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A note on cylindrical reflector design

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A note on cylindrical reflector design. In this paper we consider the problem of designing a cylindrically symmetric reflector for a linear light source such that a prescribed intensity distribution is realized. Since there are such reflectors solving the problem, one could try to meet other design specifications such as restrictions on the dimensions of the reflector. Under suitable conditions, we show how for a fixed beginpoint of the reflector, the range of possible endpoints can be determined.

Zum Entwurf zylindrischer Spiegelreflektoren. Der Entwurf von zylindrischen Spiegelreflektoren für eine lineare Lichtquelle mit vorgegebener Lichtverteilung wird studiert. Da jeweils eine große Anzahl verschiedener Reflektoren als Problemlösungen resultieren, kann man versuchen, auch andere Spezifizierungen, wie z. B. Einschränkungen in den Reflektorabmessungen, zu berücksichtigen. Es wird gezeigt, wie unter passenden Bedingungen für einen festgelegten Ausgangspunkt des Reflektors der Bereich der möglichen Endpunkte bestimmt werden kann.

1. Preliminaries and problem statement

In this paper we consider the reflector design problem for a linear light source in combination with a linearly symmetric (i.e. cylindrical) specular reflector. Under the usual assumptions of Geometrical Optics, e.g. propagation of light along straight lines, application of Snell's law, etc., this problem is well understood. In fact, it has already been solved in 1958 by Keller [1], and also by Trembač [2]. Generally, the problem has infinitely many solutions, and in practical applications one may try to meet other design specifications, concerning for instance the dimensions of the reflector. We will show how, under suitable conditions, for a fixed starting point of the reflector, the range of possible endpoints can be determined.

Let us require a luminous intensity distribution I for the reflected light on an interval $[\theta_1, \theta_2]$, with $\theta_1 < \theta_2$. We assume that I is a continuous, non-negative function. Furthermore, we assume that the reflector is situated between two angles t_1 and t_2 with $t_1 < t_2$. The problem is now to determine a cylindrical reflector that realizes the required intensity distribution. Of course, a necessary requirement of this problem is the conservation of energy, i.e.,

$$\int_{\theta_1}^{\theta_2} I(\phi)\, d\phi = \int_{t_1}^{t_2} \varrho I_s ds, \tag{1}$$

where ϱ is the reflection coefficient, and I_s is the luminous intensity of the source. In general, I_s may not be constant, and ϱ may be a function of the angle of incidence. In this paper, however, both I_s and ϱ will be constant, and we may assume that units are chosen such that $\varrho I_s = 1$.

Consider the situation illustrated in fig. 1. Here $r(t)$ is the distance between the reflector and the light source in direction t and $\theta(t)$ is the direction of the reflected ray. If the function r is differentiable at t, then Snell's law can be formulated as

$$\frac{\dot{r}(t)}{r(t)} = \tan\left(\frac{t + \theta(t)}{2}\right), \tag{2}$$

where $-\pi < t + \theta(t) < \pi$.

Now, in order to solve the design problem, we need to specify a relation $\theta(t)$ between incident and reflected rays. In general, infinitely many choices for θ are possible. Indeed, the required distribution I can be realized by each function θ: $[t_1, t_2] \to [\theta_1, \theta_2]$ satisfying

$$\int_{\varphi_1}^{\varphi_2} I(\phi)\, d\phi = \int_{S_\theta(\phi_1, \phi_2)} 1\, ds \tag{3}$$

for all $\phi_1, \phi_2 \in [\theta_1, \theta_2]$ with $\phi_1 \leqq \phi_2$, where $S_\theta(\phi_1, \phi_2)$ is the set of all s such that $\phi_1 \leqq \theta(s) \leqq \phi_2$. This can be explained as follows: the right-hand side of (3) is the size of the angular part of the reflector that reflects the light into the directions between ϕ_1 and ϕ_2. Since $\varrho I_s = 1$, the size of that angular part is proportional to the amount of light that is incident on and reflected from that part. Now, if θ is chosen, then from (2) a continuous function r_θ can be determined by

$$r_\theta(t) = r_\theta(t_1) \exp\left(\int_{t_1}^{t} \tan\left(\frac{s + \theta(s)}{2}\right) ds\right). \tag{4}$$

Here we write $r_\theta(t)$ to emphasize that r depends on the choice of θ. So when $r_\theta(t_1)$ is fixed, the shape of the reflector is completely determined by the choice of θ.

For practical applications, the impact of the choice of θ on the dimensions of the reflector is worth investigating.

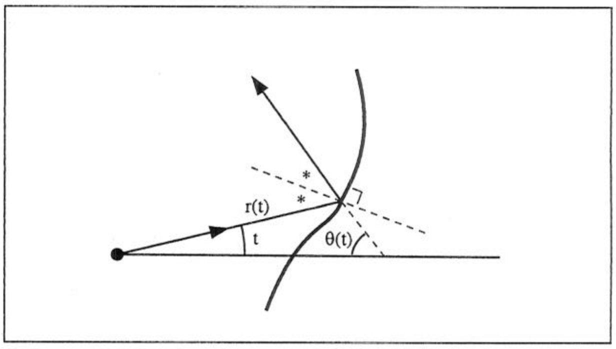

Fig. 1. Illustration of Snell's law.

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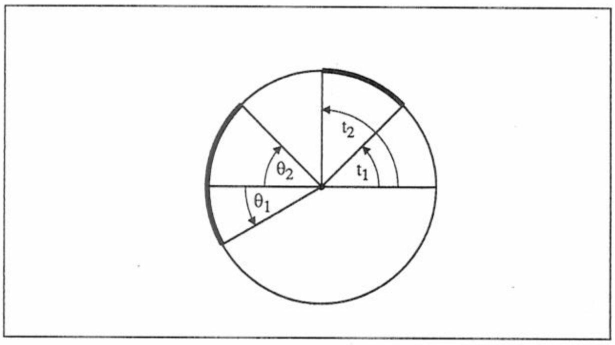

Fig. 2. An illustration of condition (6). In this figure, $t_1 = \pi/4$, $t_2 = \pi/2$, $\theta_1 = -\pi/6$ and $\theta_2 = \pi/4$. Both segments lie above the diameter in direction t_1.

In particular, design specifications will often put restrictions on the ratio

$$\frac{r_\theta(t_2)}{r_\theta(t_1)} = \exp\left(\int_{t_1}^{t_2} \tan\left(\frac{s + \theta(s)}{2}\right) ds\right) \tag{5}$$

between the extreme points of the reflector. The problem that we consider in this paper is: what is the range of values that (5) can assume? Naturally, this problem is only relevant if we consider reflectors that are described by a continuous function r_θ, so that a gap in the reflector does not occur. In this paper we will solve the problem under the condition that either

$$\theta_1 + t_1 \geqq 0 \quad \text{or} \quad \theta_2 + t_2 \leqq 0. \tag{6}$$

To understand the geometric meaning of this condition, first note that, a suggested by fig. 1, incident and reflected rays are measured counterclockwise and clockwise relative to the horizontal through the source, respectively. When we represent the sets of directions of both incident and reflected rays relative to the source as segments on a unit circle (see fig. 2), condition (6) is equivalent to the existence of a diameter of the circle such that both angular segments are on the same side of this diameter. In this situation, we will see that the extreme values of (5) will be assumed for the two special choices for θ which we will now introduce.

First, let the increasing function $\theta^+ : [t_1, t_2] \to [\theta_1, \theta_2]$ be defined by

$$\int_{\theta_1}^{\theta^+(t)} I(\phi)\, d\phi = \int_{t_1}^{t} 1\, ds = t - t_1 . \tag{7}$$

This uniquely defines θ^+, and sometimes an explicit expression for θ^+ can be found; see example 1.2 below. It is easily verified that θ^+ satisfies condition (3). Similarly we can define the decreasing function $\theta^- : [t_1, t_2] \to [\theta_1, \theta_2]$ by

$$\int_{\theta^-(t)}^{\theta_2} I(\phi)\, d\phi = \int_{t_1}^{t} 1\, ds = t - t_1 , \tag{8}$$

or equivalently, $\theta^-(t) = \theta^+(t_1 + t_2 - t)$.

Now the main result of this paper is the following theorem.

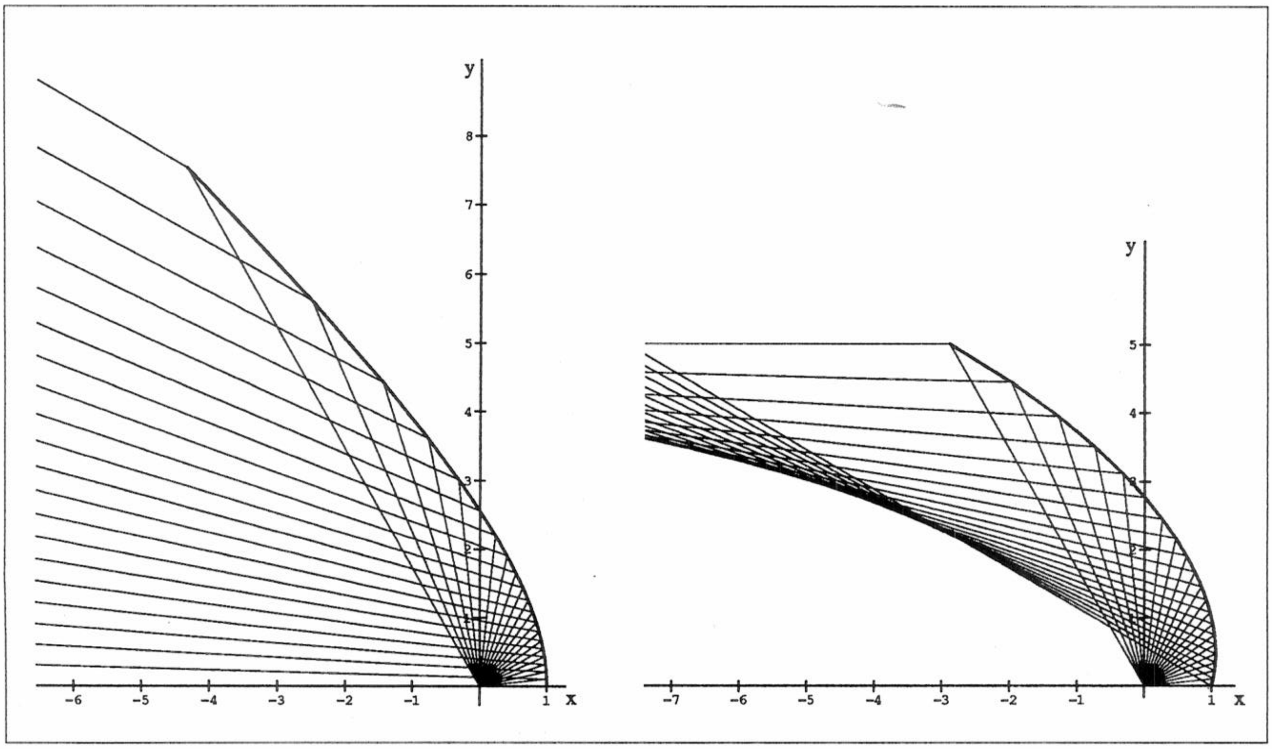

Fig. 3. The increasing and decreasing solutions to the problem in example 1.2 (a), respectively. Here the source is located in the origin of a rectangular coordinate system, such that $r(t)$ corresponds to the reflectorpoint with coordinates $(r(t) \cos t, r(t) \sin t)$.

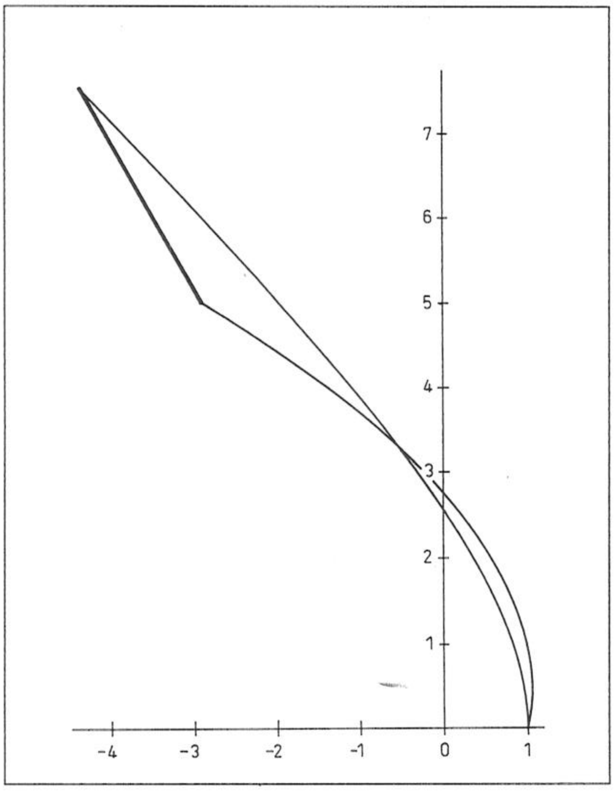

Fig. 4. The range of possible endpoints for the solutions to the problem in example 1.2 (a).

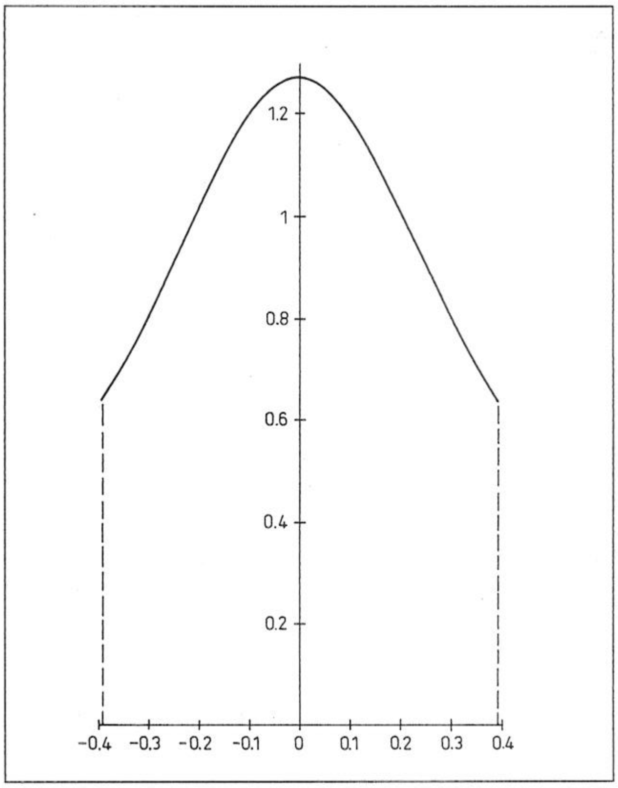

Fig. 5. Graph of the required distribution function $I(\theta) = 4\pi/(\pi^2 + 64\theta^2)$ of the problem in example 1.2 (b).

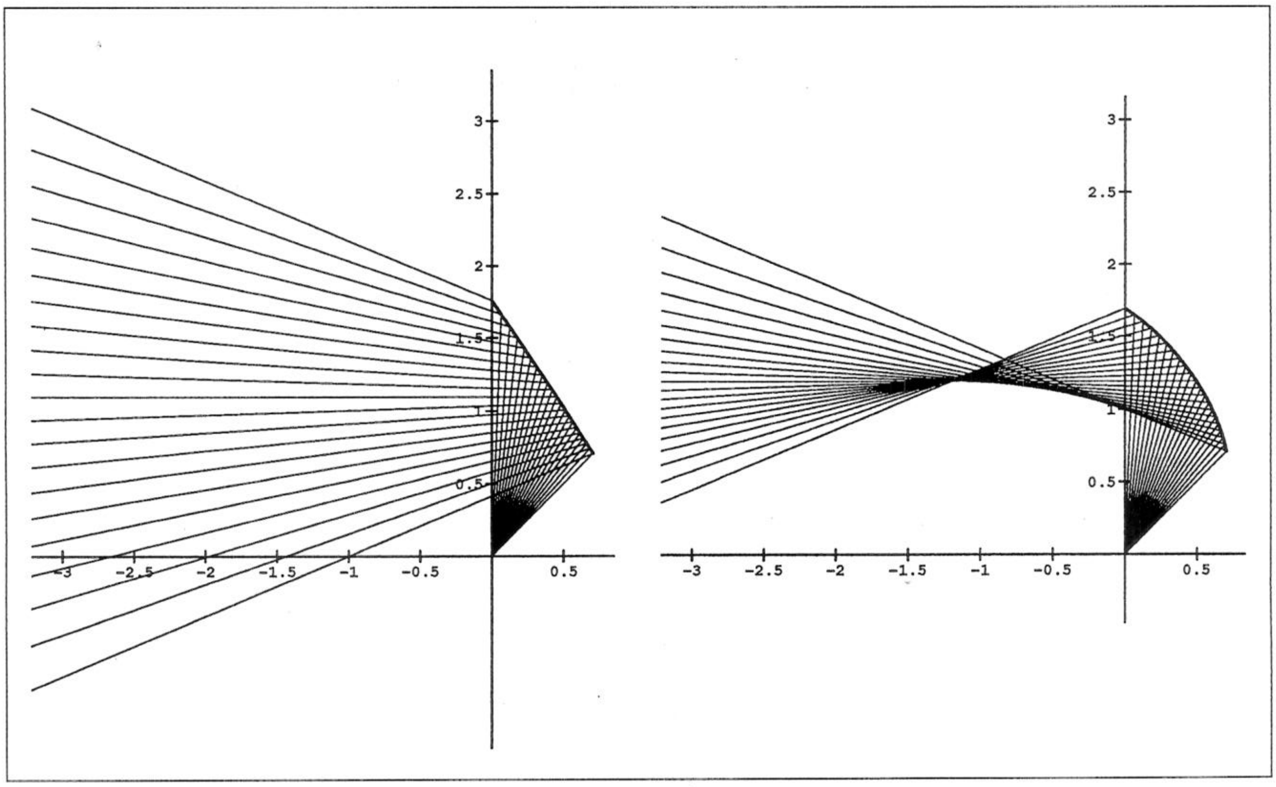

Fig. 6. The increasing and decreasing solutions to the problem in example 1.2 (b), respectively.

Theorem 1.1. For each θ satisfying (3), we have the inequalities

$$\frac{r_{\theta^-}(t_2)}{r_{\theta^-}(t_1)} \leqq \frac{r_\theta(t_2)}{r_\theta(t_1)} \leqq \frac{r_{\theta^+}(t_2)}{r_{\theta^+}(t_1)} \quad \text{if} \quad \theta_1 + t_1 \geqq 0, \tag{9}$$

and

$$\frac{r_{\theta^+}(t_2)}{r_{\theta^+}(t_1)} \leqq \frac{r_\theta(t_2)}{r_\theta(t_1)} \leqq \frac{r_{\theta^-}(t_2)}{r_{\theta^-}(t_1)} \quad \text{if} \quad \theta_2 + t_2 \leqq 0. \tag{10}$$

This theorem will be proved in the following section. First we will give an example.

Example 1.2. (a). We require a uniform intensity distribution $I(\theta) = 4$ on an interval $[\theta_1, \theta_2] = [0, \pi/6]$. Let the reflector be situated in the angular interval $[0, 2\pi/3]$, and let the beginpoint $r(0) = 1$ be fixed. Note that (1) is satisfied. From (7) and (8) it follows immediately that $\theta^+(t) = t/4$ and $\theta^-(t) = \pi/6 - t/4$. The reflectors realizing θ^+ and θ^- are shown in fig. 3. Since (6) is satisfied, the range of possible endpoints is $[r_{\theta^-}(t_2), r_{\theta^+}(t_2)] \approx [5.79, 8.69]$; see fig. 4.

(b). Let the prescribed intensity distribution be defined by $I(\theta) = 4\pi/(\pi^2 + 64\theta^2)$, for $\theta \in [-\pi/8, \pi/8]$; see fig. 5. Let $[t_1, t_2] = [\pi/4, \pi/2]$, and let $r(\pi/4) = 1$. Now, the function $f(\theta) = \frac{1}{2}\arctan(8\theta/\pi)$ is a primitive of I, and it is easily verified that (1) is satisfied. From (7) we find $f(\theta^+(t)) - f(\theta_1) = t - t_1$, so $\theta^+(t) = f^{-1}(t - t_1 + f(\theta_1)) = \pi/8 \tan(2(t - 3\pi/8))$, and $\theta^-(t) = -\theta^+(t)$. The corresponding reflectors are shown in fig. 6, and the small range [1.69, 1.75] of possible endpoints is shown in fig. 7.

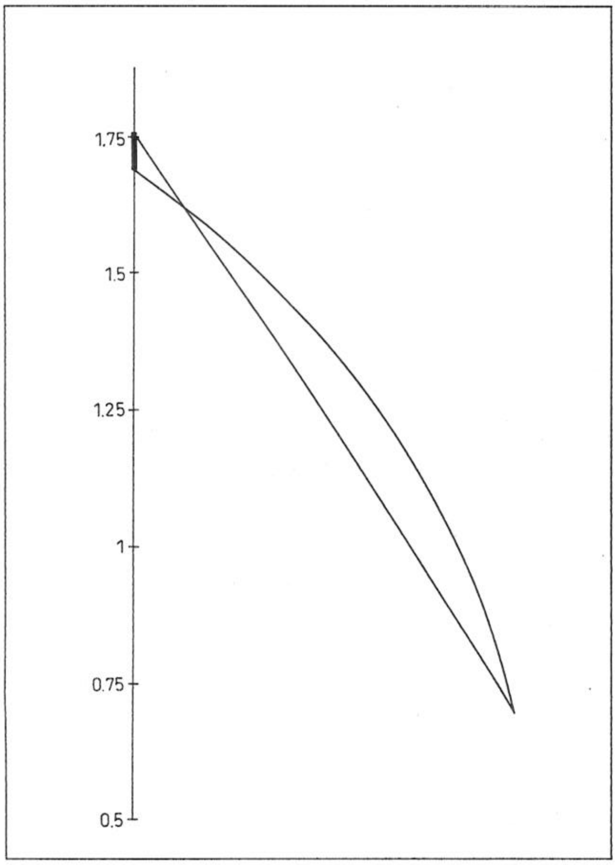

Fig. 7. The range of possible endpoints for the solutions to the problem in example 1.2 (b).

2. Proof of the main result

In this section we will prove theorem 1.1. We will use a basic mathematical result from rearrangement theory; see e.g. Hardy, Littlewood and Pólya [3, Ch. 10]. To introduce the notion of a rearrangement of a function, we first note the following. A function θ satisfies (3) for all ϕ_1, ϕ_2, if and only if

$$\int_{S_\theta(\phi_1, \phi_2)} 1\, ds = \int_{S_{\theta^+}(\phi_1, \phi_2)} 1\, ds, \tag{11}$$

i.e., if all sets $S_\theta(\phi_1, \phi_2)$ and $S_{\theta^+}(\phi_1, \phi_2)$ have equal size. (So the corresponding parts of the reflector will reflect the same amount of light in the directions between ϕ_1 and ϕ_2.) In terms of rearrangement theory, if (11) holds and if θ^+ is increasing, then we say that θ^+ is an increasing rearrangement of θ. Similarly we say that θ^- is a decreasing rearrangement of θ.

We will give the proof to (9) only; the other part of the theorem is proved completely analogously. So let $\theta_1 + t_1 \geqq 0$. By a change in the choice of zero-direction, we may even assume that both $\theta_1 \geqq 0$ and $t_1 \geqq 0$. (In fact we might even assume $t_1 = 0$.) In order to prove theorem 1.1, it follows from (5) that it suffices to prove the following proposition.

Proposition 2.1. If $\theta_1 \geqq 0$ and $t_1 \geqq 0$, then for θ satisfying (11) we have the inequalities

$$\int_{t_1}^{t_2} \tan\left(\frac{s + \theta^-(s)}{2}\right) ds \leqq \int_{t_1}^{t_2} \tan\left(\frac{s + \theta(s)}{2}\right) ds \leqq \int_{t_1}^{t_2} \tan\left(\frac{s + \theta^+(s)}{2}\right) ds. \tag{12}$$

Proof. We can write the series expansion of $\tan\left(\frac{s+\theta(s)}{2}\right)$ as

$$\tan\left(\frac{s + \theta(s)}{2}\right) = \sum_{k=1}^{\infty} c_k \left(\frac{s + \theta(s)}{2}\right)^k, \tag{13}$$

where all $c_k \geqq 0$, see Abramowitz and Stegun [4, p. 75]. Hence

$$\int_{t_1}^{t_2} \tan\left(\frac{s + \theta(s)}{2}\right) ds = \sum_{k=1}^{\infty} \sum_{l=0}^{k} \binom{k}{l} \frac{c_k}{2^k} \int_{t_1}^{t_2} s^{k-l}(\theta(s))^l\, ds. \tag{14}$$

Now for any $m, n = 0, 1, \ldots$ we have

$$\int_{t_1}^{t_2} s^m(\theta^-(s))^n\, ds \leqq \int_{t_1}^{t_2} s^m(\theta(s))^n\, ds \leqq \int_{t_1}^{t_2} s^m(\theta^+(s))^n\, ds. \tag{15}$$

This follows from the well-known rearrangement inequalities

$$\int_{t_1}^{t_2} f^+(s)\, g^-(s)\, ds \leqq \int_{t_1}^{t_2} f^+(s)\, g(s)\, ds \leqq \int_{t_1}^{t_2} f^+(s)\, g^+(s)\, ds \tag{16}$$

which result from Hardy, Littlewood and Pólya [3, p. 278], and from

$$(s^m)^+ = s^m, (\theta^n)^+ = (\theta^+)^n, \quad \text{and} \quad (\theta^n)^- = (\theta^-)^n. \quad (17)$$

The theorem then follows from (14) and (15).

3. Discussion

We have shown how, for a cylindrical reflector realizing a prescribed intensity distribution, the ratio of the two distances from its extreme points to the light source can be determined in the case that condition (6) is satisfied. For practical purposes, this result may help to meet design specifications concerning the dimensions of the reflector, or perhaps to show these specifications to be conflicting. We will now shortly discuss the extension of the problem to more general cases.

First of all suppose that condition (6) is not satisfied. In that case, the inequalities (9) and (10) are generally not true. However, for many required luminous intensity distributions one can still determine the choices for θ that will yield the extreme values for the ratio (5). This problem has been studied by the authors, and the results involve a considerable amount of mathematics, beyond the scope of this paper.

A second extension would be to drop the assumption that the reflection coefficient ϱ and the luminous intensity of the source I_s are constant. Unfortunately, these asumptions are necessary. If two different parts of the reflector, that are located within angular ranges of equal size do not reflect the same amount of light, then eq. (11) is not satisfied, and rearrangement theory does not apply. The same problem arises when one considers rotationally symmetric reflectors with point sources.

Finally, it should be mentioned that the analogue of theorem 1.1 is generally not true if the required luminous intensity distribution is given on a flat screen at a finite distance of the optical system. Several case studies however, have shown that in this case it is a good 'rule of thumb' to assume that the extreme values of the ratio (5) are still obtained for increasing and decreasing functions.

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