SIAM J. MATH. ANAL.
Vol. 15, No. 1, January 1984

0036-1410/84/1501-0014 $01.25/0

A NOTE ON HUDSON'S THEOREM ABOUT FUNCTIONS WITH NONNEGATIVE WIGNER DISTRIBUTIONS*

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Abstract. We show that a (generalized) function f has a nonnegative Wigner distribution $W(f,f)$ if and only if f is a Gauss function (possibly degenerate). We prove, more generally, that the convolution of $W(f,f)$ with certain Gauss functions is nonnegative if and only if f is of the special type mentioned. As a consequence we have that the only (generalized) functions whose Wigner distributions are concentrated on a curve of a particular type are delta functions or exponentials $\exp(-\pi\alpha t^2+2\pi\beta t+\gamma)$ with α, β, γ complex, $\operatorname{Re}\alpha=0$. The main tool used is Moyal's formula for the Wigner distribution together with Bargmann's integral transform.

1. Introduction. For $f\in L^2(\mathbb{R})$, the Wigner distribution $W(f,f)$ of f is defined as

$$W(x,y;f,f)=\int_{-\infty}^{\infty} e^{-2\pi iyt}f\left(x+\frac{1}{2}t\right)\overline{f\left(x-\frac{1}{2}t\right)}\,dt, \qquad (x\in\mathbb{R}, y\in\mathbb{R}). \tag{1.1}$$

It is known that $W(f,f)$ is a continuous, bounded, real-valued function that may take negative values. The Wigner distribution was introduced by Wigner [15] as a device that allows one to express quantum mechanical expectation values in the same form as the averages of classical statistical mechanics. By means of the Wigner distribution one can describe Weyl's correspondence [7], [14] in the following elegant form (see for this e.g. [4]). If a: $\mathbb{R}^2\to\mathbb{R}$ is an observable, then the expectation value of a in the state f is given by

$$\iint a(x,y)W(x,y;f,f)\,dx\,dy, \tag{1.2}$$

i.e., instead of substituting a particular point (x_0,y_0) of the phase plane in a (as one does in classical mechanics), one integrates a against the "density function" $W(f,f)$. More recently there has been considerable interest in the Wigner distribution as a tool for signal analysts to describe a signal in time and frequency simultaneously (cf. [3], [5]). In both quantum mechanics and signal analysis one likes to interpret $W(f,f)$ as a density function of two variables. Such an interpretation is awkward, since $W(f,f)$ may take negative values as already said. Nevertheless, there is a fairly extensive list of positivity properties of the Wigner distribution (cf. [3], [11]). These properties express that certain averages of the Wigner distribution are nonnegative. A typical example is: for any $f\in L^2(\mathbb{R})$ (cf. [2]),

$$\iint \exp\left(-2\pi\delta(x-a)^2-2\pi\gamma(y-b)^2\right)W(x,y;f,f)\,dx\,dy\geq 0, \tag{1.3}$$

for all $\delta>0$, $\gamma>0$, $a\in\mathbb{R}$, $b\in\mathbb{R}$ where $\delta\gamma\leq 1$.

It is convenient to allow in this note certain generalized functions f which we shall describe in §2. We shall show that if $f\neq 0$ has a Wigner distribution that is nonnegative everywhere (in a generalized sense), then f is necessarily of the form

$$f(t)=\exp(-\pi\alpha t^2+2\pi\beta t-\pi\gamma), \tag{1.4}$$

or

$$f(t)=d\delta_a(t), \tag{1.5}$$

* Received by the editors November 9, 1981, and in revised form June 25, 1982.
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where α, β, γ, a, d are complex numbers with $\operatorname{Re}\alpha \geq 0$. If we restrict to $f \in L^2(\mathbb{R})$, this is known as Hudson's theorem [8]. The f's in (1.4) are what we call Gabor functions (although this name is usually reserved for the case that α is real and positive). We have for the f in (1.4), by calculation,

(1.6)

$$W(x,y;f,f) = \left(\frac{2}{\operatorname{Re}\alpha}\right)^{1/2} \exp\big(-2\pi \operatorname{Re}\gamma + 2\pi(\operatorname{Re}\beta)^2/\operatorname{Re}\alpha - 2\pi(x - \operatorname{Re}\beta/\operatorname{Re}\alpha)^2 \operatorname{Re}\alpha - 2\pi(y + x\operatorname{Im}\alpha - \operatorname{Im}\beta)^2/\operatorname{Re}\alpha\big),$$

if $\operatorname{Re}\alpha > 0$, and

$$W(x,y;f,f) = \exp(-2\pi \operatorname{Re}\gamma + 4\pi x \operatorname{Re}\beta)\delta_0(y - \operatorname{Im}\beta + x\operatorname{Im}\alpha), \tag{1.7}$$

if $\operatorname{Re}\alpha = 0$. And for the f in (1.5), we have

$$W(x,y;f,f) = |d|^2 \exp(4\pi y \operatorname{Im}a)\delta_0(x - \operatorname{Re}a). \tag{1.8}$$

We shall show more generally that if $\delta\gamma > 1$, and (1.3) is nonnegative for all a and b, then f must be of the form (1.4) or (1.5). This result shows that Gabor functions and delta functions are fairly isolated objects in this kind of time-frequency analysis. As an application we show that if $W(f,f)$ is concentrated on a curve of a certain type, then f must be of the form (1.4) (with $\operatorname{Re}\alpha = 0$) or (1.5).

The key argument, due to Hudson (cf. [8]), is the observation that for $\gamma = \delta = 1$, the expression (1.3) can be written as $\exp(-\pi(a^2+b^2))|G(a-ib)|^2$, where G is an entire function of order 2 (Bargmann transform of f). Now, $f \in L^2(\mathbb{R})$, $W(f,f) \geq 0$ everywhere implies that $G(a-ib) \neq 0$ for all a and b (unless $f \equiv 0$). And Hadamard's theorem can be used to show that G, and hence f, has a special form. Since we also want to discuss f's which are not necessarily square integrable, we consider in §2 the Bargmann transform in some detail for f's in a convenient set of generalized functions.

2. Preliminaries. A convenient theory of generalized functions for discussing the Wigner distribution was elaborated by De Bruijn (cf. [4]); we describe it here briefly. We don't want to use Schwartz' theory of tempered distributions since this theory has the disadvantage that functions like $f(t) = \exp(t)$ and $f(t) = \delta_i(t)$ cannot be considered. Also, the theory used in this note arises naturally in the context of the Bargmann transform which will be used later on. Our test function space S consists of all entire functions f for which there are $A > 0$, $B > 0$ such that $f(x+iy) = O(\exp(-\pi Ax^2 + \pi By^2))$. This space can be identified with the Gelfand–Shilov space $S_{1/2}^{1/2}$ (cf. [6], [9]). We may describe S as the set of all $f \in L^2(\mathbb{R})$ for which $(f, \psi_n) = O(\exp(-n\alpha))$ for some $\alpha > 0$. Here ψ_n are the Hermite functions, given by

(2.1)

$$\psi_n(x) = (-1)^n 2^{1/4}(4\pi)^{-n/2}(n!)^{-1/2} e^{\pi x^2}\left(\frac{d}{dx}\right)^n e^{-2\pi x^2} \qquad (x \in \mathbb{R}, n = 0, 1, \cdots);$$

we have $H\psi_n = (n + \frac{1}{2})\psi_n$, where $H = (x^2 - 1/4\pi^2(d^2/dx^2))\pi$ is the Hermite operator. We denote the dual of S by S^*: an $F \in S^*$ is an antilinear continuous functional on S. We have $(F, \psi_n) = O(\exp(n\alpha))$ for all $\alpha > 0$, if $F \in S^*$. Yet another way to describe S and S^* is by means of the Bargmann transform (cf. [2], [12]): for $F \in S^*$ we let

$$(BF)(z) = e^{\pi z^2/2}(F, g_{\bar{z}}) \qquad (z \in \mathbb{C}), \tag{2.2}$$

where, for $w\in\mathbb{C}$,

(2.3) $$g_w(t)=2^{1/4}\exp\left(-\pi(t-w)^2\right) \qquad (t\in\mathbb{C}).$$

We note that $(B\psi_n)(z)=(z\sqrt{\pi})^n/\sqrt{n!}$. Now B maps $S(S^*)$ one-to-one onto the set of all entire functions of order 2, type $<\pi/2$ (order 2, type $\leq\pi/2$). For details we refer to [12].

It is important to note that

(2.4) $$(F,G_1(a,b))=\exp\left(-\frac{\pi}{2}(a^2+b^2)\right)(BF)(a-ib) \qquad (a\in\mathbb{R},b\in\mathbb{R}),$$

where $G_\gamma(a,b)$ denotes for $\gamma>0$, $a\in\mathbb{R}$, $b\in\mathbb{R}$ the Gabor function,

(2.5) $$G_\gamma(a,b)(t)=\left(\frac{2}{\gamma}\right)^{1/4}\exp\left(-\pi\gamma^{-1}(t-a)^2+2\pi ibt-\pi iab\right) \qquad (t\in\mathbb{R}),$$

whose Wigner distribution is given by

(2.6) $$W(x,y;\,G_\gamma(a,b),\,G_\gamma(a,b))=2\exp\left(-2\pi\gamma^{-1}(x-a)^2-2\pi\gamma(y-b)^2\right) \qquad (x\in\mathbb{R},y\in\mathbb{R}).$$

We further have

(2.7)
$$(BF)(z)=2^{1/4}(1+\alpha)^{-1/2}\exp\left(\frac{1}{2}\frac{1-\alpha}{1+\alpha}\pi z^2+\frac{2\pi\beta z}{1+\alpha}-\pi\left(\gamma-\beta^2(1+\alpha)^{-1}\right)\right) \qquad (z\in\mathbb{C}),$$

and

(2.8) $$(BF)(z)=2^{1/4}d\exp\left(-\frac{1}{2}\pi z^2+2\pi az-\pi a^2\right) \qquad (z\in\mathbb{C}),$$

where F is the f of (1.4) and (1.5) respectively. We conclude that if $P(z)=az^2+bz+c$ with $|a|\leq\pi/2$, $b\in\mathbb{C}$, $c\in\mathbb{C}$, then there is exactly one F of the form (1.4) or (1.5) such that $(BF)(z)=\exp(P(z))$.

We shall also need the operator $e^{-\alpha H}$, which can be defined on S and S^* for $\operatorname{Re}\alpha\geq 0$. We have

(2.9) $$B(e^{-\alpha H}F)(z)=e^{-\alpha/2}(BF)(ze^{-\alpha}) \qquad (z\in\mathbb{C}),$$

for $F\in S^*$ (cf. [2], [12]). For $\alpha>0$, $e^{-\alpha H}$ is De Bruijn's smoothing operator N_α (cf. [4]); the kernel K_α of N_α is given by

(2.10) $$K_\alpha(z,t)=(\sinh\alpha)^{-1/2}\exp\left(\frac{-\pi}{\sinh\alpha}\left((z^2+t^2)\cosh\alpha-2zt\right)\right) \qquad (z\in\mathbb{R},t\in\mathbb{R}).$$

The Wigner distribution can also be defined for $F\in S^*$; it thus becomes a generalized function of two variables. An important formula is due to Moyal (cf. [4]): if $F\in S^*$, $f\in S$, then

(2.11) $$(W(F,F),\,W(f,f))=|(F,f)|^2.$$

Note now that (1.3) follows from (2.6) and (2.11) in case $\delta=\gamma^{-1}$.

We shall also use the formula

(2.12)

$$W(x,y;\, N_\alpha f, N_\alpha f) = (2\sinh\alpha)^{-1}\exp\big(-2\pi(x^2+y^2)\tanh\alpha\big) \cdot \iint \exp\Big(-2\pi\coth\alpha\big((z-x/\cosh\alpha)^2+(w-y/\cosh\alpha)^2\big)\Big) \cdot W(z,w;\, f,f)\, dz\, dw$$

for $x\in\mathbb{R}$, $y\in\mathbb{R}$; this is just another way to write [4, Thm. 16.1]. Here $f\in S$, but it is easy to extend (2.12) so that it holds for $F\in S^*$ (cf. [10], where things like these are treated in detail).

3. The main result. In [9], a generalized function Φ of 2 variables is called nonnegative ($\Phi\geq 0$), if $(\Phi,\varphi)\geq 0$ for every nonnegative test function φ of two variables. It can be shown from the Riesz representation theorem (also cf. [9, App. 4]) that for such a Φ there is a unique Borel measure μ_Φ on $\mathbb{R}^2$, such that

$$\iint \exp\big(-\pi\varepsilon(x^2+y^2)\big)\,d\mu_\Phi(x,y)<\infty \quad \text{for all } \varepsilon>0,$$

and such that $(\Phi,\varphi)=\iint\overline{\varphi(x,y)}\,d\mu_\Phi(x,y)$ for all test functions φ. This notion of nonnegativity agrees with the familiar notion of nonnegativity, a.e., if Φ is an ordinary function.

THEOREM 1. *Let $F\in S^*$, and assume that $W(F,F)\geq 0$. Then F is of the form* (1.4) *or* (1.5).

Proof. Let $\Phi := W(F,F)$, and assume that $F\neq 0$. This implies by (2.11) that $\Phi\neq 0$, whence $\mu_\Phi\neq 0$. We conclude from (2.11) and (2.6) that

$$(3.1)\qquad |(F,G_1(a,b))|^2 = \big(W(F,F),\, W(G_1(a,b),\, G_1(a,b))\big) = \iint \exp\big(-2\pi(x-a)^2-2\pi(y-b)^2\big)\,d\mu_\Phi(x,y)>0,$$

for all $a\in\mathbb{R}$, $b\in\mathbb{R}$. That is, $(F,G_1(a,b))\neq 0$ for all $a\in\mathbb{R}$, $b\in\mathbb{R}$. We see from (2.4) that $(BF)\,(z)\neq 0$ for all $z\in\mathbb{C}$. Since BF is an entire function of order 2, type $\leq\pi/2$, we conclude that BF is of the form $(BF)(z)=\exp(P(z))$, where $P(z)=az^2+bz+c$, with $|a|\leq\pi/2$. Hence, by (2.7) and (2.8), and injectivity of B, F is of the form (1.4) or (1.5). This completes the proof. □

As an incidental note we remark that with a similar method one can show the following . Assume that $F\in S^*$ has a radially symmetric Wigner distribution. Then F is a multiple of a Hermite function ψ_n. Here we call a generalized function Φ of two variables radially symmetric if $(\Phi,\varphi\circ U_\theta)=(\Phi,\varphi)$ for all test functions φ and all $\theta\in\mathbb{R}$, where $(\varphi\circ U_\theta)(x,y)=\varphi(x\cos\theta+y\sin\theta,\, -x\sin\theta+y\cos\theta)$ for $(x,y)\in\mathbb{R}^2$. For the proof one observes that, by radial symmetry of $W(F,F)$ and $W(G_1(0,0),\, G_1(0,0))$ and (3.1), the expression $|(F,G_1(a,b))|^2$ only depends on a^2+b^2. This implies that $|(BF)(z)|$ only depends on $|z|$, whence, by the maximum modules principle, $(BF)(z)=cz^n$ for some $c\in\mathbb{C}$, $n=0,1,\cdots$. Hence $F=d\psi_n$ for some $d\in\mathbb{C}$. Also see [11], [13], where it is proved that

$$W(x,y;\,\psi_n,\psi_n)=2(-1)^n\exp\big(-2\pi(x^2+y^2)\big)L_n\big(4\pi(x^2+y^2)\big),$$

with L_n the nth Laguerre polynomial.

It is fairly easy to generalize the previous theorem as follows.

THEOREM 2. *Let* $F\in S^*$, $\delta>0$, $\gamma>0$, $\delta\gamma>1$, *and assume that* F *is not of the form* (1.4) *or* (1.5). *Then the convolution of* $W(F,F)$ *with* $\exp(-2\pi\delta x^2-2\pi\gamma y^2)$ *takes negative values.*

Proof. We see from (2.9) that $N_\alpha F$ is not of the form (1.4) or (1.5) if $\alpha>0$. Hence, by the previous theorem, $W(N_\alpha F,N_\alpha F)$ takes negative values. Then (2.12) shows that the convolution of $W(F,F)$ and $\exp(-2\pi\coth\alpha(x^2+y^2))$ takes negative values. This proves the theorem in case $\gamma=\delta=\coth\alpha$.

In general we can express, by a transformation of variables and (3.2) below, the convolution of $\exp(-2\pi\delta x^2-2\pi\gamma y^2)$ and $W(F,F)$ at the point (a,b), as the inner product of $\exp(-2\pi\rho((x-a\varepsilon)^2+(y-b\varepsilon^{-1})^2))$ and $W(Z_\varepsilon F,Z_\varepsilon F)$. Here $\rho=(\delta\gamma)^{1/2}$, $\varepsilon=(\delta/\gamma)^{1/4}$ and Z_ε is the operator defined by $(Z_\varepsilon f)(t)=\varepsilon^{-1/2}f(\varepsilon^{-1}t)$ for $f\in S$, and extended in the obvious way (cf. [10, 1.15]) to S^*. We use here that for $f\in S$, $x\in\mathbb{R}$, $y\in\mathbb{R}$,

$$W(\varepsilon^{-1}x,\varepsilon y;f,f)=W(x,y;Z_\varepsilon f,Z_\varepsilon f), \tag{3.2}$$

a formula that can be generalized straightforwardly so as to hold for $f\in S^*$ as well. It is clear that if F is not of the form (1.4) or (1.5), then neither is $Z_\varepsilon F$. Since we can find an $\alpha>0$ such that $\rho=\coth\alpha$, we conclude from the special case already treated that the proof is complete. □

4. An application. It is believed that the only curve a Wigner distribution can be concentrated on is a straight line[1]; this is true only if certain restrictions on the curve are imposed (cf. the examples at the end of this section). We shall give a proof for the following simple case. Let C be a continuously differentiable curve in the plane with parametrization $\gamma:\ \mathbb{R}\to\mathbb{R}^2$, where we assume that $|\gamma'(t)|>0$ for all t. Assume that for all $t_0\in\mathbb{R}$ there is a straight line l passing through $\gamma(t_0)$, but not tangent to C, such that there is $\varepsilon>0$, $\delta>0$, with the property that the distance between $\gamma(s)$ and $l\geq\varepsilon$, if $|\gamma(s)-\gamma(t_0)|\geq\delta$. This condition is satisfied, e.g., if C is the graph of a continuously differentiable function defined on $\mathbb{R}$. Now let $F\in S^*$ be a function whose Wigner distribution is concentrated on C in the following sense: there is a continuous function $h:\ C\to\mathbb{R}$, such that $h(\gamma(t))=O(\exp(\varepsilon|\gamma(t)|^2))$ for all $\varepsilon>0$, and

$$(W(F,F),\varphi)=\int_{-\infty}^{\infty}h(\gamma(t))\varphi(\gamma(t))|\gamma'(t)|\,dt \tag{4.1}$$

for all test functions φ of two variables. We shall show that this implies that F is of the form (1.4) (with $\operatorname{Re}\alpha=0$) or (1.5), so that, in particular, C is a straight line. To this end let $\gamma(t_0)=(a,b)$ be a point on C and consider for $\operatorname{Re}\alpha>0$ the function $g_{\alpha,a,b}$, given by

$$g_{\alpha,a,b}(t)=\exp\left(-\pi\alpha(t-a)^2+2\pi ibt-\pi iab\right)\qquad(t\in\mathbb{R}), \tag{4.2}$$

whose Wigner distribution $W_{\alpha,a,b}$ is given by

(4.3)

$$W_{\alpha,a,b}(x,y)=\left(\frac{2}{\operatorname{Re}\alpha}\right)^{1/2}\exp\left(-2\pi(x-a)^2\operatorname{Re}\alpha-2\pi(y-b+(x-a)\operatorname{Im}\alpha)^2/\operatorname{Re}\alpha\right).$$

We have by (2.11) and (4.1)

$$0\leq(W(F,F),\,W_{\alpha,a,b})=\int_{-\infty}^{\infty}h(\gamma(t))W_{\alpha,a,b}(\gamma(t))|\gamma'(t)|\,dt. \tag{4.4}$$

[1] Cf. [1]. I thank Alan Weinstein for calling my attention to this paper.

Now let l be the line through $\gamma(t_0)$ whose existence is assured by our assumptions, and take α such that $\{(x,y)|y=b-(x-a)\operatorname{Im}\alpha\}$ is the graph of l. (If l is parallel to the y-axis we can use a similar argument with

$$\left(\frac{2}{\gamma}\right)^{1/2}\exp\left(-\pi\gamma^{-1}(t-a)^2+2\pi ibt-\pi iab\right)$$

instead of $g_{\alpha,a,b}$, where we take $\gamma\to 0$). If we let $\operatorname{Re}\alpha\to 0$, the right-hand side of (4.4) tends to $C_0h(\gamma(t_0))|\gamma'(t_0)|$, where $C_0>0$ is a number that depends only on the angle between l and the tangent line at C through $\gamma(t_0)$. Hence $h(\gamma(t_0))\geq 0$. We easily see from our theorems and (1.6)–(1.8) that F is of the form (1.4) (with $\operatorname{Re}\alpha=0$) or (1.5).

Notes. The condition "h continuous" can be relaxed to "h measurable" at the expense of elegance of the proof. It is furthermore likely that the conditions on the curve C can be relaxed somewhat as well. On the other hand, consider the function $f=\Sigma_n\delta_n$, whose Wigner distribution is given by $\frac{1}{2}\Sigma_{k,l}(-1)^{kl}\delta_{k/2}\otimes\delta_{l/2}$, where the summations are over all integers (this follows from a straightforward calculation and the Poisson summation formula, written in the form $\Sigma_n\delta_n(x)=\Sigma_n e^{-2\pi inx}$). The points of the lattice $(\frac{k}{2},\frac{l}{2})$ can be joined by a smooth curve C; such a C does not satisfy our assumptions, of course. Another objection is that the function h cannot be continuous in this case. This is not a serious point, however, as can be shown as follows. Let k_0: $\mathbb{R}\to\mathbb{R}$ be continuous, and assume that k_0 vanishes outside $[-\frac{1}{8},\frac{1}{8}]$. The Wigner distribution of k_0*f (where f is as above and $*$ denotes convolution) is obtained by convolving $W(f,f)$ and $W(k_0,k_0)$ with respect to the first variable (cf. [5, 4.1]). We get

$$W(x,y;k_0*f,k_0*f)=\frac{1}{2}\sum_{k,l}(-1)^{kl}W\left(x-\frac{k}{2},y;k_0,k_0\right)\delta_{l/2}(y) \tag{4.5}$$

(this formula can also be derived by directly using the Poisson summation formula). Since $W(k_0,k_0)$ is concentrated in the strip $[-\frac{1}{8},\frac{1}{8}]\times\mathbb{R}$, we see that $W(k_0*f,k_0*f)$ is concentrated in the set $\{(x+\frac{k}{2},\frac{l}{2})||x|\leq\frac{1}{8},\ k\in\mathbb{Z},\ l\in\mathbb{Z}\}$. The components of this set can be embedded in a smooth curve, and the function h now becomes continuous, since $W(k_0,k_0)$ is continuous.

A second example showing that one has to be careful with the statement, "$W(f,f)$ cannot be concentrated on a curve unless this curve is a straight line," is the function $f(t)=\cos 2\pi t$, whose Wigner distribution equals $\frac{1}{4}(\delta_{2\pi}(y)+\delta_{-2\pi}(y)+2\delta_0(y)\cos 4\pi x)$. Now $W(f,f)$ is concentrated on the three lines $y=0$, $y=\pm 2\pi$, and these lines can be embedded in a smooth curve.

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