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LETTER TO THE EDITOR

On the asymptotics of some Pearcey-type integrals

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Abstract. In this letter we discuss the asymptotic behaviour of the Pearcey-type integral

$$I'_\alpha(X,Y) = 2\int_0^\infty u^{\alpha+1}\exp\big(\mathrm{i}(u^4 + Xu^2)\big)\, J_\alpha(Yu)\mathrm{d}u$$

for $-1 < \alpha < \frac{5}{2}$, where J_α is a Bessel function, as $X \to \pm\infty$, Y fixed, as $Y \to \infty$, X fixed, and as $Y = \rho(\frac{2}{3}|X|)^{3/2}$, $X \to -\infty$, ρ fixed. The case $\alpha = -\frac{1}{2}$ gives the classical Pearcey integral whose asymptotics has been investigated recently by Kaminski and Paris. In the case $\alpha = 0$, $I'_\alpha(X,Y)$ as a function of $Y \geqslant 0$ represents the radial part of the impulse–response function describing the image formation in high resolution electron microscopes at normalized defocus X. We use the approach of Paris by representing $I'_\alpha(X,Y)$ in terms of Weber parabolic cylinder functions, and we augment this approach by invoking the Chester–Friedman–Ursell method to obtain the leading asymptotics of $I'_\alpha(X,Y)$ around the caustic $Y^2 = (\frac{2}{3}|X|)^3$, $X \to -\infty$.

In [5, 6] the asymptotics of (the analytic continuation to complex variables of) the Pearcey integral

$$P'(X,Y) = 2\int_0^\infty \exp\big(\mathrm{i}(u^4 + Xu^2)\big)\cos Yu\,\mathrm{d}u \tag{1}$$

is presented. The Pearcey integral occurs at many places in the physics literature, especially where a short-wavelength description of the phenomena is desired; we refer to [3, 5, 6] and the references therein for surveys of existing literature on Pearcey's integral. In a recent study on the image formation in high resolution electron microscopes [4], an important role is played by the integral

$$I'(X,Y) = 2\int_0^\infty \exp\big(\mathrm{i}(u^4 + Xu^2)\big) J_0(Yu)u\,\mathrm{d}u \tag{2}$$

where J_0 is the Bessel function of order 0. Indeed, in the terminology of [4], $I'(X,\cdot)$ represents the radial part of the (undamped) impulse–response function at defocus X. Interestingly, in the hypothetical case of one-dimensional microscopy, the role of $I'(X,\cdot)$ would be taken over by $P'(X,\cdot)$ in (1).

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In this letter we are interested, more generally, in the asymptotics of the integral

$$I'_\alpha(X,Y) = 2\int_0^\infty \exp\big(\mathrm{i}(u^4 + Xu^2)\big) J_\alpha(Yu)u^{\alpha+1}\,\mathrm{d}u \tag{3}$$

with $-1 < \alpha < \frac{5}{2}$, where J_α is the Bessel function of order α. For $\alpha = 0$ we obtain (2), and we have

$$P'(X,Y) = \sqrt{\tfrac{1}{2}\pi Y} I'_{-1/2}(X,Y)\,. \tag{4}$$

It turns out that we can mimic the arguments of Paris in [6] for obtaining the asymptotics of $P'(X,Y)$ to a very large extent. To explain this, we note that with $x = X\exp(-\frac{1}{4}\pi\mathrm{i})$, $y = Y\exp(\frac{1}{8}\pi\mathrm{i})$ we have

$$I'_\alpha(X,Y) = 2\exp[\tfrac{1}{8}\pi\mathrm{i}(\alpha+2)]\int_0^\infty J_\alpha(yt)\exp(-t^4 - xt^2)t^{\alpha+1}\,\mathrm{d}t =: I_\alpha(x,y) \tag{5}$$

and that we have for $y \neq 0$ the generalized Paris integral representation, see [6, (2.6)]

$$I_\alpha(x,y) = \exp[\tfrac{1}{8}\pi\mathrm{i}(\alpha+2)]2^{-3\alpha/2-1/2}y^\alpha \mathrm{e}^{x^2/8} \times \frac{1}{2\pi\mathrm{i}}\int_C \Gamma(s)D_{s-\alpha-1}\Big(\frac{x}{\sqrt{2}}\Big)\Big(\frac{y^2}{4\sqrt{2}}\Big)^{-s}\mathrm{d}s\,. \tag{6}$$

Here C is a loop starting and finishing at $-\infty$ and encircling the origin in positive sense, and D_ν is the (analytic continuation to all $\nu \in \mathbb{C}$ of the) parabolic cylinder function admitting for $\mathrm{Re}\,\nu < 1$ the integral representation

$$D_\nu(z) = \frac{\mathrm{e}^{-z^2/4}}{\Gamma(-\nu)}\int_0^\infty \exp(-\tfrac{1}{2}\tau^2 - z\tau)\tau^{-\nu-1}\,\mathrm{d}\tau. \tag{7}$$

This enables us to derive the asymptotics of $I_\alpha(x,y)$ when $|x| \to \infty$, y fixed and when $|y| \to \infty$, x fixed.

In [5] Kaminski determines the asymptotics of $P'(X,Y)$ near the caustic $Y^2 = \frac{2}{3}|X|^3$, $X \to -\infty$, by using directly the integral representation (1) together with the method of Chester, Friedman and Ursell (CFU-method), see [1, ch 9] and [2], for the asymptotics of integrals with two nearly coalescing saddle points. The asymptotics of $P'(X,Y)$ exactly at the caustic $Y = (\frac{2}{3}|X|)^{3/2}$, $X \to -\infty$, is also determined by Paris in [6, section 6], as a check of the validity of his integral representation approach. However, for our case, the direct method of Kaminski is not applicable, and we must augment Paris' arguments of [6, section 6], by an appeal to the CFU-method to obtain the required asymptotics near the caustic. Doing so, we obtain the leading asymptotics for $I'_\alpha(X,Y)$ near the caustic (and not a full asymptotic expansion as Kaminski obtains for $P'(X,Y)$).

We shall now present our main results, and then indicate how these results can be proved by using Paris' arguments and extensions thereof. Although we could present the asymptotics of $I_\alpha(x,y)$ when $|x| \to \infty$ or $|y| \to \infty$ for general complex x, y (just as Paris does for his $P(x,y)$), we restrict to $x = X\exp(-\frac{1}{4}\pi\mathrm{i})$, $y = Y\exp(\frac{1}{8}\pi\mathrm{i})$ with real X and $Y > 0$. We thus get

$$I'_\alpha(X,Y) \sim \frac{\mathrm{i}}{X}\left(\frac{\mathrm{i}Y}{2X}\right)^\alpha \exp\left(\frac{-\mathrm{i}Y^2}{4X}\right)\sum_{m=0}^{\infty}\frac{(2m)!}{m!(\mathrm{i}X^2)^m}L_{2m}^{(\alpha)}\left(\frac{\mathrm{i}Y^2}{4X}\right) \tag{8}$$

as $X \to +\infty$, $Y > 0$, and

$$\begin{aligned}I'_\alpha(X,Y) \sim{}& \frac{\mathrm{i}}{X}\left(\frac{\mathrm{i}Y}{2X}\right)^\alpha \exp\left(\frac{-\mathrm{i}Y^2}{4X}\right)\sum_{m=0}^{\infty}\frac{(2m)!}{m!(\mathrm{i}X^2)^m}L_{2m}^{(\alpha)}\left(\frac{\mathrm{i}Y^2}{4X}\right)\\ &+ 2^{-\alpha/2}\pi^{1/2}(-X)^{\alpha/2}\exp(\tfrac{1}{4}\pi\mathrm{i} - \tfrac{1}{4}\mathrm{i}X^2)\\ &\times\sum_{m=0}^{\infty}\left(\frac{-\mathrm{i}Y^2}{8X}\right)^m\frac{1}{m!}J_{\alpha-2m}\left(Y\sqrt{-\tfrac{1}{2}X}\right)\end{aligned} \tag{9}$$

as $X \to -\infty$, $Y > 0$. Here $L_{2m}^{(\alpha)}$ is the $(2m)$th Laguerre polynomial of order α, see [7, section 5.1]. (It is observed here that the function $a_m(\chi)$ in [6, (3.4)–(3.6)] equals $(2m)!L_{2m}^{(-1/2)}(\chi)$.)

Next we have when $X \in \mathbb{R}$ is fixed and $W := \frac{1}{4}Y \to +\infty$

$$\begin{aligned}I'_\alpha(X,Y) \sim{}& \frac{W^{\alpha/3-2/3}}{2\sqrt{3}}\exp\left(-\tfrac{1}{6}\mathrm{i}X^2 + \tfrac{1}{2}\pi\mathrm{i}(1+\alpha) - 3\mathrm{i}W^{4/3} + \mathrm{i}XW^{2/3}\right)\\ &\times\left\{1 + \frac{X(\frac{1}{18}\mathrm{i}X^2 - \alpha)}{6W^{2/3}} + \mathrm{O}(W^{-4/3})\right\} + \frac{W^{\alpha/3-2/3}}{2\sqrt{3}}\\ &\times\exp\left(-\tfrac{1}{6}\mathrm{i}X^2 - \tfrac{1}{6}\pi\mathrm{i}(1+\alpha) - 3\mathrm{e}^{-\pi\mathrm{i}/6}W^{4/3} - \mathrm{e}^{\pi\mathrm{i}/6}XW^{2/3}\right)\\ &\times\left\{1 - \frac{X(\frac{1}{18}\mathrm{i}X^2 - \alpha)\mathrm{e}^{\pi\mathrm{i}/3}}{6W^{2/3}} + \mathrm{O}(W^{-4/3})\right\}.\end{aligned} \tag{10}$$

Finally, when $\rho > 0$ is fixed and $Y = \rho^{1/2}(\frac{2}{3}|X|)^{3/2}$, $X \to -\infty$, we have

$$\begin{aligned}I'_\alpha(X,Y) \sim{}& \left(\frac{\pi}{\rho}\right)^{1/2}\left(\frac{|X|}{6\rho}\right)^{\alpha/2}\exp[-\tfrac{1}{4}\pi\mathrm{i}(2\alpha+1) + \delta X^2]\\ &\times\left[\frac{c_0}{|X|^{2/3}}\mathrm{Ai}(\gamma^2|X|^{4/3}) - \frac{\mathrm{i}c_1}{|X|^{4/3}}\mathrm{Ai}'(\gamma^2|X|^{4/3})\right]\\ &+\left(\frac{1}{\rho}\right)^{1/2}\left(\frac{2|X|}{3\rho}\right)^{\alpha/2}\frac{1}{|X|}\exp[\tfrac{1}{2}\pi\mathrm{i}(\alpha+1) + \varepsilon X^2]\end{aligned} \tag{11}$$

where δ, γ, ε are independent of α and satisfy ($\beta = -\frac{2}{3}\ln\rho$)

$$\begin{aligned}\delta &= \tfrac{1}{12}\mathrm{i} - \tfrac{1}{6}\mathrm{i}\beta + \tfrac{5}{72}\mathrm{i}\beta^2 + \mathrm{O}(\beta^3)\\ \gamma &= 3^{-1/3}\mathrm{i}(\tfrac{1}{2}\beta)^{1/2} + \mathrm{O}(\beta^{3/2})\\ \varepsilon &= -\tfrac{2}{3}\mathrm{i} + \tfrac{1}{3}\mathrm{i}\beta - \tfrac{5}{36}\mathrm{i}\beta^2 + \mathrm{O}(\beta^3)\end{aligned} \tag{12}$$

and

$$c_0 = 3^{1/3} + \mathrm{O}(\beta) \qquad c_1 = (\tfrac{1}{3} + \alpha)3^{2/3} + \mathrm{O}(\beta^{1/2}). \tag{13}$$

In particular, we have at the caustic ($\rho = 1$; $\beta = \gamma = 0$)

$$\begin{aligned} I'_\alpha(X,Y) \sim{} & \frac{\exp[-\frac{1}{4}\pi\mathrm{i}(2\alpha+1) + \frac{1}{12}\mathrm{i}X^2]}{2\sqrt{\pi}}(\tfrac{1}{6}|X|)^{\alpha/2} \\ & \times \left[\frac{3^{1/6}\,\Gamma(\frac{1}{3})}{|X|^{2/3}} + \frac{3^{5/6}(\frac{1}{3}+\alpha)\Gamma(\frac{2}{3})}{|X|^{4/3}}\right] \\ & + (\tfrac{2}{3}|X|)^{\alpha/2}\frac{1}{|X|}\exp[\tfrac{1}{2}\pi\mathrm{i}(\alpha+1) - \tfrac{2}{3}\mathrm{i}X^2]. \end{aligned} \tag{14}$$

We shall next show that the representation (6) holds. To that end we observe the formulae

$$J_\alpha(z) = (\tfrac{1}{2}z)^\alpha \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{k!\,\Gamma(k+\alpha+1)} \qquad z \in \mathbb{C} \tag{15}$$

$$J_\alpha(z) = \mathrm{O}(|z|^{-1/2}\mathrm{e}^{|\operatorname{Im} z|}) \qquad |\arg z| < \pi,\ |z| \to \infty \tag{16}$$

$$J_\alpha(u), J'_\alpha(u), J''_\alpha(u) = \mathrm{O}(u^{-1/2}) \qquad u \to +\infty. \tag{17}$$

It then follows that $I'_\alpha(X,Y)$ is well defined as an improper Riemann integral for $-1 < \alpha < \frac{5}{2}$, and that (5) holds (on substituting $u = \mathrm{e}^{\pi\mathrm{i}/8}t$ and using Jordan's lemma). Next we use (15) with $z = yt$, interchange sum and integral, substitute $v = t^2$ in the integral, and obtain

$$I_\alpha(x,y) = \exp[\tfrac{1}{8}\pi\mathrm{i}(\alpha+2)](\tfrac{1}{2}y)^\alpha \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}y^2)^k}{k!\,\Gamma(\alpha+k+1)} \int_0^\infty \mathrm{e}^{-v^2-xv}\, v^{k+\alpha}\,\mathrm{d}v. \tag{18}$$

Then we use (7) and the fact that $\Gamma(s)$ has poles of order one at $s = -k = 0, -1, \ldots$ with residues $(-1)^k/k!$ to obtain (6). In (6) the contour C does not need to lie in $\operatorname{Re} s < \alpha + 1$, as would be the case when (7) were used, since $D_\nu(z)$ extends to an entire function of ν.

We next show how the asymptotic expansion (8) can be derived; note that $\arg(x) = -\frac{1}{4}\pi$ since $x = X\exp(-\frac{1}{4}\pi\mathrm{i})$, $X > 0$. Proceeding in the same (formal) way as in [6, section 3(*a*)], we insert the expansion

$$\begin{aligned} D_{s-\alpha-1}\left(\frac{x}{\sqrt{2}}\right) & \\ & \sim \exp(-\tfrac{1}{8}x^2)\left(\frac{x}{\sqrt{2}}\right)^{s-\alpha-1} \sum_{m=0}^{\infty} \frac{(s-\alpha-1)\ldots(s-\alpha-2m)}{m!\,x^{2m}}(-1)^m \\ & \qquad |\arg(x)| < \tfrac{1}{2}\pi \end{aligned} \tag{19}$$

into (6), and obtain

$$I_\alpha(x,y) \sim \exp[\tfrac{1}{8}\pi\mathrm{i}(\alpha+2)]\,(\tfrac{1}{2}y)^\alpha\, x^{-\alpha-1} \sum_{m=0}^{\infty} \frac{(-1)^m\, b_m(y^2/4x)}{m!\,x^{2m}}. \tag{20}$$

Here

$$b_m(\chi) = \frac{1}{2\pi i}\int_C \chi^{-s}\,\Gamma(s)(s-\alpha-1)\ldots(s-\alpha-2m)\,ds$$
$$= \sum_{l=0}^{\infty}\frac{(-\chi)^l}{l!}(l+\alpha+2m)\ldots(l+\alpha+1) = (2m)!\,e^{-\chi}\,L_{2m}^{(\alpha)}(\chi). \qquad (21)$$

From this (8) follows.

Similarly, when $x = X\exp(-\frac{1}{4}\pi i)$, $X < 0$ (so that $\arg(x) = \frac{3}{4}\pi$), we use

$$D_\nu(z) = e^{\pi i\nu} D_\nu(-z) + \frac{i(2\pi)^{1/2}}{\Gamma(-\nu)} e^{\pi i\nu/2} D_{-\nu-1}(-iz) \qquad z\in\mathbb{C} \qquad (22)$$

together with (6), to obtain

$$I_\alpha(x,y) = -\exp(-\tfrac{3}{2}\pi i\alpha) I_\alpha(-x, iy)$$
$$+ \exp[\tfrac{1}{8}\pi i(\alpha+2) + \tfrac{1}{8}x^2] 2^{-(3\alpha/2)-1/2} y^\alpha I_{2,\alpha}(x,y) \qquad (23)$$

where

$$I_{2,\alpha}(x,y) = \frac{(2\pi)^{1/2}}{2\pi i}\exp(-\tfrac{1}{2}\pi i\alpha)$$
$$\times \int_C \frac{\Gamma(s)}{\Gamma(-s+\alpha+1)} D_{-s+\alpha}\left(\frac{-ix}{\sqrt{2}}\right)\left(\frac{-iy^2}{4\sqrt{2}}\right)^{-s} ds. \qquad (24)$$

For the first term at the right-hand side of (23) we can use (8); for the second term we use (19), and obtain

$$I_{2,\alpha}(x,y) \sim (2\pi)^{1/2}\left(\frac{-x}{\sqrt{2}}\right)^\alpha \exp(\tfrac{1}{8}x^2)\sum_{m=0}^{\infty}\frac{1}{m!\,x^{2m}}$$
$$\times \frac{1}{2\pi i}\int_C (-\tfrac{1}{8}xy^2)^{-s}\frac{\Gamma(s)}{\Gamma(-s+\alpha-2m+1)}\,ds. \qquad (25)$$

Here we have used that $\Gamma(-x+1) = x(x+1)\cdots(x+2m-1)\,\Gamma(-x-2m+1)$. Finally, (9) follows by taking $\xi = y(-\frac{1}{2}x)^{1/2}$ in the identity

$$(\tfrac{1}{2}\xi)^{2m-\alpha} J_{\alpha-2m}(\xi) = \frac{1}{2\pi i}\int_C (\tfrac{1}{2}\xi)^{-2s}\frac{\Gamma(s)}{\Gamma(-s+\alpha-2m+1)}\,ds. \qquad (26)$$

We observe that the derivations just given can be shown to yield true asymptotic series by using the methods of [6, section 4]; in fact, such a thing is implicitly stated in [6, middle of p 422], about the asymptotics of $P^{(n)}(x,y)$, i.e. the case that $\alpha = n+\frac{1}{2}$.

We next turn to the derivation of (10). This can be done as in [6, section 5]; we just show some intermediate steps. After replacing $s-\alpha-1$ by $s-\frac{1}{2}$ in the integral at the right-hand side of (6) (so that we can conveniently use [6, equation (5.2)]), we obtain as in [6]

$$I_\alpha(x,y) = \exp[\tfrac{1}{8}\pi\mathrm{i}(\alpha+2) + \tfrac{1}{8}x^2]2^{\alpha+1/4}\pi^{-1/2}y^{-\alpha-1} \times \{\exp(-\tfrac{1}{4}\pi\mathrm{i})I_{+,\alpha}(x,y) + \exp(\tfrac{1}{4}\pi\mathrm{i})I_{-,\alpha}(x,y)\} \tag{27}$$

where

$$I_{\pm,\alpha}(x,y) = \frac{1}{2\pi\mathrm{i}}\int_C \Gamma(s+\tfrac{1}{2})\,\Gamma(s+\alpha+\tfrac{1}{2})\,D_{-s-1/2}\left(\pm\frac{\mathrm{i}x}{\sqrt{2}}\right)\left(\mp\frac{\mathrm{i}y^2}{4\sqrt{2}}\right)^{-s}\mathrm{d}s. \tag{28}$$

Using [6, equation (5.2)] and the result

$$\frac{\Gamma(s+\frac{1}{2})\,\Gamma(s+\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2}s+\frac{3}{4})} = \Gamma(\tfrac{3}{2}s+\tfrac{1}{4}+\alpha)\left(\frac{3^{3/2}}{4}\right)^{-s}2^{\alpha}\,3^{-(\alpha-1/4)}\left[1+O\left(\frac{1}{s}\right)\right] \tag{29}$$

we obtain

$$I_{\pm,\alpha}(x,y) = \frac{B_\pm}{2\pi\mathrm{i}}\int_C \Gamma(t)\,Z_\pm^{-t}(1 \pm At^{-1/2} + O(t^{-1}))\exp(\mp\mathrm{i}x\sqrt{t/3})\,\mathrm{d}t \tag{30}$$

where

$$\begin{aligned} A &= \frac{\mathrm{i}x}{2\sqrt{3}}\left(\alpha+\frac{1}{4}+\frac{x^2}{16}\right)\\ B_\pm &= \pi^{1/2}2^{-5\alpha/3+1/12}3^{-1/2}y^{4\alpha/3+1/3}\exp[\mp\tfrac{1}{6}\pi\mathrm{i}(2\alpha+\tfrac{1}{2})]\\ Z_\pm &= 3\exp(\mp\tfrac{1}{3}\pi\mathrm{i})(\tfrac{1}{4}y)^{4/3}. \end{aligned} \tag{31}$$

With the aid of the lemma in [6, section 5], we then get

$$I_\alpha(x,y) = T_{+,\alpha}(x,y) + T_{-,\alpha}(x,y) \tag{32}$$

where, with $w = \frac{1}{4}y$,

$$T_{+,\alpha}(x,y) = \frac{w^{\alpha/3-2/3}}{2\sqrt{3}}\exp(-\tfrac{1}{12}\pi\mathrm{i} - \tfrac{5}{24}\pi\mathrm{i}\alpha + \tfrac{1}{6}x^2 - 3\mathrm{e}^{-\pi\mathrm{i}/3}w^{4/3} - \mathrm{i}x\mathrm{e}^{-\pi\mathrm{i}/6}w^{2/3}) \times\left\{1 - \frac{(\alpha+\frac{1}{18}x^2)x}{6w^{2/3}}\exp(-\tfrac{1}{3}\pi\mathrm{i}) + O(w^{-4/3})\right\} \tag{33}$$

$$T_{-,\alpha}(x,y) = \frac{w^{\alpha/3-2/3}}{2\sqrt{3}}\exp(\tfrac{7}{12}\pi\mathrm{i} + \tfrac{11}{24}\pi\mathrm{i}\alpha + \tfrac{1}{6}x^2 - 3\mathrm{e}^{\pi\mathrm{i}/3}w^{4/3} + \mathrm{i}x\mathrm{e}^{\pi\mathrm{i}/6}w^{2/3}) \times\left\{1 - \frac{(\alpha+\frac{1}{18}x^2)x}{6w^{2/3}}\exp(\tfrac{1}{3}\pi\mathrm{i}) + O(w^{-4/3})\right\}. \tag{34}$$

From this (10) follows on setting $x = X\exp(-\frac{1}{4}\pi\mathrm{i})$, $y = Y\exp(\frac{1}{8}\pi\mathrm{i})$.

We finally show the main steps in deriving (11). When we follow the steps (6.1)–(6.13) in [6], we get ($Y = \rho^{1/2}(\frac{2}{3}|X|)^{3/2}$)

$$I'_\alpha(X,Y) = \exp[\tfrac{1}{4}\pi\mathrm{i}(\alpha+1) - \tfrac{1}{8}\mathrm{i}X^2]2^{-3\alpha/2-1/2}Y^\alpha\{I'_{1,\alpha}(X,Y) + I'_{2,\alpha}(X,Y)\} \tag{35}$$

where

$$I'_{1,\alpha}(X,Y) \sim -3^{3/4+2\alpha}\pi^{1/2}2^{1/2+\alpha/2}\rho^{-1/2-\alpha}|X|^{-\alpha}\exp(\tfrac{3}{8}\pi\mathrm{i} - \tfrac{1}{4}\pi\mathrm{i}\alpha) \times \frac{1}{2\pi\mathrm{i}}\int_C \frac{\tau^{-1/4+\alpha}}{(t^2-1)^{1/4}}\exp(\tfrac{2}{3}\pi\mathrm{i}\tau X^2) \times \{\exp[X^2 f_-(\tau,\beta)] - \mathrm{i}\exp[X^2 f_+(\tau,\beta)]\}\,\mathrm{d}\tau \tag{36}$$

$$I'_{2,\alpha}(X,Y) \sim 3^{3/4+2\alpha}\pi^{1/2}2^{1/2+\alpha/2}\rho^{-1/2-\alpha}|X|^{-\alpha}\exp(-\tfrac{1}{8}\pi\mathrm{i} - \tfrac{1}{4}\pi\mathrm{i}\alpha) \times \frac{1}{2\pi\mathrm{i}}\int_C \frac{\tau^{-1/4+\alpha}}{(t^2-1)^{1/4}}\exp[X^2 f_+(\tau,\beta)] \times \{\exp(\tfrac{2}{3}\pi\mathrm{i}\tau X^2) + \exp(-\tfrac{2}{3}\pi\mathrm{i}\tau X^2)\}\,\mathrm{d}\tau. \tag{37}$$

Here $t = \frac{1}{4}3^{1/2}\mathrm{e}^{-\pi\mathrm{i}/4}\tau^{-1/2}$, and

$$f_\pm(\tau,\beta) = f_\pm(\tau) + \beta\tau \qquad \beta = -\tfrac{2}{3}\ln\rho \tag{38}$$

with $f_\pm$ given in [6, (6.13)]. The main contributions to the above integrals come from saddle points; these are (in the t-plane) among the roots of

$$\tfrac{1}{2}(\tfrac{4}{3}t)^3(t \pm \sqrt{t^2-1}) = \rho^{-1} \tag{39}$$

so that

$$\tfrac{1}{4}\rho^2(\tfrac{4}{3}t)^6 - \rho t(\tfrac{4}{3}t)^3 + 1 = 0. \tag{40}$$

This equation has, for β close to 0, simple roots near $t = \pm\frac{3}{4}\mathrm{i}$ and two pairs of nearly coalescing roots near $t = \pm 3/2\sqrt{2}$; as in [6] only the roots near $t = 3/2\sqrt{2}$ (i.e. $\tau = -\frac{1}{6}\mathrm{i}$) and the roots near $t = -\frac{3}{4}\mathrm{i}$ (i.e. $\tau = \frac{1}{3}\mathrm{i}$) yield saddle points contributing to the integrals. As a consequence, the leading asymptotics of $I'_{1,\alpha}$ is determined by the integral

$$L_{1,\alpha}(X^2;\beta) = \frac{1}{2\pi\mathrm{i}}\int_C \frac{\tau^{-1/4+\alpha}}{(t^2-1)^{1/4}}\exp[\tfrac{2}{3}\pi\mathrm{i}X^2\tau + X^2 f_-(\tau,\beta)]\,\mathrm{d}\beta \tag{41}$$

with nearly coalescing saddle points near $\tau_1 = -\frac{1}{6}\mathrm{i}$, and the leading asymptotics of $I'_{2,\alpha}$ is determined by the integral

$$L_{2,\alpha}(X^2;\beta) = \frac{1}{2\pi\mathrm{i}}\int_C \frac{\tau^{-1/4+\alpha}}{(t^2-1)^{1/4}}\exp[-\tfrac{2}{3}\pi\mathrm{i}X^2\tau + X^2 f_+(\tau,\beta)]\,\mathrm{d}\tau \tag{42}$$

with saddle point near $\tau_2 = \frac{1}{3}\mathrm{i}$.

For $L_{1,\alpha}$ we must use the CFU method for which we follow the recipe given in [1, section 9.2]. We write

$$L_{1,\alpha}(X^2;\beta) = \frac{1}{2\pi\mathrm{i}}\int_C G_\alpha(\tau)\exp[X^2F_-(\tau,\beta)]\,\mathrm{d}\tau \tag{43}$$

with

$$F_-(\tau,\beta) = F_-(\tau)+\beta\tau \qquad G_\alpha(\tau) = \frac{\tau^{-(1/4)+\alpha}}{(t^2-1)^{1/4}} \tag{44}$$

where $F_-(\tau) = f_-(\tau)+\frac{2}{3}\pi\mathrm{i}\tau$ as in [6, section 6]. Next we introduce a regular variable transformation $\tau(s)$ (with s close to 0) by

$$F_-(\tau(s),\beta) = -\tfrac{1}{3}s^3+\gamma^2 s + r \tag{45}$$

that should be such that $\tau(\pm\gamma) = \tau_\pm$, with $\tau_\pm$ the two zeros of $F'(\tau,\beta)$ near τ_1. It then follows that

$$r = \tfrac{1}{2}(F_-(\tau_+,\beta)+F_-(\tau_-,\beta)) = -\tfrac{5}{4}\tau_1+\tau_1\beta-\tfrac{5}{12}\tau_1\beta^2+\mathrm{O}(\beta^{5/2}) \tag{46}$$

$$\tfrac{4}{3}\gamma^3 = F_-(\tau_+,\beta)-F_-(\tau_-,\beta) = \tfrac{8}{3}\tau_1(\tfrac{1}{2}\beta)^{3/2}+\mathrm{O}(\beta^{5/2}) \tag{47}$$

the two equalities at the far right-hand sides of (46) and (47) being a consequence of the formulas on the bottom of [6, p 419] and of

$$\tau_\pm = \tau_1 \pm \tau_1(\tfrac{1}{2}\beta)^{1/2}-\tfrac{5}{6}\tau_1\beta+\mathrm{O}(\beta^{3/2}). \tag{48}$$

The argument of γ is to be determined using the device developed after theorem 9.2.1 in [1]; this gives in the present case

$$\gamma = 3^{-1/3}\mathrm{i}(\tfrac{1}{2}\beta)^{1/2}(1+\mathrm{O}(\beta)). \tag{49}$$

The variable transformation $\tau(s)$ is used to bring the contribution to $L_{1,\alpha}$ from the saddle points near τ_1 into the form

$$-\frac{1}{2\pi\mathrm{i}}\int_{C_1} G_\alpha(\tau(s))\,\tau'(s)\exp[(-\tfrac{1}{3}s^3+\gamma^2 s+r)X^2]\,\mathrm{d}s \tag{50}$$

where C_1 is a portion of the Airy contour given in [1, figure 2.5]. The minus sign in (50) is due to the different orientations of $\tau(C_1)$ and C near τ_1. It then follows from the theory in [1] that the leading asymptotics of $L_{1,\alpha}$ is given as

$$L_{1,\alpha}(X^2;\beta) \sim -\exp(X^2 r)\left[\frac{a_0(\alpha)}{|X|^{2/3}}\mathrm{Ai}(\gamma^2|X|^{4/3})+\frac{a_1(\alpha)}{|X|^{4/3}}\mathrm{Ai}'(\gamma^2|X|^{4/3})\right] \tag{51}$$

with

$$a_0(\alpha) = \tfrac{1}{2}[G_\alpha(\tau_+)\,\tau'(\gamma)+G_\alpha(\tau_-)\,\tau'(-\gamma)] = 3^{-5/12}\,\mathrm{e}^{9\pi\mathrm{i}/8}\,\tau_1^\alpha+\mathrm{O}(\beta) \tag{52}$$

$$a_1(\alpha) = \frac{1}{2\gamma}[G_\alpha(\tau_+)\,\tau'(\gamma)-G_\alpha(\tau_-)\tau'(-\gamma)] = (\tfrac{1}{3}+\alpha)\,3^{-1/12}\,\mathrm{e}^{5\pi\mathrm{i}/8}\,\tau_1^\alpha+\mathrm{O}(\beta^{1/2}). \tag{53}$$

This then completes the analysis of $L_{1,\alpha}$.

The analysis of $L_{2,\alpha}$ requires a much simpler appeal to the steepest descent method for a saddle point near $\tau_2 = \frac{1}{3}\mathrm{i}$. To that end we set

$$F_+(\tau,\beta) = F_+(\tau) + \beta\tau \tag{54}$$

and we let $\tau_2(\beta)$ be the zero of $F'_+(\tau,\beta)$ near τ_2. Using the formulae $F_+(\tau_2) = -\frac{13}{8}\tau_2$, $F''_+(\tau_2) = 6/5\tau_2$, we find that

$$\tau_2(\beta) = \tau_2 - \tfrac{5}{6}\beta\tau_2 + \mathrm{O}(\beta^2) \tag{55}$$

while the steepest descent paths have directions $\frac{3}{4}\pi + \mathrm{O}(\beta)$, $-\frac{1}{4}\pi + \mathrm{O}(\beta)$. Hence we get

$$L_{2,\alpha}(X^2;\beta) \sim \frac{b_0(\alpha)}{|X|}\exp(X^2 v) \tag{56}$$

where

$$\begin{aligned} b_0(\alpha) &= \frac{1}{2\pi\mathrm{i}} G_\alpha(\tau_2(\beta)) \left|\frac{2\pi}{F''_+(\tau_2(\beta),\beta)}\right|^{1/2} \exp[\tfrac{3}{4}\pi\mathrm{i} + \mathrm{O}(\beta)] \\ &= \pi^{-1/2} 3^{-3/4-\alpha} \exp(\tfrac{3}{8}\pi\mathrm{i} + \tfrac{1}{2}\pi\mathrm{i}\alpha) + \mathrm{O}(\beta) \end{aligned} \tag{57}$$

and

$$v = F_+(\tau_2(\beta),\beta) = -\tfrac{13}{24}\mathrm{i} + \tfrac{1}{3}\mathrm{i}\beta - \tfrac{5}{36}\mathrm{i}\beta^2 + \mathrm{O}(\beta^3). \tag{58}$$

This completes the analysis of $L_{2,\alpha}$, and putting all results together we obtain expressions (11)–(13).

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