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Abstract—**It is shown that for any signal $x(t)$ the minimum of**

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(t-t_x)^2+(f-f_x)^2\right] S_x^{(w)}(t,f)\,dt\,df$$

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over all normalized time-windows $w(t)$ is achieved by the Gaussian window $w(t) = 2^{1/4}\exp(-\pi t^2)$. Here (t_x, f_x) is the center of gravity of the signal $x(t)$, $S_x^{(w)}(t, f)$ is the spectrogram of $x(t)$ due to the window $w(t)$, and the double integral is a measure of the spread of $S_x^{(w)}(t, f)$ around (t_x, f_x) in the time-frequency plane.

I. Introduction

When $x(t)$ is a square integrable signal and $w(t)$ is a square integrable window, the spectrogram of $x(t)$ due to $w(t)$ is defined by

$$S_x^{(w)}(t, f) = \left|\int_{-\infty}^{\infty} x(\tau)w(\tau - t)e^{-2\pi i f\tau}\, d\tau\right|^2 \tag{1.1}$$

or, equivalently, by

$$S_x^{(w)}(t, f) = \left|\int_{-\infty}^{\infty} X(g)W(f - g)e^{2\pi i t g}\, dg\right|^2 \tag{1.2}$$

with $X(f)$ and $W(f)$ being the Fourier transforms of $x(t)$ and $w(t)$, respectively. We use here the definition

$$Y(f) = \int_{-\infty}^{\infty} y(\tau)e^{-2\pi i f\tau}\, d\tau \tag{1.3}$$

for the Fourier transform $Y(f)$ of $y(\tau)$. In case $x(t)$ is a signal whose energy is well concentrated around a point (t_x, f_x) in the time-frequency plane, one would like $S_x^{(w)}(t, f)$ to be well-concentrated around (t_x, f_x) as well. As a measure for this, one could take

$$\Sigma_{x,w}^2(t_x, f_x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(t - t_x)^2 + (f - f_x)^2\right] S_x^{(w)}(t, f)\, dt\, df \tag{1.4}$$

which is a quadratic time-frequency moment around (t_x, f_x). In this correspondence we shall show the following. Assume that

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df = 1 \tag{1.5}$$

and let

$$t_x = \int_{-\infty}^{\infty} t|x(t)|^2\, dt, \qquad f_x = \int_{-\infty}^{\infty} f|X(f)|^2\, df \tag{1.6}$$

so that (t_x, f_x) is the center of gravity of $x(t)$. Then $\Sigma_{x,w}^2(t_x, f_x)$ is, among all $w(t)$ with

$$\int_{-\infty}^{\infty} |w(t)|^2\, dt = \int_{-\infty}^{\infty} |W(f)|^2\, df = 1 \tag{1.7}$$

uniquely minimized by the Gaussian

$$w(t) = 2^{1/4}c\exp(-\pi t^2) \tag{1.8}$$

with c a complex constant with $|c| = 1$.

This result is somewhat remarkable since the optimal $w(t)$ is completely independent of $x(t)$. We prove our result by relating $\Sigma_{x,w}^2(t_x, f_x)$ to a different measure of concentration in the time-frequency plane, viz., to

$$\sigma_y^2(\tau, g) = \int_{-\infty}^{\infty} (t - \tau)^2\, |y(t)|^2\, dt + \int_{-\infty}^{\infty} (f - g)^2\, |Y(f)|^2\, df \tag{1.9}$$

where y is any signal and (τ, g) is any point in the time-frequency plane. We shall show that

$$\Sigma_{x,w}^2(t_x, f_x) = \sigma_x^2(t_x, f_x) + \sigma_w^2(0, 0) \tag{1.10}$$

so that minimization of $\Sigma_{x,w}^2(t_x, f_x)$ reduces to minimization of $\sigma_w^2(0, 0)$, and that $\sigma_w^2(0, 0)$ is uniquely minimal for the Gaussian $w(t)$ of (1.8).

II. Derivations

We shall prove formula (1.10) under the assumptions (1.5) and (1.7). We have from (1.4)

$$\Sigma_{x,w}^2(t_x, f_x) = \Sigma_{x,w}^2(t_x) + \Sigma_{x,w}^2(f_x) \tag{2.1}$$

where

$$\Sigma_{x,w}^2(t_x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (t - t_x)^2 S_x^{(w)}(t, f)\, dt\, df \tag{2.2}$$

$$\Sigma_{x,w}^2(f_x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (f - f_x)^2 S_x^{(w)}(t, f)\, dt\, df \tag{2.3}$$

Consider $\Sigma_{x,w}^2(t_x)$. We have for any t[1]

$$\int_{-\infty}^{\infty} S_x^{(w)}(t, f)\, df = \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} x(\tau)w(t - \tau)e^{-2\pi i f\tau}\, d\tau\right|^2 df = \int_{-\infty}^{\infty} |x(\tau)w(\tau - t)|^2\, d\tau. \tag{2.4}$$

Here we have used Parseval's formula [1, p. 65]

$$\int_{-\infty}^{\infty} |Y(f)|^2\, df = \int_{-\infty}^{\infty} |y(\tau)|^2\, d\tau \tag{2.5}$$

with $y(\tau) = x(\tau)w(\tau - t)$. Similarly, by using (1.2), we have for any f[1]

$$\int_{-\infty}^{\infty} S_x^{(w)}(t, f)\, dt = \int_{-\infty}^{\infty} |X(g)W(f - g)|^2\, dg. \tag{2.6}$$

Inserting (2.4) and (2.6) into (2.2) and (2.3), we get after a simple manipulation

$$\Sigma_{x,w}^2(t_x) = \int_{-\infty}^{\infty} (t - t_x)^2|x(t)|^2\, dt + \int_{-\infty}^{\infty} t^2|w(t)|^2\, dt$$

$$\Sigma_{x,w}^2(f_x) = \int_{-\infty}^{\infty} (f - f_x)^2|X(f)|^2\, df + \int_{-\infty}^{\infty} f^2|W(f)|^2\, df. \tag{2.8}$$

Here we have used (1.5) and (1.7), and the fact that

$$\int_{-\infty}^{\infty} (t - t_x)|x(t)|^2\, dt = \int_{-\infty}^{\infty} (f - f_x)|X(f)|^2\, df = 0. \tag{2.9}$$

Adding (2.7) and (2.8) we readily obtain (1.10).

We next show that $w(t)$ of (1.8) uniquely minimizes $\sigma_w^2(0, 0)$. We have for any $w(t)$

$$\begin{aligned}\sigma_w^2(0, 0) &= \int_{-\infty}^{\infty} t^2|w(t)|^2\, dt + \int_{-\infty}^{\infty} f^2|W(f)|^2\, df \\ &\geqq 2\left(\int_{-\infty}^{\infty} t^2|w(t)|^2\, dt\right)^{1/2}\left(\int_{-\infty}^{\infty} f^2|W(f)|^2\, df\right)^{1/2} \\ &\geqq \frac{1}{2\pi}.\end{aligned} \tag{2.10}$$

[1]Formulas (2.4) and (2.6) were presented in [2, eq. (4.8) and (4.9)], without proof.

In the first inequality in (2.10) (which is the elementary inequality $a^2 + b^2 \geqq 2ab$, $a > 0$, $b > 0$) we have equality if and only if

$$\int_{-\infty}^{\infty} t^2 |w(t)|^2 \, dt = \int_{-\infty}^{\infty} f^2 |W(f)|^2 \, df. \qquad (2.11)$$

In the second inequality in (2.10) (which is the classical Heisenberg inequality, see [1, p. 273]) we have equality if and only if $w(t)$ is of the form $c(2\alpha)^{1/4} \exp(-\pi\alpha t^2)$ for some $\alpha > 0$ and some complex c with $|c| = 1$. Inserting this special form into (2.11), we readily obtain $\alpha = 1$, and the proof is complete.

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Refferences

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