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Spaces of Type W, Growth of Hermite Coefficients, Wigner Distribution, and Bargmann Transform

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In their famous monograph on generalized functions Gelfand and Shilov introduce the function spaces W_M, W^{Ω}, and W_M^{Ω}. These spaces consist of C^∞-functions and holomorphic functions, respectively, with growth behaviour specified by suitable convex functions M and Ω. In this paper we study the spaces $W_M^{M^\times}$, where $M^\times$ denotes Young's dual function corresponding to M. We characterize the Hermite expansion coefficients of the functions in $W_M^{M^\times}$, their Fourier transforms, Wigner distributions, and Bargmann transforms. In particular, we prove that $W_M^{M^\times} = W_M \cap W^{M^\times}$. © 1990 Academic Press, Inc.

0. PRELIMINARIES AND SUMMARY OF RESULTS

Let M denote a two times differentiable function on $\mathbb{R}^+$ with derivative $m = M'$. We assume that $m(0) = M(0) = 0$, $\lim_{x\to\infty} m(x) = \infty$ and $m'(x) > 0$ for all $x > 0$; i.e., M is a convex function with monotonously increasing derivative m. We set

$$M^\times(x) = \int_0^x m^\leftarrow(t)\, dt. \tag{0.1}$$

Then we have Young's inequality

$$xy \leqslant M(x) + M^\times(y), \qquad x, y \geqslant 0, \tag{0.2}$$

with equality if and only if $y = m(x)$. We observe that $M = M^{\times\times}$.

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According to Gelfand and Shilov [11] the space $W_M^{M^\times}$ consists of all entire functions φ for which there are positive constants a, b, and C such that

$$(0.3)\quad |\varphi(x+iy)| \leqslant C\exp[-M(a\,|x|)+M^\times(b\,|y|)], \qquad x+iy\in\mathbb{C}.$$

In fact, in [11] the wider class of spaces W_M^Ω is introduced where in (0.3) $M^\times$ is replaced by a function Ω with the same properties as M. It is proved in [11] that the Fourier transformation is a bijection from W_M^Ω into $W_{\Omega^\times}^{M^\times}$. So spaces of type $W_M^{M^\times}$ are Fourier invariant.

For $M(x)=\alpha x^{1/\alpha}$, $\frac{1}{2}\leqslant\alpha<1$, we have $M^\times(y)=(1-\alpha)\,y^{1/(1-\alpha)}$ and $W_M^{M^\times}$ equals the space S_α^α as introduced in [10]. This paper contains four characterizations of the spaces $W_M^{M^\times}$. Each characterization yields an extension of a known characterization for one (or all) of the spaces S_α^α. We list these characterizations here.

i. *Characterization Based on Hermite Coefficients*

For $n=0, 1, 2, \ldots$, the nth Hermite function is defined as

$$(0.4)\qquad \psi_n(x)=(\pi^{1/2}2^n n!)^{-1/2}\,(-1)^n e^{1/2x^2}\left(\frac{d}{dx}\right)^n [e^{-x^2}].$$

The normalization constant is taken such that $\{\psi_n\,|\,n\in\mathbb{N}\cup\{0\}\}$ constitutes an orthonormal basis in the Hilbert spaces $L_2(\mathbb{R})$. The Hermite functions are eigenfunctions of the differential operator $-d^2/dx^2+x^2$; we have

$$(0.5)\qquad \left[-\frac{d^2}{dx^2}+x^2\right]\psi_n=(2n+1)\psi_n.$$

The space S_α^α, $\alpha\geqslant\frac{1}{2}$, admits the following characterization:

$$(0.6)\quad \varphi\in S_\alpha^\alpha \quad \text{if and only if} \quad \exists_{\lambda>0}\colon (\varphi,\psi_n)_{L_2}=\mathcal{O}(\exp(-\lambda n^{1/2\alpha})).$$

For a proof of this result we refer to [14, 8]. In this paper we prove the following extension:

$$(0.7)\quad \varphi\in W_M^{M^\times} \quad \text{if and only if} \quad \exists_{\lambda>0}\colon (\varphi,\psi_n)_{L_2}=\mathcal{O}(\exp(-M(\lambda n^{1/2}))).$$

ii. *Characterization Based on the Fourier Transform*

Schwartz's space $\mathcal{S}$ of rapidly decreasing C^∞-functions may be characterized as follows:

(0.8) $\varphi\in\mathcal{S}$ if and only if φ is square integrable on $\mathbb{R}$ with $\varphi(x)=\mathcal{O}(|x|^{-k})$ and $(\mathbb{F}\varphi)(x)=\mathcal{O}(|x|^{-k})$ as $|x|\to\infty$ for all $k\in\mathbb{N}$.

Here $\mathbb{F}$ denotes the Fourier transform on $L_2(\mathbb{R})$. For a proof we refer to [12]. This characterization of $\mathcal{S}$ inspired Björck, cf. [3], to introduce the spaces $\mathcal{S}_\omega$ which consist of C^∞-functions φ with the property that $\varphi(x)$ *and* $(\mathbb{F}\varphi)(x)$ admit the growth order $\mathcal{O}(\exp(-\lambda\omega(|x|)))$ for all $\lambda>0$. Björck uses functions ω satisfying $\omega(x)\geqslant a\log x+b$ for some $a>0$, $b>0$. Partly, the results in this paper also apply to these Björck spaces.

Indeed, we show that under mild conditions on M we have the following characterization of $W_M^{M^\times}$:

(0.9) $\varphi\in W_M^{M^\times}$ if and only if φ is a square integrable function with $\varphi(x)=\mathcal{O}(\exp(-M(\lambda\,|x|)))$ and $(\mathbb{F}\varphi)(x)=\mathcal{O}(\exp(-M(\lambda\,|x|)))$ for some $\lambda>0$.

On the basis of (0.9) it follows easily that $W_M^{M^\times}=W_M\cap W^{M^\times}$. Here the space W_M consists of all C^∞-functions φ with the property that

$$\exists_{a>0}\forall_{l\in\mathbb{N}}\exists_{B_l>0}\colon |\varphi^{(l)}(x)|\leqslant B_l\exp(-M(a\,|x|)), \qquad x\in\mathbb{R}.$$

Moreover, $W^{M^\times}$ consists of all holomorphic functions φ with the property that

$$\exists_{b>0}\forall_{k\in\mathbb{N}}\exists_{A_k>0}\colon |(x+iy)^k\varphi(x+iy)|\leqslant A_k\exp(-M^\times(b\,|y|)), \qquad x+iy\in\mathbb{C}.$$

The corresponding result, viz. $S_\alpha^\alpha=S_\alpha\cap S^\alpha$, follows from [18], where it is proved that $S_\alpha^\beta=S_\alpha\cap S^\beta$.

iii. *Characterization Based on the Wigner Distribution*

The Wigner distribution plays a role in signal analysis as a tool for describing signals in time and frequency simultaneously; it is closely related to the radar ambiguity function. Also, it is closely related to harmonic analysis for the Heisenberg group. In this connection we mention [4–7, 16, 20].

The Wigner distribution $\mathbb{W}(f)$ of a square integrable function f is defined by

$$(0.10)\quad \mathbb{W}(x,y;f)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyt}f\left(x+\frac{1}{2}t\right)\overline{f\left(x-\frac{1}{2}t\right)}\,dt, \qquad x,y\in\mathbb{R}.$$

It can easily be checked that $\mathbb{W}(f)$ belongs to $L_2(\mathbb{R}^2)\cap L_\infty(\mathbb{R}^2)$.

In [4], De Bruijn studies the Wigner distribution in connection with the space $S_{1/2}^{1/2}$. The Wigner distribution extends to the generalized function space corresponding to the test function space $S_{1/2}^{1/2}$. In particular, the Wigner distribution of any tempered distribution can be defined. Here we analyse the Wigner distribution $\mathbb{W}(\varphi)$ of the functions φ in $W_M^{M^\times}$. We prove the characterization

(0.11)

$$\varphi \in W_M^{M^\times} \quad \text{if and only if} \quad \exists_{\lambda>0}: \mathbb{W}(x, y; \varphi) = \mathcal{O}(\exp(-M(\lambda(x^2+y^2)^{1/2}))).$$

iv. *Characterization Based on the Bargmann Transform*

The Bargmann–Segal–Fock space $\mathscr{F}$ consists of all entire functions h with

$$\|h\|_{\mathscr{F}}^2 := \int_{\mathbb{R}^2} |h(x+iy)|^2 \exp(-(x^2+y^2))\, dx\, dy < \infty. \tag{0.12}$$

The Bargmann transform $\mathbb{A}f$ of $f \in L_2(\mathbb{R})$ is defined by

$$(\mathbb{A}f)(z) = \pi^{-1/4} \int_{-\infty}^{\infty} \exp(-1/2(z^2+x^2) + \sqrt{2}\, zx)\, f(x)\, dx, \qquad z \in \mathbb{C}. \tag{0.13}$$

Thus we obtain a unitary operator $\mathbb{A}$ from $L_2(\mathbb{R})$ onto $\mathscr{F}$. In particular, we have

$$(\mathbb{A}\psi_n)(z) = \frac{z^n}{\sqrt{n!}}, \qquad n = 0, 1, ... \tag{0.14}$$

For generalities we refer to [1] where the Bargmann transform is extensively discussed, we refer to [2] where the Bargmann transform is employed for describing certain spaces of test functions and of generalized functions, and we refer to [17] and references therein, where the Bargmann transform serves as a tool for studying completeness properties of coherent states.

In [2] Bargmann characterizes the space $\mathbb{A}(\mathscr{S}) = \{\mathbb{A}\varphi \mid \varphi \in \mathscr{S}\}$ in terms of holomorphic functions with specific growth behaviour. Furthermore, in [9, 17] it is shown that

(0.15) $\varphi \in S_{1/2}^{1/2}$ if and only if $|(\mathbb{A}\varphi)(z)| \leqslant C \exp[d\,|z|^2]$, $z \in \mathbb{C}$, for suitable constants $C > 0$ and d, $0 < d < \frac{1}{2}$.

Here we prove the corresponding characterization

$$\begin{aligned} &\varphi \in W_M^{M^\times} \quad \text{if and only if for some} \quad C>0, \quad \lambda>0, \\ &|(\mathbb{A}\varphi)(z)| \leqslant C \exp(\tfrac{1}{2}|z|^2 - M(\lambda |z|)), \quad z \in \mathbb{C}. \end{aligned} \tag{0.16}$$

Throughout this paper we deal with the following class $\mathscr{K}$ of convex functions.

(0.17) DEFINITION. The class $\mathscr{K}$ consists of two times differentiable functions M,

$$M(x) = \int_0^x m(t)\, dt, \qquad x \geqslant 0.$$

Here m denotes a nondecreasing *concave* continuously differentiable function which satisfies the following conditions:

(p_1) $m(0) = 0$ and $m(t) \to \infty$ as $t \to \infty$,

(p_2) $m(t)/t$ decreases strictly to zero as $t \to \infty$.

For all $M \in \mathscr{K}$ we have

$$\alpha^2 M(x) \leqslant M(\alpha x) \leqslant \alpha M(x), \qquad \alpha \in [0, 1], \quad x \geqslant 0. \tag{0.18}$$

Remark. Assumption (p_2) is not very restrictive, since otherwise we end up with the space $S_{1/2}^{1/2}$ or with a trivial space $W_M^{M^\times}$ (i.e., $W_M^{M^\times}$ consists of the null function only). Cf. [10, p. 227 ff].

Not all conditions on m are required in all statements of this paper. For reasons of transparent presentation we do not aim at the utmost generality of our results; in most cases one can find more general formulations of our theorems oneself.

The plan of the paper is as follows. Let φ be a square integrable function for which there exists a constant $\lambda > 0$ such that

(a) $\varphi(x) = \mathcal{O}(\exp(-M(\lambda\,|x|)))$ and $(\mathbb{F}\varphi)(x) = \mathcal{O}(\exp(-M(\lambda\,|x|)))$.

We show that the Wigner distribution $\mathbb{W}(\varphi)$ satisfies

(b) $\mathbb{W}(x, y; \varphi) = \mathcal{O}(\exp(-M(\mu(x^2+y^2)^{1/2})))$ for some $\mu > 0$ dependent on λ.

Next, starting from square integrable functions φ with Wigner distribution $\mathbb{W}(\varphi)$ satisfying the growth order (b), we characterize their Hermite coefficients $(\varphi, \psi_n)_{L_2}$, $n = 0, 1, 2, ...$ It turns out that

(c) $(\varphi, \psi_n)_{L_2} = \mathcal{O}(\exp(-M(\nu n^{1/2})))$ for some $\nu > 0$ dependent on φ.

Finally, we show that a square integrable function with Hermite coefficients $(\varphi, \psi_n)_{L_2}$, satisfying the growth order (c), belongs to the function space $W_M^{M^\times}$.

Since $W_M^{M^\times} \subset W_M$ is a Fourier invariant space, each $\varphi \in W_M^{M^\times}$ satisfies (a). Thus the equivalences in parts i, ii, and iii follow. The equivalence in iv is shown by straightforward estimation.

We conclude this introduction by mentioning some functional analytic consequences.

For a self-adjoint operator $\mathscr{A}$ in a Hilbert space X, the analyticity domain $D^{\omega}(\mathscr{A})$ consists of all $v \in \bigcap_{n=1}^{\infty} D(\mathscr{A}^n)$ with the property that $\|\mathscr{A}^n v\|_X \leqslant ab^n n!$ for certain constants $a>0$, $b>0$, dependent on v. For a positive self-adjoint operator $\mathscr{A}$ we have

$$D^{\omega}(\mathscr{A}) = \bigcup_{t>0} e^{-t\mathscr{A}}(X);$$

see [9].

Now, let $\mathscr{Q}$ denote the self-adjoint operator of multiplication by x in $L_2(\mathbb{R})$ and let $\mathscr{P} = \mathbb{F}^*\mathscr{Q}\mathbb{F}$, viz. $\mathscr{P} = i(d/dx)$. Then according to (0.5) and (0.7) we have

(0.19) $$W_M^{M^\times} = D^{\omega}(M((\mathscr{P}^2 + \mathscr{Q}^2)^{1/2})).$$

Further, as a consequence of Theorem 1.5 we can replace equivalence (0.9) by

(0.9′) $\varphi \in W_M^{M^\times}$ if and only if there exists $\lambda > 0$ such that

$$\int_{-\infty}^{\infty} \exp(2\lambda M(|x|))\, |\varphi(x)|^2\, dx < \infty$$

and

$$\int_{-\infty}^{\infty} \exp(2\lambda M(|x|))\, |(\mathbb{F}\varphi)(x)|^2\, dx < \infty.$$

So in terms of analyticity domains we have

(0.20) $$D^{\omega}(M((\mathscr{P}^2 + \mathscr{Q}^2)^{1/2})) = D^{\omega}(M(|\mathscr{P}|)) \cap D^{\omega}(M(|\mathscr{Q}|)).$$

1. Characterization Based on the Wigner Distribution

For convenience we introduce some notation.

(1.1) Definition. Let $M \in \mathscr{K}$ and let $\lambda > 0$. The collection $\mathscr{C}_{M,\lambda}$ consists of all square integrable functions φ for which

$$\varphi(x) = \mathscr{O}(\exp[-M(\lambda\,|x|)])$$

and

$$(\mathbb{F}\varphi)(x) = \mathscr{O}(\exp[-M(\lambda\,|x|)]).$$

We set

$$\mathscr{C}_M = \bigcup_{\lambda>0} \mathscr{C}_{M,\lambda}.$$

The conditions on M imply that for each $\varphi \in \mathscr{C}_M$ the functions $x \mapsto x^k\varphi(x)$ and $x \mapsto x^k(\mathbb{F}\varphi)(x)$ are bounded on $\mathbb{R}$ for each $k \in \mathbb{N}$. So we have $\mathscr{C}_M \subset \mathscr{S}$.

The next theorem deals with the growth behaviour of the Wigner distribution of functions in $\mathscr{C}_{M,\lambda}$.

(1.2) Theorem. *Let $\varphi \in \mathscr{C}_{M,\lambda}$. Then for all $\tilde{\lambda} < \frac{1}{2}\lambda$*

$$\mathbb{W}(x, y; \varphi) = \mathscr{O}(\exp[-2M(\tilde{\lambda}(x^2+y^2)^{1/2})])$$

and

$$\mathbb{W}(x, y; \mathbb{F}\varphi) = \mathscr{O}(\exp[-2M(\tilde{\lambda}(x^2+y^2)^{1/2})]).$$

Proof. The conditions on M imply that for each μ, $0<\mu<\lambda$, the function

(*) $$x \mapsto \exp[M(\mu\,|x|)]\,\varphi(x)$$

is square integrable. For all $p, q > 0$ we have

(**) $$M(\tfrac{1}{2}(p+q)) \leqslant \tfrac{1}{2}(M(p)+M(q)).$$

Now we proceed as follows: Let $0<\mu<\lambda$. Then we have

$$\begin{aligned}
&\sup_{(x,y)\in\mathbb{R}^2} \exp[2M(\mu\,|x|)]\,|\mathbb{W}(x,y;\varphi)| \\
&\leqslant \sup_{(x,y)\in\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[M\left(\mu\left|x+\frac{1}{2}t\right|\right) + M\left(\mu\left|x-\frac{1}{2}t\right|\right)\right] \\
&\quad \cdot \left|\varphi\left(x+\frac{1}{2}t\right)\right| \left|\varphi\left(x-\frac{1}{2}t\right)\right| dt \\
&\leqslant \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \exp\,[2M(\mu\,|t|)]\,|\varphi(t)|^2\, dt < \infty,
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality.

Because of the relation

(***) $$\mathbb{W}(x, y; \varphi) = \mathbb{W}(-y, x; \mathbb{F}\varphi),$$

cf. [4, Sect. 12], and the conditions on $\mathbb{F}\varphi$ we similarly get

$$\sup_{(x,y)\in\mathbb{R}^2} \exp[2M(\mu\,|y|)]\,|\mathbb{W}(x,y;\varphi)| < \infty.$$

Now let $\tilde{\lambda} < \frac{1}{2}\lambda$. Then we have

$$\sup_{(x,y)\in\mathbb{R}^2} \exp\,[2M(\tilde{\lambda}(|x|+|y|))]\,|\mathbb{W}(x, y; \varphi)|$$

$$\leqslant \tfrac{1}{2} \sup_{(x,y)\in\mathbb{R}^2} \{\exp[2M(2\tilde{\lambda}\,|x|)] + \exp[2M(2\tilde{\lambda}\,|y|)]\}\,|\mathbb{W}(x, y; \varphi)| < \infty,$$

because $x \mapsto \exp[2M(\tilde{\lambda}x)]$, $x > 0$, is a convex function.

The assertions now follow from the inequality

$$M(\tilde{\lambda}(x^2+y^2)^{1/2}) \leqslant M(\tilde{\lambda}(|x|+|y|))$$

and relation (***). ∎

Conversely, starting from growth conditions on the Wigner distribution $\mathbb{W}(x, y; \varphi)$ we can derive growth conditions on φ and $\mathbb{F}\varphi$. To this end, consider the following lemma.

(1.3) Lemma. *Let φ be a square integrable function for which there exist positive constants $K>0$, $\lambda>0$ such that*

$$|\mathbb{W}(x, y; \varphi)| \leqslant K \exp[-2M(\lambda\,|x|)]$$

and

$$|\mathbb{W}(x, y; \varphi)| \leqslant K \exp[-2M(\lambda\,|y|)].$$

Then

$$\varphi \in \bigcap_{0<r<1} \mathscr{C}_{M,\lambda r}.$$

Proof. Let $0<r<1$. Since the function $x \mapsto \exp[2M(\lambda r\,|x|)]$, $x \geqslant 0$, is convex we have

$$\exp[2M(\lambda r\,|x| + \lambda(1-r)\,|y|)]$$
$$\leqslant r \exp[2M(\lambda\,|x|)] + (1-r)\exp[2M(\lambda\,|y|)].$$

Hence

$$|\mathbb{W}(x, y; \varphi)| \leqslant K \exp[-2M(r\lambda\,|x| + (1-r)\,\lambda\,|y|)].$$

Further, since $M(a+b) \geqslant M(a)+M(b)$, $a, b \geqslant 0$, it follows that

$$|\mathbb{W}(x, y; \varphi)| \leqslant K \exp[-2M(r\lambda\,|x|)]\exp[-2M((1-r)\lambda\,|y|)].$$

According to [4, Sect. 12],

$$|\varphi(x)|^2 = \int_{\mathbb{R}} \mathbb{W}(x, y; \varphi)\,dy.$$

Hence

$$|\varphi(x)|^2 \leqslant 2K \exp[-2M(r\lambda\,|x|)] \int_0^\infty \exp[-2M((1-r)\lambda\,|y|)]\,dy$$
$$= \frac{2K}{1-r} \exp[-2M(r\lambda\,|x|)] \int_0^\infty \exp[-2M(\lambda\,|y|)]\,dy. \quad ∎$$

(1.4) Theorem. *Let φ be a square integrable function for which there exist constants $K>0$, $\lambda>0$ such that*

$$|\mathbb{W}(x, y; \varphi)| \leqslant K \exp[-2M(\lambda(x^2+y^2)^{1/2})].$$

Then

$$\varphi \in \bigcap_{0<r<1} \mathscr{C}_{M,\lambda r}.$$

Proof. The assertion is an immediate consequence of the preceding lemma because of the inequality $M(\lambda\,|x|) \leqslant M(\lambda(x^2+y^2)^{1/2})$, $x, y \in \mathbb{R}$. ∎

On the basis of Lemma (1.3) we have the following side result.

(1.5) Theorem. *Let the function φ satisfy*

$$\int_{\mathbb{R}} \exp[2M(\lambda\,|x|)]\,|\varphi(x)|^2\,dx < \infty$$

and

$$\int_{\mathbb{R}} \exp[2M(\lambda\,|y|)]\,|(\mathbb{F}\varphi)(y)|^2\,dy < \infty.$$

Then

$$\varphi \in \bigcap_{0<r<1} \mathscr{C}_{M,\lambda r}.$$

Proof. For $x, y \in \mathbb{R}$ we have

$$\begin{aligned}
|\mathbb{W}(x, y; \varphi) \exp[2M(\lambda\,|x|)]| &\leqslant \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left[M\left(\lambda \left|x+\frac{1}{2}t\right|\right)\right] \left|\varphi\left(x+\frac{1}{2}t\right)\right| \\
&\quad \times \exp\left[M\left(\lambda \left|x-\frac{1}{2}t\right|\right)\right] \left|\varphi\left(x-\frac{1}{2}t\right)\right| dt \\
&\leqslant \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}} \exp[2M(\lambda\,|t|)]\,|\varphi(t)|^2\,dt \leqslant K < \infty,
\end{aligned}$$

and similarly, since $\mathbb{W}(x, y; \varphi) = \mathbb{W}(-y, x; \mathbb{F}\varphi)$,

$$|\mathbb{W}(x, y; \varphi)| \exp[2M(\lambda\,|y|)] \leqslant K < \infty.$$

Now the assertion follows from Lemma 1.3. ∎

2. Characterization Based on Hermite Coefficients

Let L_n denote the nth Laguerre polynomial

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k x^k}{k!}, \qquad n = 0, 1, 2, \ldots.$$

The corresponding generating function is given by

$$\sum_{n=0}^{\infty} z^n L_n(x) = (1-z)^{-1} \exp\left[\frac{xz}{1-z}\right].$$

We mention the relation (cf. [13, 16, or 19])

$$\mathbb{W}(x, y; \psi_n) = 2(-1)^n \exp[-(x^2+y^2)]\,L_n(2(x^2+y^2)),$$

where ψ_n denotes the nth Hermite function introduced in (0.4). For square integrable functions f and g we have Moyal's formula (see [4, Sect. 14]),

$$|(f, g)_{L_2}|^2 = \int_{\mathbb{R}^2} \mathbb{W}(x, y; f)\,\mathbb{W}(x, y; g)\,dx\,dy;$$

in particular,

$$|(f, \psi_n)_{L_2}|^2 = 2(-1)^n \int_{\mathbb{R}^2} \mathbb{W}(x, y; f) \exp[-(x^2+y^2)]\,L_n(2(x^2+y^2))\,dx\,dy.$$

Thus we have established a relation between the Hermite coefficients $(f, \psi_n)_{L_2}$ of f and its Wigner distribution $\mathbb{W}(f)$.

(2.1) Theorem. *Let φ be a square integrable function with the property that for some $\lambda > 0$*

$$\sup_{(x, y) \in \mathbb{R}^2} \exp[2M(\lambda(x^2+y^2)^{1/2})]\,|\mathbb{W}(x; y; \varphi)| < \infty.$$

Then there exist $\tilde{\lambda} > 0$, $C > 0$ such that

$$|(\varphi, \psi_n)_{L_2}| \leqslant C \exp[-M(\tilde{\lambda} n^{1/2})].$$

Proof. The series

$$F(z) = \sum_{n=0}^{\infty} z^n\,|(\varphi, \psi_n)|^2, \qquad |z| < 1,$$

represents an analytic function. Consider the following computation:

$$\begin{aligned}
F(z) &= \sum_{n=0}^{\infty} z^n\,|(\varphi, \psi_n)_{L_2}|^2 \\
&= 2 \int_0^{2\pi} \int_0^{\infty} \mathbb{W}(r, \theta; \varphi) \left[\sum_{n=0}^{\infty} (-1)^n z^n e^{-r^2} L_n(2r^2)\right] r\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^{\infty} \mathbb{W}(r, \theta; \varphi) \frac{2}{1+z} \exp\left[r^2 \frac{z-1}{z+1}\right] r\,dr\,d\theta.
\end{aligned}$$

Now we have

$$\begin{aligned}
\frac{2}{1+z} \exp\left[r^2 \frac{z-1}{z+1}\right] &= \left(1 + \frac{1}{2}(z-1)\right)^{-1} \exp\left[r^2 \left(\frac{1/2(z-1)}{1+1/2(z-1)}\right)\right] \\
&= \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k L_k(r^2)(z-1)^k.
\end{aligned}$$

So we derive the formula

$$\begin{aligned}
F^{(k)}(1-) &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\,|(\varphi, \psi_n)_{L_2}|^2 \\
&= \left(-\frac{1}{2}\right)^k k! \int_0^{2\pi} \int_0^{\infty} \mathbb{W}(r, \theta; \varphi)\,L_k(r^2)\,r\,dr\,d\theta.
\end{aligned}$$

Clearly, for $0 \leqslant k \leqslant l$,

$$\frac{\Gamma(l+1)}{\Gamma(l-k+1)} |(\varphi, \psi_l)_{L_2}|^2 \leqslant \sum_{n=k}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-k+1)} |(\varphi, \psi_n)_{L_2}|^2.$$

We arrive at the following estimation

$$(*) \qquad |(\varphi, \psi_l)_{L_2}|^2 \leqslant \frac{1}{\Gamma(l+1)} \min_{0 \leqslant k \leqslant l} \Gamma(k+1)\Gamma(l-k+1)\left(-\frac{1}{2}\right)^k \cdot \int_0^{2\pi}\int_0^{\infty} \mathbb{W}(r, \theta; \varphi) L_k(r^2) r\, dr\, d\theta.$$

By assumption there exists $C_0 > 0$ such that

$$|\mathbb{W}(r, \theta; \varphi)| \leqslant C_0 \exp[-2M(\lambda r)].$$

Further, we have the following crude estimate

$$|L_k(r^2)| \leqslant \sum_{j=0}^{k} \binom{k}{j} \frac{r^{2j}}{j!}.$$

So for $0 < \mu < \lambda$, $\sigma = \lambda - \mu$, we obtain

$$(**) \qquad \left(-\frac{1}{2}\right)^k \int_0^{2\pi}\int_0^{\infty} \mathbb{W}(r, \theta; \varphi) L_k(r^2) r\, dr\, d\theta$$
$$\leqslant 2\pi C_0 \left(\frac{1}{2}\right)^k \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j!} \int_0^{\infty} \exp[-2M(\lambda r)]\, r^{2j+1}\, dr$$
$$\leqslant 2\pi C_0 \left(\frac{1}{2}\right)^k \max_{r \geqslant 0} \left\{ \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j!} \exp[-2M(\mu r)] r^{2j} \right\} \times \int_0^{\infty} r \exp[-2M(\sigma r)]\, dr.$$

Combining $(*)$ and $(**)$, we see that there is a constant $C_1 > 0$ such that

$$(***) \qquad |(\varphi, \psi_l)_{L_2}|^2 \leqslant \frac{C_1}{\Gamma(l+1)} \min_{0 \leqslant k \leqslant l} \Gamma(k+1)\Gamma(l-k+1) \left(\frac{1}{2}\right)^k \sum_{j=0}^{\infty} \binom{k}{j} \frac{1}{j!} p_j(\mu),$$

where

$$p_j(\mu) = \max_{r \geqslant 0} r^{2j} \exp[-2M(\mu r)] = \max_{r \geqslant 0} \exp[-2M(\mu r) + 2j \log r].$$

We set

$$\Lambda_\mu(r, y) = -M(\mu r) + y \log r, \qquad r \geqslant 0, \quad y \geqslant 0.$$

The function $r \mapsto \Lambda_\mu(r, y)$ assumes its unique maximum at the point $r_\mu(y)$ determined by

$$(\dagger) \qquad \mu r_\mu(y) m(\mu r_\mu(y)) = y$$

(recall that $m = M'$). The function $y \mapsto r_\mu(y)$, $y \geqslant 0$, is increasing. Moreover, since $m(t) \leqslant Kt$, $t \geqslant 0$, for some $K > 0$ and $m(t) \to \infty$ as $t \to \infty$, we see that

$$(\dagger\dagger) \qquad r_\mu(y) \geqslant (y/K\mu^2)^{1/2}.$$

Differentiating $(\dagger)$ with respect to y we get for $y > 0$

$$0 \leqslant r'_\mu(y) = \frac{1}{\mu m(\mu r_\mu(y))} - \frac{r'_\mu(y) r_\mu(y) m'(\mu r_\mu(y))}{m(\mu r_\mu(y))} \leqslant \frac{1}{\mu m(\mu r_\mu(y))},$$

so that by $(\dagger\dagger)$

$$(\dagger\dagger\dagger) \qquad r'_\mu(y) \to 0 \quad \text{as} \quad y \to \infty.$$

Now we assert that there are $a > 0$, $A > 0$ such that

$$\sum_{j=0}^{k} \binom{k}{j} \frac{1}{j!} p_j(\mu) \leqslant A \frac{a^k}{k!} p_k(\mu), \qquad k = 0, 1, 2, \ldots.$$

Indeed we have

$$\frac{1}{j!} p_j(\mu) = \frac{1}{j!} \exp[2\Lambda_\mu(r_\mu(j), j)] \leqslant \frac{j+1}{(r_\mu(j))^2} \frac{p_{j+1}(\mu)}{(j+1)!}.$$

So by $(\dagger\dagger)$ we derive

$$\frac{1}{j!} p_j(\mu) \leqslant 2K\mu^2 \frac{p_{j+1}(\mu)}{(j+1)!}.$$

Thus the claim follows (the term with $j = 0$ is easily taken care of). Therefore, by $(***)$, for some constant C_2

$$|(\varphi, \psi_l)_{L_2}|^2 \leqslant \frac{C_2}{\Gamma(l+1)} \min_{0 \leqslant k \leqslant l} \left(\frac{1}{2} a\right)^k \Gamma(l-k+1)\, p_k(\mu)$$
$$= \frac{C_2}{\Gamma(l+1)} \min_{0 \leqslant k \leqslant l} \Gamma(l-k+1)\, p_k(\mu_1),$$

where $\mu_1 = 2\mu/a$.

Using that for some $D>0$

$$p^p e^{-p} \leqslant \Gamma(p+1) \leqslant D p^{p+1/2} e^{-p}, \qquad p>0$$

we get

$$\frac{\Gamma(l-k+1)}{\Gamma(l+1)} p_k(\mu_1) = \frac{\Gamma(l-k+1)}{\Gamma(l+1)} (r_{\mu_1}(k))^{2k} \exp[-2M(\mu_1 r_{\mu_1}(k))]$$

$$\leqslant D \exp\left[\left(l-k+\frac{1}{2}\right)\log(l-k)-(l-k)-l\log l + l + 2k\log r_{\mu_1}(k) - 2M(\mu_1 r_{\mu_1}(k))\right]$$

$$\leqslant D l^{1/2} \exp\left[k \log\left(\frac{e r_{\mu_1}^2(k)}{l-k}\right) - 2M(\mu_1 r_{\mu_1}(k))\right].$$

And hence, for some constant C_3

$$|(\varphi, \psi_l)_{L_2}|^2 \leqslant C_3 l^{1/2} \min_{0 \leqslant k \leqslant l} \exp\left[k \log\left(\frac{e r_{\mu_1}^2(k)}{l-k}\right) - 2M(\mu_1 r_{\mu_1}(k))\right].$$

For k_l the largest integer between 0 and l such that

$$l - k_l \geqslant e r_{\mu_1}^2(k_l),$$

we obtain

$$|(\varphi, \psi_l)_{L_2}|^2 \leqslant C_3 l^{1/2} \exp[-2M(\mu_1 r_{\mu_1}(k_l))].$$

To estimate k_l we consider the equation

$$\xi + e r_{\mu_1}^2(\xi) = l, \qquad \xi \in \mathbb{R}, \quad 0 \leqslant \xi \leqslant l.$$

This equation has a unique solution ξ_l, because r_{μ_1} is an increasing function; see (†). Hence $k_l \leqslant \xi_l < k_l + 1$. Further, by (††)

$$l = \xi_l + e r_{\mu_1}^2(\xi_l) \leqslant (K\mu_1^2 + e) r_{\mu_1}^2(\xi_l)$$

so that

$$r_{\mu_1}(\xi_l) \geqslant (l/(e+K\mu_1^2))^{1/2}.$$

By the mean value theorem, for some $\theta_l \in \mathbb{R}$, $k_l \leqslant \theta_l \leqslant \xi_l$ we have

$$r_{\mu_1}(k_l) = r_{\mu_1}(\xi_l) + (k_l - \xi_l) r'_{\mu_1}(\theta_l).$$

So by (†††) there exists a constant $b>0$ such that $r_{\mu_1}(k_l) \geqslant b l^{1/2}$ for all $l \in \mathbb{N}$. ∎

3. Characterization Based on Spaces of Type W

Let $(a_n)_{n=0}^\infty$ be a sequence of complex numbers with the following growth behaviour:

$$a_n = \mathcal{O}(\exp[-M(n^{1/2})]).$$

In this section we show that the series $\sum_{n=0}^\infty a_n \psi_n(z)$ represents a holomorphic function, which satisfies

$$\left|\sum_{n=0}^\infty a_n \psi_n(x+iy)\right| \leqslant C \exp[-M(a|x|) + M^\times(b|y|)]$$

for some positive constants a, b, and C.

Since $M(x)/x \to \infty$ as $x \to \infty$, the growth conditions on the sequence $(a_n)_{n=0}^\infty$ and the elementary estimate

$$|\psi_n(z)| \leqslant \exp[n^{1/2}|z| + \tfrac{1}{2}|z|^2], \qquad n = 0, 1, 2, \ldots$$

imply that the series $\sum_{n=0}^\infty a_n \psi_n(z)$ converges uniformly on compacta in $\mathbb{C}$. So it represents a holomorphic function φ.

It remains to determine the growth behaviour of φ. We employ Mehler's formula in a form as suggested by Hille in [15],

$$\sum_{n=0}^\infty e^{-nt} |\psi_n(x+iy)|^2 = \pi^{-1/2}(1-e^{-2t})^{1/2} \exp\left[-x^2 \tanh\frac{t}{2} + y^2 \coth\frac{t}{2}\right].$$

This formula yields the convenient separation of the variables x and y. In order to apply Mehler's formula we need a relation between $\exp[-M(n^{1/2})]$ and $\exp[-tn]$, $t>0$. This relation is given in the following lemma.

(3.1) Lemma. *There exists a function Ψ_M which possesses the following properties:*

(a) $\forall_{q>0} : \sup_{t\in[0,1]} \exp[-tq^2 - \Psi_M(t)] = \exp[-M(q)]$.

(b) $\exists_{K>0} \forall_{q>0} : \exp[-M(q)] \leqslant K(1+q^2) \int_0^1 \exp[-tq^2 - \Psi_M(t)]\, dt$.

(c) $\forall_{p>0} : \sup_{t\in[0,1]} \exp[p^2/t - \Psi_M(t)] = \exp[M^\times(2p)]$.

Here $M^\times$ denotes Young's dual function of M.

Proof. First we construct the function Ψ_M. We choose Ψ_M such that for each $q>0$ the function $t \mapsto tq^2 + \Psi_M(t)$ has $M(q)$ as its minimum value. If this minimum is attained at $t = \tau(q)$ we have the relations

$$q^2 + \Psi'_M(\tau(q)) = 0, \qquad q>0,$$

and

$$q^2\tau(q)+\Psi_M(\tau(q))=M(q), \qquad q>0.$$

A straightforward analysis yields

$$\tau(q)=\frac{m(q)}{2q}, \qquad q>0.$$

Since $\tau\colon q\mapsto m(q)/2q$ is strictly decreasing, the function τ can be inverted. We set $\xi=\tau^{\leftarrow}$, so that

$$t=\frac{m(\xi(t))}{2\xi(t)}.$$

Thus we obtain

$$\Psi_M(t)=M(\xi(t))-\tfrac{1}{2}\xi(t)m(\xi(t)).$$

We observe that the function $q\mapsto M(q)-\frac{1}{2}qm(q)$ is monotonously increasing. The functions ξ and τ decrease and $\xi(t)\to\infty$ as $t\downarrow 0$. We shall now directly verify that Ψ_M possesses the desired properties. We put

$$\eta(p,q)=q^2\tau(p)+M(p)-\frac{1}{2}pm(p)=-\frac{1}{2}\left[1-\frac{q^2}{p^2}\right]pm(p)+M(p).$$

Differentiating η with respect to p and setting the derivative equal to zero, we get

$$\eta_p(p,q)=\frac{1}{2}\left(1-\frac{q^2}{p^2}\right)(m(p)-pm'(p))=0.$$

(a) Since m is strictly concave, we have $m(p)>pm'(p)$. So the function $p\mapsto\eta_p(p,q)$ attains a minimum at $p=q$. We get

$$\begin{aligned}
&\sup_{t\in[0,1]}\exp[-tq^2-\Psi_M(t)]\\
&\quad=\sup_{t\in[0,1]}\exp[-tq^2-M(\xi(t))+\tfrac{1}{2}\xi(t)m(\xi(t))]\\
&\quad=\sup_{p\in[\xi(1),\infty)}\exp[-q^2\tau(p)-M(p)+\tfrac{1}{2}pm(p)]=\exp[-M(q)].
\end{aligned}$$

(b) Since m is concave, the function $p\mapsto m(p)-pm'(p)$ is increasing. Now let us consider $\xi(1)\leqslant q-1\leqslant p\leqslant q$. Then we have

$$\begin{aligned}
\int_q^p\eta_p(s,q)\,ds&\leqslant\frac{1}{2}\int_q^p\left(1-\frac{q^2}{s^2}\right)(m(s)-sm'(s))\,ds\\
&\leqslant\frac{1}{2}(m(q)-qm'(q))\int_p^q\left(\frac{q^2}{s^2}-1\right)ds\\
&\leqslant\frac{m(q)-qm'(q)}{2(q-1)}.
\end{aligned}$$

Since $(m(q)-qm'(q))/2q\to 0$ as $q\to\infty$ there exists a constant $K_0>0$ such that for all $q>\xi(1)$, $(m(q)-qm'(q))/2(q-1)\leqslant -\log K_0$. So we derive

$$\begin{aligned}
&\int_0^1\exp[-tq^2+\Psi_M(t)]\,dt\\
&\quad=\exp[-M(q)]\int_{\xi(1)}^{\infty}-\tau'(p)\exp\left[-\int_p^q\eta_p(s,q)\,ds\right]dp\\
&\quad\geqslant K_0\exp[-M(q)]\int_{q-1}^{q}-\tau'(p)\,dp.
\end{aligned}$$

Now for $p\geqslant\xi(1)$ we have

$$-\tau'(p)=\frac{m(p)-pm'(p)}{2p^2}\geqslant\frac{1}{2}K_1p^{-2},$$

where $K_1=m(\xi(1))-\xi(1)m'(\xi(1))$. It follows that

$$\int_{q-1}^{q}-\tau'(p)\,dp\geqslant\frac{K_1}{2q^2}, \qquad q\geqslant\xi(1)+1,$$

whence

$$\int_0^1\exp[-tq^2+\Psi_M(t)]\,dt\geqslant\tfrac{1}{2}K_0K_1q^{-2}\exp[-M(q)].$$

(c) We introduce the function

$$\chi(p,q)=\frac{p^2}{\tau(q)}-M(q)+\frac{1}{2}qm(q), \qquad \tau(q)=\frac{m(q)}{2q}.$$

Differentiating with respect to q yields

$$\chi_q(p,q)=\left[\frac{2p^2}{m^2(q)}-\frac{1}{2}\right](m(q)-qm'(q)).$$

So $\chi_q(p,q)=0$ if and only if $m(q)=2p$. Since $m(q)$ increases, the function $q\mapsto\chi(q,p)$ reaches its maximum at $q=m^{\leftarrow}(2p)$. We thus have

$$\begin{aligned}\sup_{t\in[0,1]}\exp\left[\frac{p^2}{t}-\Psi_M(t)\right]&=\sup_{t\in[0,1]}\exp\left[\frac{p^2}{t}-M(\xi(t))+\frac{1}{2}\xi(t)m(\xi(t))\right]\\&=\sup_{q\in[\xi(1),\infty)}\exp\left[\frac{p^2}{\tau(q)}-M(q)+\frac{1}{2}qm(q)\right]=\exp[M^\times(2p)].\end{aligned}$$

(We recall that $\chi(p,m^{\leftarrow}(2p))=2pm^{\leftarrow}(2p)-M(m^{\leftarrow}(2p))=M^\times(2p)$.) ∎

We arrive at the main theorem of this section.

(3.2) THEOREM. *Let $M\in\mathscr{K}$ and let $\lambda>0$. Consider a sequence $(a_n)_{n=0}^\infty$ with the property that for some $C>0$,*

$$|a_n|\leqslant C\exp[-M(\lambda n^{1/2})],\qquad n=0,1,2,\ldots.$$

Then the series $\varphi=\sum_{n=0}^\infty a_n\psi_n$ represents an element of $W_M^{M^\times}$ with growth behaviour

$$|\varphi(x+iy)|\leqslant A_\lambda\exp[-M(a|x|)+M^\times(b|y|)],\qquad x,y\in\mathbb{R},$$

where $a=\frac{1}{4}\sqrt{2}\,\gamma\lambda$, $b=a^{-1}$, and $A_\lambda>0$ a suitable constant.

Proof. We set $M_\lambda: x\mapsto M(\lambda x)$. Then M_λ belongs to the class $\mathscr{K}$. In the beginning of this section we already observe that φ is a holomorphic function. With the Hölder inequality we get

$$\begin{aligned}|\varphi(x+iy)|&\leqslant\left(\sum_{n=0}^\infty(n^2+1)|a_n|\right)^{1/2}\left(\sum_{n=0}^\infty\frac{1}{(n^2+1)}|a_n|\,|\psi_n(x+iy)|^2\right)^{1/2}\\&\leqslant C_1\left(\sum_{n=0}^\infty(n^2+1)^{-1}\exp[-M(\lambda n^{1/2})]\,|\psi_n(x+iy)|^2\right)^{1/2},\end{aligned}$$

where

$$C_1=C\left(\sum_{n=0}^\infty(n^2+1)|a_n|\right)^{1/2}.$$

The remaining estimations are based on Lemma (3.1).

There exists a constant $K_\lambda>0$ such that for all $x,y\in\mathbb{R}$

$$\begin{aligned}&\sum_{n=0}^\infty(n^2+1)^{-1}\exp[-M(\lambda n^{1/2})]\,|\psi_n(x+iy)|^2\\&\quad\leqslant K_\lambda\sum_{n=0}^\infty\left(\int_0^1\exp[-nt-\Psi_{M_\lambda}(t)]\,dt\right)|\psi_n(x+iy)|^2\\&\quad=\pi^{-1/2}K_\lambda\int_0^1(1-e^{-2t})^{-1/2}\\&\qquad\times\exp[-x^2\tanh\tfrac{1}{2}t+y^2\coth\tfrac{1}{2}t-\Psi_{M_\lambda}(t)]\,dt.\end{aligned}$$

The latter integral is estimated by

$$\left\{\int_0^1(1-e^{-2t})^{-1/2}\,dt\right\}\{\sup_{t\in[0,1]}\exp[-x^2\tanh\tfrac{1}{2}t-\tfrac{1}{2}\Psi_{M_\lambda}(t)]\}\cdot\{\sup_{t\in[0,1]}\exp[y^2\coth\tfrac{1}{2}t-\tfrac{1}{2}\Psi_{M_\lambda}(t)]\}.$$

Finally, since $\tanh\frac{1}{2}t\geqslant\gamma^2t$ with $\gamma^2=\tanh\frac{1}{2}$, we get

$$(*)\qquad |\varphi(x+iy)|\leqslant A_\lambda\sup_{t\in[0,1]}\exp\left[\frac{1}{2}(-t(\gamma x\sqrt{2})^2-\Psi_{M_\lambda}(t))\right]\cdot\sup_{t\in[0,1]}\exp\left[\frac{1}{2}\left(\frac{1}{t}\left(\frac{y\sqrt{2}}{\gamma}\right)^2-\Psi_{M_\lambda}(t)\right)\right].$$

By Lemma (3.1a) we have

$$\sup_{t\in[0,1]}\exp[\tfrac{1}{2}(-t(\gamma x\sqrt{2})^2-\Psi_{M_\lambda}(t))]=\exp[-\tfrac{1}{2}M(\gamma\lambda|x|\sqrt{2})],$$

and by Lemma (3.1c) we have

$$\sup_{t\in[0,1]}\exp\left[\frac{1}{2}\left(\frac{1}{t}\left(\frac{y\sqrt{2}}{\gamma}\right)^2-\Psi_{M_\lambda}(t)\right)\right]=\exp\left[\frac{1}{2}M_\lambda^\times\left(\frac{2|y|\sqrt{2}}{\gamma}\right)\right]=\exp\left[\frac{1}{2}M^\times\left(\frac{2|y|\sqrt{2}}{\lambda\gamma}\right)\right].\ \blacksquare$$

4. Characterization Based on the Bargmann Transform

In this section we prove the following theorem.

(4.1) THEOREM. *Let $M \in \mathscr{K}$. Then the Bargmann transform $\mathbb{A}\varphi$ of a square integrable fuction φ satisfies*

$$\exists_{C>0}\exists_{\rho>0} : |(\mathbb{A}\varphi)(z)| \leqslant C \exp[\tfrac{1}{2}|z|^2 - M(\rho\,|z|)]$$

if and only if its Hermite coefficients $(\varphi, \psi_n)_{L_2}$ satisfy

$$\exists_{D>0}\exists_{\lambda>0} : |(\varphi, \psi_n)_{L_2}| \leqslant D \exp[-M(\lambda n^{1/2})].$$

The proof of Theorem (4.1) is based on the series representation

$$(\mathbb{A}\varphi)(z) = \sum_{n=0}^{\infty} (\varphi, \psi_n)_{L_2} \frac{z^n}{\sqrt{n!}};$$

cf. (0.14), and the following two lemmas.

(4.2) LEMMA. *Let $(a_n)_{n=0}^{\infty}$ be a sequence with $|a_n| \leqslant C \exp[-M(\lambda n^{1/2})]$ for some $C>0$, $\lambda>0$. Then there are $\rho>0$, $D>0$ such that the holomorphic function*

$$h(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{n!}}, \qquad z \in \mathbb{C},$$

satisfies

$$|h(z)| \leqslant D \exp[\tfrac{1}{2}|z|^2 - M(\rho\,|z|)].$$

Proof. Let $0<\lambda_1<\lambda$ and let $z \in \mathbb{C}$ with $|z|>1$. A straightforward estimation yields

$$\left|\sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{n!}}\right| \leqslant C_{\lambda_1} \sup_{n\in\mathbb{N}} \exp\left[-M(\lambda_1 n^{1/2}) + \frac{1}{2} n \log|z|^2 + \frac{1}{2}n - \frac{1}{2} n \log n\right],$$

where

$$C_{\lambda_1} = C \sum_{n=0}^{\infty} \exp[-M((\lambda-\lambda_1)n^{1/2})].$$

We set $r=|z|$ and

$$\theta(q,r) = -M(\lambda_1 q) + \tfrac{1}{2}q^2 \log r^2 + \tfrac{1}{2}q^2 - \tfrac{1}{2}q^2 \log q^2.$$

Then we have

$$\left|\sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{n!}}\right| \leqslant C_{\lambda_1} \exp[\sup_{q\geqslant 1}\{\theta(q,r)\}].$$

Differentiation with respect to q yields

$$\theta_q(q,r) = q\left[-\lambda_1 \frac{m(\lambda_1 q)}{q} + \log r^2 - \log q^2\right].$$

Since $m(\lambda_1 q)/q \to 0$ as $q \to \infty$, the zeros of θ_q lie in the interval $[\delta r, r]$ for some $\delta>0$. So $\theta(q,r)$ attains its maximum inside this interval. It is clear that

$$\tfrac{1}{2}q^2[\log r^2 - \log q^2] + \tfrac{1}{2}q^2 \leqslant \tfrac{1}{2}r^2, \qquad q \in [\delta r, r],$$

and also that

$$M(\lambda_1 q) \geqslant M(\delta\lambda_1 r), \qquad q \in [\delta r, r].$$

Thus it follows that

$$\left|\sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{n!}}\right| \leqslant C_{\lambda_1} \exp\left[\frac{1}{2}|z|^2 - M(\delta\lambda_1\,|z|)\right]. \quad \blacksquare$$

(4.3) LEMMA. *Let g be a holomorphic function with the property that*

$$|g(z)| \leqslant D \exp[\tfrac{1}{2}|z|^2 - M(\rho\,|z|)]$$

for certain positive constants $D>0$, $\rho>0$. Then $g(z)$ can be written as a series

$$g(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{n!}},$$

where

$$|a_n| \leqslant C \exp[-M(\lambda n^{1/2})]$$

for some $C>0$, $\lambda>0$.

Proof. By Cauchy's formula

$$\frac{|a_n|}{\sqrt{n!}} = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{g(z)}{z^{n+1}} \, dz \right| \leqslant D r^{-n} \exp\left[\frac{1}{2} r^2 - M(\rho r)\right]$$

for all $r \geqslant 1$, $n = 0, 1, 2, \ldots$. In particular, for $r = n^{1/2}$ we get

$$\frac{|a_n|}{\sqrt{n!}} \leqslant D n^{-(1/2)n} e^{(1/2)n} \exp[-M(\rho n^{1/2})].$$

Because of the inequality $n! \leqslant 3n^{n+1/2} e^{-n}$ it follows that

$$|a_n| \leqslant D \sqrt{3}\, n^{1/4} \exp[-M(\rho n^{1/2})].$$

Consequently, for all λ, $0 < \lambda < \rho$, there exists $C > 0$ such that

$$|a_n| \leqslant C \exp[-M(\lambda n^{1/2})]. \quad \blacksquare$$

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