

GABOR REPRESENTATION AND WIGNER DISTRIBUTION OF SIGNALS

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ABSTRACT

We compare Gabor representation and the Wigner distribution on their merits for the time-frequency description of signals.

INTRODUCTION

We consider 2 methods of describing time signals in time and frequency simultaneously, viz. the Gabor representation method and the Wigner distribution method. In the first method, one develops a signal $f(t)$ as a series

$$f(t) = \sum_{n,m} c_{nm}(f) G_{nm}(t), \quad (1)$$

with

$$G_{nm}(t) = \exp(2\pi i m t \beta - \pi i n m \alpha^2) g(t - n\alpha), \quad (2)$$

where $\alpha > 0, \beta > 0$ and $g(t)$ is a fixed time function with unit energy. Gabor [1] considered the case $\alpha\beta = 1, g(t) = 2^{1/4} \exp(-\pi t^2)$, so that the "elementary signals" G_{nm} are concentrated around the points $(n\alpha, m\alpha^{-1})$ in the time-frequency plane. In Gabor's terminology, the time-frequency plane is partitioned into logons, rectangles of unit area, and the signal f is completely described by the data $c_{nm}(f)$ assigned to these logons. In the Wigner distribution method, the Wigner distribution of f is defined as

$$W_{f,f}(t,\omega) = \int_{-\infty}^{\infty} e^{-2\pi i s \omega} f(t+\frac{1}{2}s) f^*(t-\frac{1}{2}s) ds; \quad (3)$$

this $W_{f,f}(t,\omega)$ is to be interpreted as a kind of energy density function of f in time and frequency. In this paper we are interested in how the 2 modes of description are related, how they compare for certain test signals, which one of the 2 has the best resolution, and so on. Due to lack of space we shall dispense with all proofs, although most facts can be easily established from [2], [3] and [4].

GABOR EXPANSIONS

Gabor series, as in (1), were suggested by D. Gabor for the purpose of efficient signal transmission as early as 1946, but the questions about existence and uniqueness of the $c_{nm}(f)$'s, as well as convergence of the series have been settled only rather recently. See [2-9]. These investigations have revealed that, at least in Gabor's case $g(t) = 2^{1/4} \exp(-\pi t^2)$, one has to restrict to the $\alpha\beta = 1$ case: if $\alpha\beta > 1$, one can find well-behaved signals f that cannot be represented as a Gabor series, and if $\alpha\beta < 1$, there are many double sequences $(c_{nm}) \neq 0$ such that $c_{nm} \rightarrow 0$ rapidly as $n^2 + m^2 \rightarrow \infty$ while $\sum_{n,m} c_{nm} G_{nm}(t) \equiv 0$.

The case $\alpha\beta = 1$ is not without problems either: while it is true, in Gabor's case, that $(c_{nm}) = 0$ whenever $c_{nm} \rightarrow 0 (n^2 + m^2 \rightarrow 0)$ and $\sum_{n,m} c_{nm} G_{nm}(t) \equiv 0$, it also holds that $\sum_{n,m} (-1)^{n+m} c_{nm} G_{nm}(t) \equiv 0$ (see [2, 6]). This shows that the G_{nm} 's are rather close to being linearly dependent, so that we may expect problems in determining the Gabor coefficients.

We shall consider from now on the case $\alpha = \beta = 1$, and we allow g to be an arbitrary time function of unit energy. The expansion coefficients $c_{nm}(f)$ in (1) can be found as follows (see [4], [8], [9]). For any signal $h(t)$, define the Zak transform $(Th)(z,\omega)$ by

$$(Th)(z,\omega) = \sum_{k=-\infty}^{\infty} h(z-k) e^{-2\pi i k \omega}. \quad (4)$$

It can then be shown that the $c_{nm}(f)$'s are, at least formally, the Fourier coefficients of the function Tf/Tg . That is,

$$c_{nm}(f) = (-1)^{nm} \int_0^1 \int_0^1 e^{-2\pi i (mz + n\omega)} \frac{(Tf)(z,\omega)}{(Tg)(z,\omega)} dz d\omega. \quad (5)$$

Among the many properties of the Zak transform, the following ones are especially noteworthy:

(a) T is one-to-one and onto from $L^2(\mathbb{R})$ into $L^2([0,1]^2)$,

(b) $\int_0^1 \int_0^1 |(Th)(z,\omega)|^2 dz d\omega = \int_{-\infty}^{\infty} |h(t)|^2 dt,$

$$(c) h(z) = \int_0^1 (Th)(z, \omega) d\omega, (gh)(\omega) = \int_0^1 e^{-2\pi i z \omega} (Th)(z, -\omega) dz,$$

(d) if Th is continuous, then Th has a zero in the unit square.

Property (d) together with formula (5) reveal the kind of problems that may occur when determining the expansion coefficients. We see that, when g is smooth and decays sufficiently fast at ∞ , Tg is differentiable so that, in view of (d), 1/Tg is not square integrable over the unit square. As a consequence, one will usually find, even for smooth signals f of finite duration, expansion coefficients $c_{nm}(f)$ with $\sum_{nm} |c_{nm}(f)|^2 = \infty$. Hence, in general, there is no way to relate the signal energy $\int_{-\infty}^{\infty} |f(t)|^2 dt$ to the sum of the squares of the $c_{nm}(f)$'s. Furthermore, convergence of the series $\sum_{nm} c_{nm}(f) G_{nm}(t)$ is usually a doubtful matter. This must be considered as a serious drawback of the Gabor method. On the other hand, if Tg has only one zero, at (c,d), say, in the unit square, and Tf has a zero at (a,b), then it can be shown that $T(R_b-dT_{a-c}f)$ has a zero at (c,d) as well, so that $T(R_b-dT_{a-c}f)/Tg$ has a fair chance to be bounded around (c,d). Here $(T_x f)(t) = f(t+x)$, $(R_y f)(t) = \exp(-2\pi i y t) f(t)$ for real x and y. Hence, $R_b-dT_{a-c}f$ is likely to have a decently converging Gabor expansion. This requires however, knowledge of the zeros of Tf.

A different method that can be used to compute the $c_{nm}(f)$'s works as follows. Define, whenever this makes sense, the function g_0 by

$$g_0(z) = \int_0^1 \frac{d\omega}{(Tg)^*(z, \omega)} \quad (z \in (-\infty, \infty)). \quad (6)$$

It can then be shown that

$$(Tg_0)(z, \omega) = \frac{1}{(Tg)^*(z, \omega)}, \quad (7)$$

and the expansion coefficients can be obtained from g_0 and f as

$$c_{nm}(f) = (-1)^{nm} \int_{-\infty}^{\infty} e^{-2\pi i m t} f(t) g_0^*(t-n) dt \quad (8)$$

In certain cases, (8) is more convenient to use than (5), especially when g_0 can be evaluated explicitly.

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Formula (8) can be used to express $|c_{nm}(f)|^2$ in terms of the Wigner distribution of f. Indeed, it follows from the well-known formula of Moyal and elementary properties of the Wigner distribution that

$$|c_{nm}(f)|^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{f, f}(t, \omega) W_{g_0, g_0}^*(t-n, \omega-m) dt d\omega. \quad (9)$$

The Wigner distribution W_{g_0, g_0} will, in general, have significant contributions everywhere in the time-frequency plane. This claim can be proved, for instance, when g is smooth and rapidly decaying so that Tg has zeros to the extent that 1/Tg is not square integrable over the unit square. It can be shown that

$$W_{g_0, g_0}(x, y) = \int_0^1 \int_0^1 \frac{2 \exp(4\pi i y z) dz d\omega}{(Tg)^*(z, \omega) (Tg)(z+2x, \omega+2y)}, \quad (10)$$

and, due to the periodicity relations (integer n and m)

$$(Tg)(z+n, \omega+m) = e^{2\pi i n \omega} (Tg)(z, \omega), \quad (11)$$

we must conclude that W_{g_0, g_0} is singular at all points $(x, y) = (n/2, m/2)$ with integer n and m. It can furthermore be shown that

$$\sum_{n, m} |W_{g_0, g_0}(x+\frac{1}{2}n, y+\frac{1}{2}m)|^2 = \infty \quad (12)$$

for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. Hence, in addition to having many singularities, W_{g_0, g_0} has poor decay properties. It thus seems that the $|c_{nm}(f)|^2$ have poor resolution properties compared to $W_{f, f}$.

EXAMPLES

1. Consider the case $g(t) = 1$ or 0 according as $t \in [0, 1]$ or not. Now $(Tg)(z, \omega) = 1$ on $[0, 1]^2$, and for any signal f we have

$$c_{nm}(f) = (-1)^{nm} \int_0^1 f(z+n) e^{-2\pi i m z} dz. \quad (13)$$

Hence, we chop f into pieces $f_n(z) = f(z)$ or 0 according as $z \in [n, n+1]$ or not, and we develop each f_n in a Fourier series to obtain the c_{nm} 's.

2. Let f be periodic with period 1 so that $f(t) = \sum_k c_k e^{2\pi i k t}$. It can be shown then that

$$(Tf)(z, \omega) = f(z) \sum_{k=-\infty}^{\infty} \delta(\omega - k), \quad (14)$$

whence

$$c_{nm}(f) = (-1)^{nm} \int_0^1 \frac{f(z) e^{-2\pi i m z}}{(Tg)(z, 0)} dz. \quad (15)$$

This shows that $c_{nm}(f)$ can be computed by convolving the Fourier coefficients of f and $1/(Tg)(z, 0)$. Observe also that

$$W_{f, f}(t, \omega) = \sum_{n \text{ even}} \delta(\omega - \frac{n}{2}) \sum_{m \text{ even}} e^{2\pi i m t} c_{\frac{n+m}{2}}^* c_{\frac{n-m}{2}}^* + \sum_{n \text{ odd}} \delta(\omega - \frac{n}{2}) \sum_{m \text{ odd}} e^{2\pi i m t} c_{\frac{n+m}{2}}^* c_{\frac{n-m}{2}}^* \quad (16)$$

In this case the Gabor and Wigner method give rise to expressions of comparable complexity. If, however, f has a period $\alpha^{-1} \neq 1$, the formula for $c_{nm}(f)$ becomes more complicated. It can be shown then that for $f(t) = \sum_k c_k \exp(2\pi i k \alpha t)$ we have

$$c_{nm}(t) = (-1)^{nm} \sum_{k=-\infty}^{\infty} c_k e^{2\pi i n k a} \int_0^1 \frac{\exp(2\pi i (m+k\alpha)z)}{(\text{Tg})(z, k\alpha - Lk\alpha)} dz \quad (17)$$

with $[a]$ largest integer $\leq a$.

3. We consider in the remainder of this section the Gabor case $g(t) = 2^{1/4} \exp(-\pi t^2)$. It turns out that

$$(\text{Tg})(z, \omega) = 2^{1/4} e^{-\pi z^2} \theta_3(\omega + iz), \quad (18)$$

where θ_3 is the third theta function $\sum_n e^{-\pi n^2 + 2\pi i n t}$. Now Tg has exactly one zero in the unit square, viz. at $(1/2, 1/2)$. It can furthermore be shown that the function g_0 of (6), (7) and (8) is given by

$$g_0(t) = \delta \sum_{n-\frac{1}{2} \geq t} (-1)^n \exp(-\pi(n-\frac{1}{2}-t)(n-\frac{1}{2}+t)), \quad (19)$$

where δ is a constant. This g_0 is bounded, but not square integrable. And the Fourier coefficients of the function $1/(\text{Tg})(z, 0)$ as it occurs in (15) are $g_0(k)$. Note that $g_0(k)$ decays exponentially when $|k| \rightarrow \infty$.

The Gabor coefficients can be calculated explicitly for certain functions. We have e.g. for the chirp $f(t) = \exp(\pi i t^2)$

$$c_{nm}(t) = (-1)^{n+nm} a_{m-n}, \quad a_n = \sum_{k=0}^{\infty} b_{n+1-k} e^{-2\pi i (n+k)k} \quad (20)$$

and b_n is of the form $d(a + bc^{2n})c^n$ with a, b, d constants and $c = -ie^{-\pi/2}$. Similar, but more complicated formulas, can be given for $f(t) = \exp(\pi i \alpha t^2)$, $\alpha = p/q$ rational. For the case that α is irrational we were unable to find expressions for the c_{nm} 's. Note that $W_{f,f}(t, \omega) = \delta(\omega - \alpha t)$ for $f(t) = \exp(\pi i \alpha t^2)$. Hence, here the Wigner distribution is more convenient.

To show the slow decay properties of W_{g_0, g_0} we note that it can be shown that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi(t-x)^2 - 2\pi(t-y)^2} W_{g_0, g_0}(t, \omega) dt d\omega = \frac{F(x, y)}{x^2 + y^2}, \quad (21)$$

where F is periodic in its 2 variables, non-negative, has zeros at the lattice points (n, m) and satisfies $F(x, y) = F(y, x)$. Finally, it can be shown that W_{g_0, g_0} has logarithmic singularities at all points $(n/2, m/2)$.

Below we display g_0 as a function of t . This figure has been borrowed from M.J. Bastiaans' paper [9].

CONCLUSIONS

We draw the following conclusions from the previous sections. The idea of describing an arbitrary signal f by means of a double sequence $(c_{nm}(f))$ that gives an indication of the energy distribution of f over time and frequency is attractive from a theoretical point of view; there are several practical drawbacks, though. Such a description can be effectuated by expanding f as a series $\sum_{nm} c_{nm}(f) G_{nm}$, where G_{nm} are time-frequency translates (see (2)) of a fixed function g .

Expressions for the $c_{nm}(f)$'s are given by formulas (5) and (8). It can be observed that,

in general, $\sum_{nm} |c_{nm}(f)|^2 = \infty$, even when both f and g are well-behaved functions of finite energy. Furthermore, the calculation of the

$c_{nm}(f)$'s will be cumbersome in general, since both the double integral in (5) and the integral over an infinite interval in (8) have poor convergence properties. It appears that the Wigner

distribution behaves better in this respect: the speed of convergence of (3) is intrinsically determined by the signal f itself. Of course, as opposed to the $c_{nm}(f)$'s, the Wigner distribution has to be calculated for 2 continuous variables.

With respect to resolution it can be said that, usually, the Wigner method gives better results than does the Gabor method. This is apparent from the formulas (9-12), showing that $|c_{nm}(f)|^2$ can be obtained from the convolution of the Wigner

distribution of f with a function of poor decay properties. Since neither $W_{f,f}$ nor W_{g_0, g_0} is positive everywhere, formula (9) is slightly (and sometimes quite) misleading, though, because of

the occurrence of cancellations in the double integral. The examples show that analytical expressions for the coefficients $c_{nm}(f)$ may be quite complicated or hard to derive, even for

"easy" signals like periodic functions and chirps. It may also happen that 2 signals have convenient Gabor representations relative to 2 different lattices ($n\alpha, m\alpha^{-1}$) while there is

no α that works well for both signals at the same time. Our general conclusion is that the Gabor method has several interesting theoretical aspects, but that for practical purposes the Wigner method is likely to be more useful.

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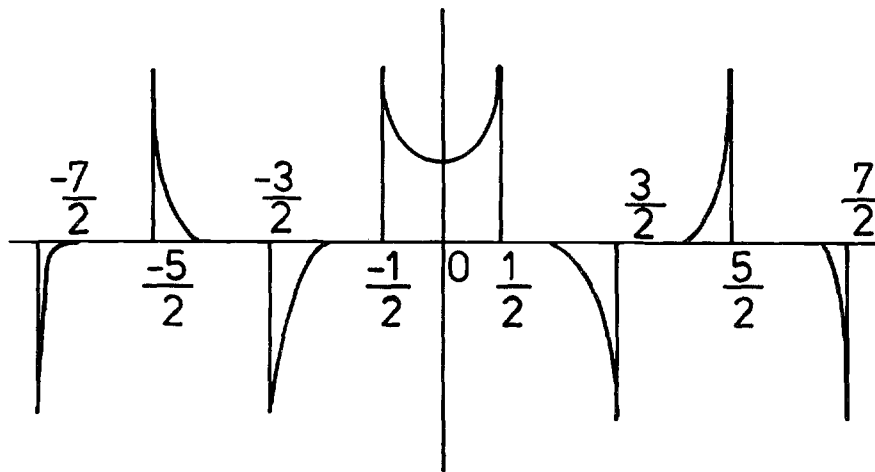


Fig. 1. The function $g_0(t)$ corresponding to the choice $g(t) = 2^{1/4} \exp(-\pi t^2)$.