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On rationally oversampled Weyl–Heisenberg frames

A.J.E.M. Janssen

Philips Research Laboratories Eindhoven WL-01, 5656 AA Eindhoven, The Netherlands

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Abstract

We relate the matrix elements of the linear systems, arising in the Zibulski–Zeevi method for computing dual functions for rationally oversampled Weyl–Heisenberg frames, to the Wexler–Raz method for computing dual functions. We give a necessary and sufficient condition for two functions g, γ having a frame upper bound to be dual in terms of their Zak transforms, we characterize the minimal dual function ${}^{\circ}\gamma$ and we present a necessary and sufficient condition, in terms of the Zak transform, for a function g so that the Tolimieri–Orr condition A is satisfied. The latter result is used to show that a g generating a rationally oversampled Weyl–Heisenberg frame and satisfying condition A has a minimal dual function that satisfies condition A as well.

Zusammenfassung

Wir beschreiben eine Beziehung zwischen den Matrixelementen der linearen Systeme, die in der Zibulski–Zeevi-Methode zur Berechnung dualer Funktionen für rational überabgetastete Weyl–Heisenberg-Frames auftreten, und der Wexler–Raz-Methode zur Berechnung dualer Funktionen. Wir geben mittels der Zak-Transformation eine notwendige und hinreichende Bedingung dafür an, daß zwei Funktionen g, γ, die eine obere Frame-Schranke besitzen, dual sind. Wir charakterisieren die minimale duale Funktion ${}^{\circ}\gamma$, und wir formulieren mit Hilfe der Zak-Transformation eine notwendige und hinreichende Bedingung dafür, daß eine Funktion g die Tolimieri–Orr-Bedingung A erfüllt. Unter Verwendung des letzteren Ergebnisses wird gezeigt, daß eine Funktion g, die einen rational überabgetasteten Weyl–Heisenberg-Frame erzeugt und Bedingung A erfüllt, eine minimale duale Funktion hat, die ebenfalls Bedingung A erfüllt.

Résumé

Nous mettons en correspondance dans cet article les éléments des matrices de systèmes linéaires apparaissant dans la méthode de Zibulski–Zeevi pour le calcul des fonctions duales pour des trames de Weyl–Heisenberg suréchantillonnées rationnellement, avec la méthode de Wexler–Raz pour le calcul des fonctions duales. Nous fournissons une condition nécessaire et suffisante pour que deux fonctions g et γ ayant une borne supérieure de trame soient duales en termes de leur transformées de Zak, nous caractérisons la fonction duale minimale ${}^{\circ}\gamma$, et nous présentons une condition nécessaire et suffisante, en termes de la transformation de Zak, sur la fonction g, de sorte que la condition A de Tolimieri–Orr soit satisfaite. Ce dernier résultat est utilisé pour montrer qu'un fonction g générant une trame de Weyl–Heisenberg sur-échantillonnée rationnellement et satisfaisant la condition A a une fonction duale minimale qui satisfait A également.

Keywords: Weyl–Heisenberg frame; Rational oversampling; Zak transform; Gabor expansion

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1. Introduction and announcement of results

1.1. Introduction

Let $a > 0$, $b > 0$ and consider for $x, y \in \mathbb{R}$ the time-frequency shift operators defined by

$$f \in L^2(\mathbb{R}) \to f_{x,y}(t) = e^{2\pi i y t} f(t-x), \quad t \in \mathbb{R}. \tag{1.1}$$

We say that a $g \in L^2(\mathbb{R})$ generates a (Weyl–Heisenberg) frame (for the parameters a, b) when there are $A > 0$, $B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{n,m} |(f, g_{na,mb})|^2 \leqslant B\|f\|^2, \quad f \in L^2(\mathbb{R}). \tag{1.2}$$

The numbers A, B are called frame lower, upper bound for g, and we say that g has a frame lower, upper bound when the left, right inequality in (1.2) holds for some $A > 0$, $B < \infty$, respectively. When g has a frame upper bound, we define the frame operator S^g associated with g by

$$S^g f = \sum_{n,m} (f, g_{na,mb}) g_{na,mb}, \quad f \in L^2(\mathbb{R}). \tag{1.3}$$

This S^g maps $L^2(\mathbb{R})$ into itself.

When $g \in L^2(\mathbb{R})$ generates a frame, there is for any $f \in L^2(\mathbb{R})$ the $L^2(\mathbb{R})$-convergent expansion (stable Gabor expansion)

$$f = \sum_{n,m} (f, {}^\circ\gamma_{na,mb}) g_{na,mb}, \tag{1.4}$$

where ${}^\circ\gamma = (S^g)^{-1} g$. For any $f \in L^2(\mathbb{R})$ the expansion (1.4) is minimal in the sense that for all double sequences $\underline{\alpha} \in l^2(\mathbb{Z} \times \mathbb{Z})$ with

$$f = \sum_{n,m} \alpha_{nm} g_{na,mb}, \tag{1.5}$$

we have

$$\sum_{n,m} |(f, {}^\circ\gamma_{na,mb})|^2 \leqslant \sum_{n,m} |\alpha_{nm}|^2 \tag{1.6}$$

with equality if and only if $\alpha_{nm} = (f, {}^\circ\gamma_{na,mb})$, $n, m \in \mathbb{Z}$. We call ${}^\circ\gamma$ the minimal dual function for g (when g generates a frame and $ab < 1$ there are many $\gamma \in L^2(\mathbb{R})$ such that an $L^2(\mathbb{R})$-convergent expansion (1.4) with ${}^\circ\gamma$ replaced by γ holds for all $f \in L^2(\mathbb{R})$). We refer to [1, Sections I. A-B-C, II. A-B-C] and [2, Sections 3.2, 3.4, 4.1, 4.2.2] for the general theory of Weyl–Heisenberg frames until 1992.

It is well known that no $g \in L^2(\mathbb{R})$ can generate a frame when $ab > 1$, that a $g \in L^2(\mathbb{R})$ that generates a frame with $ab = 1$ cannot simultaneously be smooth and decay rapidly, and that there are many very well-behaved $g \in L^2(\mathbb{R})$ that generate a frame when $ab < 1$. We refer to the cases $ab > 1$, $= 1$, < 1 as undersampled, critically sampled, oversampled, respectively. We consider in this paper the cases that $ab < 1$.

As to the problem of computing (minimal) dual functions there are several methods. For the computation of ${}^\circ\gamma$ by using the well-known frame algorithm we refer to [1, Section II.A] and [2, Section 3.2]. An alternative to this frame algorithm, which offers an opportunity to compute other dual functions as well, is provided by the following result of Wexler and Raz [11] (here we give the version formulated precisely and proved rigorously in [4]). For any $g, \gamma \in L^2(\mathbb{R})$ having a frame upper bound there holds

$$\forall_{f \in L^2(\mathbb{R})} \left[f = \sum_{n,m} (f, \gamma_{na,mb})\, g_{na,mb} \right] \tag{1.7}$$

$$\Leftrightarrow \quad \forall_{k,l \in \mathbb{Z}} [(\gamma, g_{k/b,l/a}) = ab\, \delta_{ko}\, \delta_{lo}].$$

That is, for such g, γ the duality for the parameters a, b, as expressed by the first member of (1.7), is the same as the biorthogonality for the parameters $1/b$, $1/a$, as expressed by the second member of (1.7). Hence the construction of dual functions consists of finding $\gamma \in L^2(\mathbb{R})$ with a finite frame upper bound such that the linear constraint

$$T^g \gamma = ab\, \delta_{ko}\, \delta_{lo};$$
$$T^g f = ((f, g_{k/b,l/a}))_{k,l \in \mathbb{Z}}, \quad f \in L^2(\mathbb{R}), \tag{1.8}$$

is satisfied. In the oversampling case $ab < 1$, there are many γ that satisfy (1.8) when g generates a frame. One possibility to force uniqueness is to look for the γ satisfying (1.8) with minimal energy. This minimum energy biorthogonal function, ${}^{\circ\circ}\gamma$, is known as the Wexler–Raz biorthogonal function. Other possibilities to force uniqueness are considered in [3, 6, 7, 11].

It was discovered (independently, simultaneously and by using different methods) by Janssen [5], by Daubechies et al. [3], and by Ron and Shen [9] that the minimal ${}^\circ\gamma = (S^g)^{-1} g$ and the Wexler–Raz ${}^{\circ\circ}\gamma$ actually coincide. A key observation in [3, 5, 9] is that $g \in L^2(\mathbb{R})$ generates a frame with frame bounds A, B if and only if (I identity operator of $l^2(\mathbb{Z}^2)$)

$$AI \leqslant (ab)^{-1} M^g \leqslant BI; \quad M^g = T^g(T^g)^*. \tag{1.9}$$

Accordingly, one can compute

$$^\circ\gamma = {}^{\circ\circ}\gamma = (T^g)^* (M^g)^{-1} \underline{\sigma}; \quad \underline{\sigma} = (ab\, \delta_{ko}\, \delta_{lo})_{k,l \in \mathbb{Z}}, \tag{1.10}$$

or, more explicitly,

$$^\circ\gamma = {}^{\circ\circ}\gamma = ab \sum_{k,l} ((M^g)^{-1})_{kl;oo}\, g_{k/b,\, l/a}, \tag{1.11}$$

where we note that the matrix elements of M^g with respect to the standard basis of $l^2(\mathbb{Z} \times \mathbb{Z})$ are given by

$$(M^g)_{kl;k'l'} = (g_{k'/b,\, l'/a},\, g_{k/b,\, l/a}), \quad k, l, k', l' \in \mathbb{Z}. \tag{1.12}$$

Some further results obtained in [5] are

- when $g, \gamma \in L^2(\mathbb{R})$ then g, γ are biorthogonal if and only if $(1/ab)T^\gamma$ is a left-inverse of $(T^g)^*$, i.e.

$$\frac{1}{ab} T^\gamma (T^g)^* = I; \tag{1.13}$$

- when g generates a frame then $(1/ab)T^{\circ\gamma}$ is the generalized inverse of $(T^g)^*$, i.e.

$$\frac{1}{ab} T^{\circ\gamma} = (M^g)^{-1} T^g, \qquad \frac{1}{ab} M^{\circ\gamma} = \left(\frac{1}{ab} M^g\right)^{-1}; \tag{1.14}$$

- when g generates a frame then the frame operator S^g, see (1.3), has the representation

$$S^g = \frac{1}{ab} \sum_{k,l} (g, g_{k/b,\, l/a})\, U_{kl};$$

$$U_{kl} f = f_{k/b,\, l/a}, \quad f \in L^2(\mathbb{R}), \tag{1.15}$$

in the sense that for any $f, h \in L^2(\mathbb{R})$ with $\sum_{k,l} |(U_{kl} f, h)|^2 < \infty$ we have

$$(S^g f, h) = \frac{1}{ab} \sum_{k,l} (g, g_{k/b,\, l/a})(U_{kl} f, h). \tag{1.16}$$

The representation (1.15) of S^g is particularly convenient when g satisfies Tolimieri and Orr's condition A, see [10],

$$\sum_{k,l} |(g, g_{k/b,\, l/a})| < \infty, \tag{1.17}$$

for then the series in (1.15) is unconditionally convergent. Especially, when ab is small and condition A is satisfied, one can compute $S^g f$ for $f \in L^2(\mathbb{R})$ more easily via (1.15) than via (1.3) since in the former case, as opposed to the latter case, only a few terms should be considered. In [5] it was conjectured (and proved for the case that $(ab)^{-1} \in \mathbb{N}$) that when g generates a frame and satisfies condition A, then so does $^\circ\gamma$. It is one of the purposes of the present paper to establish the latter result for the case that $ab \in \mathbb{Q}$, $ab < 1$.

For the case that $ab = p/q$, $p \in \mathbb{Z}$, $q \in \mathbb{Z}$, $p \leqslant q$, $(p, q) = 1$, the frame operator S^g and the computation of the minimal dual $^\circ\gamma$ have been studied in detail by Zibulski and Zeevi in [12] by using the Zak transform (also see [1, pp. 978, 981]). When $\lambda > 0$ one defines the Zak transform $Z_\lambda f$ of an $f \in L^2(\mathbb{R})$ by means of the $L^2_{\text{loc}}(\mathbb{R}^2)$-convergent series

$$(Z_\lambda f)(x, \Omega) = \lambda^{1/2} \sum_{k=-\infty}^{\infty} f(\lambda(x + k)) \mathrm{e}^{-2\pi i k \Omega}. \tag{1.18}$$

A convenient choice for λ here is $\lambda = b^{-1}$, in which case one writes $\hat{f}$ instead of $Z_\lambda f$. Now Zibulski and Zeevi show that when g generates a frame and $\psi \in L^2(\mathbb{R})$ one has

$$\widehat{S^g \psi}\left(x, \Omega + \frac{k}{p}\right) = \sum_{r=0}^{p-1} A_{kr}^{gg}(x, \Omega)\, \hat{\psi}\left(x, \Omega + \frac{r}{p}\right),$$

$$k = 0, \ldots, p-1, \tag{1.19}$$

where we have set for $f, h \in L^2(\mathbb{R})$

$$A_{kr}^{fh}(x, \Omega) = \frac{1}{p} \sum_{l=0}^{q-1} \hat{f}\left(x - l\frac{p}{q}, \Omega + \frac{k}{p}\right) \times \hat{h}^*\left(x - l\frac{p}{q}, \Omega + \frac{r}{p}\right) \tag{1.20}$$

for $x, \Omega \in \mathbb{R}$, $k, r = 0, \ldots, p-1$. Hence, by the periodicity relations of the Zak transform and the inversion formula for the Zak transform, see Proposition 2.1, the computation of $^\circ\gamma = S^{-1} g$ consists of solving for $0 \leqslant x < 1$, $0 \leqslant \Omega < p^{-1}$ the linear system

$$\sum_{r=0}^{p-1} A_{kr}^{gg}(x, \Omega)\, {}^\circ\hat{\gamma}\left(x, \Omega + \frac{r}{p}\right) = \hat{g}\left(x, \Omega + \frac{k}{p}\right),$$

$$k = 0, \ldots, p-1. \tag{1.21}$$

1.2. Results

The purpose of the present paper is to present a closer connection between the approaches based on the Wexler–Raz result and the Zibulski–Zeevi linear systems for computing $^\circ\gamma$ than is done so far. More explicitly we show the following result.

Proposition 1.1. *Let $f, h \in L^2(\mathbb{R})$ and let $k, r = 0, \ldots, p-1$. Then the matrix elements $A_{kr}^{fh}(x, \Omega)$ as defined in (1.20) are p/q-periodic in x and 1-periodic in Ω, with Fourier series*

$$A_{kr}^{fh}(x,\Omega) \sim \sum_{n,m} d_{nm}\, \mathrm{e}^{-2\pi i n x q/p}\, \mathrm{e}^{-2\pi i m \Omega}, \tag{1.22}$$

where

$$d_{nm} = \begin{cases} \dfrac{q}{p}\, \mathrm{e}^{-2\pi i m r/p} (f, h_{m/b, -n/a}), & nq \equiv (r-k) \bmod p, \\ 0, & \text{otherwise.} \end{cases} \tag{1.23}$$

Setting for $f, h \in L^2(\mathbb{R})$

$$\Phi^f(x,\Omega) = \left(p^{-1/2} \hat{f}\left(x - l\frac{p}{q}, \Omega + \frac{k}{p}\right)\right)_{k=0,\ldots,p-1;\, l=0,\ldots,q-1}, \tag{1.24}$$

$$A^{fh}(x,\Omega) = (A_{kr}^{fh}(x,\Omega))_{k,r=0,\ldots,p-1} = \Phi^f(x,\Omega)\,(\Phi^h(x,\Omega))^*, \tag{1.25}$$

we have the following consequences of (1.22) and (1.23).

Theorem 1.1.[1] *We have that $g, \gamma \in L^2(\mathbb{R})$ are biorthogonal if and only if (compare (1.13))*

$$A^{\gamma g}(x,\Omega) = \Phi^\gamma(x,\Omega)\,(\Phi^g(x,\Omega))^* = I_{p\times p}, \quad \text{a.e. } x, \Omega. \tag{1.26}$$

Theorem 1.2.[2] *When g generates a frame then we have (compare (1.14))*

$$\begin{aligned} \Phi^{\circ\gamma}(x,\Omega) &= (A^{gg}(x,\Omega))^{-1}\, \Phi^g(x,\Omega), \\ A^{\circ\gamma\circ\gamma}(x,\Omega) &= (A^{gg}(x,\Omega))^{-1}, \quad \text{a.e. } x, \Omega. \end{aligned} \tag{1.27}$$

[1] After completion of this paper, the author was kindly informed by M.J. Bastiaans and M. Zibulski (independently) that they have found versions of Theorems 1.1 and 1.2 as well.

[2] See footnote 1.

Theorem 1.3. *A $g \in L^2(\mathbb{R})$ satisfies condition A, see (1.17), if and only if $A^{gg}(x,\Omega)$ has an absolutely convergent Fourier series.*

Theorem 1.4. *When g generates a frame and satisfies condition A, then so does $^\circ\gamma$.*

2. Derivations

We start this section by presenting the (well-known) properties of the Zak transform, as far as relevant for our purposes.

Proposition 2.1. *Let $\lambda > 0$, $g, f \in L^2(\mathbb{R})$. For $h \in L^2(\mathbb{R})$ the series*

$$(Z_\lambda h)(x,\Omega) := \lambda^{1/2} \sum_{k=-\infty}^{\infty} h(\lambda(x+k))\, \mathrm{e}^{-2\pi i k \Omega} \tag{2.1}$$

is $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ – convergent. There holds

(a) $\displaystyle\int_{-\infty}^{\infty} g(x) f^*(x)\,\mathrm{d}x = \int_0^1\!\!\int_0^1 (Z_\lambda g)(x,\Omega)(Z_\lambda f)^*(x,\Omega)\,\mathrm{d}x\,\mathrm{d}\Omega,$

(b) $\displaystyle\lambda^{1/2} g(\lambda x) = \int_0^1 (Z_\lambda g)(x,\Omega)\,\mathrm{d}\Omega, \quad$ a.e. x,

(c) $(Z_\lambda g)(x+1,\Omega) = \mathrm{e}^{2\pi i \Omega}(Z_\lambda g)(x,\Omega), \quad$ a.e. x, Ω,

(d) $(Z_\lambda g)(x, \Omega+1) = (Z_\lambda g)(x,\Omega), \quad$ a.e. x, Ω,

(e) *for any $Z \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that $Z(x+1,\Omega) = \mathrm{e}^{2\pi i \Omega} Z(x,\Omega)$, $Z(x,\Omega+1) = Z(x,\Omega)$,* a.e. x, Ω, *there is a unique $g \in L^2(\mathbb{R})$ such that $Z = Z_\lambda g$.*

Now let $ab = p/q$ with $p, q \in \mathbb{Z}$, $0 < p \leqslant q$, $(p,q) = 1$. We consider in the remainder of this paper the choice $\lambda = b^{-1}$, and we write, as Zibulski and Zeevi do in [12],

$$\hat{h} = Z_{\frac{1}{b}} h, \qquad h \in L^2(\mathbb{R}). \tag{2.2}$$

We shall next be more precise about formula (1.19) in case that g has a frame upper bound. The derivation given in [12] of (1.19) is entirely correct as long as we consider g, ψ smooth and of bounded support; in that case (1.19) holds pointwise as an identity between two smooth and bounded functions of $x, \Omega \in \mathbb{R}$. Accordingly, there holds, see [12], formula (13)

$$(S^g \psi, \psi) = \frac{1}{p} \int_0^1 \int_0^{1/p} \sum_{l=0}^{q-1} \left| \sum_{r=0}^{p-1} \hat{g}\left(x - l\frac{p}{q}, \Omega + \frac{r}{p}\right) \times \hat{\psi}^*\left(x, \Omega + \frac{r}{p}\right)\right|^2 \mathrm{d}x \, \mathrm{d}\Omega \qquad (2.3)$$

for such functions g, ψ. Next, when g has a frame upper bound and ψ is smooth and of bounded support, formula (1.19) holds as an identity between an L^2_{loc}-function on the left-hand side and an L^1_{loc}-function on the right-hand side. This is so since $\hat{g} \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$, $\hat{\psi} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$, and g and $S^g \psi$ can be approximated arbitrarily well by h and $S^h \psi$ with h smooth and of bounded support. Also, formula (2.3) is valid in this case. Now since the sets of all ψ and all $\hat{\psi}_{[0,1]^2}$ with ψ smooth and of bounded support are dense in $L^2(\mathbb{R})$ and $L^2([0,1]^2)$, respectively, it follows from (2.3) that g has a frame upper bound if and only if

$$\operatorname{ess\,sup} |\hat{g}| < \infty. \qquad (2.4)$$

Hence, when g has a frame upper bound and $\psi \in L^2(\mathbb{R})$, the formula (1.19) holds as an identity between two $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ functions, since ψ can be approximated arbitrarily well by smooth functions of bounded support and $\hat{g} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$. Note that in the notation of (1.2) and (1.25) we can express the identity (2.3) as

$$(S^g \psi, \psi) = \int_0^1 \int_0^{1/p} (A^{gg}(x, \Omega) v^\psi(x, \Omega), v^\psi(x, \Omega)) \, \mathrm{d}x \, \mathrm{d}\Omega \qquad (2.5)$$

with $v^\psi(x, \Omega) = (\hat{\psi}(x, \Omega + r/p))_{r=0,\ldots,p-1} \in \mathbb{C}^p$ and $(\,,\,)$ the usual inner product in $\mathbb{C}^p$.

Proof of Proposition 1.1. From Proposition 2.1 it is clear that A^{fh}_{kr} is well-defined as an $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ function when $f, h \in L^2(\mathbb{R})$, $k, r = 0, \ldots, p-1$. The 1-periodicity of $A^{fh}_{kr}(x, \Omega)$ in Ω also follows from Proposition 2.1. Next we compute for a.e. x, Ω

$$\begin{aligned} A^{fh}_{kr}\left(x - \frac{p}{q}, \Omega\right) &= \frac{1}{p} \sum_{l=0}^{q-1} \hat{f}\left(x - (l+1)\frac{p}{q}, \Omega + \frac{k}{p}\right) \times \hat{h}^*\left(x - (l+1)\frac{p}{q}, \Omega + \frac{r}{p}\right) \\ &= \frac{1}{p} \sum_{l=1}^{q-1} \hat{f}\left(x - l\frac{p}{q}, \Omega + \frac{k}{p}\right) \times \hat{h}^*\left(x - l\frac{p}{q}, \Omega + \frac{r}{p}\right) \\ &\quad + \hat{f}\left(x - p, \Omega + \frac{k}{p}\right) \times h^*\left(x - p, \Omega + \frac{r}{p}\right). \end{aligned} \qquad (2.6)$$

Now since by Proposition 2.1(c) for any $\psi \in L^2(\mathbb{R})$, $s \in \mathbb{Z}$

$$\begin{aligned} \hat{\psi}\left(x - p, \Omega + \frac{s}{p}\right) &= \mathrm{e}^{-2\pi\mathrm{i}(\Omega + s/p)p} \hat{\psi}\left(x, \Omega + \frac{s}{p}\right) \\ &= \mathrm{e}^{-2\pi\mathrm{i}\Omega p} \hat{\psi}\left(x, \Omega + \frac{s}{p}\right), \end{aligned} \qquad (2.7)$$

we see that $A^{fh}_{kr}(x - p/q, \Omega) = A^{fh}_{kr}(x, \Omega)$ for a.e. x, Ω.

For the computation of the Fourier coefficients

$$d_{nm} = \frac{q}{p} \int_0^{p/q} \int_0^1 A^{fh}_{kr}(x, \Omega) \, \mathrm{e}^{2\pi\mathrm{i}nxq/p} \, \mathrm{e}^{2\pi\mathrm{i}m\Omega} \, \mathrm{d}x \, \mathrm{d}\Omega, \qquad (2.8)$$

we first restrict to smooth functions f, h of bounded support. Then

$$\begin{aligned} d_{nm} &= \frac{q}{p^2} \sum_{l=0}^{q-1} \int_0^{p/q} \int_0^1 \hat{f}\left(x - l\frac{p}{q}, \Omega + \frac{k}{p}\right) \times \hat{h}^*\left(x - l\frac{p}{q}, \Omega + \frac{r}{p}\right) \times \mathrm{e}^{2\pi\mathrm{i}nxq/p} \, \mathrm{e}^{2\pi\mathrm{i}m\Omega} \, \mathrm{d}x \, \mathrm{d}\Omega \\ &= \frac{q}{p^2} \int_0^p \int_0^1 \hat{f}\left(x, \Omega + \frac{k}{p}\right) \hat{h}^*\left(x, \Omega + \frac{r}{p}\right) \times \mathrm{e}^{2\pi\mathrm{i}nxq/p} \, \mathrm{e}^{2\pi\mathrm{i}m\Omega} \, \mathrm{d}x \, \mathrm{d}\Omega. \end{aligned} \qquad (2.9)$$

We next insert the definition (2.1) and (2.2) of $\hat{f}$, $\hat{h}$ into the far right-hand side of (2.9) and perform the integration over Ω to obtain

$$d_{nm} = \frac{q \, \mathrm{e}^{-2\pi\mathrm{i}rm/p}}{p^2 b} \sum_{l=-\infty}^{\infty} \int_0^p f\left(\frac{x+l}{b}\right) h^*\left(\frac{x+l-m}{b}\right) \times \mathrm{e}^{2\pi\mathrm{i}(r-k)l/p} \, \mathrm{e}^{2\pi\mathrm{i}nxq/p} \, \mathrm{d}x. \qquad (2.10)$$

Next the $\sum_{l=-\infty}^{\infty} \int_0^p$ at the right-hand side of (2.10) is written as $\sum_{l=-\infty}^{\infty} \sum_{j=0}^{p-1} \int_0^1$, the summations over l and j are interchanged, in the summation over l the $l+j$ is replaced by l, and we get

$$d_{nm} = \frac{q\,\mathrm{e}^{-2\pi i r m/p}}{p^2 b} \sum_{j=0}^{p-1} \sum_{l=-\infty}^{\infty} \int_0^1 f\left(\frac{x+l}{b}\right) \times h^*\left(\frac{x+l-m}{b}\right) \times \mathrm{e}^{2\pi i(r-k)l/p}\,\mathrm{e}^{2\pi i n x q/p}\,\mathrm{e}^{2\pi i(nq-r+k)j/p}\,\mathrm{d}x. \tag{2.11}$$

Now we have

$$\sum_{j=0}^{p-1} \mathrm{e}^{2\pi i(nq-r+k)j/p} = p \text{ or } 0, \tag{2.12}$$

according as $nq - r + k \equiv 0 \bmod p$ or not. In the former case we have

$$\exp(2\pi i(r-k)l/p + 2\pi i n x q/p) = \exp(2\pi i n q(x+l)/p). \tag{2.13}$$

Hence $d_{nm} = 0$ when $nq - r + k \not\equiv 0 \bmod p$, and when $nq - r + k \equiv 0 \bmod p$ we get

$$d_{nm} = \frac{q}{pb}\mathrm{e}^{-2\pi i r m/p} \sum_{l=-\infty}^{\infty} \int_0^1 f\left(\frac{x+l}{b}\right) \times h^*\left(\frac{x+l-m}{b}\right)\mathrm{e}^{2\pi i n q(x+l)/p}\,\mathrm{d}x$$
$$= \frac{q}{pb}\mathrm{e}^{-2\pi i r m/p} \int_{-\infty}^{\infty} f\left(\frac{x}{b}\right) h^*\left(\frac{x}{b} - \frac{m}{b}\right)\mathrm{e}^{2\pi i n q x)p}\,\mathrm{d}x$$
$$= \frac{q}{p}\mathrm{e}^{-2\pi i r m/p}\,(f, h_{m/b,\,-n/a}), \tag{2.14}$$

where we have used that $q/p = 1/ab$. This completes the proof of (1.23) for the case that f, h are smooth and of compact support so that the summations over l in (2.10) (2.11) and (2.14) are really finite.

For general $f, h \in L^2(\mathbb{R})$ we observe that $A_{kr}^{\varphi\psi} \to A_{kr}^{fh}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$-sense and that $(\varphi, \psi_{m/b,-n/a}) \to (f, h_{m/b,-n/a})$ for all $m, n \in \mathbb{Z}$ when $\varphi \to f$, $\psi \to h$ in $L^2(\mathbb{R})$-sense, see Proposition 2.1. Hence we get (2.8) by taking smooth functions φ, ψ of bounded support with $\varphi \to f$, $\psi \to h$ in $L^2(\mathbb{R})$-sense and using what we just have proved for such functions.

Proof of Theorem 1.1. Let $g, \gamma \in L^2(\mathbb{R})$, and consider the Fourier expansion

$$A_{kr}^{\gamma g}(x, \Omega) \sim \sum_{n,m} d_{nm}\,\mathrm{e}^{-2\pi i n x q/p}\,\mathrm{e}^{-2\pi i m \Omega} \tag{2.15}$$

for $k, r = 0, \ldots, p-1$. When g, γ are biorthogonal, so that

$$(\gamma, g_{t/b,\,s/a}) = ab\,\delta_{to}\,\delta_{so}, \quad t, s \in \mathbb{Z}, \tag{2.16}$$

we have by (1.23) that

$$d_{nm} = \frac{q}{p}ab = 1, \quad n = m = r - k = 0, \tag{2.17}$$

and $d_{nm} = 0$ otherwise. This shows (1.26). The converse is equally easy.

Proof of Theorem 1.2. It is easily established from (2.5) and the discussion preceeding (2.5) that g generates a frame, with frame bounds $A > 0, B < \infty$, if and only if[3]

$$AI_{p\times p} \leqslant A^{gg}(x, \Omega) \leqslant BI_{p\times p}, \quad \text{a.e. } x, \Omega. \tag{2.18}$$

Here Proposition 2.1(a),(e) have also been used. Now when g generates frame we have that for a.e. x, Ω, see (1.19)–(1.21),

$$\sum_{r=0}^{p-1} A_{kr}^{gg}(x, \Omega)\,{}^{\circ}\hat{\gamma}\left(x, \Omega + \frac{r}{p}\right) = \hat{g}\left(x, \Omega + \frac{k}{p}\right),$$
$$k = 0, \ldots, p-1. \tag{2.19}$$

Since $A_{kr}^{gg}(x, \Omega)$ is p/q-periodic in x, we can conclude that for a.e. x, Ω we have

$$\sum_{r=0}^{p-1} A_{kr}^{gg}(x, \Omega)\,{}^{\circ}\hat{\gamma}\left(x - l\frac{p}{q}, \Omega + \frac{r}{p}\right) = \hat{g}\left(x - l\frac{p}{q}, \Omega + \frac{k}{p}\right),$$
$$k = 0, \ldots, p-1, \quad l = 0, \ldots, q-1. \tag{2.20}$$

[3] One can show that $\det(A^{gg}(x, \Omega))$ is (q^{-1}, p^{-1})-periodic in (x, Ω), whence for checking that g generates a frame, it is sufficient to consider $A^{gg}(x, \Omega)$ for $(x, \Omega) \in (0, q^{-1}) \times [0, p^{-1})$. More explicitly there holds $A^{gg}(x, \Omega + p^{-1}) = JA^{gg}(x, \Omega)\,J^{-1}$ and $A^{gg}(x + q^{-1}, \Omega) = FA^{gg}(x, \Omega)F^{-1}$ with J the permutation matrix corresponding to the permutation $0 \to 1 \to \cdots \to p-1 \to 0$, and F the diagonal matrix with entries $\exp(-2\pi i m_0 k/p)$, $k = 0, \ldots, p-1$, where $m_0 \in \mathbb{Z}$ only depends on p, q.

That is, in the notation of (1.24) and (1.25), for a.e. x, Ω

$$A^{gg}(x,\Omega)\Phi^{\circ\gamma}(x,\Omega) = \Phi^{g}(x,\Omega), \tag{2.21}$$

and the first formula in (1.27) follows. The second formula in (1.27) is an easy consequence of this and of the definitions (1.24) and (1.25).

Proof of Theorem 1.3. We note that

$$\sum_{k,r=0}^{p-1} \sum_{n,m;\, nq\,\equiv\,(r-k)\bmod p} \frac{q}{p}\,|(g, g_{m/b,-n/a})| = q \sum_{t,s=-\infty}^{\infty} |(g, g_{t/b,s/a})|, \tag{2.22}$$

and the result follows from (1.23).

Proof of Theorem 1.4. We assume that g generates a frame and satisfies condition A. By the second formula in (1.27) we have

$$A^{\circ\gamma\circ\gamma}(x,\Omega) = \det{}^{-1}(A^{gg}(x,\Omega))\operatorname{adj}(A^{gg}(x,\Omega)), \quad \text{a.e. } x,\Omega. \tag{2.23}$$

Now both $\det(A^{gg}(x,\Omega))$ and the elements of $\operatorname{adj}(A^{gg}(x,\Omega))$ have an absolutely convergent Fourier series as finite sums of finite products of functions having such a Fourier series, see Theorem 2.3. In particular, $\det(A^{gg}(x,\Omega))$ is a continuous function, bounded below by A^p, see (2.18), with $A > 0$ a lower frame bound for g. It therefore follows that $\det^{-1}(A^{gg}(x,\Omega))$ has an absolutely convergent Fourier series by Wiener's $1/f$-theorem, see [8], Section 150. Hence the elements of $A^{\circ\gamma\circ\gamma}(x,\Omega)$, as products of two functions having an absolutely convergent Fourier series, have such a Fourier series as well. Now Theorem 1.4 follows from Theorem 1.3.

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