Bilinear phase-plane distribution functions and positivity

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There is a theorem of Wigner that states that phase-plane distribution functions involving the state bilinearly and having correct marginals must take negative values for certain states. The purpose of this paper is to support the statement that these phase-plane distribution functions are for hardly any state everywhere non-negative. In particular, it is shown that for certain generalized Wigner distribution functions there are no smooth states (except the Gaussians for the Wigner distribution function itself) whose distribution function is everywhere non-negative. This class of Wigner-type distribution functions contains the Margenau–Hill distribution. Furthermore, the argument used in the proof of Wigner's theorem is augmented to show that under mild conditions one can find for any two states f, g with non-negative distribution functions a linear combination h of f and g whose distribution function takes negative values, unless f and g are proportional.

I. INTRODUCTION AND PRELIMINARIES

The formulation of quantum mechanics by means of phase-plane distribution functions involving the states bilinearly allows one to exhibit quantum mechanical expectation values as averages over the phase-plane of classical observables. Given a bilinear map[1] $(f,g) \to C_{f,g}$ mapping pairs of states onto functions of position q and momentum p, one can formulate a correspondence principle between bounded self-adjoint linear operators T of $L^2(\mathbb{R})$ and functions $a(q,p)$ as follows: T and a are said to correspond to each other when

$$(Tf,g) = \iint a(q,p) C_{f,g}(q,p)\,dq\,dp\,, \tag{1}$$

for all $f \in L^2(\mathbb{R})$, $g \in L^2(\mathbb{R})$. Here $(\ ,\ )$ denotes the usual inner product in $L^2(\mathbb{R})$. Of course, in order that to any T (or a) there is a unique $a = a_T$ (or $T = T_a$) such that (1) holds, the mapping $(f,g) \to C_{f,g}$ should satisfy certain properties. When one takes $f = g$ in (1) the left-hand side equals the expectation of T in the state f while the right-hand side equals an average of the classical observable a corresponding to T, where $C_{f,f}$ is used as the weight function.

In view of the interpretation of $C_{f,f}$ as a distribution function, one would like the following properties to be satisfied: (a) correct marginals, i.e., for all states f one has

$$\int C_{f,f}(q,p)\,dp = |f(q)|^2, \quad q \in \mathbb{R}\,, \tag{2}$$

$$\int C_{f,f}(q,p)\,dq = |F(p)|^2, \quad p \in \mathbb{R}\,, \tag{3}$$

where F is the Fourier transform[2] of f, given by

$$F(p) = \int e^{-2\pi i q p} f(q)\,dq, \quad p \in \mathbb{R}\,; \tag{4}$$

(b) positivity, i.e., for all states f one has

$$C_{f,f}(q,p) \geqslant 0, \quad q \in \mathbb{R}, \quad p \in \mathbb{R}\,. \tag{5}$$

It has been shown by Wigner[3] that the requirements (a) and (b) (together with the bilinearity) are incompatible. What one can distill from the arguments in his proofs is this: when f_1 and f_2 are two compactly supported states with $f_1 \not\equiv 0$, $f_2 \not\equiv 0$, $f_1(q) f_2(q) = 0$ for all q while the requirements (a) and (b) are met, then $C_{f,f}(q,p)$ takes negative values, where $f = f_1 + f_2$. Hence, there is an abundance of states whose distribution functions take negative values. A particular strong result of this type, known as Hudson's theorem,[4] holds for the Wigner distribution[5] [See (11) below with $\alpha = 0$]: the only square-integrable states for which the Wigner distribution is everywhere non-negative are the Gaussians.

It is the purpose of this paper to give generalizations on both Wigner's theorem and Hudson's theorem. The starting point we take in this paper differs slightly from the one taken by Wigner in Ref. 3. Wigner assumes the existence of self-adjoint operators $M(q,p)$ of $L^2(\mathbb{R})$ such that for all states f,

$$C_{f,f}(q,p) = (f, M(q,p) f), \quad q \in \mathbb{R}, \quad p \in \mathbb{R}\,. \tag{6}$$

We consider in this paper Cohen's class of phase-plane distribution functions.[6–8] This class is parametrized by means of a function Φ of two variables as follows: for any state f one defines

$$C^{(\Phi)}_{f,f}(q,p) = \iiint \exp(-2\pi i(\theta q + \tau p - \theta u)) \times \Phi(\theta,\tau) f(u + \tfrac{1}{2}\tau)\overline{f(u - \tfrac{1}{2}\tau)}\,d\theta\,d\tau\,du\,,$$
$$q \in \mathbb{R}, \quad p \in \mathbb{R}\,. \tag{7}$$

The distributions in Cohen's class all have the shift properties

$$\begin{aligned} C^{(\Phi)}_{T_a f, T_a f}(q,p) &= C^{(\Phi)}_{f,f}(q+a,p), \quad q \in \mathbb{R}, \quad p \in \mathbb{R}\,,\\ C^{(\Phi)}_{R_b f, R_b f}(q,p) &= C^{(\Phi)}_{f,f}(q,p+b), \quad q \in \mathbb{R}, \quad p \in \mathbb{R}\,, \end{aligned} \tag{8}$$

where for all $a \in \mathbb{R}$, $b \in \mathbb{R}$ and all states $f \in L^2(\mathbb{R})$,

$$(T_a f)(q) = f(q+a), (R_b f)(q) = e^{-2\pi i b q} f(q), \quad q \in \mathbb{R}\,. \tag{9}$$

In this paper we pay particular attention to the choice

$$\Phi_\alpha(\theta,\tau) = \exp(2\pi i \alpha \theta \tau), \quad \theta \in \mathbb{R}, \quad \tau \in \mathbb{R}\,, \tag{10}$$

where $\alpha \in \mathbb{R}$. This yields what may be called generalized Wigner distributions $C^{(\alpha)}_{f,f}$ that can be brought into the form

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$$C^{(\alpha)}_{f,f}(q,p)\equiv C^{(\Phi_\alpha)}_{f,f}(q,p)$$
$$= \int e^{-2\pi ipt} f\Big(q+\Big(\frac{1}{2}-\alpha\Big)t\Big) \times \overline{f\Big(q-\Big(\frac{1}{2}+\alpha\Big)t\Big)}dt. \quad (11)$$

The choice $\alpha = 0$ yields the Wigner distribution $W_{f,f}$, and the choice $\alpha = -\frac{1}{2}$ yields what is known in signal analysis as Rihaczek's distribution[9] $R_{f,f}$. The latter distribution can be written as

$$R_{f,f}(q,p) = e^{2\pi iqp}\,\overline{f(q)}F(p), \quad q\in\mathbb{R}, \quad p\in\mathbb{R}, \quad (12)$$

with F given in (4). The real part of Rihaczek's distribution was considered by Margenau and Hill.[10] We observe here that $R_{f,f}(q,p)$ cannot be brought in the form (6) with a bounded linear operator $M(q,p)$ of $L^2(\mathbb{R})$.

As is well known[11] every $C_{f,f}$ can be expressed in terms of $W_{f,f}$ as

$$C^{(\Phi)}_{f,f}(q,p) = \iint \varphi(q-a,p-b)W_{f,f}(a,b)\,da\,db, \quad q\in\mathbb{R}, \quad p\in\mathbb{R}, \quad (13)$$

with

$$\varphi(q,p) = \iint e^{-2\pi i(\theta q+\tau p)}\Phi(\theta,\tau)d\theta\, d\tau, \quad q\in\mathbb{R}, \quad p\in\mathbb{R}. \quad (14)$$

As an aside we note that this φ can be used to relate the operators $M(q,p)$ of Wigner's approach in (6) and the function Φ in Cohen's approach. To that end we assume that Φ is such that $|C^{(\Phi)}_{f,f}(0,0)|\leqslant M\,\|f\|^2$ for some $M>0$. Then the operator $M(0,0)$, determined by

$$(f,M(0,0)g) = C^{(\Phi)}_{f,g}(0,0), \quad f,g\in L^2(\mathbb{R}), \quad (15)$$

is bounded, and we have, according to (13),

$$(f,M(0,0)f) = \iint \varphi(-a,-b)W_{f,f}(a,b)\,da\,db, \quad f\in L^2(\mathbb{R}). \quad (16)$$

Hence, $\varphi(-a,-b)$ is what is called the Weyl symbol[12] of the operator $M(0,0)$, i.e., $\varphi(-a,-b)$ and $M(0,0)$ correspond to each other as in (1) when we take $C_{f,g} = W_{f,g}$ (Weyl correspondence[13]). Observe that, because of the shift properties, we have

$$M(q,p) = R_{-p}\,T_{-q}\,M(0,0)T_q\,R_p, \quad q\in\mathbb{R}, \quad p\in\mathbb{R}. \quad (17)$$

We shall restrict ourselves in this paper to functions Φ that are bounded. The boundedness of Φ ensures that $C^{(\Phi)}_{f,f}\in L^2(\mathbb{R}^2)$ for all $f\in L^2(\mathbb{R}^2)$. In fact we have the estimate[14]

$$\iint |C^{(\Phi)}_{f,f}(q,p)|^2\,dq\,dp \leqslant \|f\|^4 \sup|\Phi|^2, \quad (18)$$

for all $f\in L_2(\mathbb{R})$. It does not follow from boundedness of Φ that $C^{(\Phi)}_{f,f}(0,0)$ can be expressed as in (15) with the aid of a bounded operator $M(0,0)$. As a counterexample we have $R_{f,f}(0,0)$ in (12). However, when (15) does hold with a bounded $M(0,0)$ it can be shown from (16) and the fact that $C^{(\Phi)}_{f,f}(q,p) = (f,M(q,p)f)$ that all $C^{(\Phi)}_{f,f}$'s are bounded, continuous functions[15] of (q,p).

A second restriction is that we want real $C^{(\Phi)}_{f,f}$'s only. This is guaranteed when the φ in (14) is real valued[16] (possibly as a generalized function). A third restriction is that we want the $C^{(\Phi)}_{f,f}$'s to have correct marginals. It is well known[17] that $C^{(\Phi)}_{f,f}$ has correct marginals for all $f\in L^2(\mathbb{R})$ if and only if

$$\Phi(0,\tau) = \Phi(\theta,0) = 1, \quad \theta\in\mathbb{R}, \quad \tau\in\mathbb{R}. \quad (19)$$

We now summarize the results of this paper. It is shown in Sec. II, under a regularity condition on Φ, that for any two smooth functions f, g with $C^{(\Phi)}_{f,f}>0$, $C^{(\Phi)}_{g,g}>0$ everywhere we can find an $a\in\mathbb{C}$ such that $C^{(\Phi)}_{f+ag,f+ag}$ takes negative values, unless f and g are proportional. In Sec. III we consider smooth states f for which Re $C^{(\alpha)}_{f,f}(q,p)\geqslant 0$ in (q,p) sets of the form $(a,b)\times\mathbb{R}$. It is shown, for example, that for such an f we have $|f(q)| = \exp(\psi(q))$ with ψ concave on (a,b) [it is assumed here that $f(q)\neq 0$ for $q\in(a,b)$]. We also show that, if $\alpha\neq 0$, there is no smooth state $f\not\equiv 0$ such that Re $C^{(\alpha)}_{f,f}(q,p)\geqslant 0$ for all $q\in\mathbb{R}$, $p\in\mathbb{R}$. The restriction to smooth states is not entirely necessary but makes the proofs run smoothly; a class of functions which are sufficiently smooth is the set $\mathscr{S}$ of Schwartz [it is, however, sufficient to require the states to be sufficiently often differentiable and to decay as rapidly as $(1+q^2)^{-k}$ with k sufficiently large]. For $|\alpha| = \frac{1}{2}$ we have a stronger result, viz. there is no $f\in L^2(\mathbb{R})$, $f\not\equiv 0$ such that Re $R_{f,f}\geqslant 0$ almost everywhere [see (12)]. A remarkable phenomenon here is that there do exist generalized functions $f\not\equiv 0$ with Re $R_{f,f}\geqslant 0$ (in generalized sense). This is remarkable since the sets of smooth and generalized functions f for which $W_{f,f} = C^{(0)}_{f,f}\geqslant 0$ everywhere are essentially the same, i.e., (degenerate) Gaussians. We conjecture stronger results than the ones proved in this paper. This is based on the fact that we have not been able to find any Φ (other than $\Phi\equiv 1$, Wigner distribution case) and any $f\in L^2(\mathbb{R})$ (other than Gaussians) for which $C^{(\Phi)}_{f,f}\geqslant 0$ everywhere.

II. A RESULT FOR GENERAL BILINEAR PHASE-PLANE DISTRIBUTION FUNCTIONS

We consider in this section bounded Φ's for which $C^{(\Phi)}_{f,f}$ is real valued and has correct marginals for all $f\in L^2(\mathbb{R})$. The arguments used in the proof of the theorem below are somewhat similar to those used by Wigner in Ref. 1.

Theorem 1: Assume that the set $\{(\theta,\tau)|\Phi(\theta,\tau)\neq 0\}$ is dense in $\mathbb{R}^2$, and let $f\not\equiv 0$, $g\not\equiv 0$ be smooth states such that $C^{(\Phi)}_{f,f}>0$, $C^{(\Phi)}_{g,g}>0$ everywhere. Then there is an $a\in\mathbb{C}$, $b\in\mathbb{C}$ such that $C^{(\Phi)}_{af+bg,af+bg}$ takes negative values, unless f and g are proportional.

Proof: Suppose that

$$C^{(\Phi)}_{af+bg,af+bg}(q,p)\geqslant 0, \quad q\in\mathbb{R}, \quad p\in\mathbb{R}, \quad a\in\mathbb{C}, \quad b\in\mathbb{C}. \quad (20)$$

We shall show that f and g are proportional by using the following steps.

(1) We show that

$$|f(q)|^2C^{(\Phi)}_{g,g}(q,p) = |g(q)|^2C^{(\Phi)}_{f,f}(q,p), \quad q\in\mathbb{R}, \quad p\in\mathbb{R}. \quad (21)$$

Let $q\in\mathbb{R}$, and take $a_0\in\mathbb{C}$, $b_0\in\mathbb{C}$ such that $a_0f(q)+b_0g(q) = 0$. Then

$$C^{(\Phi)}_{a_0f+b_0g,a_0f+b_0g}(q,p) = 0, \quad p\in\mathbb{R}, \quad (22)$$

because of (20) and the correct marginals condition (2) applied to a_0f+b_0g. Hence we have for all $a\in\mathbb{C}$, $b\in\mathbb{C}$, $p\in\mathbb{R}$ (see Ref. 18),

$$|a|^2C^{(\Phi)}_{f,f}(q,p) + |b|^2C^{(\Phi)}_{g,g}(q,p) + 2\,\mathrm{Re}\,a\bar{b}C^{(\Phi)}_{f,g}(q,p) \geqslant 0\,, \tag{23}$$

with equality when $a = a_0$, $b = b_0$. Let $p \in \mathbb{R}$, $a_0 = -g(q)/f(q)$, $b = 1$, and write $A = C^{(\Phi)}_{f,f}(q,p)$, $B = C^{(\Phi)}_{g,g}(q,p)$, $C = C^{(\Phi)}_{f,g}(q,p)$, $a = x + iy$, $a_0 = x_0 + iy_0$. Now (23) can be written as

$$P(x,y) := A(x^2 + y^2) + B + 2x\,\mathrm{Re}\,C - 2y\,\mathrm{Im}\,C \geqslant 0\,, \tag{24}$$

with equality when $x = x_0$, $y = y_0$. Since for all $x \in \mathbb{R}$, $y \in \mathbb{R}$,

$$P(x,y) = A(x + A^{-1}\,\mathrm{Re}\,C)^2 + A(y - A^{-1}\,\mathrm{Im}\,C)^2 + B - A^{-1}|C|^2 \geqslant 0\,, \tag{25}$$

and $P(x_0, y_0) = 0$ it follows that $AB = |C|^2$, $C = -A(x_0 - iy_0)$, i.e.,

$$|C^{(\Phi)}_{f,g}(q,p)|^2 = C^{(\Phi)}_{f,f}(q,p)C^{(\Phi)}_{g,g}(q,p)\,,$$
$$C^{(\Phi)}_{f,g}(q,p) = \frac{\overline{g(q)}}{\overline{f(q)}}\,C^{(\Phi)}_{f,f}(q,p)\,. \tag{26}$$

Now elimination of $C_{f,g}(q,p)$ gives (21).

(2) We have

$$|F(p)|^2C^{(\Phi)}_{g,g}(q,p) = |G(p)|^2C^{(\Phi)}_{f,f}(q,p), \quad q \in \mathbb{R}, \quad p \in \mathbb{R}\,. \tag{27}$$

This is proved in a similar way as (21).

(3) There is a constant $D > 0$ such that

$$C^{(\Phi)}_{g,g}(q,p) = DC^{(\Phi)}_{f,f}(q,p), \quad q \in \mathbb{R}, \quad p \in \mathbb{R}\,. \tag{28}$$

Indeed from (21) and (27) we get

$$\frac{C^{(\Phi)}_{g,g}(q,p)}{C^{(\Phi)}_{f,f}(q,p)} = \frac{|G(p)|^2}{|F(p)|^2} = \frac{|g(q)|^2}{|f(q)|^2}, \quad q \in \mathbb{R}, \quad p \in \mathbb{R}\,, \tag{29}$$

and (28) follows.

(4) The proof is completed as follows. We see from (13) and (28) that $\varphi * (W_{g,g} - DW_{f,f}) = 0$. Here $*$ denotes two-dimensional convolution. Performing the inverse double Fourier transform and using the convolution theorem we get by (14)

$$\Phi(\theta,\tau)[A_{g,g}(\theta,\tau) - DA_{f,f}(\theta,\tau)] = 0, \quad \theta \in \mathbb{R}, \quad \tau \in \mathbb{R}\,. \tag{30}$$

Here

$$A_{g,g}(\theta,\tau) = \int e^{2\pi i\theta t} g\Big(t + \frac{1}{2}\tau\Big)\overline{g\Big(t - \frac{1}{2}\tau\Big)}dt, \quad \theta \in \mathbb{R}, \quad \tau \in \mathbb{R}\,, \tag{31}$$

and $A_{f,f}(\theta,\tau)$ is defined similarly. Since $A_{g,g}, A_{f,f}$ are continuous functions and $\Phi(\theta,\tau) \neq 0$ in a set of (θ,τ), which is dense in $\mathbb{R}^2$, we conclude that $A_{g,g}(\theta,\tau) = DA_{f,f}(\theta,\tau)$ for all $(\theta,\tau) \in \mathbb{R}^2$. It follows easily that g is a multiple of f.

Notes: (1) When we have a Φ such that Moyal's formula

$$\iint C^{(\Phi)}_{f,f}(q,p)C^{(\Phi)}_{g,g}(q,p)dq\,dp = |(f,g)|^2 \tag{32}$$

holds for all $f \in L^2(\mathbb{R})$, $g \in L^2(\mathbb{R})$, then $|\Phi(\theta,\tau)| = 1$ (see Ref. 19).

(2) When we take $\Phi(\theta,\tau) = \cos \pi\alpha\theta\tau$, so that $C^{(\Phi)}_{f,f} = \mathrm{Re}\,C^{(\alpha)}_{f,f}$ [cf. (11)], the condition that $\{(\theta,\tau) \mid \Phi(\theta,\tau) \neq 0\}$ is dense in $\mathbb{R}^2$ is satisfied. We shall show in Sec. III that there are no smooth states for which $\mathrm{Re}\,C^{(\alpha)}_{f,f} \geqslant 0$ everywhere.

(3) The condition of having correct marginals forces $\Phi(\theta,0) = \Phi(0,\tau) = 1 \neq 0$ for $\theta \in \mathbb{R}$, $\tau \in \mathbb{R}$. The proof of Theorem 1 shows that

$$|g(q)|^2 = D|f(q)|^2, \quad |G(p)|^2 = D|F(p)|^2\,, \quad q \in \mathbb{R}, \quad p \in \mathbb{R}\,, \tag{33}$$

when $C^{(\Phi)}_{af+bg,af+bg} \geqslant 0$ everywhere for all $a \in \mathbb{C}$, $b \in \mathbb{C}$. It does not follow, however, from (33) that f and g are proportional. [As a counterexample, take an $f \in L^2(\mathbb{R})$ with $|f(q)| = |f(-q)|$ for all $q \in \mathbb{R}$, and let $g(q) = \overline{f(-q)}$. Then $|g(q)| = |f(q)|$ for all $q \in \mathbb{R}$ and $G(p) = \overline{F(p)}$ so that $|G(p)| = |F(p)|$ for all $p \in \mathbb{R}$.]

Corollary: The condition $C^{(\Phi)}_{f,f} > 0$, $C^{(\Phi)}_{g,g} > 0$ everywhere can be replaced by the condition $C^{(\Phi)}_{f,f} \geqslant 0$, $C^{(\Phi)}_{g,g} \geqslant 0$ everywhere when at least one of the functions $f \cdot g$ and $F \cdot G$ does not vanish identically. To show this, we only have to provide a new proof of formula (28). To this end we let I and J be two open intervals where f and F, respectively, have no zeros. We claim that there is a constant $D_{I,J}$ such that $C_{g,g} = D_{I,J}C_{f,f}$ in the set $S_{I,J} = I \times \mathbb{R} \cup \mathbb{R} \times J$, while $|g/f|^2 = D_{I,J}$ in I, $|G/F|^2 = D_{I,J}$ in J. To see this we let $(q,p) \in S_{I,J}$ such that $C^{(\Phi)}_{f,f}(q,p) > 0$. Since $C_{f,f}$ is continuous, there are open intervals $I_q \subset I$, $J_p \subset J$ such that $C^{(\Phi)}_{f,f}(q,p) > 0$ in $I_q \times J_p$. Now formulas (21) and (27) show that there is a $D_{q,p} \geqslant 0$ such that $C^{(\Phi)}_{g,g}(q,p) = D_{q,p}C^{(\Phi)}_{f,f}(q,p)$ in $I_q \times J_p$, while $|g/f|^2 = D_{q,p}$ in I_q, $|G/F|^2 = D_{q,p}$ in J_p. Hence, $|g/f|^2$ is a continuous function on I and for every $q \in I$ there is an open interval I_q containing q where $|g/f|^2$ is constant. Hence $|g/f|^2$ is constant on I. Similarly, $|G/F|^2$ is constant on J, and our initial claim follows. When I, J and K,L are four open intervals such that f and F have no zeros in I,K and J,L, respectively, we find four constants $D_{I,J}$, $D_{I,L}$, $D_{K,J}$, $D_{K,L}$. These constants must all be equal since, e.g., $D_{I,J} = D_{I,L} = |g(q)/f(q)|^2$, when $q \in I$. It thus follows that there is a constant $D \geqslant 0$ such that $C^{(\Phi)}_{g,g}(q,p) = DC^{(\Phi)}_{f,f}(q,p)$ for all $(q,p) \in \mathbb{R}^2$ with $C^{(\Phi)}_{f,f}(q,p) > 0$. Similarly, there is a constant $E \geqslant 0$ such that $C^{(\Phi)}_{f,f}(q,p) = EC^{(\Phi)}_{g,g}(q,p)$ for all $(q,p) \in \mathbb{R}^2$ with $C^{(\Phi)}_{g,g}(q,p) > 0$. Now when there is a point q with $f(q)g(q) \neq 0$ or a point p with $F(p)G(p) \neq 0$ we see that $D \neq 0 \neq E$ and $E = D^{-1}$. And then $C^{(\Phi)}_{g,g} = DC^{(\Phi)}_{f,f}$ everywhere, as was required to prove. We note that there exist smooth $f \in L^2(\mathbb{R})$, $g \in L^2(\mathbb{R})$ with $f \cdot g = 0$, $F \cdot G = 0$. These examples can be found by properly smoothing, multiplying, and shifting the generalized function $f_0 = \Sigma^{\infty}_{n=-\infty}\delta_n$, whose Fourier transform equals $F_0 = \Sigma^{\infty}_{m=-\infty}\delta_m$.

III. WIGNER-TYPE PHASE-PLANE DISTRIBUTION FUNCTIONS THAT ARE NON-NEGATIVE

We consider in this section phase-plane distribution functions $\mathrm{Re}\,C^{(\alpha)}_{f,f}$ with $\alpha \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$ a smooth state. We are particularly interested in consequences for the state f of non-negativity of $\mathrm{Re}\,C^{(\alpha)}_{f,f}$ in certain strips in the phase plane. Our main result is that there are no smooth states f with $\mathrm{Re}\,C^{(\alpha)}_{f,f} \geqslant 0$ everywhere, except when $\alpha = 0$. This section is divided into four subsections. In Sec. III A we examine the consequence of non-negativity of $\mathrm{Re}\,C^{(\alpha)}_{f,f}$ in strips for the (smooth) state f. The results of this subsection are based on the inversion formulas

$$h(q,s):=f(q+(\tfrac{1}{2}-\alpha)s)\overline{f(q-(\tfrac{1}{2}+\alpha)s)}$$
$$=\int e^{2\pi isp}\,C^{(\alpha)}_{f,f}(q,p)dp\,, \tag{34}$$
$$H(p,s):=F(p-(\tfrac{1}{2}+\alpha)s)\overline{F(p+(\tfrac{1}{2}-\alpha)s)}$$
$$=\int e^{2\pi isq}\,C^{(\alpha)}_{f,f}(q,p)dq\,, \tag{35}$$

or rather

$$g(q,s):=\tfrac{1}{2}h(q,s)+\tfrac{1}{2}\overline{h(q,-s)}$$
$$=\int e^{2\pi isp}\,\mathrm{Re}\,C^{(\alpha)}_{f,f}(q,p)dp\,, \tag{36}$$
$$G(p,s):=\tfrac{1}{2}H(p,s)+\tfrac{1}{2}\overline{H(p,-s)}$$
$$=\int e^{2\pi isq}\,\mathrm{Re}\,C^{(\alpha)}_{f,f}(q,p)dq\,. \tag{37}$$

Another important fact is $C^{(\alpha)}_{F,F}(q,p)=C^{(\alpha)}_{f,f}(-p,q)$, $q\in\mathbb{R}$, $p\in\mathbb{R}$. In Sec. III B we give the proof of our main result for smooth functions and $0\neq|\alpha|\neq\frac{1}{2}$, and in Sec. III C we treat the case of $f\in L^2(\mathbb{R})$, $\alpha=\pm\frac{1}{2}$. The case $\alpha=0$ has been considered in Refs. 4 and 20 and does not need a proof here. Finally, Sec. III D contains some examples of generalized functions f and numbers α such that $\mathrm{Re}\,C^{(\alpha)}_{f,f}\geqslant 0$ everywhere.

A. States with $\mathrm{Re}\,C^{(\alpha)}_{f,f}\geqslant 0$ in a strip

We assume in this subsection that both f and F are smooth functions of q and p respectively so that all manipulations below can be justified (it is not hard to give more specific conditions that guarantee this).

Theorem 2: Let $-\infty<a<b<\infty$, and assume that $f(q)\neq 0$, $\mathrm{Re}\,C^{(\alpha)}_{f,f}(q,p)\geqslant 0$ for $q\in(a,b)$, $p\in\mathbb{R}$. Write $|f(q)|=\exp(\psi(q))$ with ψ smooth. Then

$$(\tfrac{1}{4}+\alpha^2)\,\psi''(q)+2\alpha^2(\psi'(q))^2\leqslant 0,\quad q\in(a,b)\,. \tag{38}$$

Proof: Write $f(q)=\exp(\psi(q)+i\varphi(q))$ with φ smooth on (a,b), and insert the expansions

$$\varphi(q+u)=\varphi(q)+u\varphi'(q)+\tfrac{1}{2}u^2\varphi''(q)+o(u^2),\quad u\to 0\,, \tag{39}$$
$$\psi(q+u)=\psi(q)+u\psi'(q)+\tfrac{1}{2}u^2\psi''(q)+o(u^2),\quad u\to 0\,, \tag{40}$$

into the definition (34) of $h(q,s)$ and $\overline{h(q,-s)}$. We get

$$\begin{aligned}h(q,s)&=\exp(2\psi(q)-2\alpha s\psi'(q)+(\tfrac{1}{4}+\alpha^2)s^2\psi''(q)\\&\quad+i[s\varphi'(q)-\alpha s^2\varphi''(q)]+o(s^2)),\\\overline{h(q,-s)}&=\exp(2\psi(q)+2\alpha s\psi'(q)+(\tfrac{1}{4}+\alpha^2)s^2\psi''(q)\\&\quad+i[s\varphi'(q)+\alpha s^2\varphi''(q)]+o(s^2)),\quad s\to 0\,.\end{aligned} \tag{41}$$

Hence

$$\begin{aligned}g(q,s)&=\tfrac{1}{2}h(q,s)+\tfrac{1}{2}\overline{h(q,-s)}\\&=\tfrac{1}{2}\exp(2\psi(q)+(\tfrac{1}{4}+\alpha^2)s^2\psi''(q)+is\varphi'(q))\\&\quad\times\{\exp(-2\alpha s\psi'(q)-i\alpha s^2\varphi''(q))\\&\quad+\exp(2\alpha s\psi'(q)+i\alpha s^2\varphi''(q))\}(1+o(s^2))\\&=\exp(2\psi(q)+(\tfrac{1}{4}+\alpha^2)s^2\psi''(q)+is\varphi'(q))\\&\quad\times\{\cosh(2\alpha s\psi'(q)+i\alpha s^2\varphi''(q))\}(1+o(s^2)),\quad s\to 0\,.\end{aligned} \tag{42}$$

Now

$$\cosh(2\alpha s\psi'(q)+i\alpha s^2\varphi''(q))=1+2\alpha^2s^2(\psi'(q))^2+o(s^2),\quad s\to 0\,, \tag{43}$$

so that

$$\begin{aligned}|g(q,s)|&=\exp(2\psi(q)+(\tfrac{1}{4}+\alpha^2)s^2\psi''(q))\\&\quad\times\{1+2\alpha^2s^2(\psi'(q))^2+o(s^2)\}(1+o(s^2))\\&=\exp(2\psi(q))\{1+(\tfrac{1}{4}+\alpha^2)s^2\psi''(q)\\&\quad+2\alpha^2s^2(\psi'(q))^2+o(s^2)\}(1+o(s^2))\,,\quad s\to 0\,.\end{aligned} \tag{44}$$

Since $\mathrm{Re}\,C^{(\alpha)}_{f,f}(q,p)\geqslant 0$, $p\in\mathbb{R}$, we see from (36) that

$$|g(q,s)|\leqslant g(q,0)=|f(q)|^2=\exp(2\psi(q)),\quad s\in\mathbb{R}\,. \tag{45}$$

This can only be true when (38) holds, and the proof is complete.

Notes: (1) When $-\infty<c<d<\infty$ and $F(p)\neq 0$, $\mathrm{Re}\,C^{(\alpha)}_{f,f}(q,p)\geqslant 0$ for $q\in\mathbb{R}$, $p\in(c,d)$, we have

$$(\tfrac{1}{4}+\alpha^2)\Psi''(p)+2\alpha^2(\Psi'(p))^2\leqslant 0,\quad p\in(c,d)\,, \tag{46}$$

where $|F(p)|=\exp(\Psi(p))$ and Ψ is smooth on (c,d).

(2) The condition (38) [and (46)] is weakest for $\alpha=0$. When $\alpha=0$ we see that (38) [(46)] means that $\psi(\Psi)$ is concave on (a,b) [(c,d)].

We next study the behavior of an f with $\mathrm{Re}\,C^{(\alpha)}_{f,f}(q,p)\geqslant 0$, where q is a zero of f.

Theorem 3: Let $|\alpha|\neq\frac{1}{2}$, $q\in\mathbb{R}$, and assume that $f(q)=0$, $\mathrm{Re}\,C^{(\alpha)}_{f,f}(q,p)\geqslant 0$, $p\in\mathbb{R}$. Then $f^{(n)}(q)=0$, $n=1,2,\ldots$.

Proof: We have as in the proof of Theorem 2 that

$$|g(q,s)|=|\tfrac{1}{2}h(q,s)+\tfrac{1}{2}\overline{h(q,-s)}|\leqslant|g(q,0)|=|f(q)|^2=0\,. \tag{47}$$

Let $n\in\mathbb{N}$ be the smallest number with $f^{(n)}(q)\neq 0$. Then

$$\begin{aligned}h(q,s)&=f(q+(\tfrac{1}{2}-\alpha)s)\overline{f(q-(\tfrac{1}{2}+\alpha)s)}\\&=[(\tfrac{1}{2}-\alpha)^n(s^n/n!)f^{(n)}(q)+o(s^n)]\\&\quad\times[(-(\tfrac{1}{2}+\alpha))^n(s^n/n!)\overline{f^{(n)}(q)}+o(s^n)]\\&=[(-1)^n(\tfrac{1}{4}-\alpha^2)^n/(n!)^2]|f^{(n)}(q)|^2s^{2n}\\&\quad+o(s^{2n}),\quad s\to 0\,.\end{aligned} \tag{48}$$

Similarly,

$$\overline{h(q,-s)}=[(-1)^n(\tfrac{1}{4}-\alpha^2)^n/(n!)^2]|f^{(n)}(q)|^2s^{2n}+o(s^{2n}),\quad s\to 0\,. \tag{49}$$

Hence

$$g(q,s)=[(-1)^n(\tfrac{1}{4}-\alpha^2)^n/(n!)^2]|f^{(n)}(q)|^2s^{2n}+o(s^{2n}),\quad s\to 0\,. \tag{50}$$

This contradicts (47), and therefore $f^{(n)}(q)=0$.

Theorem 4: Let $\alpha\neq 0$, $|\alpha|\neq\frac{1}{2}$. Assume that $-\infty<a<b<\infty$, that $\mathrm{Re}\,C^{(\alpha)}_{f,f}(q,p)\geqslant 0$ for $q\in(a,b)$, $p\in\mathbb{R}$, and that $f(a)\neq 0\neq f(b)$. Then the set $\{q\,|\,f(q)\neq 0\}$ is dense in (a,b).

Proof: Suppose we can find an interval $(c_0,d_0)\subset(a,b)$ such that $f(q)=0$ for $q\in(c_0,d_0)$. We can assume that both c_0

and d_0 are accumulation points of the set $\{q\,|\,f(q)\neq 0\}$. When $q\in(c_0,d_0)$ we have for $s\in\mathbb{R}$

$$f(q+(\tfrac12-\alpha)s)\overline{f(q-(\tfrac12+\alpha)s)} + \overline{f(q-(\tfrac12-\alpha)s)}f(q+(\tfrac12+\alpha)s)=0\,. \tag{51}$$

We consider the following cases.

(1) $0<\alpha<\frac12$. We can find $c_n<c_0$, $d_n>d_0$ with $f(c_n)\neq 0 \neq f(d_n)$, $c_n\uparrow c_0$, $d_n\downarrow d_0$. Now if we let

$$q_n=\tfrac12(c_n+d_n)+\alpha(d_n-c_n),\quad s_n=d_n-c_n,\quad n=1,2,\ldots, \tag{52}$$

then we have $q_n+(\frac12-\alpha)s_n=d_n$, $q_n-(\frac12+\alpha)s_n=c_n$ while $q_n-(\frac12-\alpha)s_n\in(c_0,d_0)$ for large n. We conclude that the left-hand side of (51) with $q=q_n$, $s=s_n$ equals $f(d_n)\overline{f(c_n)}\neq 0$ for large n. We thus have a contradiction.

(2) $-\frac12<\alpha<0$. This case is similar to the previous one.

(3) $\alpha>\frac12$. Let $c_n<c_0$ be such that $c_n\uparrow c_0, f(c_n)\neq 0$. Now if we let

$$q_{n,m}=\tfrac12(c_n+c_m)+\alpha(c_m-c_n),\quad s_{n,m}=c_m-c_n,\quad n,m=1,2,\ldots, \tag{53}$$

we have $q_{n,m}+(\frac12-\alpha)s_{n,m}=c_m$, $q_{n,m}-(\frac12+\alpha)s_{n,m}=c_n$. We can take n so large that $q_{n,\infty}+(\frac12+\alpha)s_{n,\infty}\in(c_0,d_0)$. Here $q_{n,\infty}=\frac12(c_n+c_0)+\alpha(c_0-c_n)$, $s_{n,\infty}=c_0-c_n$. Observe that $q_{n,\infty}>c_0$, and that $q_{n,m}<q_{n,\infty}$, $s_{n,m}<s_{n,\infty}$, $q_{n,m}\uparrow q_{n,\infty}$ when $m\to\infty$. Hence, when m is large enough, $q_{n,m}>c_0$ while $q_{n,m}+(\frac12+\alpha)s_{n,m}<d_0$. Therefore, the left-hand side of (51), with $q=q_{n,m}$, $s=s_{n,m}$, equals $f(c_m)\overline{f(c_n)}\neq 0$ when m is large enough. We thus have a contradiction.

(4) $\alpha<-\frac12$. This case is similar to the case $\alpha>\frac12$.

B. Smooth states $f\neq 0$ with Re $C^{(\alpha)}_{f,f}\geqslant 0$ everywhere do not exist: case $0\neq|\alpha|\neq\frac12$

We now start the proof of the statement that there are no smooth states f with Re $C^{(\alpha)}_{f,f}\geqslant 0$ everywhere for the case $0\neq|\alpha|\neq\frac12$; the case $\alpha=0$ is covered by Hudson's theorem and the case $|\alpha|=\frac12$ will be considered in Sec. III C. The proof is lengthy and consists of several steps; it can be outlined as follows. Suppose that we have a smooth $f\not\equiv 0$ with Re $C^{(\alpha)}_{f,f}\geqslant 0$ everywhere. It will be shown in Lemma 1 below that the zero set of f must be unbounded above and below; at the same time it will be shown that we must have $|\alpha|<\frac12$. In Lemma 2 below it will be shown that f cannot vanish identically on any interval. The remainder of the proof consists of a careful analysis of f around its zeros. When $f(a)=0$, we have

$$f(a+(\tfrac12-\alpha)s)\,\overline{f(a-(\tfrac12+\alpha)s)} = -\overline{f(a-(\tfrac12-\alpha)s)}f(a+(\tfrac12+\alpha)s),\quad s\in\mathbb{R}\,. \tag{54}$$

This identity can be used to show the following (Lemma 3): Let a be a zero of f for which there is a $\delta>0$ such that $f(q)\neq 0$ for $q\in(a-\delta,a)$. Then there is a smooth function k_a: $(0,\infty)\to\mathbb{C}$ such that

$$k_a(q)=-k_a([(\tfrac12+\alpha)/(\tfrac12-\alpha)]q)\,,\quad q>0\,, \tag{55}$$

while

$$f(a+q)=k_a(q)f(a-q),\quad q>0\,. \tag{56}$$

Then it can be shown that the zero set of f has the form $\{a+bl\,|\,l\in\mathbb{Z}\}$, where $a\in\mathbb{R}$, $b\in\mathbb{R}$. We finally derive a contradiction by showing the identity

$$k_a(q)=k_{a+b}(q-b)\,\overline{k_a(2b-q)}k_{a-b}(q),\quad b<q<2b\,, \tag{57}$$

in which the left-hand side behaves smoothly as $q\downarrow b$ whereas the right-hand side oscillates violently as $q\downarrow b$ because of the factor $k_{a+b}(q-b)$ and (55).

Lemma 1: (i) Let $0<|\alpha|<\frac12$, and assume Re $C^{(\alpha)}_{f,f}\geqslant 0$ everywhere. Then the zero set of f is unbounded below and above.

(ii) Let $|\alpha|>\frac12$. Then Re $C^{(\alpha)}_{f,f}$ takes negative values.

Proof: (i) Suppose there is a $b\in\mathbb{R}$ with $f(q)\neq 0$ for $q<b$. Then we can write $|f(q)|=\exp(\psi(q))$, where ψ is smooth and satisfies (38) for $q<b$. In particular ψ is concave on $(-\infty,b)$. Since $\int|f(q)|^2\,dq<\infty$ we must have that $\psi(q)\to-\infty$ as $q\to-\infty$. Furthermore we can find a $q_0<b$ such that $\psi'(q_0)>0$. Now $\psi'(q)\geqslant\psi'(q_0)$ for $q<q_0$, and, according to (38),

$$1/\psi'(q)-1/\psi'(q_0)=\int_q^{q_0}\frac{\psi''(r)}{(\psi'(r))^2}\,dr\leqslant -c_\alpha(q_0-q),\quad q\leqslant q_0\,, \tag{58}$$

where $c=2\alpha^2/(\frac14+\alpha^2)>0$. Therefore

$$1/\psi'(q)\leqslant 1/\psi'(q_0)-c_\alpha(q_0-q),\quad q\leqslant q_0\,. \tag{59}$$

The left-hand side of (59) is positive for $q\leqslant q_0$ whereas the right-hand side of (59) tends to $-\infty$ when $q\to-\infty$. Contradiction. Hence, f must have zeros in $(-\infty,b)$. In a similar way we conclude that f has zeros in any interval (a,∞).

(ii) Suppose Re $C^{(\alpha)}_{f,f}\geqslant 0$ everywhere and $|\alpha|>\frac12$. In the proof of (i) it was not used that $0<|\alpha|<\frac12$. Since $f\not\equiv 0$ we can find a $q_0\in\mathbb{R}$ with $f(q_0)\neq 0$. Now let $a:=\inf\{q_1\,|\,f(q)\neq 0$ for $q\in(q_1,q_0)\}$. Then $a>-\infty$, $f(a)=0$, and there is a $b>a$ such that $f(q)\neq 0$ for $q\in(a,b)$. As $f(q)\neq 0$ for $q\in(a,b)$ we have, according to (38),

$$|f|''(q)=[\psi''(q)+(\psi'(q))^2]\exp(\psi(q)) \leqslant -\frac{\alpha^2-\frac14}{\alpha^2+\frac14}(\psi'(q))^2\exp(\psi(q))\leqslant 0,\quad q\in(a,b)\,. \tag{60}$$

Hence $|f|$ is concave on (a,b) and $\lim_{q\downarrow a}|f|'(q)>0$ (this limit may be $+\infty$). But

$$|f|'(q)=\exp(-i\arg f(q))[f'(q)-i(\arg f)'(q)f(q)] = \exp(-i\arg f(q))f'(q) - i|f(q)|\,(\arg f)'(q),\quad q\in(a,b)\,, \tag{61}$$

and the real part of the right-hand side tends to 0 on account of Theorem 3. This results in a contradiction, and the proof is complete.

We assume from now on that $0\neq|\alpha|<\frac12$.

Lemma 2: The set $\{q\,|\,f(q)\neq 0\}$ is dense in $\mathbb{R}$.

Proof: We proceed according to the following steps.

(1) f cannot have compact support. For otherwise F is an analytic function. According to Lemma 1, F has zeros, and at a zero, p, of F we have according to Theorem 3 that $F^{(n)}(p)=0$. This is impossible in view of the analyticity of F and the fact that $f\not\equiv 0$.

(2) f cannot vanish identically on a semi-infinite interval. For suppose that $f(q)=0$ for $q\leqslant a$, where $a:=\max\{q_1\,|\,f(q)=0,\ q\leqslant q_1\}$. We know from step (2) that

the support of f is unbounded above. Hence, in view of Theorem 4 there are no intervals $[c,d]$ with $d>c>a$ such that f vanishes on $[c,d]$. By Lemma 1 we can find a $q>a$ with $f(q)=0$. When $\alpha<0$, we see from (54) that $\overline{f(q+(\frac{1}{2}-\alpha)s)} \times f(q-(\frac{1}{2}+\alpha)s)=0$ whenever $s \geqslant (\frac{1}{2}-\alpha)^{-1}(q-a)$. However, when $(\frac{1}{2}-\alpha)^{-1}(q-a)<s<(\frac{1}{2}+\alpha)^{-1}(q-a)$ we have $q+(\frac{1}{2}-\alpha)s>a$, $q-(\frac{1}{2}+\alpha)s>a$, so that neither $f(q+(\frac{1}{2}-\alpha)s)$ nor $f(q-(\frac{1}{2}+\alpha)s)$ can vanish identically for s in any subinterval of $(\frac{1}{2}-\alpha)^{-1}(q-a)$, $(\frac{1}{2}+\alpha)^{-1}(q-a)$. This contradicts the continuity of f. When $\alpha>0$ we can derive a contradiction in a similar way. It follows therefore that the support of f cannot be bounded below, and in a similar way it can be shown that the support of f is not bounded above. This completes the proof of step (2).

The proof of the lemma is now easily completed by using Theorem 4.

We assume now that $-\frac{1}{2}<\alpha<0$.

Lemma 3: Let a be a zero of f for which there is a $\delta>0$ such that $f(q)\neq 0$ for $q\in(a-\delta,a)$. Then there is a smooth function $k_a:(0,\infty)\mapsto\mathbb{C}$ such that $k_a(q)=-k_a(\beta q)$, $f(a+q)=k_a(q)f(a-q)$, $q>0$. Here $\beta=(\frac{1}{2}+\alpha)/(\frac{1}{2}-\alpha)<1$.

Proof: With β as given above we can write (54) as

$$f(a+q)\overline{f(a-\beta q)}=-\overline{f(a-q)}f(a+\beta q),\quad q\in\mathbb{R}. \tag{62}$$

Therefore, when $f(a-q)\neq 0\neq f(a-\beta q)$ (in particular when $0<q<\delta$)

$$f(a+q)/\overline{f(a-q)}=-f(a+\beta q)/\overline{f(a-\beta q)}. \tag{63}$$

Define

$$k_a(q):=f(a+q)/f(a-q),\quad 0<q<\delta, \tag{64}$$

and extend the domain of k_a to $(0,\infty)$ by setting for $q\geqslant\delta$

$$k_a(q):=(-1)^n k_a(q\beta^n), \tag{65}$$

where $n=1,2,...$ is such that $q\beta^n\in[\beta\delta,\delta)$. We claim that this k_a satisfies the requirements. Indeed, we have the identity $k_a(q)=-k_a(\beta q)$ for $0<q<\delta$ because of (63), and the extension of k_a according to (65) is such that this identity remains valid for $q\geqslant\delta$. It is also clear that k_a is smooth. Finally, let $q_0>0$. We want to show that $f(a+q_0)=k_a(q_0)\times\overline{f(a-q_0)}$. When $0<q_0<\delta$ this follows at once from the definition (64). When $q_0\geqslant\delta$ we take $n=1,2,...$ such that $q_0\beta^n\in[\beta\delta,\delta)$. The set $\{q\,|\,f(q)\neq 0\}$ is open and dense in $\mathbb{R}$ according to Lemma 2. Hence we can find a sequence $(q_k)_{k=1,2,...}$ with $q_k\beta^n\in[\beta\delta,\delta)$, $q_k\to q_0$, and $f(a-q_k)\neq 0$, $f(a-\beta q_k)\neq 0,...,f(a-\beta^{n-1}q_k)\neq 0$. It follows from (63)–(65) that $f(a+q_k)=k_a(q_k)\overline{f(a-q_k)}$ for all $k=1,2,...$. By taking the limit $k\to\infty$ and using continuity of f and k_a we conclude that $f(a+q_0)=k_a(q_0)\overline{f(a-q_0)}$, and the proof is complete.

Corollaries: (1) Let a be a zero of f for which there is a $\delta>0$ such that $f(q)\neq 0$ for $q\in(a,a+\delta)$. Then there is a smooth function l_a: $(0,\infty)\mapsto\mathbb{C}$ such that $l_a(q)=-l_a(\beta q)$, $\overline{f(a-q)}=l_a(q)f(a+q)$, $q>0$. Here $\beta=(\frac{1}{2}+\alpha)/(\frac{1}{2}-\alpha)<1$.

(2) Let a be a zero of f for which there is a $\delta>0$ such that $f(q)\neq 0$ for $q\in(a,a+\delta)\cup(a-\delta,a)$. Then the function k_a of Lemma 3 has no zeros. Indeed, $k_a(q)\neq 0$ for $0<q<\delta$, and hence, by (65), $k_a(q)\neq 0$ for $q>0$.

Lemma 4: The zero set of f is of the form $\{a+lb\,|\,l\in\mathbf{Z}\}$ for some $a\in\mathbb{R}$, $b\in\mathbb{R}$.

Proof: We can find an $a\in\mathbb{R}$, $b>0$ such that $f(a)=0=f(a+b)$ while $f(q)\neq 0$ for $q\in(a,a+b)$. Assume the interval $(a-b,a)$ contains a zero $a-c_0$ of f with $0<c_0<b$. With l_a as in Corollary 1 we see that $l_a(c_0)=0$, as $f(a+c_0)\neq 0$. Since $l_a(q)=-l_a(\beta q)$ for $q>0$ we get that $f(a-c_0\beta^n)=0$ for $n=1,2,...$. Hence a is an accumulation point of the zeros of f less than a. We can find zeros c and d of f with $a-\frac{1}{2}b<c<d<a$ such that the interval (c,d) is zero-free [otherwise f would vanish identically on a subinterval of $(a-\frac{1}{2}b,a)$, which cannot happen by Theorem 4]. According to Lemma 3 we have $f(d+q)=k_d(q)\overline{f(d-q)}$ for all $q>0$. Since $f(d+q)\neq 0$ for $q\in(a-d,a+b-d)$ we see that $f(d-q)\neq 0$ for $q\in(a-d,a+b-d)$. In particular $f(a-b)=f(d-(d-(a-b)))\neq 0$ since $d-(a-b)\in(a-d,a+b-d)$. However, by Corollary 1, $\overline{f(a-b)}=l_a(b)f(a+b)=0$. Thus, we have a contradiction. Hence, f has no zeros in $(a-b,a)$ while $f(a-b)=0$. It follows now by induction that $f(a-2b)=f(a-3b)=\cdots=0$, while $f(c)\neq 0$ for $c<a, c\neq a-b, a-2b,...$. Similarly, $f(a+2b)=f(a+3b)=\cdots=0$ while $f(c)\neq 0$ for $c>a, c\neq a+b, a+2b, a+3b,...$, and the proof is complete.

We shall now finish the proof of the main result of this subsection. We assume for convenience that $a=0$, $b=1$. According to Corollary 2 to Lemma 3 the functions k_l with integer l have no zeros, are smooth and satisfy $k_l(q)=-k_l(\beta q)$, $f(l+q)=k_l(q)\overline{f(l-q)}$ for $q>0$. Therefore, $k_l(q)$ is bounded away from 0 and oscillates violently when $q\downarrow 0$. We shall show that

$$k_1(q-1)\overline{k_0(2-q)}k_{-1}(q)=k_0(q),\quad 1<q<2. \tag{66}$$

Indeed, we have for $1<q<2$ that $f(q)=k_0(q)\overline{f(-q)}$, and at the same time

$$\begin{aligned} f(q)&=k_1(q-1)\overline{f(1-(q-1))}=k_1(q-1)\overline{f(2-q)}\\ &=k_1(q-1)\overline{k_0(2-q)\overline{f(q-2)}}\\ &=k_1(q-1)\overline{k_0(2-q)}f(-1+(q-1))\\ &=k_1(q-1)\overline{k_0(2-q)}k_{-1}(q-1)\overline{f(-q)}. \end{aligned} \tag{67}$$

Since f has no zeros in $(-2,-1)$ we conclude that (66) holds by equating the two expressions for $f(q)$. From (66) we easily derive a contradiction: $\lim_{q\downarrow 1}k_0(q)$, $\lim_{q\downarrow 1}k_0(2-q)$, $\lim_{q\downarrow 1}k_{-1}(q)$ exist and are unequal to 0 whereas $\lim_{q\downarrow 1}k_1(q-1)$ does not exist.

This completes the proof of our main result for the case $-\frac{1}{2}<\alpha<0$. The proof for the case $0<\alpha<\frac{1}{2}$ is practically the same.

Note: We can ask ourselves how close we can get to proving Hudson's theorem for the Wigner distribution by employing the same techniques as in this section. It is not hard to show that any smooth f with $W_{f,f}\geqslant 0$ everywhere has no zeros, and that $|f(q)|=\exp(\psi(q))$, $|F(p)|=\exp(\Psi(p))$, where ψ and Ψ are concave functions defined on $\mathbb{R}$ with $\psi(q)\to-\infty$ as $q\to\pm\infty$, $\Psi(p)\to-\infty$ as $p\to\pm\infty$. We have not been able to find examples f [other than Gaussians $f(q)=\exp(-\pi\gamma q^2+2\pi\delta q-\pi\epsilon)$ with $\operatorname{Re}\gamma>0$, $\delta\in\mathbb{C}$, $\epsilon\in\mathbb{C}$]

such that both $\log|f|$ and $\log|F|$ are well defined and concave everywhere on $\mathbb{R}$.

C. Square-integrable states f with Re $C_{f,f}^{(\alpha)}\geqslant 0$ everywhere do not exist: case $|\alpha| = \frac{1}{2}$

In this subsection we shall show that there is no $f\in L^2(\mathbb{R})$ such that Re $C_{f,f}^{(1/2)}\geqslant 0$ almost everywhere (unless $f=0$ almost everywhere). The reason why the proof in Sec. III B for (smooth) f's fails when $\alpha = \pm\frac{1}{2}$ is that formula (54) does not provide useful information when $f(a)=0$. [The result of Theorem 2 is still valid and shows that $\psi''(q)+(\psi'(q))^2\leqslant 0$ on any interval where f has no zeros; this Theorem 2 implies that $|f|$ is concave when f is smooth on such an interval.] We therefore have to resort to entirely different methods. We shall show that the main result in this case also holds for all $f\in L^2(\mathbb{R})$, which are not necessarily smooth. Furthermore, the proof is more constructive in the sense that one can more explicitly indicate the regions where Re $C_{f,f}^{(\pm 1/2)}$ takes negative values.

Since $C_{f,f}^{(1/2)}(q,p) = \overline{C_{f,f}^{(-1/2)}(q,p)}$ it is sufficient to consider the case $\alpha=\frac{1}{2}$ only. We have

$$\text{Re}\, C_{f,f}^{(1/2)}(q,p) = \text{Re}\big[e^{2\pi iqp}\,\overline{f(q)}F(p)\big]\ . \tag{68}$$

The proof can be outlined as follows. Assume for a while that $f(q)$ is positive on an interval $[a,b]$. Then non-negativity of (68) for all q and p implies that $\text{Re}[e^{2\pi iqp}F(p)]\geqslant 0$ for $q\in[a,b]$, $p\in\mathbb{R}$. When $|p|$ is sufficiently large, $\{e^{2\pi iqp}\,|\,q\in[a,b]\} = \{z\,|\,|z|=1\}$, leaving for $F(p)$ no other possibility than $F(p)=0$. Hence $F(p)=0$ for $|p|$ sufficiently large. When f is continuous and nonzero in an interval a slightly more sophisticated argument gives the same conclusion. When F is continuous and nonzero in an interval as well, we can argue in a similar fashion that $f(q)=0$ for $|q|$ sufficiently large. This then gives a contradiction since f and F cannot both be compactly supported. The reasoning is essentially the same but gets more technical when $f\in L^2(\mathbb{R})$ since one has now to consider Lebesgue points[21] of f and F, instead of continuity points.

We proceed with the proof according to the following steps, where we denote the sets of Lebesgue points of f and F by LP_f and LP_F, respectively. We suppose that $\text{Re}[e^{2\pi iqp}\,\overline{f(q)}F(p)]\geqslant 0$ almost everywhere, where $f\not\equiv 0$.

Step 1: When $q_0\in\text{LP}_f$, $p_0\in\text{LP}_F$, and $f(q_0)\neq 0\neq F(p_0)$, then $\text{Re}[e^{2\pi iq_0p_0}\,\overline{f(q_0)}F(p_0)]\geqslant 0$. Indeed if this is not true, we can find a $\delta>0$ such that $\text{Re}[e^{2\pi iqp}\bar{z}w]<0$ for all $z\in\mathbb{C}$, $w\in\mathbb{C}$, $q\in\mathbb{R}$, $p\in\mathbb{R}$ with $|z-f(q_0)|<\delta$, $|w-F(p_0)|<\delta$, $|q-q_0|<\delta$, $|p-p_0|<\delta$. Since $q_0\in\text{LP}_f$ we can find[21] an ϵ_1, $0<\epsilon_1<\delta$, with $\mu(\{q\in[q_0-\epsilon_1,q_0+\epsilon_1]\,|\,|f(q)-f(q_0)|<\delta\})>\epsilon_1$. Similarly, we can find an ϵ_2, $0<\epsilon_2<\delta$ such that $\mu(\{p\in[p_0-\epsilon_2,p_0+\epsilon_2]\,|\,|F(p)-F(p_0)|<\delta\})>\epsilon_2$. With $\epsilon:=\min(\epsilon_1,\epsilon_2)$ we get $\mu(\{(q,p)|\text{Re}[e^{2\pi iqp}\,\overline{f(q)}F(p)]<0\})>\epsilon^2$, and this contradicts the assumption that $\text{Re}[e^{2\pi iqp}\,\overline{f(q)}F(p)]\geqslant 0$ almost everywhere.

Step 2: Let $q\in\text{LP}_f$ with $q>0, f(q)\neq 0$, let C be the conic set $\{z|\arg z\in(\pi a,\pi b)\}$ with $b-a<1$, and let $F^{\leftarrow}(C)=\{p\in\mathbb{R}|F(p)\in C\}$. Then we have, for $n=0,1,\dots,$

$$I_n(q)\cap\text{LP}_F\cap F^{\leftarrow}(C)=\phi\ , \tag{69}$$

where $I_n(q)$ is the interval

$$I_n(q) = \Big[\frac{2n+\frac{1}{2}+\pi^{-1}\arg f(q)-a}{2q},\ \frac{2n+\frac{3}{2}+\pi^{-1}\arg f(q)-b}{2q}\Big]\ , \tag{70}$$

whose midpoint $(2n+1+(1/\pi)\arg f(q)-\frac{1}{2}(a+b))/2q$ and length $[1-(b-a)]/2q$ are denoted by $m_n(q)$ and $l(q)$, respectively. Indeed when $p\in I_n(q)\cap\text{LP}_F\cap F^{\leftarrow}(C)$ we have $\text{Re}[e^{2\pi iqp}\,\overline{f(q)}F(p)]<0$, and this contradicts the result of step 1.

Step 3: Let $q_0\in\text{LP}_f$ with $q_0>0, f(q_0)\neq 0$, and let

$$V_{\delta,\epsilon} := \{q\in(q_0-\epsilon,q_0+\epsilon)\,|\,|f(q)-f(q_0)|<\delta\}\cap\text{LP}_f \tag{71}$$

for $\delta>0$, $\epsilon>0$, $\epsilon<\frac{1}{2}q_0$. Furthermore let $J_n(q)$ be the closed interval with midpoint $(2n/2q)+m_0(q_0)$ and length $\frac{1}{4}l(q_0)$. We can find $\delta>0$, $\epsilon>0$ such that $J_n(q)\subset I_n(q)$ for $q\in V_{\delta,\epsilon}$. Indeed this is achieved when δ and ϵ are so small that $|m_0(q)-m_0(q_0)|<\frac{1}{8}l(q_0)$.

Step 4: Let $\epsilon(n)=4q_0/(2n+1)$. Then the set of midpoints of $J_n(q)$ with $|q-q_0|<\frac{1}{2}\epsilon(n)$ equals $(m_n(q_0)-(1/q_0)\times[2n/(2n+3)],\ m_n(q_0)+(1/q_0)[2n/(2n-1)])$.

Step 5: We have[21] $\lim_{\epsilon\downarrow 0}\mu(V_{\delta,\epsilon})/2\epsilon=1$ for every $\delta>0$. We can take n so large that $(q_0-\epsilon(n),q_0+\epsilon(n))\setminus V_{\delta,\epsilon(n)}$ contains no intervals of length $\geqslant\frac{1}{2}q_0\,l(q_0)\,\epsilon(n)$. The latter number is $<\frac{1}{2}\epsilon(n)$, and when $q_1,\ q_2\in(q_0-\epsilon(n),\ q_0+\epsilon(n))$, $|q_1-q_2|<\frac{1}{2}q_0l(q_0)\epsilon(n)$, then $J_n(q_1)\cap J_n(q_2)\neq\phi$. Hence,

$$S_n(q_0) := \bigcup_{q\in V_{\delta,\epsilon(n)}} J_n(q)\supset\bigcup_{|q-q_0|<\frac{1}{2}\epsilon(n)} J_n(q)\ , \tag{72}$$

and this set contains the interval of length $=4q_0^{-1}\times(2n+3)^{-1}n$ with midpoint $m_n(q_0)$. Hence the $S_n(q_0)$'s overlap when n is large enough.

Step 6: When $p\in S_n(q_0)\cap\text{LP}_F$, we have $F(p)\notin C$. For otherwise $p\in I_n(q)\cap\text{LP}_F\cap F^{\leftarrow}(C)$ for some $q\in V_{\delta,\epsilon(n)}$, which contradicts (69). Hence $F(p)\notin C$ when $p>0$ is sufficiently large.

Step 7: By taking three different conic sets C_1, C_2, C_3 as in step 2 with $C_1\cup C_2\cup C_3=\mathbb{C}\setminus\{0\}$, we see that $F(p)=0$ when $p>0$ is sufficiently large. Similarly, $F(p)=0$ when $-p>0$ is sufficiently large.

Step 8: By interchanging q and p we see that $f(q)=0$ when $|q|$ is sufficiently large. The proof is now completed by noting that f and F cannot both be compactly supported.

Note: As the proof shows, it is not necessary to require $f\in L^2(\mathbb{R})$; the requirement that f, F can be identified with locally integrable functions is sufficient. For instance, when $f\in L^1(\mathbb{R})$ we have that F is continuous and bounded, and $\text{Re}[e^{2\pi iqp}\,\overline{f(q)}F(p)]$ takes negative values.

D. Examples of generalized functions f with Re $C_{f,f}^{(\alpha)}\geqslant 0$ everywhere

In this subsection we give examples of generalized functions f and $\alpha\in\mathbb{R}$ such that Re $C_{f,f}^{(\alpha)}\geqslant 0$ everywhere (in the generalized sense). It was already noted in Sec. I that the existence of such f's and α's is remarkable in view of the fact that the sets of smooth and generalized functions f with $W_{f,f}=C_{f,f}^{(0)}\geqslant 0$ everywhere are essentially the same. In constructing examples of f's with Re $C_{f,f}^{(\alpha)}\geqslant 0$ we are led by what may be called the interference formula

$$|C^{(\alpha)}_{f,f}(q,p)|^2 = \left(\frac{1}{2}-\alpha\right)^2 \iint W_{f,f}\left(\frac{q}{\frac{1}{2}+\alpha}-a, \frac{p}{\frac{1}{2}-\alpha}-b\right) \times W_{f,f}\left(\frac{\frac{1}{2}+\alpha}{\frac{1}{2}-\alpha}a, \frac{\frac{1}{2}-\alpha}{\frac{1}{2}+\alpha}b\right) da\, db. \quad (73)$$

The way this formula is used is as follows. Assume we have an f, which can be thought of as a sum of functions each of which is "coherent" in the sense that its Wigner distribution is concentrated (and positive) in a rather small region of the phase-plane. Due to the presence of cross terms in $C^{(\alpha)}_{f,f}$, each pair of components of f will produce what is called a ghost. Although averages of these cross terms over sufficiently large regions are small, provided that the regions to which the Wigner distributions of the components are confined are more or less disjoint, the amplitude is not. Hence, negative values of Re $C^{(\alpha)}_{f,f}$ are likely to be found in the regions where the ghosts appear. What formula (73) tells us is how the regions where we can expect ghosts vary with α. For example, when $f=f_1+f_2$ and W_{f_1,f_1}, W_{f_2,f_2} is concentrated around (q_1,p_1) and (q_2,p_2), respectively, one can show from (73) that $C^{(\alpha)}_{f,f}$ has a ghost around the point

$$(\tfrac{1}{2}(q_1+q_2), \tfrac{1}{2}(p_1+p_2)) + \alpha(q_1-q_2, p_2-p_1). \quad (74)$$

Hence, if one wants to construct examples $f=\Sigma_n f_n$, with each f_n "coherent" in the above sense, such that Re $C^{(\alpha)}_{f,f}\geqslant 0$ everywhere, one should take care that for each pair f_n, f_m producing their ghost according to (74), there is an f_k whose Wigner distribution is positive in the region where the ghost appears. It should be noted that all Re $C^{(\alpha)}_{f_k,f_k}$ tend to be concentrated and non-negative in approximately the same regions as $W_{f_k f_k}$; this can be seen from formula (13), with $\Phi(\theta,\tau)=\cos\pi\alpha\theta\tau, \varphi(q,p)=\alpha^{-1}\cos\pi\alpha^{-1}qp$, which exhibits Re $C^{(\alpha)}_{f_k,f_k}$ as the convolution of W_{f_k,f_k} and a function φ, which is positive and slowly varying near (0,0) and rapidly oscillating far away from (0,0). In this way we achieve that the negative values of Re $C^{(\alpha)}_{f_n,f_m}$ are masked by the positive values of Re $C^{(\alpha)}_{f_k,f_k}$.

We start with the case $\alpha=\frac{1}{2}$. Although formula (73) degenerates in this case, the above noted principles are still valid. As is strongly suggested by the proof in Sec. III C, we should look for f's for which either f or F is supported by very small sets. We consider below f's that are supported by discrete sets.

Example 1: Let $f=\delta_a$, where $a\in\mathbb{R}$. Define for $b\in\mathbb{R}$

$$e_b(p) := \exp(-2\pi ibp), \quad p\in\mathbb{R}. \quad (75)$$

Then $F(p)=e_a(p)$, and

$$C^{(1/2)}_{f,f}=\delta_a \otimes e_0 \geqslant 0. \quad (76)$$

Example 2: Let $f=\delta_a+\delta_b$, where $a\in\mathbb{R}, b\in\mathbb{R}, a\neq b$. Then $F(p)=e_a(p)+e_b(p)$, and

$$\text{Re}[C^{(1/2)}_{f,f}(q,p)] = (1+\cos 2\pi(a-b)p)(\delta_a(q)+\delta_b(q)), \quad q\in\mathbb{R}, \quad p\in\mathbb{R}, \quad (77)$$

which is non-negative everywhere. Note that the ghosts of δ_a and δ_b (whose Wigner distributions are $\delta_a\otimes e_0$ and $\delta_b\otimes e_0$, respectively) appear on the lines $q=a$ and $q=b$.

Example 3: Let $f=\Sigma^{\infty}_{n=-\infty}\delta_n$. Then $F(p)=\Sigma^{\infty}_{m=-\infty}\delta_m$, and thus

$$C^{(1/2)}_{f,f}=\sum_{n,m=-\infty}^{\infty}\delta_n\otimes\delta_m\geqslant 0. \quad (78)$$

Notes: (1) The following can be shown. When $f=\Sigma^{\infty}_{n=-\infty}c_n\delta_n$ and Re $C^{(1/2)}_{f,f}\geqslant 0$, then either (a) infinitely many of the c's are $\neq 0$ (b) only one c is $\neq 0$, or (c) only two c's $\neq 0$. In case (c) the two c's that are $\neq 0$ have equal modules. If F is smooth, only the last two options can occur.

(2) The only square-integrable states that have non-negative Wigner distributions are the Gaussians. When one passes from square integrable to generalized states, the situation remains the same, except that one has to allow certain degeneracies (delta functions and exponentials, cf. Ref. 20). Such a thing does not hold for the distributions Re $C^{(1/2)}_{f,f}$. A second deviation is found when one considers the behavior of the distributions under smoothing by means of Gaussians. It has been shown in Ref. 20 that a (generalized) function f for which $G_\gamma * W_{f,f}\geqslant 0$ everywhere must be a (degenerate) Gaussian when $\gamma>1$. Here

$$G_\gamma(q,p)=\exp(-2\pi\gamma(q^2+p^2)), \quad q\in\mathbb{R}, \quad p\in\mathbb{R}. \quad (79)$$

When we consider as an example $f=\delta_a+z\delta_b$, where $a\in\mathbb{R}$, $b\in\mathbb{R}$, $z\in\mathbb{R}$, $z>1$, we have

$$\text{Re}[C^{(1/2)}_{f,f}(q,p)] = \delta_a(q)(1+z\cos 2\pi(a-b)p) + z\delta_b(q)(z+\cos 2\pi(a-b)p). \quad (80)$$

The second term at the right-hand side of (80) is non-negative; the convolution of the first term with G equals

$$\frac{1}{\sqrt{2\gamma}}\exp(-2\pi\gamma q^2) \times\left(1+z\exp\left(-\frac{\pi}{2\gamma}(a-b)^2\right)\cos 2\pi(a-b)p\right) \quad (81)$$

and is non-negative everywhere when $z\exp(-(\pi/2\gamma)(a-b)^2)\leqslant 1$. Hence, when $|b-a|$ is sufficiently large, a small amount of smoothing will turn Re $C^{(1/2)}_{f,f}$ into an everywhere non-negative distribution.

We finally consider some examples with $\alpha=-k+\frac{1}{2}$, where k is an integer $\neq 0,1$ and f is a sum of delta functions. We note that the ghosts of δ_a and δ_b manifest themselves in $C^{(\alpha)}_{f,f}$ on the lines $q=a+k(b-a), q=b-k(b-a)$. Hence, we must consider sums consisting of either one or infinitely many terms.

Example 4: Let $f=\delta_a$, where $a\in\mathbb{R}$. Then

$$C^{(\alpha)}_{f,f}=\delta_a\otimes e_0\geqslant 0. \quad (82)$$

Example 5: When $f=\Sigma^{\infty}_{n=-\infty}\delta_n$, then

$$C^{(\alpha)}_{f,f}=\sum_{n,m=-\infty}^{\infty}\delta_n\otimes\delta_m\geqslant 0. \quad (83)$$

Example 6: When $f=\Sigma^{\infty}_{n=-\infty}(\delta_{nk}+\delta_{nk+1})$, we have

$$C^{(\alpha)}_{f,f}(q,p)=k^{-1}\sum_{n,m=-\infty}^{\infty}[\delta_{nk^{-1}}(p)\delta_{mk}(q)(1+e^{2\pi iq}) + \delta_{nk^{-1}}(p)\delta_{mk+1}(q)(1+e^{-2\pi iq})], \quad (84)$$

and the real part of this distribution equals

$$k^{-1}\sum_{n,m=-\infty}^{\infty}\delta_{nk^{-1}}(p)(1+\cos 2\pi q) \times[\delta_{mk}(q)+\delta_{mk+1}(q)]\geqslant 0. \quad (85)$$

Note that when $V:=\{nk+l\,|\,l=0,1; n\in\mathbf{Z}\}$, we have $a\in V$,

$b \in V,\ a + k(b-a) \in V,\ b - k(b-a) \in V$. It can furthermore be shown that $\mathrm{Re}\, C^{(\alpha)}_{g,g}$ takes negative values when $g = \Sigma_{n=-\infty}^{\infty} (\delta_{nk-1} + \delta_{nk} + \delta_{nk+1})$.

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[1]We have $C_{af_1+bf_2,cg_1+dg_2} = a\bar{c}C_{f_1,g_1} + a\bar{d}C_{f_1,g_2} + b\bar{c}C_{f_2,g_1} + b\bar{d}C_{f_1,g_2}$.

[2]We designate states by lowercase symbols and their Fourier transforms by the corresponding capitals.

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[8]A. J. E. M. Janssen, J. Math. Phys. **25**, 2240 (1984). We write in the present paper $C^{(\varphi)}_{f,f}$ instead of $C^{(\varphi)}_{f}$ to accommodate some of our proofs notationally, and to better emphasize the bilinear dependence on the state f.

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[14]This can be deduced from the equality of the left-hand sides of (21) and (24) in A. J. E. M. Janssen, Philips J. Res. **37**, 79 (1982), Sec. 3. Also, see J. G. Krüger and A. Poffyn, Physica A **85**, 84 (1976), Sec. 12.

[15]This follows from the fact that for any $f \in L^2(\mathbf{R})$ and any $\epsilon > 0$ one can find a step function s with $\|f - s\| < \epsilon$. Now $C^{(\varphi)}_{s,s}(q,p) = (s, M(q,p)s)$ is uniformly continuous by (17) and boundedness of $M(0,0)$ and $|C^{(\varphi)}_{s,s}(q,p) - C^{(\varphi)}_{f,f}(q,p)| < \|M(0,0)\|\epsilon(\|f\| + \|s\|)$ for all $q \in \mathbf{R}, p \in \mathbf{R}$. Here $\|\ \|$ denotes the $L^2(\mathbf{R})$ norm.

[16]It can be shown that $C^{(\varphi)}_{f,f}$ is real valued for all $f \in L^2(\mathbf{R})$ if and only if φ is real. This is done by choosing $f(q) = 2^{1/4}\exp(-\pi q^2)$ so that $W_{f,f}(q,p) = 2\exp(-2\pi(q^2 + p^2))$ and observing that $\psi * W_{f,f} = 0 \Leftrightarrow \psi = 0$. The distributions $C^{(\alpha)}_{f,f}$ with $\alpha \neq 0$ are, in general, complex valued. We consider therefore $\mathrm{Re}\, C^{(\alpha)}_{f,f}$, which can be brought into the form (7) by taking $\Phi(\theta,\tau) = \cos \pi\alpha\theta\tau$, i.e., $\varphi(q,p) = \alpha^{-1}\cos \pi\alpha^{-1}qp$.

[17]See, e.g., Refs. 7–9.

[18]We use here that $C^{(\varphi)}_{f,g}(q,p) = \overline{C^{(\varphi)}_{g,f}(q,p)}$. This follows from the fact that $W_{f,g}(q,p) = \overline{W_{g,f}(q,p)}$, formula (13), and the fact that φ is real by assumption.

[19]See the paper quoted in Ref. 14.

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[21]When $f: \mathbf{R} \to \mathbf{C}$ is locally integrable we say that $q \in \mathbf{R}$ is a Lebesgue point of f when

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} |f(q+u) - f(q)|\,du = 0\,.$$

The set of $q \in \mathbf{R}$ that are not Lebesgue points of f has zero measure. When q is a Lebesgue point of f we have

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\mu(\{u \in [-\epsilon,\epsilon]\,|\,|f(q+u) - f(q)| < \delta\}) = 1$$

for every $\delta > 0$. Here μ is ordinary Lebesgue measure.