)$, and let $f(t) = \delta_a(t)$, the delta function at a. We know that $P_T^{(0)}(t,\omega) = V(t)$, and it can be checked that $H_{f,f}^{(0)}(t,\omega) = \delta_a(t)$. Also, $(Tf)(t) = V(t) = V(a)\delta_a(t)$ so that $H_{Tf,Tf}^{(0)}(t,\omega) = |V(a)|^2 \delta_a(t) = |P_T^{(0)}(t,\omega)|^2 H_{f,f}^{(0)}(t,\omega)$.

(2) Let T be a linear time-invariant system with transmission function g, and let $f(t) = \exp(2\pi\,\mathrm{i}\,bt)$. We know that $P_T^{(0)}(t,\omega) = g(\omega)$, and it can be checked that $H_{f,f}^{(0)}(t,\omega) = \delta_b(\omega)$. Also, $(Tf)(t) = g(b)\exp(2\pi\,\mathrm{i}\,bt)$, and as in example (1) it can be shown that $H_{Tf,Tf}^{(0)}(t,\omega) = |P_T^{(0)}(t,\omega)|^2 H_{f,f}^{(0)}(t,\omega)$.

The approximation result can also be used to find a clue as to where the Wigner distribution of a function f satisfying an equation $Tf = 0$ is located. We formulate the following principle. If T is a self-adjoint linear operator ($T = T^*$) and f is a function such that $Tf = 0$, then $H_{f,f}^{(\alpha)}$ has its main contribution near the set $\{(t,\omega) \mid P_T^{(\alpha)}(t,\omega) = 0\}$. This can be seen from the fact that $(P_T^{(\alpha)})^2 H_{f,f}^{(\alpha)}$ is an approximation for $H_{Tf,Tf}^{(\alpha)} = 0$ so that $H_{f,f}^{(\alpha)}$ can only have significant contributions in regions where $P_T^{(\alpha)}$ is small. Indeed, far away from the set $\{(t,\omega) \mid P_T^{(\alpha)}(t,\omega) = 0\}$ we shall often find that $H_{f,f}^{(\alpha)}$ is small, or that $H_{f,f}^{(\alpha)}$ is not necessarily small but rapidly oscillating, which in either case implies that integrals of $H_{f,f}^{(\alpha)}$ over regions of sufficiently large area are small. At the end of section 6 we shall give more details for a special case.

In order to indicate in what sense the above formulated principle holds we should say to what extent composition of linear operators and multiplication of symbols agree. This is a quite involved subject to which a whole branch of mathematics, the calculus of pseudo-differential operators, is devoted. For details we refer to refs 14 and 15 ($\alpha = 0$) and ref. 4 ($\alpha = -\frac{1}{2}$). We stress again the fact that validity of the principle is more convincing for the case $\alpha = 0$ than for the case $\alpha = -\frac{1}{2}$; in fact, much of this paper is meant to verify this claim.

Now let us give some examples. Consider $f(t) = \exp(2\pi i \varphi(t))$, where φ is a smooth real-valued function. This f satisfies $(1/(2\pi i)\,d/dt - \varphi'(t))f(t) = 0$. Hence $Tf = 0$, where $T = 1/(2\pi i)\,d/dt - \varphi'$. According to the examples given earlier we have $P_T^{(\alpha)}(t,\omega) = \omega - \varphi'(t)$ for all α. Hence we may expect $H_{f,f}^{(\alpha)}$ to be concentrated near the curve $(t, \varphi'(t))$ in the plane. This was verified by Rihaczek [22]) in case $\alpha = \frac{1}{2}$, and we will verify it in the next sections, mainly for $\alpha = 0$. As a second example we consider the Hermite functions ψ_n. These ψ_n's satisfy $((1/(2\pi i)\,d/dt)^2 + t^2 - (n+\frac{1}{2})/\pi)\,\psi_n = 0$. Hence. the principle applied with $T = ((1/(2\pi i)\,d/dt)^2 + t^2 - (n+\frac{1}{2}))/\pi$ (so that $P_T^{(\alpha)} = \omega^2 + t^2 - (n+\frac{1}{2})/\pi$) and $f = \psi_n$ gives that $H_{\psi_n,\psi_n}^{(\alpha)}$ should be concentrated near the circle around the origin with radius $((n+\frac{1}{2})/\pi)^{\frac{1}{2}}$. The formulas

$$H_{\psi_n,\psi_n}^{(0)}(t,\omega) = 2(-1)^n \exp(-2\pi\,(t^2+\omega^2))\,L_n(4\pi\,(t^2+\omega^2)), \tag{31}$$

$$H_{\psi_n,\psi_n}^{(-\frac{1}{2})}(t,\omega) = \exp(2\pi\, i\, t\omega)\,(-i)^n\,\psi_n(t)\,\psi_n(\omega) \tag{32}$$

show that $\alpha = 0$ gives better results than $\alpha = \frac{1}{2}$ (ref. 16, sec. 2; we have $\mathscr{F}\psi_n = (-i)^n\psi_n$, and L_n is the n^{th} Laguerre polynomial of order 0). Indeed, it can be seen from the asymptotic properties of Laguerre polynomials (ref. 24, chap. 8) that $H_{\psi_n,\psi_n}^{(0)}$ decays quickly for $(n+\frac{1}{2})/\pi < t^2+\omega^2 \to \infty$, oscillates rapidly for $t^2+\omega^2 < (n+\frac{1}{2})/\pi$, and is positive near the circle $t^2+\omega^2 = (n+\frac{1}{2})/\pi$. In fact, for the neighbourhood of the circle $t^2+\omega^2 = (n+\frac{1}{2})/\pi$ we can use the following precise result (ref. 24, chap. 8; the relation between the Airy function A used there and our Ai used in sec. 6 is $A(z) = 3^{-\frac{1}{3}}\pi\, Ai(-3^{-\frac{1}{3}}z)$)

$$\exp(-x/2)\,L_n(x) = (-1)^n\,(2n)^{-\frac{1}{3}}\,Ai(u) + 0(1/n), \tag{33}$$

where u is given by $x = 4n + 2 + 2(2n)^{\frac{1}{3}}u$ (the 0-term holds uniformly in any bounded set of u's). We refer to ref. 5 where cases with more general Hamiltonians than $t^2+\omega^2$ are considered.

To illustrate the principle for $\alpha = 0$°) we have included pictures of the Wigner distributions of $f(t) = \exp(2\pi i\varphi(t))$ with $\varphi(t) = at,\ bt^2,\ ct^3,\ dt^4$ and of $f(t) = \psi_n(t)$ (n^{th} Hermite function). We make the following observations. The principle holds exactly for $f(t) = \exp(2\pi i\varphi(t))$ where $\varphi(t) = at, bt^2$. We also see that the Wigner distributions of $f(t) = \exp(2\pi i\varphi(t))$, where $\varphi(t) = ct^3, dt^4$ are indeed concentrated near the curve $(t, \varphi'(t))$, and that they decay fast at the convex side of the curve and oscillate rapidly at the concave side of the curve. A similar thing can be said about the Wigner distribution of $f(t) = \psi_n(t)$ which oscillates rapidly inside the curve $t^2+\omega^2 = (n+\frac{1}{2})/\pi$. In sec. 6 we consider the Wigner distribution of $\exp(2\pi i\varphi(t))$ in detail; it turns out that the typical behaviour just observed in the examples (fast decay vs. rapid oscillations) is a more general feature.

In figs 1–4 we have used the pseudo-Wigner distribution (see ref. 9, part II),

°) Coloured displays of Rihaczek's density function for several functions f can be found in ref. 19.

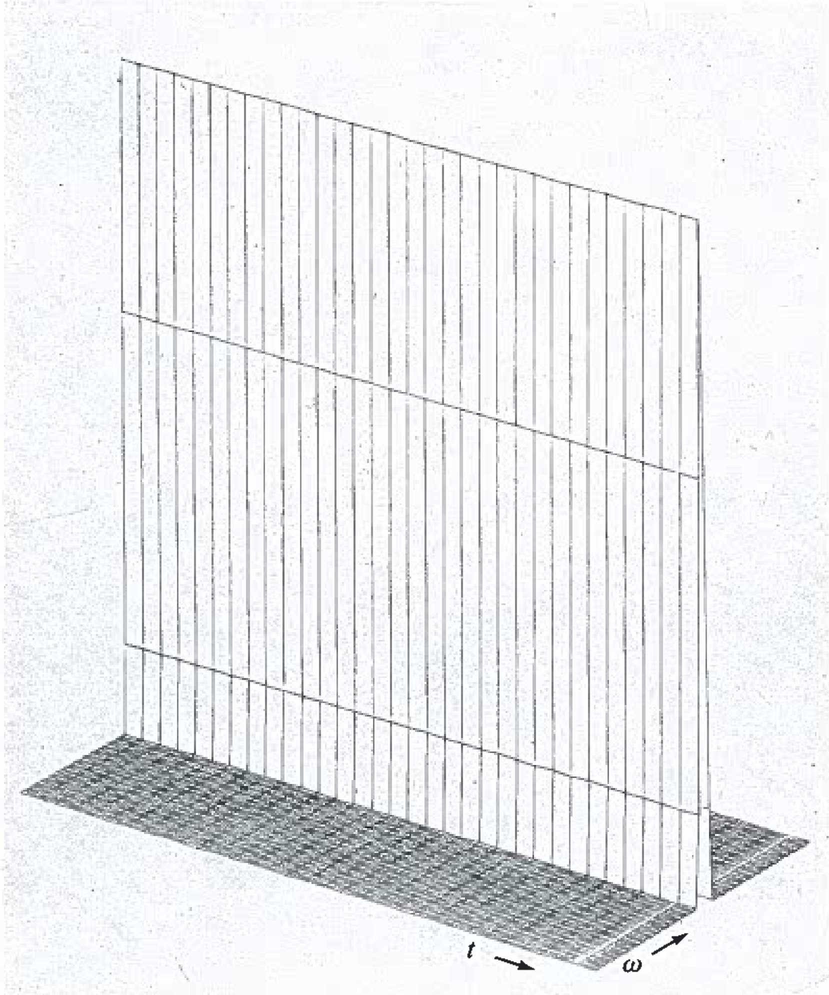


Fig. 1. Wigner distribution of $f(t) = \exp(10\pi i t)$ for $(t,\omega) \in [0,1] \times [0,10]$.

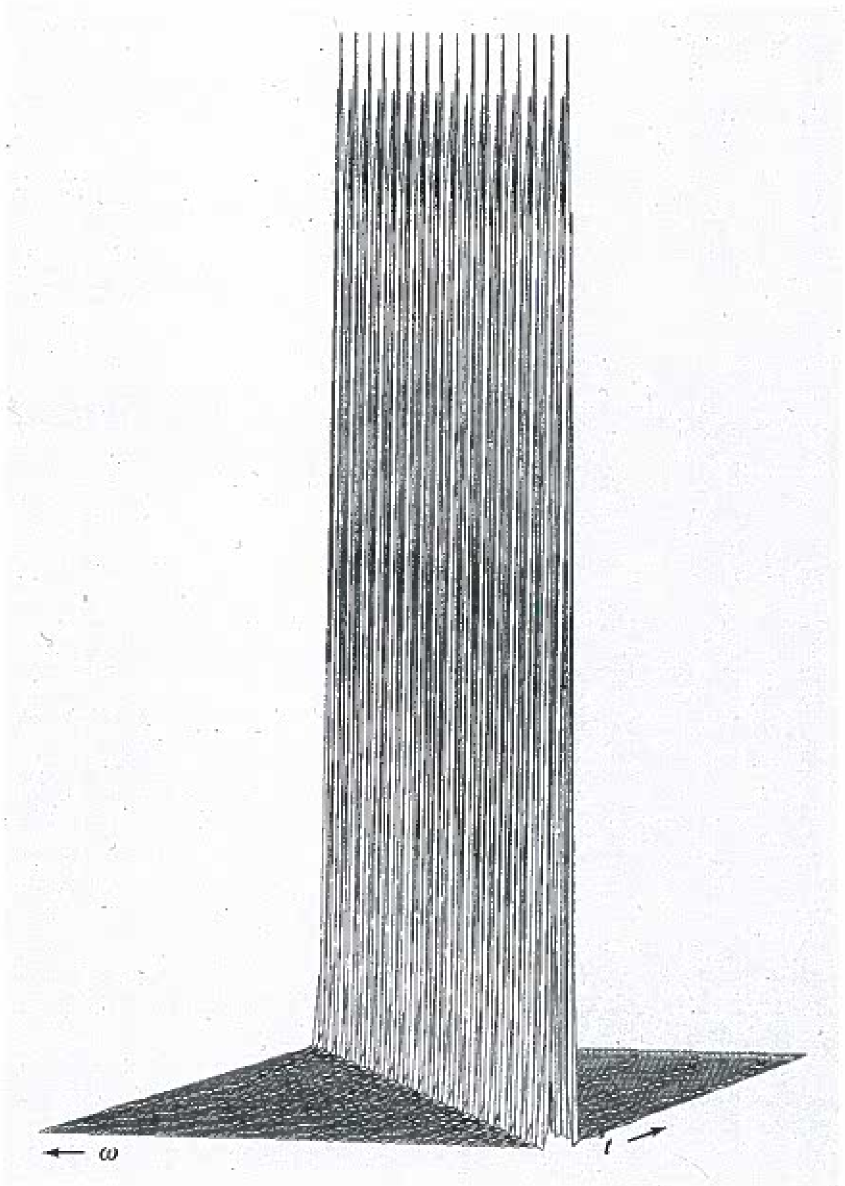


Fig. 2. Wigner distribution of $f(t) = \exp(10\pi \mathrm{i}\, t^2)$ for $(t, \omega) \in [0,1] \times [0,10]$.

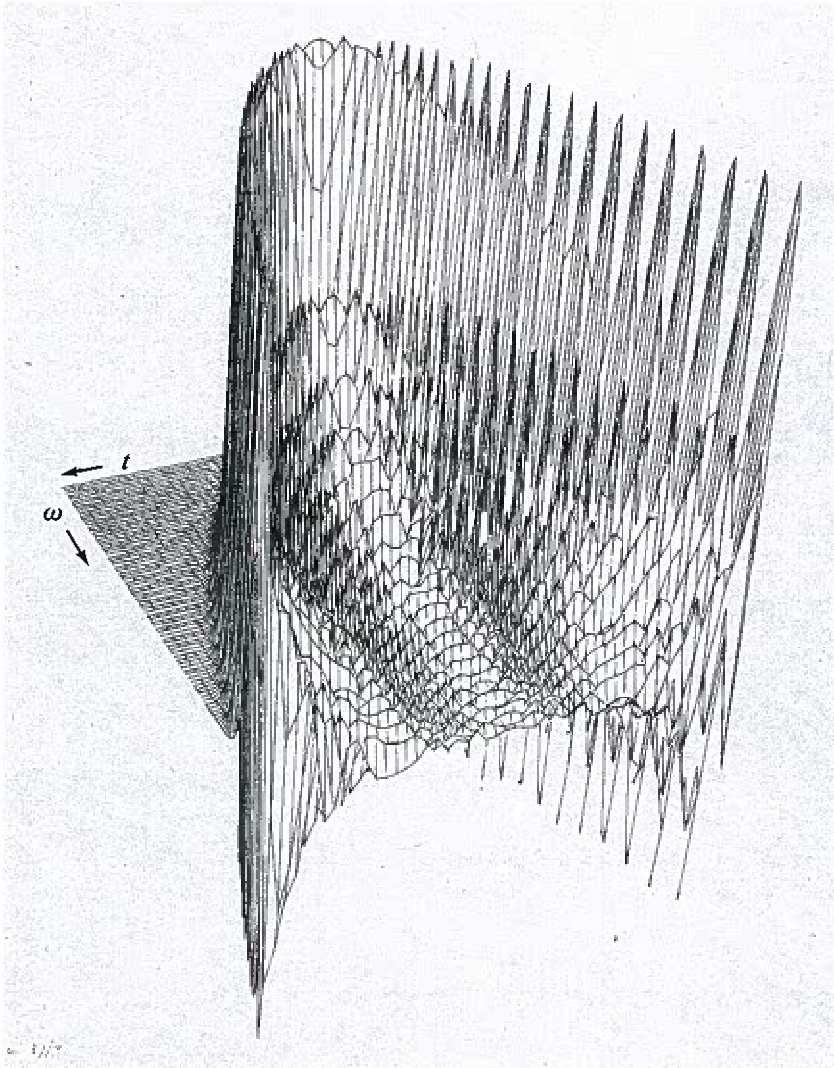


Fig. 3. Wigner distribution of $f(t) = \exp(10\pi \mathrm{i}\, t^3)$ for $(t, \omega) \in [-1,1] \times [-\frac{1}{4},15]$.

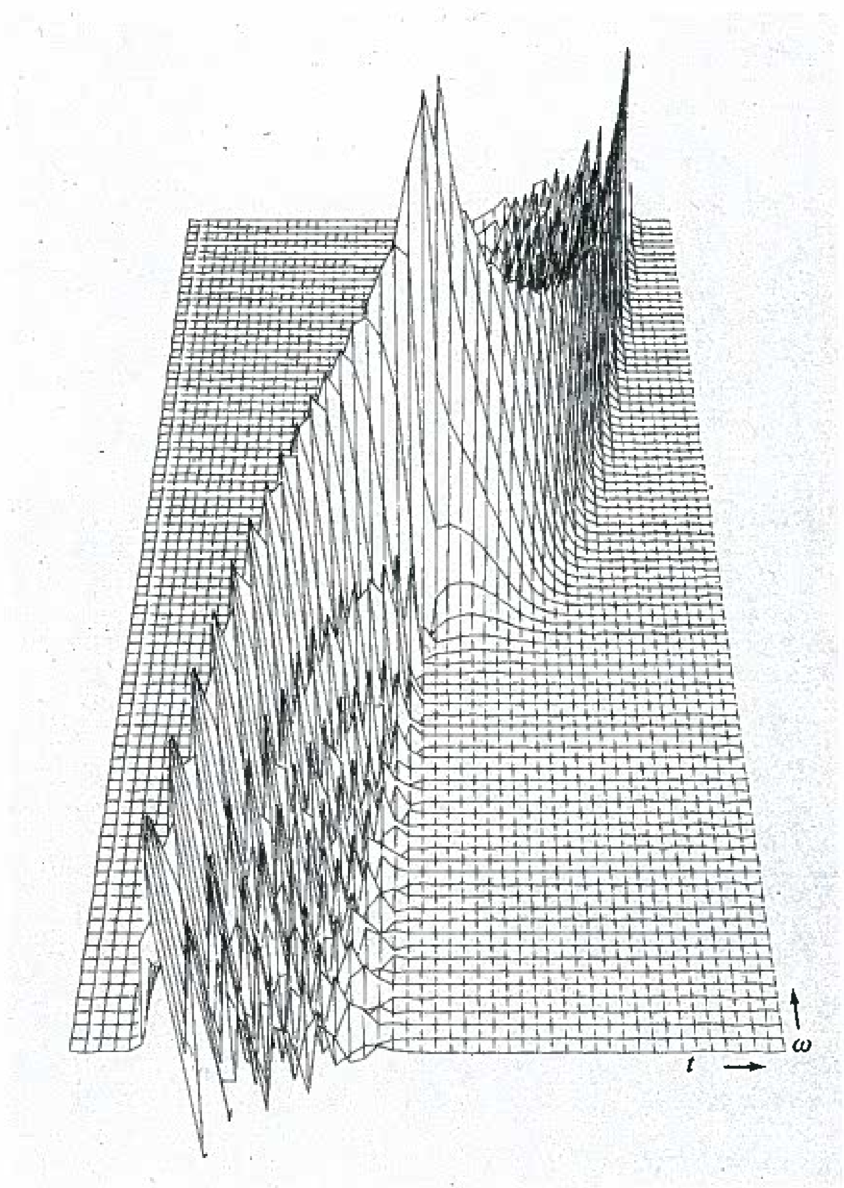


Fig. 4. Wigner distribution of $f(t) = \exp(5\pi i t^4)$ for $(t,\omega) \in [-1,1] \times [-5,6\frac{1}{4}]$.

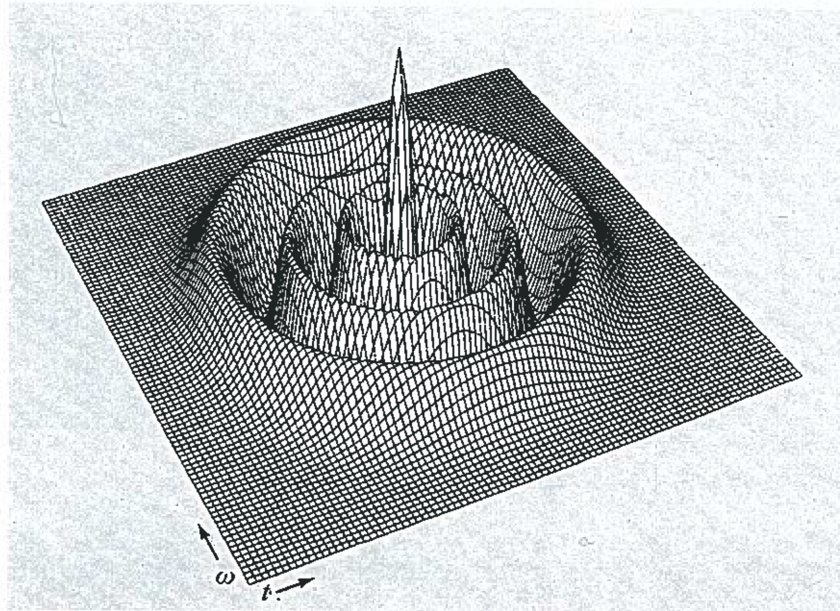


Fig. 5. Wigner distribution of $f(t) = \psi_6(t)$, the 6[th] Hermite function, for $(t,\omega) \in [-2,2] \times [-2,2]$.

i.e. the Wigner distribution of a windowed version of the signal involved; for fig. 5 we have used the explicit formula (31).

5. Global and local spread for time-frequency pseudo-density functions

Let f be a sufficiently well-behaved complex-valued function defined on $\mathbb{R}$, and let $\alpha \in \mathbb{R}$, $|\alpha| \leq \frac{1}{2}$. If $(t_0, \omega_0) \in \mathbb{R}^2$ then we call

$$\sigma_f^{(\alpha)}(t_0, \omega_0) = \iint [(t - t_0)^2 + (\omega - \omega_0)^2] \, |H_{f,f}^{(\alpha)}(t, \omega)|^2 \, dt \, d\omega \qquad (34)$$

the global spread of $|H_{f,f}^{(\alpha)}|^2$ around the point (t_0, ω_0), and we call

$$\sigma_{f,t_0}^{(\alpha)}(\omega_0) = \int (\omega - \omega_0)^2 \, |H_{f,f}^{(\alpha)}(t_0, \omega)|^2 \, d\omega, \qquad (35)$$

the local spread of $|H_{f,f}^{(\alpha)}|^2$ at time t_0 around frequency ω_0.

We shall show now that for any f and any $(t_0, \omega_0) \in \mathbb{R}^2$ the global spread $\sigma_f^{(\alpha)}(t_0, \omega_0)$ is minimal for $\alpha = 0$. It will also follow that the optimal performance of $\alpha = 0$ is most apparent when we take for t_0 and ω_0 the centres of gravity of f and $\mathscr{F}f$, defined respectively by

$$t_0 = \int t\,|f(t)|^2\,\mathrm{d}t \Big/ \int |f(t)|^2\,\mathrm{d}t, \tag{36}$$

$$\omega_0 = \int \omega\,|(\mathcal{F}f)(\omega)|^2\,\mathrm{d}\omega \Big/ \int |(\mathcal{F}f)(\omega)|^2\,\mathrm{d}\omega. \tag{37}$$

To that end we evaluate the first few moments of $|H_{f,f}^{(\alpha)}|^2$. This can be done by using Moyal's formula (cf. sec. 3, in particular (20)), Parseval's formula and the identities

$$t\,H_{f,f}^{(\alpha)}(t,\omega) = \tfrac{1}{2}H_{Qf,f}^{(\alpha)}(t,\omega) + \tfrac{1}{2}H_{f,Qf}^{(\alpha)}(t,\omega) + \alpha \int s\exp(-2\pi\,\mathrm{i}\,s\omega)\,f(t+s(\tfrac{1}{2}-\alpha))\,\overline{f(t-s(\tfrac{1}{2}+\alpha))}\,\mathrm{d}s, \tag{38}$$

$$\omega\,H_{f,f}^{(\alpha)}(t,\omega) = (\tfrac{1}{2}-\alpha)\,H_{Pf,f}^{(\alpha)}(t,\omega) + (\tfrac{1}{2}+\alpha)\,H_{f,Pf}^{(\alpha)}(t,\omega), \tag{39}$$

which follow easily from the definition of $H_{f,f}^{(\alpha)}$. Here we have

$$(Qf)(t) = t\,f(t),\quad (Pf)(t) = f'(t)/2\pi\,\mathrm{i}.$$

We get

$$\iint |H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega = \|f\|^4, \tag{40}$$

$$\iint t\,|H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega = \|f\|^2(Qf,f), \tag{41}$$

$$\iint \omega\,|H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega = \|f\|^2(Pf,f), \tag{42}$$

$$\iint t^2\,|H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega = \frac{1+4\alpha^2}{2}\|Qf\|^2\|f\|^2 + \frac{1-4\alpha^2}{2}|(Qf,f)|^2, \tag{43}$$

$$\iint \omega^2\,|H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega = \frac{1+4\alpha^2}{2}\|Pf\|^2\|f\|^2 + \frac{1-4\alpha^2}{2}|(Pf,f)|^2, \tag{44}$$

$$\iint t\omega\,|H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega = \frac{1+4\alpha^2}{2}(Qf,f)(Pf,f) + \frac{1-4\alpha^2}{2}\|f\|^2\,\mathrm{Re}(Pf,Qf). \tag{45}$$

Compare ref. 8, sec. 14. We further note that $(Qf,f) = (f,Qf)$ and $(Pf,f) = (f,Pf) = (Q\mathcal{F}f,\mathcal{F}f)$ are real numbers.

Before we consider the global spread of $|H_{f,f}^{(\alpha)}|^2$ we observe that the centre of gravity of $H_{f,f}^{(\alpha)}$, i.e. the point (a,b) given by

$$a = \iint t\,|H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega \Big/ \iint |H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega \tag{46}$$

and

$$b = \iint \omega\,|H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega \Big/ \iint |H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega, \tag{47}$$

coincides with (t_0,ω_0), where t_0 and ω_0 are given by (36) and (37) respectively. This follows easily from (41) and (42).

Using (43) and (44) we get

$$\iint (t^2+\omega^2)\,|H_{f,f}^{(\alpha)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega = \frac{1+4\alpha^2}{2}(\|Qf\|^2+\|Pf\|^2)\|f\|^2 + \frac{1-4\alpha^2}{2}(|(Qf,f)|^2+|(Pf,f)|^2). \tag{48}$$

Clearly, $|(Qf,f)|^2 \leqslant \|Qf\|^2\|f\|^2$, $|(Pf,f)|^2 \leqslant \|Pf\|^2\|f\|^2$ by Schwarz' inequality, and hence (48) is minimal for $\alpha = 0$. This proves the claim made at the beginning of this section for $(t_0,\omega_0) = 0$.

The general case can be treated by noting that

$$H_{T_{t_0}R_{\omega_0}f,\,T_{t_0}R_{\omega_0}f}^{(\alpha)}(t,\omega) = H_{f,f}^{(\alpha)}(t+t_0,\omega+\omega_0) \tag{49}$$

(cf. (2)). Now (48) can be applied with $T_{t_0}R_{\omega_0}f$ instead of f, and this completes the proof.

We have proved, more precisely, that

$$\sigma_f^{(\alpha)}(t_0,\omega_0) = \sigma_f^{(0)}(t_0,\omega_0) + 2\alpha^2 + (\|Qf_1\|^2\|f_1\|^2 + \|Pf_1\|^2\|f_1\|^2 - |(Qf_1,f_1)|^2 - |(Pf_1,f_1)|^2), \tag{50}$$

where $f_1 = T_{t_0}R_{\omega_0}f$. We can assume, without loss of generality, that the centres of gravity of f and $\mathcal{F}f$ equal zero. This implies that $(Qf,f) = (Pf,f) = 0$. Now it can be shown that the expression in (50) between parentheses is independent of t_0 and ω_0 so that it equals $\|Qf\|^2\|f\|^2 + \|Pf\|^2\|f\|^2$. This shows that for any $\alpha\in\mathbb{R}$ the ratio $\sigma_f^{(0)}(t_0,\omega_0)/\sigma_f^{(\alpha)}(t_0,\omega_0)$ is minimal (as a function of t_0 and ω_0) at the centre of gravity of $H_{f,f}^{(0)}$ (for $\sigma_f^{(0)}(t_0,\omega_0)$ is minimal at that point).

A time-frequency pseudo-density function which is not of the form $H_{f,f}^{(\alpha)}$ with $\alpha\in\mathbb{R}$, $|\alpha| \leqslant \tfrac{1}{2}$ is the real part of Rihaczek's function $\mathrm{Re}\,H_{f,f}^{(-\frac{1}{2})}$. The expression (48), with $\mathrm{Re}\,H_{f,f}^{(-\frac{1}{2})}$ instead of $H_{f,f}^{(\alpha)}$, gets a bit complicated, and it is not easy to see whether it is always larger than the corresponding expression for the Wigner distribution. We have calculated $\iint (t^2+\omega^2)\,|H_{g,g}^{(0)}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega$ and $\iint (t^2+\omega^2)\,|\mathrm{Re}\,H_{g,g}^{(-\frac{1}{2})}(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega$ for the first two Hermite functions $g = 2^{\frac{1}{4}}\exp(-\pi t^2)$ and $g = 2^{\frac{1}{4}}\pi^{-\frac{1}{2}}\,t\exp(-\pi t^2)$ (the centre of gravity is the origin for both cases). We found $1/4\pi$, $(1+1/(2\sqrt{2}))/4\pi$ for the zeroth Hermite function and $1/\pi^2$, $(1+9/32\pi^2\sqrt{2}))/\pi^2$ for the first Hermite function. Hence, the Wigner distribution performs for certain functions only marginally better°) than the real part of Rihaczek's function if we use the global spread around the centre of gravity as our criterion. It will turn out, however, that the Wigner distribution performs definitely better when the local spread is used with signals f of the form $f(t) = \exp(2\pi\,\mathrm{i}\,\varphi(t))$.

Let us consider the local spread $\sigma_{f,t_0}^{(\alpha)}$ defined by (35) in more detail. We would like to evaluate $\sigma_{f,t_0}^{(\alpha)}$ with $f(t) = \exp(2\pi\,\mathrm{i}\,\varphi(t))$, where φ is a smooth real-valued function. This is not possible since $H_{f,f}^{(\alpha)}$ is for these f's in general a distribution, and the definition of $\sigma_{f,t_0}^{(\alpha)}$ involves the square of $H_{f,f}^{(\alpha)}$. Instead

°) It is more proper, though, to compare the ratios $\iint (t^2+\omega^2)\,|H(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega \Big/ \iint |H(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega$ with $H = H_{g,g}^{(0)}$ and $\mathrm{Re}\,H_{g,g}^{(-\frac{1}{2})}$.

we consider f's of the form $f(t) = \exp(2\pi \mathrm{i}\varphi(t) - \pi\varepsilon t^2)$, and we study the behaviour of $\sigma_{f,t_0}^{(\alpha)}$ as $\varepsilon \to 0$. According to the general principle stated in sec. 4, we expect $\sigma_{f,t_0}^{(\alpha)}(\omega_0)$ to be small for ω_0 close to the instantaneous frequency $\varphi'(t_0)$. This expectation is supported by formula (5) showing that the average frequency at time t equals $\varphi'(t)$. It was also noted by Rihaczek in ref. 23 that $H_{f,f}^{(-\frac{1}{2})}$ is concentrated around the curve $(t, \varphi'(t))$ in the sense that integrals of $H_{f,f}^{(-\frac{1}{2})}$ over regions far away from that curve are negligible.

We shall evaluate the first few local moments of $|H_{f,f}^{(\alpha)}|^2$ for $\alpha = 0, -\frac{1}{2}$ and of $|\mathrm{Re}\, H_{f,f}^{(-\frac{1}{2})}|^2$ with $f(t) = \exp(2\pi \mathrm{i}\varphi(t) - \pi\varepsilon t^2)$. We can do that for $\alpha = 0, -\frac{1}{2}$ by using formula (39) together with

$$\int H_{f_1,g_1}^{(\alpha)}(t,\omega)\, \overline{H_{f_2,g_2}^{(\alpha)}(t,\omega)}\, \mathrm{d}\omega$$
$$= \int f_1(t + s(\tfrac{1}{2} - \alpha))\, \overline{f_2(t + s(\tfrac{1}{2} - \alpha))}\, \overline{g_1(t - s(\tfrac{1}{2} + \alpha))}\, g_2(t - s(\tfrac{1}{2} + \alpha))\, \mathrm{d}s \quad (51)$$

which holds for all f_1, f_2, g_1, g_2 by Parseval's formula. We get, after using the identity $(Pf)(t) = (\varphi'(t) + \mathrm{i}\varepsilon t) f(t)$, for $\alpha = 0$

$$\int |H_{f,f}^{(0)}(t,\omega)|^2\, \mathrm{d}\omega = \varepsilon^{-\frac{1}{2}} \exp(-4\pi\varepsilon t^2), \quad (52)$$

$$\int \omega\, |H_{f,f}^{(0)}(t,\omega)|^2\, \mathrm{d}\omega = \exp(-4\pi\varepsilon t^2) \int \exp(-4\pi\varepsilon s^2) \times (\varphi'(t+s) + \varphi'(t-s))\, \mathrm{d}s, \quad (53)$$

$$\int \omega^2 |H_{f,f}^{(0)}(t,\omega)|^2\, \mathrm{d}\omega = \exp(-4\pi\varepsilon t^2) \left\{ \frac{\varepsilon^{\frac{1}{2}}}{2\pi} + \int [\varphi'(t+s) + \varphi'(t-s)]^2 \exp(-4\pi\varepsilon s^2)\, \mathrm{d}s \right\}. \quad (54)$$

And for $\alpha = -\frac{1}{2}$ we get

$$\int |H_{f,f}^{(-\frac{1}{2})}(t,\omega)|^2\, \mathrm{d}\omega = |f(t)|^2 \|f\|^2, \quad (55)$$

$$\int \omega\, |H_{f,f}^{(-\frac{1}{2})}(t,\omega)|^2\, \mathrm{d}\omega = |f(t)|^2 (Pf, f), \quad (56)$$

$$\int \omega^2 |H_{f,f}^{(-\frac{1}{2})}(t,\omega)|^2\, \mathrm{d}\omega = |f(t)|^2 \|Pf\|^2, \quad (57)$$

(these formulas hold for general f; the formulas for general f and general α are unfortunately much more complicated).

To calculate the local moments for $\mathrm{Re}\, H_{f,f}^{(-\frac{1}{2})}$ we write

$$|\mathrm{Re}\, H_{f,f}^{(-\frac{1}{2})}(t,\omega)|^2 = \tfrac{1}{2} |H_{f,f}^{(-\frac{1}{2})}(t,\omega)|^2 + \tfrac{1}{2} \mathrm{Re}\, (\overline{f^2(t)}\, ((\mathscr{F}f)(\omega))^2 \exp(4\pi \mathrm{i}\, t\, \omega)),$$

and we use the convolution theorem, the formula $Q\mathscr{F} = \mathscr{F}P$ and (55)–(57) to get

$$\int |\mathrm{Re}\, H_{f,f}^{(-\frac{1}{2})}(t,\omega)|^2\, \mathrm{d}\omega = \tfrac{1}{2}\{|f(t)|^2 \|f\|^2 + \mathrm{Re}\, \overline{f^2(t)}\, (f * f)(2t)\}, \quad (58)$$

$$\int \omega\, |\mathrm{Re}\, H_{f,f}^{(-\frac{1}{2})}(t,\omega)|^2\, \mathrm{d}\omega = \tfrac{1}{2}\{|f(t)|^2 (Pf, f) + \mathrm{Re}\, \overline{f^2(t)}\, (Pf * f)(2t)\}, \quad (59)$$

$$\int \omega^2 |\mathrm{Re}\, H_{f,f}^{(-\frac{1}{2})}(t,\omega)|^2\, \mathrm{d}\omega = \tfrac{1}{2}\{|f(t)|^2 \|Pf\|^2 + \mathrm{Re}\, \overline{f^2(t)}\, (Pf * Pf)(2t)\}. \quad (60)$$

Here $(f_1 * f_2)(s)$ denotes the convolution $\int f_1(s-x) f_2(x)\, \mathrm{d}x$ of f_1 and f_2 at the point s.

The formulas (52)–(60) are especially interesting for the case that the instantaneous frequency $\varphi'(t)$ is equal to zero, for then (54), (57), (60) give the spread around the instantaneous frequency. The following table gives formulas (52)–(60) for $f(t) = \exp(2\pi \mathrm{i}\varphi(t) - \pi\varepsilon t^2)$ omitting terms of order $\varepsilon^{\frac{1}{2}}$, and with $t = 0$, $\varphi'(0) = \varphi(0) = 0$.

TABLE I

	$\lvert H_{f,f}^{(0)}\rvert^2$	$\lvert H_{f,f}^{(-\frac{1}{2})}\rvert^2$	$\lvert \mathrm{Re}\, H_{f,f}^{(-\frac{1}{2})}\rvert^2$
ω^0	$\varepsilon^{-\frac{1}{2}}$	$(2\varepsilon)^{-\frac{1}{2}}$	$\frac{1}{2}\{(2\varepsilon)^{-\frac{1}{2}} + \int \exp(-2\pi\varepsilon s^2) \cos 2\pi\, \varphi_e(s)\, \mathrm{d}s\}$
ω^1	$\int \exp(-4\pi\varepsilon s^2)\, \varphi_0'(s)\, \mathrm{d}s$	$\int \exp(-2\pi\varepsilon s^2)\, \varphi'(s)\, \mathrm{d}s$	$\frac{1}{2}\int \exp(-2\pi\varepsilon s^2)\, \varphi_e'(s) \cos^2 \pi\, \varphi_e(s)\, \mathrm{d}s$
ω^2	$\frac{1}{2}\int \exp(-4\pi\varepsilon s^2) (\varphi_0'(s))^2\, \mathrm{d}s$	$\int \exp(-2\pi\varepsilon s^2) (\varphi'(s))^2\, \mathrm{d}s$	$\frac{1}{4}\int \exp(-2\pi\varepsilon s^2) (\varphi_0'(s))^2\, \mathrm{d}s + -\int \exp(-2\pi\varepsilon s^2)\, \varphi'(s)\, \varphi'(-s) \sin^2 \pi\, \varphi_e(s)\, \mathrm{d}s$

Here $\varphi_e(s) = \varphi(s) + \varphi(-s)$ and $\varphi_0(s) = \varphi(s) - \varphi(-s)$.

The formulas (52)–(60) and the table give rise to the following remarks (we ignore terms of order $\varepsilon^{\frac{1}{2}}$).

(1) In order to compare the Wigner distribution to (the real part of) Rihaczek's function we normalize the expressions (53), (54), (56), (57), (59), (60). Thus we consider

$$R_i^{(1)} \equiv R_i^{(1)}(t,\varepsilon) = \int \omega\, |H_i(t,\omega)|^2\, \mathrm{d}\omega \Big/ \int |H_i(t,\omega)|^2\, \mathrm{d}\omega, \quad (61)$$

$$R_i^{(2)} \equiv R_i^{(2)}(t,\varepsilon) = \int \omega^2 |H_i(t,\omega)|^2\, \mathrm{d}\omega \Big/ \int |H_i(t,\omega)|^2\, \mathrm{d}\omega, \quad (62)$$

for $i = 1, 2, 3$, where $H_1 = H_{f,f}^{(0)}$, $H_2 = H_{f,f}^{(-\frac{1}{2})}$, $H_3 = \mathrm{Re}\, H_{f,f}^{(-\frac{1}{2})}$. Observe that (according to (25)–(27)) $R_2^{(1)}$ and $R_2^{(2)}$ are independent of t. Hence $H_{f,f}^{(-\frac{1}{2})}$ spreads uniformly around the curve $(t, \varphi'(t))$.

(2) Assume that $\varphi_0'(s) = \varphi'(s) + \varphi'(-s)$ is bounded, and that

$$L_1 := \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi_0'(s)\, \mathrm{d}s$$

exists. By a Tauberian theorem in ref. 17, chap. 4, sec. 2 we know that

$$\lim_{\varepsilon\downarrow 0} \tfrac{1}{2} \varepsilon^{\frac{1}{2}} \int_0^\infty \exp(-\pi\varepsilon s^2)\, \varphi_0'(s)\, \mathrm{d}s$$

exists and equals L_1. It follows (since φ_0' is even) from the table that

$$\lim_{\varepsilon\downarrow 0} R_1^{(1)}(0,\varepsilon) = \lim_{\varepsilon\downarrow 0} R_2^{(1)}(0,\varepsilon) = L_1.$$

Similarly, if

$$M_1 := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (\varphi_0'(s))^2\, \mathrm{d}s$$

exists, then

$$\lim_{\varepsilon\downarrow 0} R_1^{(2)}(0,\varepsilon) = \tfrac{1}{4}M_1,$$

and if

$$M_2 := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (\varphi'(s))^2\,\mathrm{d}s$$

exists, then

$$\lim_{\varepsilon\downarrow 0} R_2^{(2)}(0,\varepsilon) = M_2.$$

Hence, if φ_0' is small compared to φ', $H_{f,f}^{(0)}$ will have much less spread than $H_{f,f}^{(-\frac{1}{2})}$.

(3) Consider the case that $\varphi(t) = \frac{1}{2}\alpha t^2 (\alpha \in \mathbb{R})$. Then it can be checked that $H_{f,f}^{(0)}(t,\omega) = \delta_0(\omega - \alpha t)$ and that $H_{f,f}^{(-\frac{1}{2})}(t,\omega) = (\mathrm{i}\alpha)^{-\frac{1}{2}}\exp(-\pi\,\mathrm{i}\alpha^{-1}(\omega - \alpha t)^2)$ or $\delta_0(\omega)$ according as $\alpha \neq 0$ or $\alpha = 0$. Hence $H_{f,f}^{(0)}$ performs considerably better in this case. More generally, assume that $\varphi'(t)$ and $\varphi'(-t)$ have opposite sign if $t > 0$ and stay away from zero if $t \to \infty$. Define for $\delta > 0$

$$L(\delta) = \delta^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-\pi\,\delta\, s^2)\,(\varphi_0'(s))^2\,\mathrm{d}s.$$

Since $|\varphi_0'(s)| \leqslant |\varphi'(s)|$ we see from the table that

$$R_1^{(2)}(0,\varepsilon)/R_2^{(2)}(0,\varepsilon) \leqslant L(4\varepsilon)/4L(2\varepsilon).$$

As to $R_3^{(2)}$ the situation is slightly more complicated. By assumption, $\varphi_e'(s) = \varphi'(s) - \varphi'(-s)$ stays away from zero if $s > 0$. Hence

$$\frac{1}{2T}\int_{-T}^{T} \cos 2\pi\,\varphi_e(s)\,\mathrm{d}s \to 0$$

if $T \to \infty$ since $\cos 2\pi\,\varphi_e$ oscillates rapidly, and this implies that

$$\int_{-\infty}^{\infty} \exp(-2\pi\,\varepsilon\, s^2)\cos 2\pi\,\varphi_e(s)\,\mathrm{d}s = o(\varepsilon^{-\frac{1}{2}})$$

as $\varepsilon \downarrow 0$ by the Tauberian theorem in ref. 17, chap. 4, sec. 2. As

$$\varphi'(s)\,\varphi'(-s)\sin^2\pi\,\varphi_e(s) \leqslant 0,$$

we see from the table that $R_1^{(2)}(0,\varepsilon)/R_3^{(2)}(0,\varepsilon) = (\frac{1}{2} + 0(1))\,L(4\varepsilon)/L(2\varepsilon)$ as $\varepsilon\downarrow 0$. If e.g. $\lim_{\delta\downarrow 0} L(\delta)$ exists, $\neq 0$ then $R_1^{(2)}$ is at least four times as small as $R_2^{(2)}$ and at least twice as small as $R_3^{(2)}$ (asymptotically). Usually, though, $R_1^{(2)}$ will be much smaller than $R_3^{(2)}$ since the integral involving $\varphi'(s)\,\varphi'(-s)\sin^2\varphi_e(s)$ will be a negative number of large magnitude.

(4) The case where $\varphi'(s)$ and $\varphi'(-s)$ have the same sign for all $s > 0$ can be analyzed in a similar way. We find now, under proper assumptions, that $R_1^{(2)}$ is not larger than $R_2^{(2)}$ or $R_3^{(2)}$ (asymptotically).

A second way to get an idea as to how far the various pseudo-density functions spread around the curve $(t, \varphi'(t))$ is the stationary phase method. We have

$$H_{f,f}^{(0)}(t,\omega) = \int \exp(-2\pi\,\mathrm{i}\,\omega s + 2\pi\,\mathrm{i}(\varphi(t+\tfrac{1}{2}s) - \varphi(t-\tfrac{1}{2}s)))\,\mathrm{d}s, \quad (63)$$

$$H_{f,f}^{(-\frac{1}{2})}(t,\omega) = \int \exp(-2\pi\,\mathrm{i}\,\omega s + 2\pi\,\mathrm{i}(\varphi(t+s) - \varphi(t)))\,\mathrm{d}s \quad (64)$$

(these integrals may or may not converge as oscillatory integrals). If t is fixed, the integral in (63) has stationary points for all ω in the set

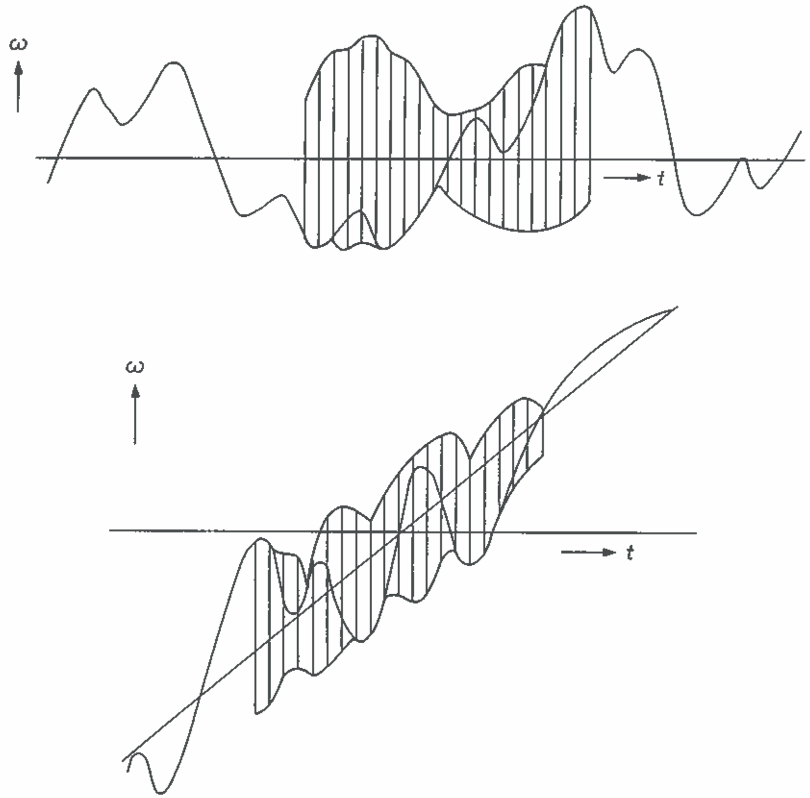
$$\{\tfrac{1}{2}(\varphi'(t+u) + \varphi'(t-u))\,|\,u \in \mathbb{R}\} := E_1(t),$$

and the integral in (64) has stationary points for all ω in the set

$$\{\varphi'(u)\,|\,u\in\mathbb{R}\} := E_2(t).$$

Note that $E_1(t) \subset E_2(t)$ and that E_2 is independent of t. Usually, E_2 will be significantly larger than $E_1(t)$.

In the figs 6 and 7 we have drawn portions of two smooth curves $(t, \varphi'(t))$



Figs 6 and 7. Portion of critical curve $(t, \varphi'(t))$ for the Wigner distribution of $f(t) = \exp(2\pi\,\mathrm{i}\,\varphi(t))$; the shaded region shows the points (t, ω) for which the integral (63) has stationary points.

together with a portion of the set of (t, ω)'s (shaded region) for which the oscillatory integral (63) defining the Wigner distribution of $\exp(2\pi i \varphi(t))$ has stationary points. The pictures are slightly misleading since the regions shown are obtained by using only a finite piece of the curves. The corresponding sets for the integral (64) would consist of the strip

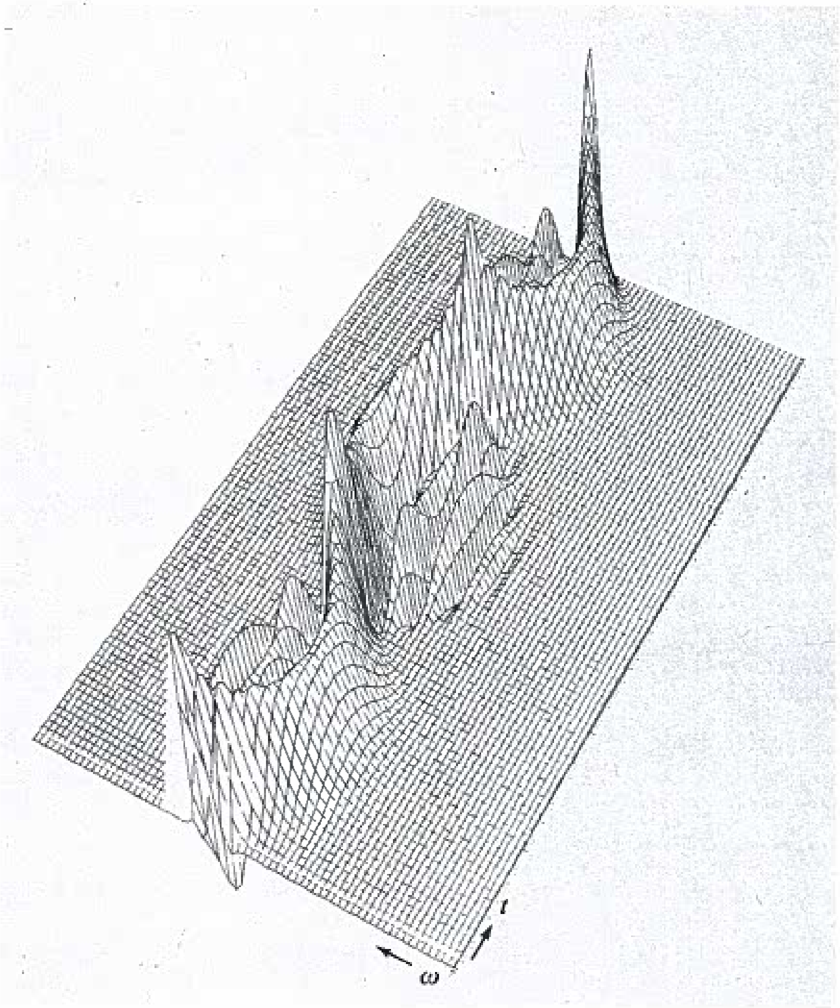


Fig. 8. Wigner distribution of $f(t) = \exp(10\pi i t + \frac{3}{4}\pi i \sin \pi t)$ for $(t, \omega) \in [-\frac{3}{2}, \frac{3}{2}] \times [0,10]$.

$$\{(t, \omega) \mid \min_s \varphi'(s) \leqslant \omega \leqslant \max_s \varphi'(s)\}.$$

Especially fig. 7 (also cf. table I and point (3) following table I) shows the advantage of the Wigner distribution $H^{(0)}_{f,f}$ over (the real part of) Rihaczek's function.

We have also included fig. 8 showing the pseudo-Wigner distribution of an FM signal $f(t) = \exp(2\pi i \alpha t + \pi i \beta \sin \pi \gamma t)$ which was also considered in ref. 9. The curve $(t, \varphi'(t)) = (t, \alpha + \frac{1}{2}\pi\gamma\beta \cos \pi \gamma t)$ has some interesting features. If k is an integer and $t = (k + \frac{1}{2})\gamma^{-1}$ then the set $E_2(t)$ consists of the single point (t, α), and if $t = k/\gamma$ then $E_2(t)$ is the set $\{(t, \alpha + \frac{1}{2}\pi\gamma\beta a) \mid -1 \leqslant a \leqslant 1\}$. Indeed, $H^{(0)}_{f,f}((k + \frac{1}{2})\gamma^{-1}, \omega)$ is concentrated completely at the point $\omega = \alpha$ as can be seen from ref. 9, (27) and fig. 1. This does not contradict the theorem proved in sec. 6 about Wigner distributions concentrated on curves.

6. More details for the Wigner distribution of $\exp(2\pi i \varphi(t))$

We consider in this section the behaviour of the Wigner distributions of signals f of the form $f(t) = \exp(2\pi i \varphi(t))$ along the curve $(t, \varphi'(t))$ in more detail. Such an analysis is facilitated quite a bit by the fact that the Wigner distribution, compared to other pseudo-density functions, has a particularly pleasant behaviour under certain area-preserving transformations (symplectic transformations) of the time-frequency plane. Using these transformations we can rotate, shift or dilate the Wigner distribution of a function in any position or direction we want; the result is always the Wigner distribution of a function which is determined by the initial function and the particular transformation.

Following ref. 8, sec. 27.3 we can describe any symplectic transformation by six real numbers $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ with $a_{11}a_{22} - a_{12}a_{21} = 1$: the point (t, ω) is mapped onto $(t', \omega') = (a_{11}t + a_{12}\omega + a_{13}, a_{21}t + a_{22}\omega + a_{23})$. Together with this transformation we consider the matrix

$$A = \begin{bmatrix} a_{22} & -i\,a_{21} & -i\,a_{23} \\ i\,a_{21} & a_{11} & a_{13} \\ 0 & 0 & 1 \end{bmatrix}. \qquad (65)$$

In ref. 8, sec. 27.3 an inner product preserving operator has been associated with every symplectic transformation in such a way that multiplication of the matrices corresponds to composition of the operators (apart from a factor of modulus unity which is of no importance here). Denoting the operator corresponding to A in (65) by Γ_A we have by ref. 8, sec. 27.12.2

$$(\Gamma_A f, \Gamma_A g) = (f, g), \qquad (66)$$

$$H^{(0)}_{\Gamma_A f, \Gamma_A g}(t', \omega') = H^{(0)}_{f,g}(t, \omega) \qquad (67)$$

for all f and g and all t and ω ((t', ω') has been defined above).

The following table gives the elementary symplectic transformations together with the associated operators; from this table all symplectic transformations and associated operators can be obtained by multiplication of matrices and composition of operators as indicated above. In this table $a, b, \alpha, \beta, \theta$ are real and λ is positive.

TABLE II

Symplectic transformation	Associated operator
$(t', \omega') = (t - a, \omega - b)$	$f \to \exp(-\pi i a b - 2\pi i b t) f(t + a)$
$(t', \omega') = (\lambda^{-1} t, \lambda \omega)$	$f \to \lambda^{\frac{1}{2}} f(\lambda t)$
$(t', \omega') = (t \cos\theta + \omega \sin\theta, \omega \cos\theta - t \sin\theta)$	$T(\theta)$
$(t', \omega') = (\omega, -t)$; $(t', \omega') = (-\omega, t)$	$\mathcal{F}$; $\mathcal{F}^{-1}$
$(t', \omega') = (-t, -\omega)$	$f \to f(-t)$
$(t', \omega') = (t, \omega - \beta t)$	$f \to \exp(-\pi i \beta t^2) f(t)$
$(t', \omega') = (t - \alpha \omega, \omega)$	$f \to (i\alpha)^{-\frac{1}{2}} \int \exp(\pi i \alpha^{-1}(z - t)^2) f(z)\, dz$

The operator $T(\theta)$ needs some further explanation. If θ is a multiple of π, then $T(\theta)$ is, apart from a sign, the identity operator ($\theta = 2k\pi$) or the operator given by $f \to f(-t)$ ($\theta = (2k+1)\pi$). If θ is not a multiple of π, then $T(\theta)$ is given by

$$(T(\theta) f)(t) = (i \sin\theta)^{-\frac{1}{2}} \int \exp\left(\frac{\pi i}{\sin\theta} ((z^2 + t^2)\cos\theta - 2zt)\right) f(z)\, dz \qquad (68)$$

with some choice for the square root.

If we take, e.g. $f(t) = 1$, then it can be verified that

$$(T(\theta) f)(t) = (\cos\theta)^{-\frac{1}{2}} \exp(-\pi i t^2 \tan\theta).$$

Since the Wigner distribution of f equals $\delta_0(\omega)$ we see from (67) that the Wigner distribution of $(\cos\theta)^{-\frac{1}{2}} \exp(-\pi i t^2 \tan\theta)$ is a delta function concentrated on the line $\{\lambda(-\sin\theta, \cos\theta) \mid \lambda \in \mathbb{R}\}$.

We have the following consequence of the table. If we have an f of the form $f(t) = \exp(2\pi i \varphi(t))$ whose Wigner distribution we want to consider near a point (t_0, ω_0), we may assume that $\varphi'(t_0) = \varphi''(t_0) = 0$. For otherwise we consider $f_1(t) = \exp(2\pi i(\varphi(t) - \varphi'(t_0)(t - t_0) - \frac{1}{2}\varphi''(t_0)(t - t_0)^2))$ which has the same form as f and for which we have

$$H^{(0)}_{f_1, f_1}(t, \omega) = H^{(0)}_{f,f}(t, \omega + \varphi'(t_0) + t\varphi''(t_0)).$$

To get some further insight into the behaviour of Wigner distributions around certain curves we consider the following question. What can we say about functions whose Wigner distributions are completely concentrated on

a curve (or set of curves)? It was shown in ref. 18 that, under rather severe restrictions on the curve, only chirps, δ-functions and exponentials have Wigner distributions concentrated on a curve. If one relaxes the assumptions on the curve then, as a rule, one can say that at all points of the curve where the radius of curvature is finite the Wigner distribution cannot have "mass".

Let us give a not too rigorous proof of this statement for a rather general case (for a rigorous treatment we need to consider the Wigner distribution as a generalized function). Let f be a function whose Wigner distribution is concentrated on a smooth curve C in the plane. That is, we have a smooth function γ: $\mathbb{R} \to \mathbb{R}^2$ and a continuous function g defined on $C = \{\gamma(s) \mid s \in \mathbb{R}\}$ such that $\iint H^{(0)}_{f,f}(t, \omega)\, G(t, \omega)\, dt\, d\omega = \int g(\gamma(s)\, G(\gamma(s)) \mid \gamma'(s) \mid ds$ for all smooth functions G of compact support. Let $s_0 \in \mathbb{R}$ be such that $g(\gamma(s_0)) > 0$ and such that the tangent line at C through $\gamma(s_0) = (t_0, \omega_0)$ is perpendicular to the t-axis. This is no serious restriction since we can apply a rotation of the plane if necessary. Also assume there is a $\delta > 0$ such that the vertical $C_t = \{(t, \omega) \mid \omega \in \mathbb{R}\}$ intersects C in at most 2 points if $|t - t_0| < \delta$. (This restriction simplifies the presentation of the proof; however, the arguments can be modified so as to apply as well if the verticals intersect C in finitely or, if proper assumptions on convergence are made, in infinitely many points.) Finally assume that C has in $\gamma(s_0)$ a finite radius of curvature. We get the following picture.

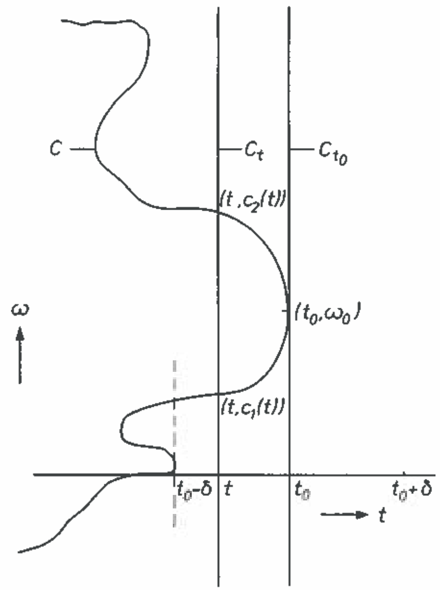


Fig. 9.

We shall derive a contradiction now. We get by Fourier inversion (integration along C_t; see picture)

$$f(t+\tfrac{1}{2}v)\overline{f(t-\tfrac{1}{2}v)} = \int \exp(2\pi\,\mathrm{i}\,\omega\,v)\,H^{(0)}_{f,f}(t,\omega)\,\mathrm{d}\omega$$
$$= d_1(t)\,g(c_1(t))\exp(2\pi\,\mathrm{i}\,c_1(t)\,v) + d_2(t)\,g(c_2(t))\exp(2\pi\,\mathrm{i}\,c_2(t)\,v) \qquad (69)$$

for all t with $t_0-\delta<t<t_0$ and all v, and

$$f(t+\tfrac{1}{2}v)\overline{f(t-\tfrac{1}{2}v)} = 0 \qquad (70)$$

for all t with $t_0<t<t_0+\delta$ and all v. Here $d_i(t)$ is the modulus of the derivative of the tangent at C through $c_i(t)$. Since g is continuous and non-zero at (t_0,ω_0) and since $c_i(t)\to\omega_0(t\uparrow t_0)$, $d_i(t)\to\infty(t\uparrow t_0)$ we see that

$$f(t+\tfrac{1}{2}v)\overline{f(t-\tfrac{1}{2}v)} \neq 0$$

for (t,v) in a set $(t_0-\delta_1,t_0)\times(-\delta_2,\delta_2)$. This implies that $f(t)\neq 0$ in an interval $(t_0-\delta_3,t_0+\delta_3)$. But this contradicts (70) °).

We now consider approximations of the Wigner distribution of $f(t)=\exp(2\pi\,\mathrm{i}\varphi(t))$. It is not unreasonable, in view of the statement just proved, to expect a large value for the spread of the Wigner distribution around the curve $(t,\varphi'(t))$ at those points t_0 where $\varphi'''(t_0)$ is large. Let $t_0\in\mathbb{R}$, and assume that $\varphi'''(t_0)\neq 0$. Replace φ in the integral (63) by its third order Taylor approximation around t_0. We get as an approximation for $H^{(0)}_{f,f}(t_0,\omega)$ the integral °°)

$$\int \exp(2\pi\,\mathrm{i}(\varphi'(t_0)-\omega)\,s + \frac{\pi\,\mathrm{i}}{12}\varphi'''(t_0)\,s^3)\,\mathrm{d}s. \qquad (71)$$

We can express (71) in terms of Airy's function

$$Ai(z) = \frac{1}{2\pi}\int \exp(\tfrac{1}{3}\mathrm{i}\,t^3 + \mathrm{i}\,z\,t)\,\mathrm{d}t \qquad (72)$$

as $Ai((\varphi'(t_0)-\omega)\beta^{-\frac{1}{3}})\beta^{-\frac{1}{3}}$ where, $\beta=\beta(t_0)=\varphi'''(t_0)/32\pi^2$. A similar approximation was used by M. V. Berry in ref. 5 to obtain a semi-classical limiting form of the Wigner distribution of the bound energy eigenstates for integrable systems.

Although we have to keep in mind that (71) only provides an approximation of $H^{(0)}_{f,f}$ we make the following observations °°°). Also see fig. 10.

°) A similar argument is used in ref. 3. The conclusion in ref. 3 that this implies that a Wigner distribution cannot be concentrated on curves other than straight lines is wrong [18].

°°) Formula (71) is useful as an approximation for $H^{(0)}_{f,f}(t_0,\omega)$ if (t_0,ω) lies not too far away from the curve $(t,\varphi'(t))$ and if the integral (63) (with t_0 instead of t) has at most two stationary points s close to zero. When (t_0,ω) is not close to the critical curve, or if there are more than two stationary points, the stationary phase method is more appropriate.

°°°) Similar observations were made by M. V. Berry in ref. 5.

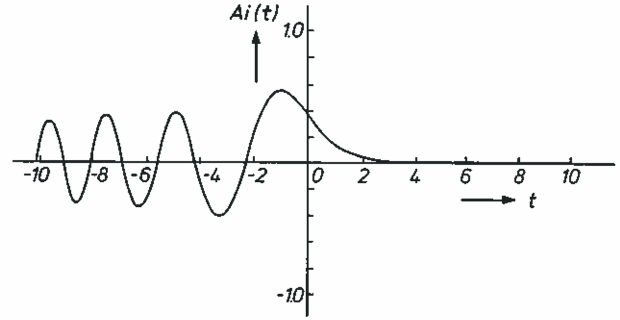


Fig. 10. Sketch of Airy's function (72) for $t\in[-10,10]$.

(1) Assume that $\varphi'''(t_0)$ varies only slowly with t_0. Then (71) depends to a large extent only on $\varphi'(t_0)-\omega$. Hence the level lines of (71) tend to be replicas of the curve $(t,\varphi'(t))$ shifted in vertical direction. The larger the value of $|\varphi'''(t_0)|$ the larger the distance between the level lines when the difference between the various levels is kept fixed.

(2) There is a dramatic difference between the behaviour of $Ai(z)$ for $z<0$ and $z>0$. We have $Ai(z)>0$ for $z>0$, $Ai(z)$ oscillates for $z<0$, Ai takes its maximum for a z close to -1, Ai has an asymptotic expansion

$$Ai(z) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\exp(-\tfrac{2}{3}z^{\frac{3}{2}})\sum_0^\infty(-1)^k c_k(\tfrac{2}{3}z^{\frac{3}{2}})^{-k} \qquad (73)$$

for $z>0$, $z\to\infty$, and Ai has an asymptotic expansion

$$Ai(z) \sim \pi^{-\frac{1}{2}}(-z)^{-\frac{1}{4}}\{\sin(\tfrac{2}{3}(-z)^{\frac{3}{2}}+\tfrac{\pi}{4})\sum_0^\infty(-1)^k c_{2k}(\tfrac{2}{3}(-z)^{\frac{3}{2}})^{-2k}$$
$$-\cos(\tfrac{2}{3}(-z)^{\frac{3}{2}}+\tfrac{\pi}{4})\sum_0^\infty(-1)^k c_{2k+1}(\tfrac{2}{3}(-z)^{\frac{3}{2}})^{-2k-1} \qquad (74)$$

for $z<0$, $z\to-\infty$. Here $c_k=\Gamma(3k+\frac{1}{2})/54^k k!\,\Gamma(k+\frac{1}{2})$ (these formulas are taken from ref. 1, sec. 10.4). Hence, $Ai(z)$ decays fast for $z>0$ and rather slowly for $z<0$. Since (71) involves $(\varphi'''(t_0))^{-\frac{1}{3}}$ we have fast decay of (71) at the convex side and oscillatory behaviour of (71) at the concave side of the curve $(t,\varphi'(t))$ (assumed that this curve is indeed convex or concave).

(3) When $\varphi'''(t_0)$ is close to zero, the set of all ω for which (71) is significantly unequal to zero in a large lopsided interval around $\varphi'(t_0)$. In practice, however, we always deal with windowed signals. If the window-length is rather large, but still small compared to $|1/\varphi'''(t_0)|$, it is better to approximate φ by a second order Taylor expansion. Then we get $\delta_0(\omega-\varphi'(t_0))$ instead of (71).

The slow decay of $H_{f,f}^{(0)}$ at the concave side of the curve $(t, \varphi'(t))$ must be considered as a kind of interference phenomenon. A relevant formula in this connection is

$$4 \iint H_{f,f}^{(0)}(2a - t, 2b - \omega)\, H_{f,f}^{(0)}(t, \omega)\, \mathrm{d}t\, \mathrm{d}\omega = (H_{f,f}^{(0)}(a,b))^2, \qquad (75)$$

which holds for general f and which can be derived from Moyal's formula in a straightforward manner. What the formula shows is this. If we have an f for which $H_{f,f}^{(0)}$ is large and positive around two points (t_1, ω_1) and (t_2, ω_2), then we can expect $|H_{f,f}^{(0)}|$ to be large around the point $(\frac{1}{2}(t_1 + t_2), \frac{1}{2}(\omega_1 + \omega_2)) = (a,b)$ since the neighbourhood of the point $(t, \omega) = (t_1, \omega_1)$ gives a large contribution to the double integral in (75). Note that (75) does not give information about the sign of $H_{f,f}^{(0)}$ at (a,b). Hence if $f(t) = \exp(2\pi \mathrm{i} \varphi(t))$ and (a,b) is a point of the form $(\frac{1}{2}(t+s), \frac{1}{2}(\varphi'(t) + \varphi'(s)))$ then the value of $H_{f,f}^{(0)}$ at (a,b) will not be negligible in general by "interference" of the "mass" at the points $(t, \varphi'(t))$ and $(s, \varphi'(s))$.

In sec. 4 we claimed that integrals of $H_{f,f}^{(0)}$ over regions far away from the curve $(t, \varphi'(t))$ are negligible. This claim will now be substantiated. To that end we consider Ψ, defined by

$$\Psi(t, \omega) = 2 \iint H_{f,f}^{(0)}(t+s, \omega + \lambda) \exp(-2\pi (s^2 + \lambda^2))\, \mathrm{d}s\, \mathrm{d}\lambda. \qquad (76)$$

It is well-known that (see ref. 8, sec. 27.12.1)

$$\Psi(t, \omega) = |2^{\frac{1}{4}} \int f(s) \exp(-2\pi\, \mathrm{i}\, \omega\, s - \pi (s - t)^2)\, \mathrm{d}s|^2; \qquad (77)$$

in particular, $\Psi(t, \omega) \geqslant 0$ for all $(t, \omega) \in \mathbb{R}^2$. Hence the oscillations of $H_{f,f}^{(0)}$ around the curve $(t, \varphi'(t))$ can be appeased by convoluting with the Gaussian $2 \exp(-2\pi(s^2 + \lambda^2))$. Consider the case $t = 0$ and let the integral $I(\omega)$ be defined by

$$I(\omega) = 2^{\frac{1}{4}} \int \exp(2\pi\, \mathrm{i}\, \varphi(s) - 2\pi\, \mathrm{i}\, \omega\, s - \pi\, s^2)\, \mathrm{d}s, \qquad (78)$$

and assume that $\varphi(0) = \varphi'(0) = 0$ (note that $\Psi(0, \omega) = |I(\omega)|^2$). Now $I(\omega)$ can be approximated by inserting the 3rd order Taylor polynomial $\frac{1}{2}as^2 + \frac{1}{6}bs^3$ of φ around 0 in the exponential. Some elementary manipulations show that the resulting approximation $J(\omega)$ can be expressed as

$$J(\omega) = 2^{\frac{1}{4}}(\pi b)^{-\frac{1}{3}} \exp(\tfrac{2}{3}\pi\, \mathrm{i}\, b\, \varrho^3 + 2\pi\, \mathrm{i}\, \varrho\, \omega)\, Ai(-2\pi\, (\pi b)^{-\frac{1}{3}} (\omega + \tfrac{1}{2} b\, \varrho^2)), \qquad (79)$$

where $\varrho = (a + \mathrm{i}) b^{-1}$. The approximation $J(\omega)$ for $I(\omega)$ is far more accurate than the one given by (71) is for $H_{f,f}^{(0)}$ since we are now dealing with rapidly convergent integrals.

Let us assume $b > 0$. Using the asymptotic expansions (74) and (73) (valid in $|\arg z - \pi| < \frac{2}{3}\pi$ and $|\arg z| < \pi$ respectively) one can show that

$$J(\omega) = 0(\omega^{-\frac{1}{4}} \exp(-2\pi\, b^{-1}\, \omega + \pi\, |a|\, (2/b)^{\frac{1}{2}}\, \omega^{\frac{1}{2}})) \qquad (80)$$

for $\omega \to \infty$, and that

$$J(\omega) = 0(\omega^{-\frac{1}{4}} \exp(-\tfrac{4}{3}\pi (2/b)^{-\frac{1}{2}} |\omega|^{\frac{3}{2}} + 2\pi b^{-1} |\omega| - \tfrac{1}{2}(a^2 - 1)\, \pi (2/b)^{\frac{1}{2}} |\omega|^{\frac{1}{2}})) \qquad (81)$$

for $\omega \to -\infty$. Note that $J(\omega)$ decays fast at both sides of the critical curve $(t, \varphi'(t))$. The decay is faster at the convex side of the curve than at the concave side, but the difference is far less dramatic than for $H_{f,f}^{(0)}$. See also fig. 2 in ref. 9, Part I where an FM signal is considered.

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