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Signal Analytic Proofs of Two Basic Results on Lattice Expansions

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We present new and short proofs of two theorems in the theory of lattice expansions. These proofs are based on a necessary and sufficient condition, found by Wexler and Raz, for biorthogonality. The first theorem is the Lyubarskii–Seip–Wallstén theorem for lattices, according to which the set of Gaussians $2^{1/4}\exp(-\pi(t-na)^2+2\pi imbt), n, m \in \mathbb{Z}$, constitutes a frame when $a>0, b>0, ab<1$. In addition, we display dual functions for this case. The second theorem is the result that a set $g_{na,mb}(t)=g(t-na)\exp(2\pi imbt), n, m \in \mathbb{Z}$ of time–frequency translates of a $g \in L^2(\mathbb{R})$ cannot be a frame when $a>0, b>0, ab>1$. © 1994 Academic Press, Inc.

1. INTRODUCTION AND RESULTS

Consider for $g \in L^2(\mathbb{R})$ the set of functions

$$g_{x,y}(t) = e^{2\pi iyt} g(t-x), \quad t \in \mathbb{R}, \tag{1.1}$$

where $x, y \in \mathbb{R}$. In this paper, we present new and short proofs of two basic theorems in the theory of lattice expansions.

The first theorem reads as follows. For $g \in L^2(\mathbb{R})$ and $a>0, b>0$, we call the set $g_{na,mb}, n, m \in \mathbb{Z}$ a frame when there are constants $A, B, 0<A \leqslant B<\infty$ such that for all $f \in L^2(\mathbb{R})$

$$A\,||f||^2 \leqslant \sum_{n,m} |(f, g_{na,mb})|^2 \leqslant B\,||f||^2. \tag{1.2}$$

It was conjectured in [1] by Daubechies and Grossmann that $g_{na,mb}$ is a frame when $g(t)$ is the Gaussian $2^{1/4}\exp(-\pi t^2)$ and $ab<1$; for numerical evidence and further background, we refer to [2, pp. 980–982; 3, Sect. 3.4.4.B]. The conjecture was proved by Lyubarskii [4] and, independently, by Seip and Wallstén [5] using advanced methods from entire function theory (in fact, both Lyubarskii and Seip and Wallstén prove the result for more general sets of time–frequency translates of g). We shall refer below to this result as Theorem 1.

The second theorem reads as follows: when $ab>1$ and $g \in L^2(\mathbb{R})$, then the set $g_{na,mb}, n, m \in \mathbb{Z}$ cannot constitute a frame. We shall refer below to this result as Theorem 2. In [2, p. 978], it is noted that one can show that there is an $f \neq 0, f \in L^2(\mathbb{R})$ such that $(f, g_{na,mb}) = 0$ when $ab>1, g \in L^2(\mathbb{R})$ (this implies that the $g_{na,mb}$ do not constitute a frame). As said in [2], the latter result was shown by Howe and Steger by applying results of Rieffel in [6] to the computation of the coupling constants of the von Neumann algebra spanned by the time–frequency shift operators corresponding to the parameters na, mb with $n, m \in \mathbb{Z}$. A different proof of Theorem 2, in which the asymptotic number (as $\Omega \to \infty, T \to \infty$) of orthogonal functions bandlimited to $[-\Omega, \Omega]$ and having at least a fixed fraction γ of their energy in $[-T, T]$ (also see [7, 8]) is determined, was found by Landau [9] under certain decay and smoothness conditions on g. This proof, which is signal analytic in nature, depends on a clever analysis of the eigenvalues of certain time-frequency limiting operators.

The proofs of the two theorems we present here are based on Proposition A below, which is a version of a result found by Wexler and Raz [10]. The proof of Proposition A uses a technique found by Tolimieri and Orr in [11], and is presented in Section 2. We first need two definitions.

DEFINITION 1. Let $h \in L^2(\mathbb{R})$. We say that h has a finite upper frame bound if there is a $B<\infty$ such that

$$\sum_{n,m} |(f, h_{na,mb})|^2 \leqslant B\,||f||^2, \quad f \in L^2(\mathbb{R}). \tag{1.3}$$

DEFINITION 2. Let $g, \gamma \in L^2(\mathbb{R})$. We say that g and γ are dual if both g and γ have a finite upper frame bound and

$$(f, h) = \sum_{n,m} (f, \gamma_{na,mb})\,(g_{na,mb}, h), \quad f, h \in L^2(\mathbb{R}). \tag{1.4}$$

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PROPOSITION A. *Let $g, \gamma \in L^2(\mathbb{R})$ both have a finite upper frame bound. Then g and γ are dual if and only if*

$$(g_{k/b,l/a}, \gamma) = ab\delta_k\delta_l, \quad k, l \in \mathbb{Z}, \tag{1.5}$$

where δ denotes Kronecker's delta.

The condition on g, γ of having a finite upper frame bound represents a mild restriction on decay and smoothness of g, γ (it is satisfied, for instance, when $\sum_{k,l} |(h, h_{k/b,l/a})| < \infty$ for $h = g, \gamma$; see [11, Sect. 3]). However, it cannot be deleted: for the case $ab = 1/k, k = 1, 2, \ldots$, one can construct, by using Zak transforms (see [2, pp. 976 and 981]), an example of a pair $g, \gamma \in L^2(\mathbb{R})$ such that (1.5) holds while neither g nor γ has a finite upper frame bound (so that even convergence of the right-hand series in (1.4) is questionable). We note that any h in the Schwartz space $\mathscr{S}$ of smooth and rapidly decaying functions has a finite upper frame bound.

We shall now outline how Proposition A is used for proving the two theorems. For Theorem 1, we argue as follows. Assume that $g, \gamma \in L^2(\mathbb{R})$ are dual. Then it follows from the Cauchy–Schwarz inequality used in (1.4) with $f = h \in L^2(\mathbb{R})$ that

$$1 \leqslant \left\{ \frac{1}{\|f\|^2} \sum_{n,m} |(f, \gamma_{na,mb})|^2 \right\} \times \left\{ \frac{1}{\|f\|^2} \sum_{n,m} |(f, g_{na,mb})|^2 \right\}. \tag{1.6}$$

Hence we see that (1.2) holds with frame bounds $A = B_\gamma^{-1}, B = B_g$ where B_g, B_γ are upper frame bounds for g, γ. Therefore, by Proposition A, it follows that the $g_{na,mb}$ constitute a frame if we can find a $\gamma \in L^2(\mathbb{R})$ with a finite upper frame bound such that the biorthogonality condition (1.5) holds. Now when $g(t) = 2^{1/4}\exp(-\pi t^2), ab < 1$, it is well known to those familiar with the Bargmann transform, see [1], that such a γ indeed exists. For selfcontainedness of the present paper we show, more explicitly, the following result in Section 3, for which we follow the methods in [12, Sect. 2].

PROPOSITION B. *For any $\varepsilon > 0, \varepsilon < 1 - ab$, the function γ_ε defined by*

$$\gamma_\varepsilon(t) = 2^{-1/4} b K^{-1} e^{\pi t^2} \sum_{k=-\infty}^{\infty} (-1)^k e^{-\pi a(k+1/2)^2/b} \times \operatorname{erfc}\left[\left(t - \left(k + \frac{1}{2}\right)a\right)\sqrt{\pi/\varepsilon}\right] \tag{1.7}$$

with

$$K = \sum_{k=-\infty}^{\infty} (-1)^k (2k+1) e^{-\pi a(k+1/2)^2/b};$$
$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-s^2} ds \tag{1.8}$$

provides a γ satisfying (1.5). Moreover, $\gamma_\varepsilon \in \mathscr{S}$.

Interestingly, and consistently with the theory in [12, 2.7 and 2.14], this $\gamma_\varepsilon(t)$ tends to

$$\gamma_0(t) = 2^{-1/4} b K^{-1} e^{\pi t^2} \sum_{k+1/2 \geqslant t} (-1)^k \times \exp\left(-\pi a \left(k + \frac{1}{2}\right)^2 \Big/ b\right) \tag{1.9}$$

as $\varepsilon \downarrow 0$. For the special case that $ab = 1$, this γ_0 has become known as Bastiaans' singular function; see [13].

As to the Theorem 2, we proceed as follows. Suppose that $ab > 1$ and $g \in L^2(\mathbb{R})$ is such that the $g_{na,mb}$ constitute a frame. In [2, Chap. II], a dual function γ is constructed as follows. Define the operator $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$Sf = \sum_{n,m} (f, g_{na,mb})\, g_{na,mb}, \quad f \in L^2(\mathbb{R}). \tag{1.10}$$

Then

$$\gamma^0 = S^{-1} g \tag{1.11}$$

is a dual function, so that it satisfies (1.4) for all $f, h \in L^2(\mathbb{R})$. This γ^0 is special among all dual functions in the following sense: for any $f \in L^2(\mathbb{R})$ the minimum of $\sum_{n,m} |\alpha_{nm}|^2$ over all $\underline{\alpha} \in l^2(\mathbb{Z}^2)$ (not necessarily of the form $((f, \gamma_{na,mb}))_{n,m}$) satisfying

$$f = \sum_{n,m} \alpha_{nm} g_{na,mb} \tag{1.12}$$

is uniquely assumed by

$$\alpha_{nm} = \left(f, \gamma^0_{na,mb}\right), \quad n, m \in \mathbb{Z}. \tag{1.13}$$

The latter result was mentioned in [1, p. 155], and a proof of it (for general frames) can be found in [3, p. 61].

The proof of Theorem 2 can then be completed as follows. Take $f = g$ in (1.12). Since we have for this f the trivial representation

$$f = g = \sum_{n,m} \delta_n \delta_m g_{na,mb}, \tag{1.14}$$

it follows that

$$|(g,\gamma^0)|^2 \leqslant \sum_{n,m} \left|\left(g,\gamma^0_{na,mb}\right)\right|^2 \leqslant \sum_{n,m} |\delta_n\delta_m|^2 = 1. \quad (1.15)$$

However, from (1.5) with $k = l = 0$, we see that

$$(g,\gamma^0) = ab > 1. \quad (1.16)$$

Contradiction.

2. PROOF OF PROPOSITION A

In this section, we present the proof of our version of the Wexler–Raz result (Proposition A). Wexler and Raz prove their result in [10, Appendix A] by formally using the Poisson summation formula twice without stating further restrictions on g and γ. For the proof of Theorem 1 (in which case g and γ are in $\mathscr{S}$), this formal approach can be easily justified. Moreover, for the proof of Theorem 2 we do not need the full Proposition A: it is sufficient to show that (1.5) holds for $k = l = 0$ only. This would simplify the work to be done, but we consider Proposition A as stated as interesting result in its own right. We found it somewhat cumbersome to make the Wexler–Raz argument rigourous so as to yield the precise statement of Proposition A. Instead, we use a different approach, based on some results of windowed Fourier tranforms; see [14, Sect. 1.4]. The technique used in the proof presented here is due to Tolimieri and Orr [11].

It is well known that for $u, v \in L^2(\mathbb{R})$ the function

$$STF_{u,v}(x,y) = \left(u, v_{x,y}\right) = \int_{-\infty}^{\infty} u(t)\, v^*(t-x)\, e^{-2\pi i y t}\, dt \quad (2.1)$$

is bounded, continuous, and in $L^2(\mathbb{R})$. Moreover, there is the resolution-of-identity formula, valid for $u, v, w, z \in L^2(\mathbb{R})$,

$$\iint STF_{u,v}(x,y)\,(STF_{w,z}(x,y))^*\,dxdy = (u,w)(z,v). \quad (2.2)$$

We use (2.2) with $u = f_{-\beta,\alpha}, v = \gamma_{-\beta,\alpha}, w = h, z = g$ where $\alpha, \beta \in \mathbb{R}$, and there results

$$\iint \left(f,\gamma_{x,y}\right)\left(g_{x,y},h\right) e^{2\pi i\alpha x + 2\pi i\beta y}\,dxdy = \left(f_{-\beta,\alpha},h\right)\left(g,\gamma_{-\beta,\alpha}\right). \quad (2.3)$$

Now let $g, \gamma \in L^2(\mathbb{R})$ have a finite upper frame bound, and consider the function H defined by the pointwise absolutely convergent series

$$H(x,y) = \sum_{n,m} \left(f_{-x,-y},\gamma_{na,mb}\right)\left(g_{na,mb},h_{-x,-y}\right), \quad x,y \in \mathbb{R}. \quad (2.4)$$

Since for $u, v \in L^2(\mathbb{R})$ we have

$$\left(u_{-x,-y}, v_{t,s}\right) = e^{2\pi i(y+s)x}\left(u, v_{t+x,s+y}\right), \quad x,y,t,s \in \mathbb{R}, \quad (2.5)$$

we have

$$H(x,y) = \sum_{n,m} \left(f,\gamma_{na+x,mb+y}\right)\left(g_{na+x,mb+y},h\right). \quad (2.6)$$

It thus follows that H is periodic in x and y with periods a and b, respectively. Moreover, by using that $||f_{-x,-y} - f_{-t,-s}|| \to 0, ||h_{-x,-y} - h_{-t,-s}|| \to 0$ as $(x,y) \to (t,s)$ together with the finite upper frame bound condition and (2.4), it is seen that H is continuous.

We compute the Fourier coefficients c_{kl} in

$$H(x,y) \sim \frac{1}{ab}\sum_{k,l} c_{kl} e^{-2\pi ikx/a - 2\pi ily/b} \quad (2.7)$$

as follows. We have

$$\begin{aligned} c_{kl} &= \int_0^a \int_0^b H(x,y)\, e^{2\pi ikx/a + 2\pi ily/b}\,dxdy \\ &= \sum_{n,m} \int_0^a \int_0^b \left(f,\gamma_{na+x,mb+y}\right)\left(g_{na+x,mb+y},h\right) \\ &\quad \times e^{2\pi ikx/a+2\pi ily/b}\,dxdy \\ &= \iint \left(f,\gamma_{x,y}\right)\left(g_{x,y},h\right) e^{2\pi ikx/a + 2\pi ily/b}\,dxdy, \end{aligned} \quad (2.8)$$

where we have used Lebesgue's bounded convergence theorem and the fact that $\exp(2\pi ikx/a + 2\pi ily/b)$ is (a,b)-periodic in (x,y). Hence, by (2.3), we obtain

$$c_{kl} = \left(f_{-l/b,k/a},h\right)\left(g,\gamma_{-l/b,k/a}\right). \quad (2.9)$$

Now assume that (1.5) holds. Then we have from (2.7), (2.9) and continuity of H that

$$H(x,y) = (f,h), \quad x,y \in \mathbb{R}. \quad (2.10)$$

In particular,

$$\sum_{n,m} (f, \gamma_{na,mb})\,(g_{na,mb}, h) = H(0,0) = (f,h). \quad (2.11)$$

Conversely, assume that g, γ satisfy (1.4). Then

$$H(x,y) = \left(f_{-x,-y}, h_{-x,-y}\right) = (f,h), \quad x,y \in \mathbb{R}, \quad (2.12)$$

so that by (2.9),

$$(f_{-l/b,k/a}, h)\,(g, \gamma_{-l/b,k/a}) = ab\delta_k\delta_l\,(f,h), \quad k,l \in \mathbb{Z}. \quad (2.13)$$

By taking $f, h \in L^2(\mathbb{R})$ such that $(f_{x,y}, h)$ never vanishes (e.g., $f(t) = h(t) = 2^{1/4}\exp(-\pi t^2)$), we conclude that g, γ satisfy (1.5), and the proof is complete.

3. PROOF OF PROPOSITION B

In this section, we prove Proposition B, for which we follow the approach of [12, Sect. 2]. With

$$\varphi(t) = 2^{1/4}e^{-\pi t^2}\gamma^*(t) = g(t)\gamma^*(t), \quad t \in \mathbb{R}, \quad (3.1)$$

it is seen that the condition (1.5) is equivalent to

$$(\mathscr{F}^*\varphi)(l/a + ik/b) = ab\delta_k\delta_l, \quad k,l \in \mathbb{Z}. \quad (3.2)$$

Here $\mathscr{F}^*$ denotes the inverse Fourier transform

$$(\mathscr{F}^*\varphi)(z) = \int e^{2\pi izt}\varphi(t)\,dt, \quad z \in \mathbb{C}. \quad (3.3)$$

Now consider the theta function

$$G(z) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-\pi a(k+1/2)^2/b} e^{(2k+1)\pi iaz}, \quad z \in \mathbb{C}; \quad (3.4)$$

see [15, Chap. 21]. This G has simple zeros at all lattice points $l/a + ik/b, k,l \in \mathbb{Z}$, as follows from the elementary properties of theta functions. This suggests taking φ such that

$$(\mathscr{F}^*\varphi)(z) = \frac{ab}{G'(0)}\frac{G(z)}{z}e^{-\pi\varepsilon z^2}, \quad z \in \mathbb{C}, \quad (3.5)$$

with $\varepsilon > 0$ such that the resulting γ, see (3.1), is in $\mathscr{S}$.

With $\varphi' = d\varphi/dt$, we can write (3.5) as

$$(\mathscr{F}^*\varphi')(z) = -2\pi iz\,(\mathscr{F}^*\varphi)(z) = DG(z)\,e^{-\pi\varepsilon z^2}; \quad D = \frac{-2\pi iab}{G'(0)}. \quad (3.6)$$

Equation (3.6) can be solved for φ' by taking the Fourier transform at both sides. We find from (3.4) that

$$\varphi'(t) = D\varepsilon^{-1/2}\sum_{k=-\infty}^{\infty}(-1)^k \times e^{-\pi a(k+1/2)^2/b-\pi\varepsilon^{-1}(t-(k+1/2)a)^2} =: \chi(t). \quad (3.7)$$

With

$$c = \frac{1}{ab+\varepsilon}, \quad (3.8)$$

the function χ in (3.7) can be written as

$$\chi(t) = D\varepsilon^{-1/2}e^{-\pi ct^2}\sum_{k=-\infty}^{\infty}(-1)^k \times \exp\left(\frac{-\pi a}{\varepsilon bc}\left(k + \frac{1}{2} - bct\right)^2\right), \quad (3.9)$$

showing that

$$\chi(t) = O\left(\exp\left(-\pi ct^2\right)\right), \quad t \in \mathbb{R}. \quad (3.10)$$

Moreover, since (see (3.6))

$$\int_{-\infty}^{\infty}\chi(s)\,ds = DG(0) = 0, \quad (3.11)$$

we have that

$$\varphi(t) = -\int_t^{\infty}\chi(s)\,ds = O\left(e^{-\pi ct^2}\right), \quad t \in \mathbb{R}. \quad (3.12)$$

It then follows from smoothness of φ and (3.1) that γ^* is a smooth function satisfying

$$\gamma^*(t) = 2^{-1/4}e^{\pi t^2}\varphi(t) = O\left(e^{-\pi(c-1)t^2}\right), \quad t \in \mathbb{R}. \quad (3.13)$$

Hence, $\gamma \in \mathscr{S}$ whenever $c > 1$, i.e., $\varepsilon < 1 - ab$, as required.

Finally, we get explicitly from (3.7), (3.12), (3.13) that

$$\gamma^*(t) = -D\varepsilon^{-1/2}2^{-1/4}e^{\pi t^2}\sum_{k=-\infty}^{\infty}(-1)^k \times e^{-\pi a(k+1/2)^2/b}\int_t^{\infty}e^{-\pi\varepsilon^{-1}(s-(k+1/2)a)^2}ds. \quad (3.14)$$

Recalling the definitions of K, erfc, G, D in (1.8), (3.4), (3.6), we can put γ in the form of (1.7), and the proof of Proposition B is complete.

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