We consider the causal channel

$$h_k = \begin{cases} (k+1)e^{-(k+1)} & k \geq 0 \\ 0 & k < 0. \end{cases}$$

We truncate the channel response so that $N_a = 0$ and $N_c = 20$. As before, the channel input is an i.i.d. sequence, and the noise is stationary and white with $\text{SNR}_{\text{chan}} = 20$ dB. We assume the number of feedforward taps is fixed at $\eta = 10$.

Fig. 3 shows the optimal decision delay (defined as that which maximizes SNR_{DFE}) versus the number of feedback taps. When the number of feedback taps is small, the optimal feedforward filter is two-sided; that is, it contains both causal and anticausal taps. As expected, however, the optimal Δ converges to $\eta - 1$ as the number of feedback taps get large. Note that when $\Delta = \eta - 1$, the feedforward filter is anticausal; that is, $\{w_k\} = \{w_{-\Delta}, \cdots, w_0\}$.

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Some Counterexamples in the Theory of Weyl-Heisenberg Frames

A. J. E. M. Janssen

***Abstract*— We present an example of a positive function g with a positive Fourier transform $\hat{g}$ and reasonable smoothness and decay properties such that**

$$(-1)^{nm} \exp(\pi itm)\, g(t-n),\ n, m \in \mathbb{Z}$$

does not constitute a frame for $L^2(\mathbb{R})$. We also give counterexamples for the statement that one can tell (in)definiteness of a Weyl–Heisenberg frame operator from (in)definiteness of its Weyl symbol.

***Index Terms*—Weyl–Heisenberg frame, Zak transform, Weyl symbol.**

I. Introduction

In her pioneering paper [1, p. 981], Daubechies raises the question whether a positive function g having a positive Fourier transform $\hat{g}$ always generates a Weyl–Heisenberg frame $g_{na,mb}$, $n, m \in \mathbb{Z}$, when $a, b > 0$, $ab < 1$. Here

$$g_{x,y}(t) = \exp(-\pi ixy + 2\pi iyt)\, g(t-x),\ t \in \mathbb{R}$$

for real x and y. This is motivated by the fact that this holds true for Gaussian g, see [2] for an elementary proof for this special case and for references to earlier proofs.

We shall show in this correspondence that the above statement does not hold when $a = 1$, $b = \frac{1}{2}$. That is, we shall display a smooth function g with decay like $1/t^2$ such that both g and $\hat{g}$ are positive everywhere, while the functions $\exp(\pi itm)\, g(t-n)$, $n, m \in \mathbb{Z}$, do not constitute a frame for $L^2(\mathbb{R})$.

The two other examples concern the question whether perhaps a frame operator (corresponding to a $g \in L^2(\mathbb{R})$) is positive definite when, or only when, its Weyl symbol is positive (almost) everywhere. We refer to [3]–[5] for motivating material concerning the Weyl symbol and its role in time–frequency analysis. Although it is well known that the notions of positivity of linear operators of $L^2(\mathbb{R})$ and positivity of their Weyl symbols are quite different, they are not totally unrelated. For instance, the Weyl symbol of a positive-definite linear operator becomes everywhere positive upon appropriate Gaussian smoothing. And since frame operators constitute a very special class of linear operators of $L^2(\mathbb{R})$, it could well be that the notions of positivity of frame operators and that of their Weyl symbols do agree. Our two examples show that they do not.

II. The Examples

According to [1, p. 981], the lower and upper frame bounds A and B of the collection

$$(-1)^{nm} \exp(\pi itm)\, g(t-n),\ n, m \in \mathbb{Z}$$

are given by

$$A = \text{ess inf}\, |(U_z\, g)(t,s)|^2 + |(U_z\, g)(t - \tfrac{1}{2}, s)|^2 \tag{2.1}$$

$$B = \text{ess sup}\, |(U_z\, g)(t,s)|^2 + |(U_z\, g)(t - \tfrac{1}{2}, s)|^2 \tag{2.2}$$

where for a $g \in L^2(\mathbb{R})$ we have denoted

$$(U_z\, g)(t,s) = \sum_{l \in \mathbb{Z}} e^{2\pi itl}\, g(s-l),\ s, t \in \mathbb{R} \tag{2.3}$$

for the Zak transform of g. When $\hat{g}$ denotes the Fourier transform

$$\hat{g}(\nu) = \int e^{-2\pi i\nu s}\, g(s)\, ds,\ \nu \in \mathbb{R} \tag{2.4}$$

we can as well consider

$$|(U_z\, \hat{g})(s,t)|^2 + |(U_z\, \hat{g})(s, t + \tfrac{1}{2})|^2 \tag{2.5}$$

for the computation of the frame bounds A, B. This is so since

$$(U_z\, g)(t+1, s) = (U_z\, g)(t,s),\ (U_z\, g)(t, s+1) = e^{2\pi it}(U_z\, g)(t,s) \tag{2.6}$$

and

$$(U_z\, \hat{g})(t,s) = e^{2\pi its}(U_z\, g)(s,-t). \tag{2.7}$$

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We shall find a positive g with a positive $\hat{g}$ such that

$$(U_z\,\hat{g})(\frac{1}{2},\frac{1}{4}) = \sum_{l\in\mathbb{Z}} (-1)^l \hat{g}(\tfrac{1}{4}-l) = \left\{\hat{g}(\frac{1}{4})-\hat{g}(\frac{3}{4})-\hat{g}(\frac{5}{4})+\hat{g}(\frac{7}{4})\right\} + \left\{\hat{g}(\frac{9}{4})-\hat{g}(\frac{11}{4})-\hat{g}(\frac{13}{4})+\hat{g}(\frac{15}{4})\right\}+\dots \tag{2.8}$$

vanishes. (Observe that both g and $\hat{g}$ are even functions.) Now since

$$(U_z\,\hat{g})(\frac{1}{2},\frac{3}{4}) = -(U_z\,\hat{g})(\frac{1}{2},\frac{1}{4}) \tag{2.9}$$

it is concluded that (2.5) vanishes for $s=\frac{1}{2}$, $t=\frac{1}{4}$. And since we construct $\hat{g}$ such that $U_z\,\hat{g}$ is continuous, we see that A in (2.1) is zero, so that $(-1)^{nm}\exp(\pi itm)\,g(t-n)$ does not generate a frame.

We construct $\hat{g}$ in the form

$$\hat{g}(\nu) = \hat{\varphi}(\nu) + \sum_{k=0}^{\infty} a_k\,\hat{\psi}\left(\frac{\nu}{2k+7/4}\right),\ \nu\in\mathbb{R}. \tag{2.10}$$

Here we have set $a_k = e^{-k}$ and

$$\hat{\varphi}(\nu) = (e^{-|\nu|}-e^{-\frac{1}{4}})\,\chi_{(-\frac{1}{4},\frac{1}{4})}(\nu)$$
$$\hat{\psi}(\nu) = (1-|\nu|)\,\chi_{(-1,1)}(\nu). \tag{2.11}$$

Obviously, $\hat{g}$ is positive, continuous, and rapidly decaying. Also

$$\varphi(t) = 2\int_0^{1/4} (e^{-\nu}-e^{-1/4})\cos 2\pi\nu t\,d\nu = \frac{1}{2\pi^2t^2}\left((1-\cos\frac{1}{2}\pi t)\,e^{-1/4} + \int_0^{1/4}(1-\cos 2\pi\nu t)\,e^{-\nu}\,d\nu\right) > 0 \tag{2.12}$$

for all $t\in\mathbb{R}$, and

$$\psi(t) = 2\int_0^1 (1-\nu)\cos 2\pi\nu t\,d\nu = \mathrm{sinc}^2(\pi t) \geq 0 \tag{2.13}$$

for all $t\in\mathbb{R}$. Hence g is positive and continuous everywhere, and decays like $1/t^2$. Finally

$$(U_z\,\hat{g})\left(\frac{1}{2},\frac{1}{4}\right) = 0$$

since $\hat{\varphi}$ vanishes outside $(-\frac{1}{4},\frac{1}{4})$, while each of the functions

$$\hat{\psi}\left(\frac{\nu}{2k+7/4}\right)$$

is linear on each of the intervals

$$\left[\frac{1}{4},\frac{7}{4}\right],\left[\frac{9}{4},\frac{15}{4}\right],\dots$$

so that

$$\hat{\psi}\left(\frac{1/4}{2k+7/4}\right)-\hat{\psi}\left(\frac{3/4}{2k+7/4}\right)-\hat{\psi}\left(\frac{5/4}{2k+7/4}\right)+\hat{\psi}\left(\frac{7/4}{2k+7/4}\right) = \hat{\psi}\left(\frac{9/4}{2k+7/4}\right)-\hat{\psi}\left(\frac{11/4}{2k+7/4}\right)-\hat{\psi}\left(\frac{13/4}{2k+7/4}\right)+\hat{\psi}\left(\frac{15/4}{2k+7/4}\right) = \dots = 0. \tag{2.14}$$

Since $U_z\,\hat{g}$ is continuous by continuity and rapid decay of $\hat{g}$, this completes the construction of the example.

We now turn to the two examples concerning Weyl symbols of frame operators. The Weyl symbol of a projection operator $f\in L^2(\mathbb{R})\to(f,g)\,g$, where $g\in L^2(\mathbb{R})$, equals the Wigner distribution W_g of g, given by

$$W_g(t,\nu) = \int e^{-2\pi i\nu s}\,g(t+\frac{1}{2}s)\,g^*(t-\frac{1}{2}s)\,ds,\ t,\nu\in\mathbb{R}. \tag{2.15}$$

When $a>0$, $b>0$ and $g\in L^2(\mathbb{R})$ we have

$$W_{g_{na,mb}}(t,\nu) = W_g(t-na,\nu-mb),\ n,m\in\mathbb{Z}. \tag{2.16}$$

Hence, the Weyl symbol S^W of the frame operator S

$$f\in L^2(\mathbb{R})\to Sf = \sum_{n,m}(f,g_{na,mb})\,g_{na,mb} \tag{2.17}$$

is given, at least formally, by

$$S^W(t,\nu) = \sum_{n,m} W_g(t-na,\nu-mb). \tag{2.18}$$

Now let $a>0$, $b>0$, $ab>1$, and take $g(t)=\exp(-\pi t^2)$. As is well known $(g_{na,mb})_{n,m\in\mathbb{Z}}$ is not a frame for $L^2(\mathbb{R})$, since there are well-behaved $0\neq f\in L^2(\mathbb{R})$ such that $Sf=0$. Nevertheless, $S^W>0$ everywhere since

$$W_g(t,\nu) = 2\exp(-2\pi t^2-2\pi\nu^2),\ t,\nu\in\mathbb{R}. \tag{2.19}$$

Hence (strict) positivity of the Weyl symbol S^W does not imply that the frame operator S is positive-definite.

We use this opportunity to point out an error in [6]. There it is stated on p. 586 that for the operator " A_μ the Szegö theorem holds." In the present case we have $A_\mu = S$ when we choose

$$d\mu(x,y) = \frac{1}{ab}\sum_{k,l}(g,g_{k/b,l/a})\,\delta_{l/b}(x)\,\delta_{k/a}(-y). \tag{2.20}$$

This is so since by [7, Proposition 2.8] we have the following representation for S:

$$Sf = \frac{1}{ab}\sum_{k,l}(g,g_{k/b,l/a})\,f_{k/b,l/a},\ f\in L^2(\mathbb{R}), \tag{2.21}$$

where $f_{x,y}(t) = e^{-\pi ixy+2\pi iyt}\,f(t-x)$ as above. Without going into all the details of the argument, the statement "the Szegö theorem holds for A_μ" would imply that 0 is not in the spectrum of $A_\mu = S$ since S^W is bounded away from 0. Our example shows that this statement is incorrect.

For our final example we consider $a>0$, $b>0$, $ab\leq 1$ and

$$g(t) = e^{-\alpha t}\,\chi_{[0,\infty)}(t),\ t\in\mathbb{R} \tag{2.22}$$

where $\alpha>0$. It has been shown in [8, sec. 4], that $(g_{na,mb})_{n,m\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. An elementary calculation shows that

$$W_g(t,\nu) = e^{-2\alpha t}\,\frac{\sin 4\pi\nu t}{\pi\nu}\,\chi_{[0,\infty)}(t), t,\nu\in\mathbb{R}. \tag{2.23}$$

By using the formula (for $y>0$)

$$\sum_{m=-\infty}^{\infty}\frac{\sin 4\pi(\nu-mb)\,y}{\pi(\nu-mb)} = \frac{1}{b}\,\frac{\sin(2\pi(\lfloor 2yb\rfloor+\frac{1}{2})\,\nu/b)}{\sin(\pi\nu/b)} \tag{2.24}$$

with $\lfloor x\rfloor$ = largest integer $\leq x$, we obtain

$$S^W(t,\nu) = \sum_{n,m} W_g(t-na,\nu-mb) = \frac{1}{b}\sum_{n\,;\,t-na\geq 0} e^{-2\alpha(t-na)}\,\frac{\sin(2\pi(\lfloor 2(t-na)\,b\rfloor+\frac{1}{2})\,\nu/b)}{\sin(\pi\nu/b)} \tag{2.25}$$

for the Weyl symbol of S.

Now consider the case that $a = b = 1$, so that

$$S^W(t,\nu) = \sum_{n \le t} e^{-2\alpha(t-n)} \frac{\sin(2\pi(\lfloor 2(t-n) \rfloor + \frac{1}{2})\nu)}{\sin \pi\nu}. \quad (2.26)$$

When $m \in \mathbb{Z}$, $\frac{1}{2} < \delta < 1$, and $t = m + \delta$, we see that for large α, only the term with $n = m$ needs to be considered in the right-hand side of (2.26), so that

$$S^W(t,\nu) \approx e^{-2\delta\alpha} \frac{\sin 3\pi\nu}{\sin \pi\nu} \quad (2.27)$$

which becoms negative when ν varies. Hence the Weyl symbol of S takes negative values, and it does so on a large subset of $\mathbb{R}^2$.

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Irregular Sampling for Spline Wavelet Subspaces

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***Abstract*— Spline wavelets $\psi_m(t)$ are important in time–frequency localization due to**

i) ψ_m can be arbitrarily close to the optimal case as m is sufficiently large,

ii) ψ_m has compact support and simple analytic expression, which lead to effective computation.

Although the spline wavelet subspaces are so simple, Walter's well-known sampling theorem does not hold if the order of spline m is even. Moreover, when irregular sampling is considered in these spaces, it is hard to determine the sampling density, which is a serious problem in applications. In this correspondence, a general sampling theorem is obtained for $m \ge 3$ in the sense of iterative construction and the sampling density δ_m is estimated.

***Index Terms*—Sampling, spline wavelets, algorithm.**

I. Introduction and Preliminaries

A. Introduction

The classical Shannon sampling theorem plays an important role in signal analysis. Unfortunatly this theorem is not appropriate for some signals. Walter [1] established a more general sampling theorem in wavelet subspaces, which does not contain spline wavelet subspaces with even order. Janssen [2] obtained one sampling theorem for quadratic spline wavelet subspace by using samples $\{(n+\frac{1}{2})\}$ instead of the integer set $\{n\}$. Although Walter's main theorem in [1] was extended to the irregular case [3], this involved a hard to determine δ (with this δ, t_n's can be used as samples to recover a signal if $|t_n - n| \le \delta$) even in the spline case. The more serious problem in applications is the complexity of error estimation . Recently, Feichtinger and Grochenig [4] designed a new iterative algorithm to recover a band-limited signal, which uses only the function values on a sequence and thus can be thought of as a sampling theorem. By this sampling theorem, a band-limited signal f can be recoved by any samples $\{f(t_n)\}$ if the density of $\{t_n\}$, which is defined as $\sup_n (t_{n+1} - t_n)$, is less than 1 and the more important thing is that the truncation error $||f_n - f||$ is easily estimated.

B. The Algorithm of Feichtinger and Grochenig

We review the Feichtinger–Grochenig algorithm in this section. Given

$$f \in B_\pi = \{f \in L^2(R),\ \mathrm{supp}\,\hat{f} \subseteq [-\pi, \pi]\}$$

define

$$f_0(t) = Af = P\left(\sum_{n \in Z} f(x_n)\chi_n\right)$$

and

$$f_{k+1} = f_k + A(f - f_k)$$

for $k \ge 0$, where P is the projection from $L^2(R)$ onto B_π,

$$\cdots < x_n < x_{n+1} < \cdots$$

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