INTERPOLATION OF BAND-LIMITED DISCRETE-TIME SIGNALS BY MINIMIZING OUT-OF-BAND ENERGY

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ABSTRACT

An interpolation method for restoring burst errors in discrete-time, band-limited signals is presented. The restoration is such that the restored signal has minimal out-of-band energy. The filter coefficients depend only on the burst length and on the size of the band to which the signal is assumed to be band-limited. The influence of additive noise and the effect of violation of the band-limitedness assumption is analysed with the aid of discrete prolate spheroidal sequences and wave functions. It is indicated for what combinations of values for noise power, burst length and bandwidth the method is still stable enough to be practicable.

INTRODUCTION

The aim of this paper is to analyse an interpolation algorithm for restoring burst errors in discrete-time signals that have (almost) all spectral energy in a given baseband. A possible application consists of restoration of burst errors in the audio signals from Compact Disc. These signals usually have spectra that are reasonably well contained in a baseband whose length is about 2/3 of the total bandwidth, and the burst lengths that occur are usually between 1 and 6.

Choose the sampling time equal to unity, and consider signals $s(k)$, $-\infty < k < \infty$ for which the Fourier spectrum $S(\theta)$, given by

$$S(\theta) = \sum_{k=-\infty}^{\infty} s(k)\, e^{2\pi j k\theta} \qquad (1)$$

is "negligibly small" outside the interval $|\theta| \leq \alpha/2$. Here α is a fixed number (i.e. independent of the particular signal) between 0 and 1. Assume now that the signal $s(k)$ is unknown for $k=0,1,\ldots,m-1$, and that one has to reconstruct the unknown samples from the known ones (m is a fixed integer). The restoration method investigated here is based on the following principle : choose the missing samples $s(0)$, $s(1)\ldots,s(m-1)$ in such a way that the restored signal has a minimal amount of out-of-band energy. That is, minimize

$$\int_{\frac{\alpha}{2}\leq|\theta|\leq\frac{1}{2}} |S(\theta)|^2 d\theta \qquad (2)$$

as a function of the missing samples. This method has been worked out in detail in [1], with special emphasis on mathematical rigor. In the present paper the results of [1] are presented, conclusions are drawn and the performance of the method in practice is shown.

Minimizing (2) with respect to the unknown samples $s(0)$, $s(1),\ldots,s(m-1)$ gives the following solution. Consider the low-pass matrix M, given by

$$M = \left(\frac{\sin \pi(k-l)\alpha}{\pi(k-l)}\right)_{-\infty<k,l<\infty} \qquad (3)$$

This M is such that

$$(Ms)(k) = \sum_{l=-\infty}^{\infty} \frac{\sin\pi(k-l)\alpha}{\pi(k-l)}\, s(l) = s(k) \qquad (4)$$

for all k when s is indeed band-limited to $\alpha/2$. Now the estimate $\hat{\underline{z}} = (\hat{z}(0), \hat{z}(1), \ldots, \hat{z}(m-1))^T$ for the vector of missing samples obtained by minimizing (2) is

$$\hat{\underline{z}} = (I-M_0)^{-1}\underline{y}_0, \qquad (5)$$

where

$$\underline{y}_0 = ((Ms_{tr})(0),(Ms_{tr})(1),\ldots,(Ms_{tr})(m-1))^T, \qquad (6)$$

$s_{tr}(k) = 0$ or $s(k)$ according as $0 \leqslant k \leqslant m-1$ or not, and M_0 is the square Toeplitz matrix, given by

$$M_0 = \left(\frac{\sin\pi(k-l)\alpha}{\pi(k-l)}\right)_{k,l=0,1,\ldots,m-1} . \qquad (7)$$

In view of (4) it is interesting to note that the solution of the interpolation problem is such that

$$\sum_{k=-\infty}^{\infty} |(Ms)(k) - s(k)|^2 \qquad (8)$$

is minimal as a function of the missing samples for $\hat{\underline{z}}=(s(0), s(1),\ldots,s(m-1))^T$. In particular, if s is indeed band-limited to $\alpha/2$, one obtains perfect restoration with this method.

As an aside it is noted that $\hat{\underline{z}}$ can be obtained iteratively from $\underline{y}_0$ (see (6)) and M_0 (see (7)) by means of the series expansion

$$\underline{z} = \underline{y}_0 + M_0\underline{y}_0 + M^2_0\underline{y}_0 + \ldots \qquad . (9)$$

Hence the solution to the interpolation problem is of the same mathematical form as the one to the extrapolation problem discussed by Sabri and Steenaart in [2].

Since the interpolation method is to be used for "real-life" signals (such as audio signals from Compact Disc) containing (quantization) noise that

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are almost never completely restricted to a fixed frequency band, one has to find out how the method performs when noise or small out-of-band components are present. A further point is the feasibility of the matrix inversion in (5), and the effect of windowing the signal for the calculation of $\underline{y}_0$ in (6). All these issues can be dealt with by using asymptotic properties of the eigenvalues and eigenvectors of the matrix M_0 that have been studied extensively by Slepian in [3]. This will be done in the next section where also interpolation results are given for certain test signals as well as "real-life" signals.

PERFORMANCE ANALYSIS

Let M_0 be the Toeplitz matrix given in (7), and denote the eigenvalues and normalized eigenvectors of M_0 by λ_k and $\underline{v}_k = (v_k(0), v_k(1), \ldots, v_k(m-1))^T$ respectively $(k=0,1,\ldots,m-1)$. It is understood here that $1 > \lambda_0 > \lambda_1 > \cdots > \lambda_{m-1} > 0$. Note that both λ_k and $\underline{v}_k$ depend on m and α. The following properties are crucial : when k is small and fixed and $m \to \infty$, then (see [1, 2.17)] , [3] or [4] , section B)

$$1-\lambda_k \sim \frac{\sqrt{2\pi}}{k!}\left[\frac{8m\sin\pi\frac{\alpha}{2}}{\cos^2\pi\frac{\alpha}{2}}\right]^{k+\frac{1}{2}}\left[\frac{\cos\pi\frac{\alpha}{2}}{1+\sin\pi\frac{\alpha}{2}}\right]^{2m}, \qquad (10)$$

and (see [1, section 3])

$$v_0(l) > 0, \quad l = 0,1,\ldots,m-1. \qquad (11)$$

It is known that (10) is accurate already for small values of m (see the plots in [3]); e.g. when $\alpha = 1/2$, it gives good results for $m \geq 5$. One can see from (10) that, for small k,

$$\frac{1-\lambda_{k+1}}{1-\lambda_k} \sim \frac{8m\sin\pi\frac{\alpha}{2}}{(k+1)\cos^2\pi\frac{\alpha}{2}} \qquad (12)$$

when $m \to \infty$. Thus, when α is not too small, $1-\lambda_0 \ll 1-\lambda_1 \ll \cdots$.

It can now be indicated when inversion of the matrix $I-M_0$ is feasible. In fig. 1 level curves of trace $(I-M_0)^{-1}$ as a function of m and α are plotted, i.e. graphs $(m, f_c(m))$, where $\alpha = f_c(m)$ is such that trace $(I-M_0)^{-1} = c$, and c takes the values 10^2–10^7. Since trace $(I-M_0)^{-1} \approx (1-\lambda_0)^{-1}$, it is concluded from (10) that

$$f_c(m) \sim \frac{\log c}{\pi m} \qquad (m \to \infty). \qquad (13)$$

Indeed, the graphs $(m, f_c(m))$ resemble hyperboles. Since trace $(I-M_0)^{-1}$ is a measure for feasibility of inversion of $I-M_0$, it is seen that the band to which the signals must be assumed to be limited decreases roughly as 1/m.

Next consider the effect on the interpolation results of addition of white noise to a signal band-limited to $\alpha/2$. In [1 , section 4] the following has been proved : let $s(k) = x(k) + n(k)$ with $x(k)$ band-limited to $\alpha/2$ and $n(k)$ white noise with zero mean and variance σ^2. Then the interpolation error $\underline{\varepsilon} = (s(0)-\hat{z}(0), s(1)-\hat{z}(1), \ldots, s(m-1)-\hat{z}(m-1))^T$ is a random vector of the form

$$\underline{\varepsilon} = \sum_{k=0}^{m-1} p_k \underline{v}_k , \qquad (14)$$

where p_k, $k=0,1,\ldots,m-1$ are random variables with

$$E[p_k]=0, E[p_k p_l]=\sigma^2\lambda_k(1-\lambda_k)^{-1}\delta_{kl}. \qquad (15)$$

In the sum $\sum_{k=0}^{m-1} p_k \underline{v}_k$, the term with k=0 is usually highly dominant, for $E[p_k]^2 = \sigma^2\lambda_k(1-\lambda_k)^{-1}$ (see (12) and (15)). Hence, in view of property (11), the interpolation error tends to be pulse-shaped. Depending on the application, this must be considered as a drawback of the interpolation method (for application in Compact Disc, pulse-shaped errors are certainly undesirable).

The asymptotic formula (10) can now be used to find what combinations of values for σ^2, m and α are allowed if one wants to keep the interpolation error below a fraction a of the average signal energy $E = \sum_{k=0}^{m-1} |s(k)|^2$. The result is that m and α should roughly satisfy

$$m\alpha \leq \frac{1}{\pi}\log\left[\frac{aE}{\sigma^2}\right]. \qquad (16)$$

Fig. 2 shows the interpolation result for a sine to which white noise has been added (S/N ratio 40 dB). The power of the interpolation error is 1.3 x signal power. Also notice that the error is pulse-shaped.

To study the influence of the presence of out--of-band components in the signals to be restored one can use the following property (see [1 , Theorem 4.2]): let $|\theta| > \frac{\alpha}{2}$ and let $s_\theta(k) = e^{-2\pi jk\theta}$. The interpolation error $\underline{\varepsilon}_\theta = (s_\theta(0)-\hat{z}_\theta(0), s_\theta(1)-\hat{z}_\theta(1), \ldots, s_\theta(m-1)-\hat{z}_\theta(m-1))^T$ is of the form

$$\underline{\varepsilon}_\theta = \sum_{k=0}^{m-1} c_k(\theta)\underline{v}_k , \qquad (17)$$

where, for small k, $c_k(\theta)$ is a rapidly varying function with envelope of the order $(1-\lambda_k)^{-1/2}$ in $|\theta| \in (\frac{\alpha}{2}, \frac{1}{2})$.

For an arbitrary signal s(k) the interpolation error $\underline{\varepsilon} = (s(0)-\hat{z}(0), s(1)-\hat{z}(1), \ldots, s(m-1)-\hat{z}(m-1))^T$ can be expressed as (see (1) for the definition of $S(\theta)$)

$$\underline{\varepsilon} = \sum_{k=0}^{m-1}\left(\int_{\frac{\alpha}{2}<|\theta|<\frac{1}{2}} c_k(\theta)S(\theta)d\theta\right)\underline{v}_k . \qquad (18)$$

Note that the term with k=0 is usually the most important one.

Figs. 3a-f show interpolation results for some audio signals from Compact Disc with m = 4 and α = 0.65. It is seen that the method does not perform very well when the signal contains significant high frequency components.

Consider finally the effect of windowing the signal s(k) on the calculation of $\hat{\underline{z}}$ in (5). Note that $\hat{\underline{z}}$ can be written as

$$\hat{\underline{z}} = \sum_{l=-\infty}^{\infty} s_{tr}(l)(I-M_0)^{-1}\underline{r}_l , \qquad (19)$$

where $\underline{r}_l$ is the m-vector $\left(\frac{\sin\pi(k-l)\alpha}{\pi(k-l)}\right)_{k=0,1,\ldots,m-1}$. It has been shown in [1], section 4 that $\|(I-M_0)^{-1}\underline{r}_l\|$ decays slowly as $l \to \infty$, and that $\|(I-M_0)^{-1}\underline{r}_l\|$ is maximal for $|l-(m-1)/2| \sim \frac{m}{2\cos\pi\frac{\alpha}{2}}$! Hence, in general a large signal segment will be needed to restore the burst.

CONCLUSIONS

An interpolation method has been investigated for the restoration of burst errors in discrete-time signals that can be considered to be band-limited. The performance of this method depends very critically on how realistic the band-limitedness assumption is. It has been observed that the presence of noise or out-of-band components of very low power can already produce significant interpolation errors; this effect becomes worse as the product $m\alpha$ (where m is the burst length and α the width of the band to which the signal is assumed to be limited) increases. It is therefore concluded that the method is only practicable for modest values of m and for situations where the signals to be interpolated obey the band-limitedness assumption quite well.

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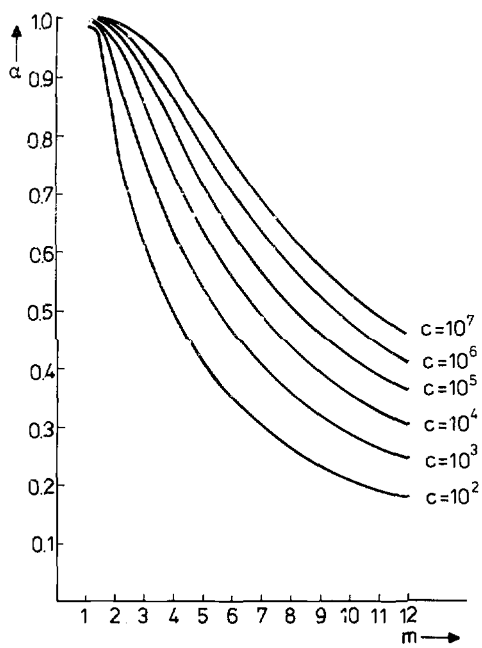

Fig. 1 Level curves of trace $(I-M_0)^{-1}$ as a function of burst length m and bandwidth α for the levels $c = 10^2, 10^3, 10^4, 10^5, 10^6, 10^7$.

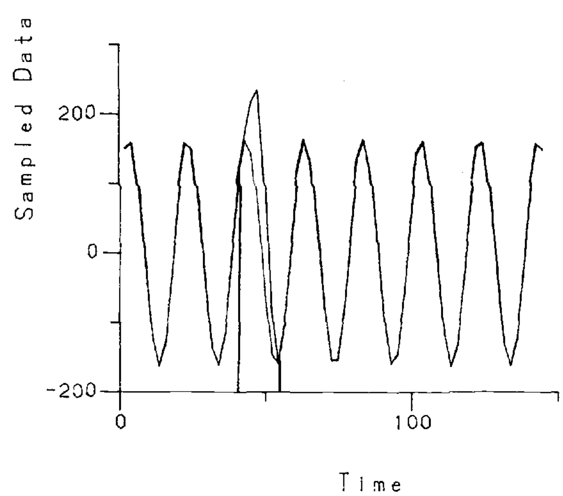

Fig. 2 Interpolation result for a sine with additive white noise; amplitude of sine $2^{14}-1$, frequency of sine 5/22, noise power 13422 (-40 dB), burst length m=4, bandwidth α =15/22.

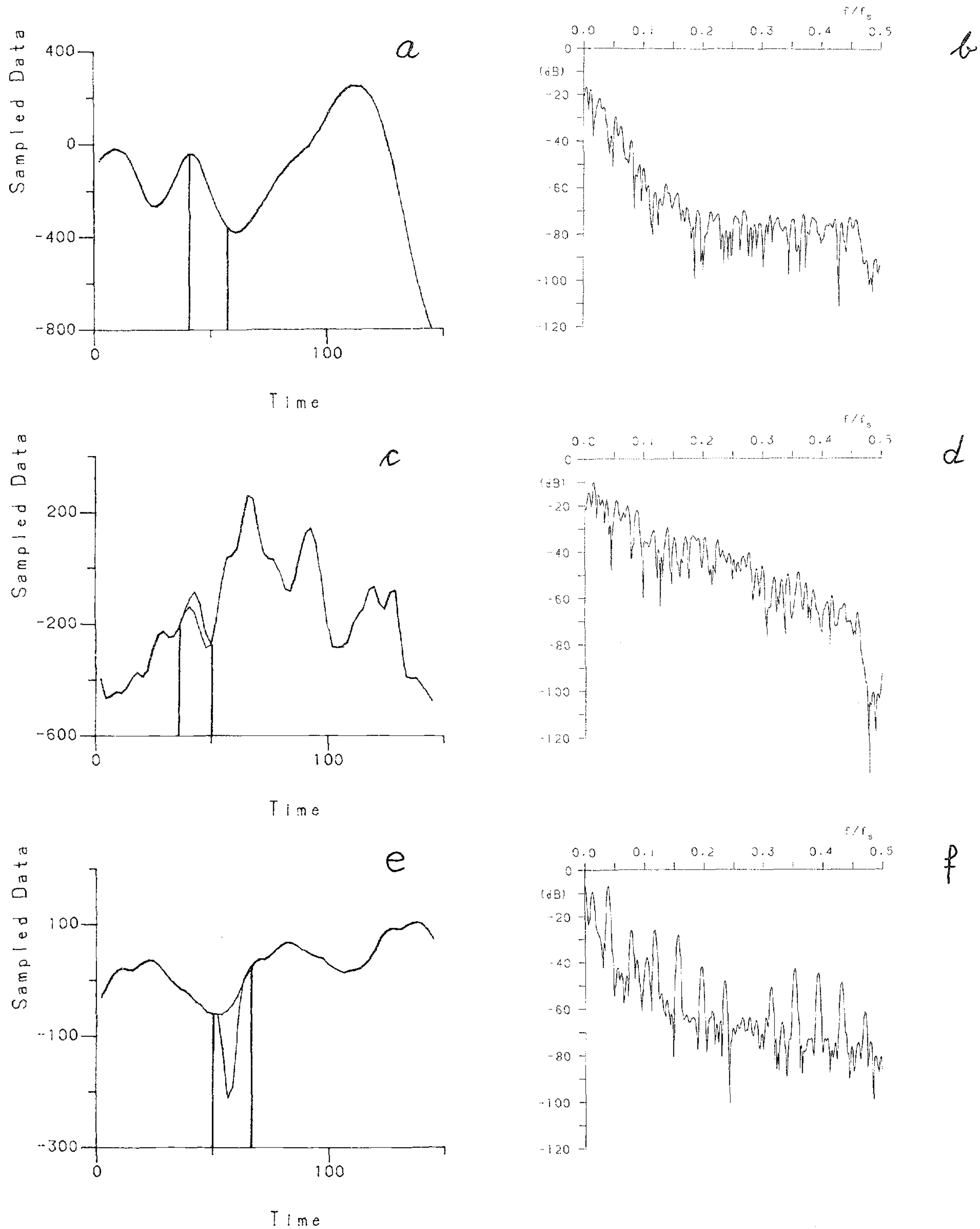

Fig. 3 Interpolation results for three pieces of music from Compact Disc; burst length m=4 and bandwidth α = 15/22. Figs. 3a (from Piano Concert of Beethoven), 3c (from Great Gates of Kiev) and 3e(from Violin Concert of Beethoven) show the original signals and the restored mutilated signals. Figs. 3b, 3d, 3f show the respective frequency spectra for figs. 3a, 3c, 3e (for which a neighbourhood of 512 samples was used).