A Note on "Positive Time-Frequency Distributions"

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Abstract—We discuss the performance of certain recently proposed positive time-frequency distribution functions with correct marginals. It turns out that for signals of the FM type, satisfactory results can only be obtained with a strongly signal dependent choice of the parameters in these distribution functions.

I. Introduction

For some time there has been a discussion in the signal analysis community on how to deal with the occurrence of negative values in certain popular time-frequency distributions. This discussion was recently revived by the introduction, by Cohen and Zaparovanny [1], of a class of distributions with many desirable properties such as positivity and the correct marginals property. In [2] and the above paper,[1] these distributions are recommended as an alternative to, e.g., the Wigner distribution, since the latter one cannot be interpreted as a true probability density. There is agreement that the Wigner distribution can be of great help in displaying time-frequency characteristics of time-varying signals and signals with transient behavior (see, e.g., [3], [4]), but the application of this distribution requires some skill. This has to do with the fact that the Wigner distribution involves the signal bilinearly, so that for multicomponent signals, disturbing crossterms leading to negative values must be expected [6]. In this correspondence we raise the question of whether the positive distributions proposed in [2] and the above paper[1] can be equally successful in displaying signal characteristics of time-varying signals. It turns out this is not so for signals of the FM type.

II. Bilinear Distributions and Positive Distributions

The Wigner distribution is a member of a large class of distributions which can be parameterized by means of a function Φ of two real variables,

$$C_{f,f}^{(\Phi)}(t, \omega) = \iiint \exp\left(-2\pi i(\theta t + \tau\omega - \theta u)\right) \Phi(\theta, \tau) \times f(u + \tfrac{1}{2}\tau) f^*(u - \tfrac{1}{2}\tau)\, d\theta\, d\tau\, du \qquad \text{(II.1)}$$

where f is the signal under study, and Φ is independent of f. The distributions so obtained have been studied extensively, especially with respect to the point how various "natural" requirements are reflected in restrictions on Φ. The Wigner distribution results on taking $\Phi(\theta, \tau) = 1$ in (II.1), and it satisfies a long list of desirable mathematical properties [6].

Among these properties are the following:

a) correct marginals, so that integration over t and ω yields the power spectral density and the instantaneous power,

b) time-frequency translations of the signal are reflected by corresponding translations of the distribution, and

c) vanishing of power spectral density or instantaneous power outside intervals implies vanishing of the distribution outside the corresponding strips.

The class described by (II.1) is a subclass of Cohen's class introduced in [7], where the kernel Φ may depend on f. To obtain everywhere positive distributions satisfying a), one must take Φ's

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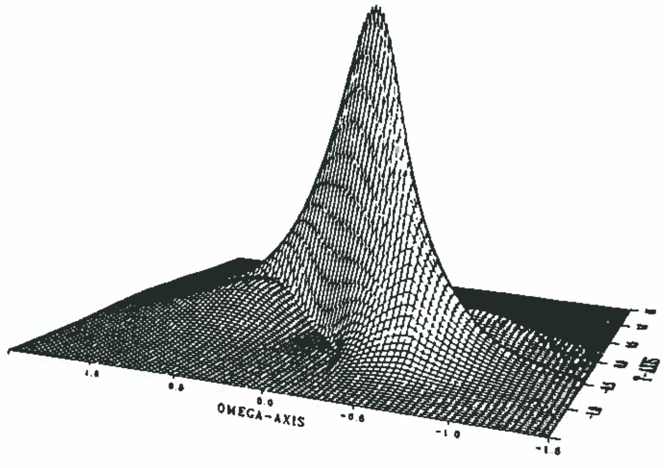
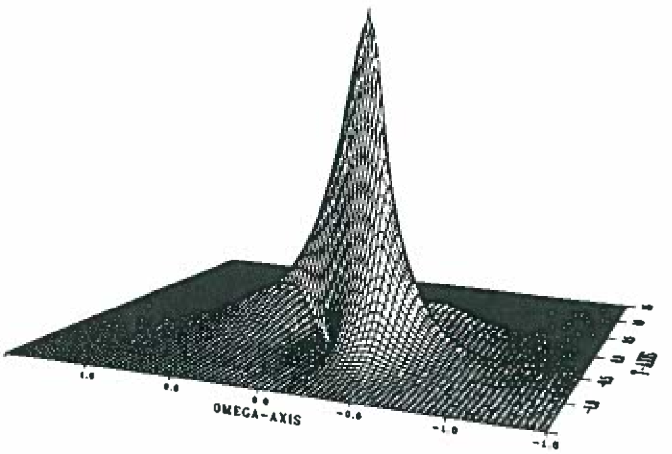


(a)



(b)

Fig. 1. Cohen–Posch–Zaparovanny distribution for the f of (III.1) with (a) $\epsilon = 1$, $\beta = 1$ and (b) $\epsilon = 1$, $\beta = 0$, respectively, and with h and c as specified.

depending on the signal since by Wigner's theorem and extensions of it [5], [8], for any choice of Φ such that a) holds, there is an abundance of signals f for which $C_{f,f}^{(\Phi)}$ takes negative values.

The positive distributions introduced by Cohen and Zaparovanny in [1] can be generated according to the formula

$$C_f(t, \omega) = |f(t)|^2 \, |F(\omega)|^2 \left(1 + c\rho(x(t), y(\omega))\right), \quad \text{(II.2)}$$

where, with $E = \int |f(\tau)|^2 \, d\tau$,

$$x(t) = \frac{1}{E} \int_{-\infty}^{t} |f(\tau)|^2 \, d\tau, \; y(\omega) = \int_{-\infty}^{\omega} |F(\lambda)|^2 \, d\lambda. \quad \text{(II.3)}$$

The function ρ and the constant c act as parameters in (II.3). They must be chosen such that a) is satisfied and that $1 + c\rho(x, y) \geq 0$ for all $x, y \in [0, 1]$, but are otherwise unrestricted. The general form of ρ and the manner in which ρ and the Φ of (II.1) are related are given in [1], [2], and the above paper.[1] The distributions (II.3), with ρ, c chosen as stated above, satisfy conditions a), b), and c); in addition, they satisfy d) positivity.

III. Positive Time-Frequency Distributions for FM Signals

Although the mathematical properties of Section II are important, the signal analysts want something else. They have quite often an intuitive idea or advance knowledge where the signal energy of a particular signal should be concentrated. When dealing with an application yielding signals of a certain type, they need a distribution that exhibits the expected behavior for signals on which they can check, so that they can expect meaningful results for signals that are only known to be of that particular type. To achieve this end, they are willing to sacrifice quite a few of the mathematical properties. It has been demonstrated in various applications that the Wigner distribution is an excellent distribution to start from. It can be modified and enhanced so that the information one looks for can be better obtained from it. The enhancement techniques needed may depend on the particular application, but remain essentially fixed for all the signals encountered in one application.

The set of distributions generated by formula (II.3) could offer an alternative to this procedure. It is therefore interesting to see whether one can make choices for the parameters c and ρ such that all signals of a particular type are accommodated by the distribution (II.3) with this c and this ρ. We discuss here the performance for signals $\exp(2\pi i\varphi(t))$ of the FM type.

According to signal analysts' intuition, the signal energy of $\exp(2\pi i\varphi(t))$ should be concentrated around the curve $(t, \varphi'(t))$. Consider the chirplike signal

$$f(t) = (2\epsilon)^{1/4} \exp(-\pi\epsilon t^2 + \pi i\beta t^2), \quad \text{(III.1)}$$

where $\epsilon > 0$. Now

$$F(\omega) = (2\epsilon)^{1/4} (\epsilon - i\beta)^{-1/2} \exp\left(\frac{-\pi\epsilon\omega^2}{\epsilon^2 + \beta^2} - \frac{\pi i\beta\omega^2}{\epsilon^2 + \beta^2}\right). \quad \text{(III.2)}$$

It follows from the definition of $x(t)$ and $y(\omega)$ in (2.3) that

$$x(t) = \tfrac{1}{2} + I(t\sqrt{2\epsilon}),$$

$$y(\omega) = \tfrac{1}{2} + I\left(\omega[2\epsilon/(\epsilon^2 + \beta^2)]^{1/2}\right), \quad \text{(III.3)}$$

where $I(a) = \int_0^a \exp(-\pi u^2) \, du$.

When β is not too small, compared to ϵ, both $|f(t)|$ and $|F(\omega)|$ are very flat functions of t and ω, respectively. Therefore, $C_f(t, \omega)$ can only exhibit the desired behavior when one succeeds in choosing the parameters c and ρ such that $1 + c\rho(x(t), y(\omega))$ is large near the straight line $(t, \beta t)$, $t \in R$ and small away from this line.

Assume $\beta > 0$. When $\epsilon > 0$ is small and ω is close to βt, we have that $y(\omega)$ is close to $\frac{1}{2} + I(t\sqrt{2\epsilon})$. Hence, when $C_f(t, \omega)$ exhibits the desired behavior, $1 + c\rho(x, y)$ must be large only in a small neighborhood of a curve that passes through $(\frac{1}{2}, \frac{1}{2})$ and that coincides with the diagonal $\{(x, x) | x \in [0, 1]\}$ when $\epsilon \downarrow 0$. (A good choice in this case for ρ would be $\rho(x, y) = \delta(x - y) - 1$, $c = 1$; this ρ can be derived in the manner of (2.2) in the paper[1] from $h(x, y) = \delta(x - y)$.)

Assume $\beta < 0$. When $\epsilon > 0$ is small and ω is close to βt, we have that $y(\omega)$ is close to $\frac{1}{2} - I(t\sqrt{2\epsilon})$. Now $1 + c\rho(x, y)$ must be large only in a small neighborhood of a curve that passes through $(\frac{1}{2}, \frac{1}{2})$ and that coincides with the cross diagonal $\{(x, 1 - x) | x \in [0, 1]\}$ when $\epsilon \downarrow 0$. (Now $\rho(x, y) = \delta(x + y - 1) - 1$, $c = 1$ would be a good choice.)

The two conditions on c, ρ thus obtained are clearly incompatible so that one cannot accommodate chirps f with positive and negative sweep rate β with a single choice of the parameters c, ρ.

More dramatically, when one chooses c and ρ such that the $C_f(t, \omega)$ for chirps f with positive sweep rate are satisfactory, then for any other signal g one will get a distribution that looks a bit like the distribution of that chirp. This is so since $C_g(t, \omega)$ is large near the set $\{(t, \omega) | x(t) = y(\omega)\}$, and both $x(t)$ and $y(\omega)$ increase in t and ω, respectively, whence $C_g(t, \omega)$ exhibits a ridge extending from the third quadrant into the first quadrant of the time-frequency plane.

We have included the pictures of the distributions of f_1 and f_2 obtained from (III.1) by taking $\epsilon = 1$, $\beta = 1$ and $\epsilon = 1$, $\beta = 0$, respectively. To accommodate f_1, we take a ρ that is derived from

$$h(x, y) = \begin{cases} b(1 - a^{-1}|y - p(x)|), & |y - p(x)| \leq a \\ 0, & |y - p(x)| > a \end{cases} \quad \text{(III.4)}$$

in the manner of (2.2) in the paper.[1] Here $p(x) = \frac{1}{2} + \frac{1}{2}\sqrt{2}(x - \frac{1}{2})$, $a = \frac{1}{4}$, b is such that h is properly normalized, and c is such that min $1 + c\rho(x, y) = 0$. The choice of h is such that near $(t, \omega) = (0, 0)$, the distribution of f_1 exhibits a ridge along the line $t = \omega$, and the resulting picture is quite satisfactory. However, the picture that results for f_2 exhibits a similar ridge along the line $t = \omega/\sqrt{2}$, which should definitely not be the case.

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