The Zak Transform and Sampling Theorems for Wavelet Subspaces

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Abstract—**The Zak transform is used for generalizing a sampling theorem of G. Walter for wavelet subspaces. Cardinal series based on signal samples $f(a + n)$, $n \in \mathbb{Z}$ with a possibly unequal to 0 (Walter's case) are considered. The condition number of the sampling operator and worst-case aliasing errors are expressed in terms of Zak transforms of scaling function and wavelet. This shows that the stability of the resulting interpolation formula depends critically on a.**

I. Introduction

In [1] G. Walter presents a version of the classical Shannon sampling theorem for wavelet subspaces. The setting is a multiscale analysis $((V_m)_{m\in\mathbb{Z}}; \varphi)$ for $L^2(\mathbb{R})$, where the closed linear subspaces V_m of $L_2(\mathbb{R})$, $m \in \mathbb{Z}$, and the real scaling function φ satisfy the usual properties, so that in particular $(\varphi(t - n))_{n\in\mathbb{Z}}$ is an orthonormal base for V_0. Also, as usual, W_m is the orthogonal complement of V_m in V_{m+1}, ψ is the associated wavelet whose integer translates $(\psi(t - n))_{n\in\mathbb{Z}}$ span W_0, and φ and ψ are related according to

$$\varphi(t) = 2 \sum_n h_n\, \varphi(2t - n), \qquad \psi(t) = 2 \sum_n (-1)^n h_{1-n}\, \varphi(2t - n) \tag{1}$$

where $2^{1/2} h_n$ are the expansion coefficients of $\varphi \in V_0 \subset V_1$ with respect to the orthonormal basis $(2^{1/2} \varphi(2t - n))_{n\in\mathbb{Z}}$ of V_1. The question that is raised in [1] is whether one can find a function $s(t) \in V_0$ such that any $f \in V_0$ can be represented as a cardinal series involving the integer translates of s and the samples of f at integer points, i.e.

$$f(t) = \sum_n f(n)\, s(t - n), \qquad t \in \mathbb{R}. \tag{2}$$

This question is dealt with by Walter under the assumption that φ is continuous and that for some $\epsilon > 0$

$$\varphi(t) = O((1 + |t|)^{-1-\epsilon}), \qquad t \in \mathbb{R}. \tag{3}$$

Then, he shows that when the discrete Fourier transform

$$\Phi_d(w) = \sum_{n=-\infty}^{\infty} \varphi(n)\, e^{-2\pi i n w} \tag{4}$$

has no zeros, there is indeed such an s and there is uniform convergence in (2) for all $f \in V_0$. The Fourier transform S of s,

$$S(w) = \int_{-\infty}^{\infty} e^{-2\pi i w t}\, s(t)\, dt \tag{5}$$

is expressed by Walter in terms of the Fourier transform Φ of φ and Φ_d in (4) as

$$S(w) = \Phi(w)/\Phi_d(w). \tag{6}$$

Although it was not mentioned in [1], it can be shown that this s satisfies $s(m) = \delta_{m0}$, just as one would expect from an interpolating function. Walter then proceeds by estimating the aliasing error for functions $f = f_0 + f_1$ with $f_0 \in V_0, f_1 \in W_0$ in terms of the norm of the "out-of-space" component f_1 as

$$|e_f(t)|^2 = |f(t) - \sum_n f(n)\, s(t - n)|^2 \le C\, \|f_1\|^2 \tag{7}$$

where C is a constant independent of f and t.

The aim of this paper is to show that the Zak transform is a very appropriate tool to discuss this matter, especially when one is interested in sharp frame bounds and sharp estimates for the aliasing error e_f. We shall work under the slightly weaker assumption that (3) φ is bounded, and that

$$\sum_n |\varphi(t - n)| \text{ converges uniformly in } t \in [0, 1]. \tag{8}$$

We do not require continuity of φ until Section III. As a consequence of these assumptions we have that $\varphi \in L_1(\mathbb{R})$. Under assumption (8) we have also that, [see (1)] $\Sigma_n |h_n| < \infty$, and that $\Sigma_n |\psi(t - n)|$ converges uniformly in $t \in [0, 1]$.

The Zak transform of an $f \in L_2(\mathbb{R})$ is defined as

$$(Zf)(t, w) = \sum_{k=-\infty}^{\infty} e^{-2\pi i k w} f(t + k), \qquad t, w \in \mathbb{R}. \tag{9}$$

Actually, (9) should be interpreted in an $L^2(\mathbb{R}^2)$-sense, but for the functions $f = \varphi, \psi, s$, it turns out that we can also interpret the right-hand side of (9) pointwise as an absolutely and locally uniformly convergent series. We refer to [2], [3] for the various properties and names of the mapping Z. See Section III, in particular the quasi-periodicity relations (43).

We shall, also more generally, consider cardinal series

$$f(t) = \sum_n f(a + n)\, s_a(t - n), \qquad t \in \mathbb{R} \tag{10}$$

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with $a \in \mathbb{R}$ and f, $s_a \in V_0$. This extension is significant since, contrary to the classical Shannon case, the function s in (2) cannot be expected to be symmetric about the point 0. Also, it turns out that the condition number of the sampling operator $f \in V_0 \to (f(n+a))_{n\in\mathbb{Z}} \in l^2$ as well as the aliasing errors depend critically on the choice of a.

The Fourier transform $S_a(w)$ of $s_a(t)$ in (10) turns out to be given by

$$S_a(w) = \Phi(w)/(Z\varphi)(a, w) \tag{11}$$

provided that $(Z\varphi)(a, w) \neq 0$, $w \in \mathbb{R}$. Now it is an interesting property of Zak transforms that they have zeros in any unit square provided that they are continuous. (Under assumption (8) we have that $Z\varphi$ is continuous under the weak condition that φ is continuous.) Hence one should choose a such that the set $\{a\} \times \mathbb{R}$ avoids the zeros of $Z\varphi$.

The above observation is made more specific as follows. We shall show that

$$\sup_{f\in V_0, \|f\|=1}/\inf \sum_{n=-\infty}^{\infty} |f(a+n)|^2 = \max_w/\min |(Z\varphi)(a, w)|^2. \tag{12}$$

As to the aliasing errors, we show the following results. When $f \in V_1$, we define the aliasing error $e_{f,a}(t)$ by

$$e_{f,a}(t) = f(t) - \sum_n f(a+n)\, s_a(t-n). \tag{13}$$

Then, we have

$$\sup_{f\in W_0, \|f\|=1}/\inf \|e_{f,a}\|^2 = 1 + \max_w/\min \left|\frac{(Z\psi)(a, w)}{(Z\varphi)(a, w)}\right|^2 \tag{14}$$

while for any $t \in \mathbb{R}$

$$\sup_{f\in W_0, \|f\|=1} |e_{f,a}(t)|^2 = \int_{-1/2}^{1/2} \left|(Z\psi)(t, w) - \frac{(Z\psi)(a, w)}{(Z\varphi)(a, w)} (Z\varphi)(t, w)\right|^2 dw, \tag{15}$$

$$\min_{f\in W_0, \|f\|=1} |e_{f,a}(t)|^2 = 0. \tag{16}$$

For (13)–(16) it is assumed that $(Z\varphi)(a, w) \neq 0$, $w \in \mathbb{R}$. Note that, even in the case $a = 0$, the Zak transform arises naturally in the expression (15) for the worst-case aliasing error.

The explicit expressions for the bounds in (12), (14), (15), should be used to guide the choice of a. For instance, one could look for that particular a for which

$$\frac{\max_w |(Z\varphi)(a, w)|^2}{\min_w |(Z\varphi)(a, w)|^2} \quad \text{or} \quad \max_w \left|\frac{(Z\psi)(a, w)}{(Z\varphi)(a, w)}\right|^2 \tag{17}$$

is as low as possible so as to obtain the best bound for the condition number of the sampling operator or the lowest value for the worst-case L^2-aliasing error.

In Section III we shall show that

$$\begin{bmatrix} (Z\varphi)(t, w) \\ (Z\psi)(t, w) \end{bmatrix} = U_h(w) \begin{bmatrix} (Z\varphi)(2t, \frac{1}{2} w) \\ (Z\varphi)(2t, \frac{1}{2}(w+1)) \end{bmatrix} \tag{18}$$

where $U_h(w)$ is the unitary matrix

$$U_h(w) = \begin{bmatrix} h(\frac{1}{2} w) & h(\frac{1}{2}(w+1)) \\ -e^{-\pi i w} h^*(\frac{1}{2}(w+1)) & e^{-\pi i w} h^*(\frac{1}{2} w) \end{bmatrix} \tag{19}$$

and with $h(w)$ defined as, see (1),

$$h(w) = \sum_n h_n e^{-2\pi i n w}. \tag{20}$$

Since, [see (43)],

$$(Z\varphi)(2t, \tfrac{1}{2}(w+1)) = -\exp(\pi i w)(Z\varphi) \cdot (2t-1, \tfrac{1}{2}(w+1)) \tag{21}$$

we thus see that there is a remarkable connection between Zak transforms of scaling functions and wavelets on one hand and the well-known baker's transformation

$$(t, w) \in [0, 1]^2 \to \begin{cases} (2t, \frac{1}{2} w), & 2t \leq 1 \\ (2t-1, \frac{1}{2}(w+1)), & 2t > \frac{1}{2} \end{cases} \tag{22}$$

on the other.

Formula (18) shall be used to show that, when $Z\varphi$ is continuous, there are points where $Z\varphi$ vanishes while $Z\psi$ does not. This implies that the second quantity in (17) is unbounded, so that the choice of a really matters.

II. Derivations

In this section we present the proofs of the results just announced. We start with the proof of (12). Let $f \in V_0$ and write

$$f(t) = \sum_k \alpha_k \varphi(t-k), \qquad \sum_k |\alpha_k|^2 = \|f\|^2. \tag{23}$$

Since $(\varphi(t-k))_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$ for all t, we can interpret (23) pointwise, and we have

$$\sum_n |f(a+n)|^2 = \sum_{k,l} \alpha_k \alpha_l^* \left[\sum_n \varphi(a+n-k) \cdot \varphi^*(a+n-l)\right] \tag{24}$$

with an absolutely convergent triple series at the right-hand side. (That this latter series converges absolutely follows from the fact that $(\Sigma_k |\alpha_k|\, |\varphi(a+n-k)|)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ as a convolution of an $l^2(\mathbb{Z})$-sequence and an $l^1(\mathbb{Z})$-sequence.) Hence the required sup and inf can be expressed as the maximum and minimum of the spectrum of the infinite Toeplitz matrix

$$(b_{k-l})_{k,l\in\mathbb{Z}}; \qquad b_k = \sum_n \varphi(a+n-k)\, \varphi^*(a+n). \tag{25}$$

It is well known, and easy to prove, that this maximum and minimum are the ess sup and ess inf over [0, 1] of the function

$$\sum_k b_k e^{2\pi ikw} = |(Z\varphi)(a, w)|^2. \tag{26}$$

The proof of (12) is now completed by noting that $(Z\varphi)(a, w)$ is periodic in w by (43) and continuous in w by (8).

Before starting the proofs of (14)–(16) we collect some useful facts about the Zak transform and the expansion coefficients of $s_a \in V_0$.

Lemma 1: Suppose that $a \in \mathbb{R}$ is such that $(Z\varphi)(a, w) \neq 0$, $w \in \mathbb{R}$. Then we have

$$s_a(t) = \sum_n \sigma_{n,a}\, \varphi(t - n) \tag{27}$$

where

$$\sigma_{n,a} = \int_0^1 \frac{e^{2\pi inw}}{(Z\varphi)(a, w)}\, dw. \tag{28}$$

Furthermore, $\Sigma_n |\sigma_{n,a}| < \infty$, the series $\Sigma_n |s_a(t + n)|$ converges uniformly, and

$$(Zs_a)(t, w) = (Z\varphi)(t, w)/(Z\varphi)(a, w), \qquad t, w \in \mathbb{R}. \tag{29}$$

Proof: We have as in the proof of [1, Section III, Theorem] that

$$S_a(w) = \Phi(w)/(Z\varphi)(a, w) = \Phi(w) \sum_n \sigma_{n,a}\, e^{-2\pi inw}. \tag{30}$$

Here we have $\Sigma_n |\sigma_{n,a}| < \infty$ by Wiener's theorem since $(Z\varphi)(a, w) \neq 0$, $w \in \mathbb{R}$, has an absolutely convergent Fourier series (8). This implies (27) and the uniform convergence of the series $\Sigma_n |s_a(t + n)|$. Also formula (29) follows easily.

Consequences:

1) For any $f \in V_0$ the cardinal series at the right-hand side of (10) is uniformly convergent.

2) By taking $f(t) = \varphi(t - m)$ in (10) one obtains

$$s_a(a + m) = \delta_{m0}, \qquad m \in \mathbb{Z}. \tag{31}$$

We now show (14)–(16). When $f \in W_0$ we have in a similar fashion as in (12) that

$$\sum_n |f(n + a)|^2 \le \|f\|^2 \max_w |(Z\psi)(a, w)|^2 < \infty. \tag{32}$$

Hence $e_{f,a}(t)$ is pointwise well-defined as an absolutely convergent series. For the proof of (14) we restrict ourselves first to

$$f(t) = \sum_k \beta_k\, \psi(t - k) \tag{33}$$

where $\beta_k \neq 0$ for only finitely many k, so that $\Sigma_n |f(a + n)| < \infty$. Since f and $s(t - n)$ are orthogonal for all $n \in \mathbb{Z}$ we have

$$\|e_{f,a}\|^2 = \|f\|^2 + \sum_{n,m} f(n + a)\, f^*(m + a)\, r_{nm} \tag{34}$$

where

$$r_{nm} = \int_{-\infty}^{\infty} s(t - n)\, s(t - m)\, dt = \sum_k \sigma_{k-n,a}\, \sigma^*_{k-m,a}. \tag{35}$$

In (35) the orthonormality of the $\varphi(t - n)$'s and (27) have been used. It readily follows from Parseval's formula for Fourier series and (27) and (28) that

$$\begin{aligned}\|e_{f,a}\|^2 &= \|f\|^2 + \sum_k \left| \sum_m \sigma_{k-m,a}\, f(m + a) \right|^2 \\ &= \|f\|^2 + \int_0^1 \left| \frac{(Zf)(a, w)}{(Z\varphi)(a, w)} \right|^2 dw \end{aligned} \tag{36}$$

with pointwise defined Zf. Also, from (33)

$$(Zf)(a, w) = (Z\psi)(a, w) \sum_k \beta_k\, e^{-2\pi ikw}. \tag{37}$$

The set of all functions

$$\left| \sum_k \beta_k\, e^{-2\pi ikw} \right|^2 \tag{38}$$

with $\Sigma_k |\beta_k|^2 = 1$, $\beta_k \neq 0$ for only finitely many k, is dense in the set of all nonnegative L^1-functions with unit L^1-norm. It then follows from continuity of $(Z\psi)(a, w)$, $(Z\varphi)(a, w)$ as a function of w that

$$\begin{aligned}\|f\|^2 \left(1 + \min_w \left| \frac{(Z\psi)(a, w)}{(Z\varphi)(a, w)} \right|^2 \right) &\le \|e_{f,a}\|^2 \\ \le \|f\|^2 \left(1 + \max_w \left| \frac{(Z\psi)(a, w)}{(Z\varphi)(a, w)} \right|^2 \right) \end{aligned} \tag{39}$$

for all f of the considered type. It is now elementary Hilbert space theory to conclude that (39) holds for all $f \in W_0$. This completes the proof of (14).

We finally prove (15) and (16). We have for $f = \Sigma_k \beta_k \psi(\cdot - k) \in W_0$ where $\beta \in l^2$ by the Cauchy–Schwarz inequality

$$\begin{aligned}|e_{f,a}(t)|^2 &= \left| \sum_k \beta_k \left[\psi(t - k) - \sum_n \psi(n + a - k) \cdot s_a(t - n) \right] \right|^2 \\ &\le \|f\|^2 \sum_k \left| \psi(t - k) - \sum_n \psi(n + a - k) \cdot s_a(t - n) \right|^2 \end{aligned} \tag{40}$$

with equality when

$(\beta_k)_{k\in}$ and

$$\left(\gamma_k := \psi(t - k) - \sum_n \psi(n + a - k)\, s_a(t - n) \right)_{k \in \mathbb{Z}} \tag{41}$$

are linearly dependent. We next calculate

$$\sum_k |\gamma_k|^2 = \int_0^1 |(Z\psi)(t, w) - (Z\psi)(a, w)(Zs_a)(t, w)|^2\, dw$$
$$= \int_0^1 \left|(Z\psi)(t, w) - \frac{(Z\psi)(a, w)}{(Z\varphi)(a, w)}(Z\varphi)(t, w)\right|^2 dw \tag{42}$$

where we have used (29). This proves (15). Finally (16) follows by taking $(\beta_k)_{k\in\mathbb{Z}}$ perpendicular to $(\gamma_k)_{k\in\mathbb{Z}}$ in (41).

Note: When $t = a$ we have $e_{f,a}(t) = 0$ for all $f \in W_0$.

III. Zak Transforms of Scaling Functions and Wavelets

We shall consider now the Zak transforms of φ and ψ under the additional assumption that φ is continuous. Then ψ is also continuous, and so are $Z\varphi$ and $Z\psi$. It follows from the quasi-periodicity relations:

$$(Zf)(t + 1, w) = e^{2\pi i w}(Zf)(t, w);$$
$$(Zf)(t, w + 1) = (Zf)(t, w) \tag{43}$$

satisfied for any f for which the series in (9) converge, that $Z\varphi$ and $Z\psi$ have zeros in any unit square, [2, Section V]. In fact, since φ is real it follows from

$$(Z\varphi)(t, \tfrac{1}{2}) = \sum_n (-1)^n \varphi(t + n) \in \mathbb{R};$$
$$(Z\varphi)(1, \tfrac{1}{2}) = -(Z\varphi)(0, \tfrac{1}{2}) \tag{44}$$

that $Z\varphi$ has at least one zero in the set $\{(t, 1/2) | 0 \le t \le 1\}$.

We shall now show (18) and draw some conclusions relevant to the worst case aliasing errors. Explicitly we have to show that

$$(Z\varphi)(t, w) = h(\tfrac{1}{2}w)(Z\varphi)(2t, \tfrac{1}{2}w)$$
$$+ h(\tfrac{1}{2}(w + 1))(Z\varphi)(2t, \tfrac{1}{2}(w + 1)) \tag{45}$$

and

$$(Z\psi)(t, w) = e^{-\pi i w}[h^*(\tfrac{1}{2}w)(Z\varphi)(2t, \tfrac{1}{2}(w + 1))$$
$$- h^*(\tfrac{1}{2}(w + 1))(Z\varphi)(2t, \tfrac{1}{2}w)]. \tag{46}$$

To show (45), we insert the series expansion (1) for φ into formula (9) for $Z\varphi$, change the order of summation, and split the resulting series over n into one over even n and one over odd n. Each of the two double series so obtained can be written as a product of one factor involving the sum/difference of $h(1/2\, w)$ and $h(1/2\, (w + 1))$ and another factor involving the sum/difference of $(Z\varphi)(2t, 1/2\, w)$ and $(Z\varphi)(2t, 1/2\, (w + 1))$. The proof of (45) is then completed by combining the resulting eight terms appropriately. The proof of (46) is similar.

The unitarity of the matrix $U_h(w)$ in (19) follows from

$$|h(v)|^2 + |h(v + \tfrac{1}{2})|^2 = 1. \tag{47}$$

It follows from (45) with $w = 0$ that

$$(Z\varphi)(t, 0) = (Z\varphi)(2t, 0), \qquad t \in \mathbb{R} \tag{48}$$

where it has been used that $h(0) = 1$, $h(1/2) = 0$. Consequently, by continuity of $(Z\varphi)(t, 0)$ at $t = 0$, we have that $(Z\varphi)(t, 0)$ is constant. This constant must be equal to one since $1 = \Phi(0) = \int_0^1 (Z\varphi)(t, 0)\, dt$. Hence,

$$(Z\varphi)(t, 0) = 1, \qquad t \in \mathbb{R}. \tag{49}$$

In a similar fashion it follows that

$$(Z\psi)(t, 0) = (Z\varphi)(2t, \tfrac{1}{2}) \tag{50}$$

so that $(Z\psi)(t, 0) = 0$ for at least two $t \in [0, 1]$, (40), (41).

We next show from (18) that there is an (a, w) such that $Z\varphi$ vanishes at (a, w) while $Z\psi$ does not. To that end we suppose that $Z\psi$ vanishes everywhere where $Z\varphi$ does, and derive a contradiction. When (a, w) is such that $(Z\varphi)(a, w) = (Z\psi)(a, w) = 0$, then

$$(Z\varphi)(2a, \tfrac{1}{2}w) = (Z\varphi)(2a, \tfrac{1}{2}(w + 1)) = 0 \tag{51}$$

by unitarity of $U_h(w)$ in (19). Repeating this argument, we can find sequences $(a_n, w_n) \in \mathbb{R}^2$ such that $(Z\varphi)(a_n, w_n) = 0$ while $w_n \to 0$, $n \to \infty$. This, however, contradicts (49) and the uniform continuity of $Z\varphi$.

IV. Examples

A. Example 1

Consider [1, Section IV, Example 3] with $n = 2$. Hence the Fourier transform Φ of the scaling function φ is given by

$$\Phi(w) = \frac{\Theta_2(w)}{\Sigma_2^{1/2}(w)} \tag{52}$$

where Θ_2 is the Fourier transform of the second order basic spline ϑ_2, and the orthogonalizing function Σ_2 is given by

$$\Sigma_2(w) = \sum_{k=-\infty}^{\infty} |\Theta_2(w + k)|^2. \tag{53}$$

We have explicitly,

$$\Theta_2(w) = \left(\frac{1 - e^{-2\pi i w}}{2\pi i w}\right)^3,$$
$$\vartheta_2(t) = \begin{cases} 0, & t \le 0 \text{ or } t \ge 3 \\ \frac{1}{2}t^2 & 0 \le t \le 1 \\ \frac{3}{4} - (t - \frac{3}{2})^2, & 1 \le t \le 2 \\ \frac{1}{2}(3 - t)^2, & 2 \le t \le 3 \end{cases} \tag{54}$$

and, using the method explained in [4, Section 5.4],

$$\Sigma_2(w) = (\sin \pi w)^6 \sum_{k=-\infty}^{\infty} \left(\frac{1}{\pi(w + k)}\right)^6$$
$$= \frac{11}{20} + \frac{13}{30}\cos 2\pi w + \frac{1}{60}\cos 4\pi w. \tag{55}$$

By periodicity of Σ_2 we have

$$(Z\varphi)(a, w) = (Z\vartheta_2)(a, w)/\Sigma_2^{1/2}(w) \tag{56}$$

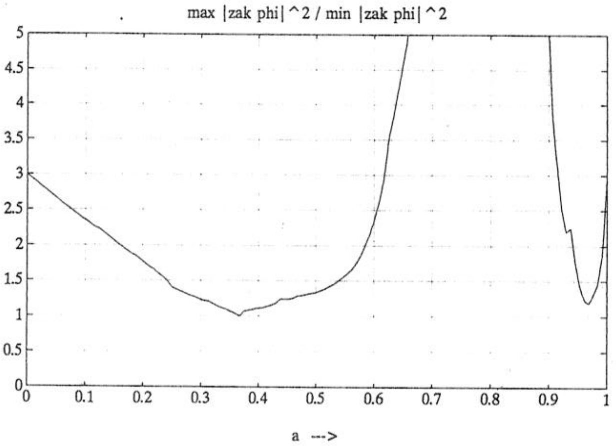

Fig. 1. The quantity $\max_w |(Z\varphi)(a, w)|^2/\min_w |(Z\varphi)(a, w)|^2$ as a function of $a \in [0, 1]$ for $\varphi(t) = {}_2\varphi(3 - t)$ (Daubechies scaling function).

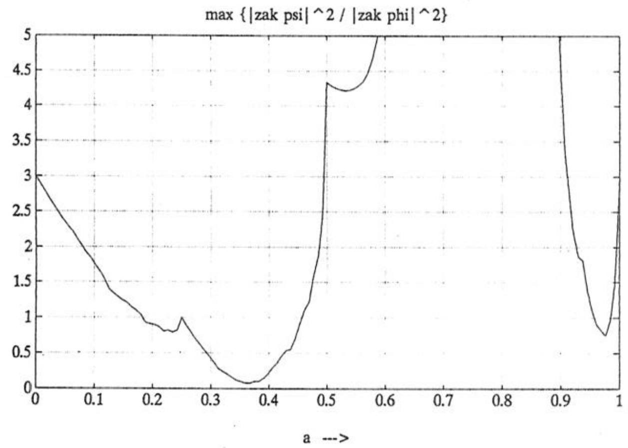

Fig. 2. The quantity $\max_w |(Z\psi)(a, w)|^2/|(Z\varphi)(a, w)|^2$ as a function of $a \in [0, 1]$ for $\varphi(t) = {}_2\varphi(3 - t) = {}_2\psi(3 - t)$ (Daubechies scaling function and wavelet).

so that, see (11),

$$S_a(w) = \frac{\Phi(w)}{(Z\varphi)(a, w)} = \frac{\Theta_2(w)}{(Z\vartheta_2)(a, w)}. \tag{57}$$

Furthermore, by (54), we have for $a \in [0, 1]$

$$(Z\vartheta_2)(a, w) = \tfrac{1}{2} a^2 + (\tfrac{3}{4} - (a - \tfrac{1}{2})^2)\, e^{-2\pi i w} + \tfrac{1}{2} (1 - a)^2\, e^{-4\pi i w}. \tag{58}$$

Hence when $a = 0$ we see that

$$(Z\varphi)(0, \tfrac{1}{2}) = (Z\vartheta_2)(0, \tfrac{1}{2})/\Sigma_2^{1/2}(\tfrac{1}{2}) = 0. \tag{59}$$

We shall show that

$$(Z\psi)(0, \tfrac{1}{2}) \neq 0. \tag{60}$$

Indeed, when $(Z\psi)(0, 1/2) = 0$, we would find, see (51), that $(Z\varphi)(0, 1/4) = 0$ as well. However, from (58) we have

$$(Z\vartheta_2)(0, \tfrac{1}{4}) = -\tfrac{1}{2} - \tfrac{1}{2} i \neq 0. \tag{61}$$

(In fact it can be shown that $(Z\psi)(0, 1/2) = -\sqrt{15/8}$.)

When $a = 1/2$, we obtain

$$(Z\vartheta_2)(\tfrac{1}{2}, w) = e^{-2\pi i w}(\tfrac{3}{4} + \tfrac{1}{2}\cos 2\pi w) \tag{62}$$

which vanishes for no real value of w. Hence, we can base a stable interpolation formula on the sample values $f(1/2 + n)$ of signals $f \in V_0$ using the interpolating function $s_{1/2}(t)$ whose Fourier transform $S_{1/2}(w)$ is given by (57). Such a thing is not possible when $a = 0$: the interpolating function $s_0(t)$ has a singular Fourier transform $S_0(w)$, and the quantities in (17) are infinite.

B. Example 2

Consider [1, Section IV, Example 5] with $v = +1/\sqrt{3}$. In Figs. 1 and 2 we have plotted the quantities (17) for the condition number of the sampling operator and the worst-case L^2-aliasing error as a function of $a \in [0, 1]$. Clearly, the first quantity cannot be less than unity and the second one cannot be less than zero; for $a \approx 0.37$ we see that both quantities come very close to their lower bounds. Hence, $a \approx 0.37$ would give very good interpolation results. On the other hand, $a \approx 0.8$ gives very large values for these quantities, and thus bad interpolation results.

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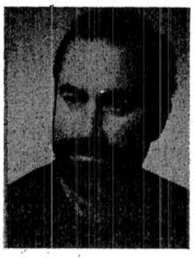

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