Validity of WH-frame bound conditions depends on lattice parameters

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Abstract.
In the study of Weyl-Heisenberg frames the assumption of having a finite frame upper bound appears recurrently. In this note it is shown that it actually depends critically on the time-frequency lattice used. Indeed, for any irrational $\alpha > 0$ we can construct a smooth $g \in L^2(\mathbb{R})$ such that for any two rationals $a > 0$ and $b > 0$ the collection $(g_{na,mb})_{n,m\in\mathbb{Z}}$ of time-frequency translates of g has a finite frame upper bound, while for any $\beta > 0$ and any rational $c > 0$ the collection $(g_{nc\alpha,m\beta})_{n,m\in\mathbb{Z}}$ has no such bound.

It follows from a theorem of I. Daubechies, as well as from the general atomic theory developed by Feichtinger and Gröchenig, that for any non-zero $g \in L^2(\mathbb{R})$ which is sufficiently well-behaved, there exists $a_c > 0$, $b_c > 0$ such that $(g_{na,mb})_{n,m\in\mathbb{Z}}$ is a frame whenever $0 < a < a_c$, $0 < b < b_c$. We present two examples of a non-zero $g \in L^2(\mathbb{R})$, bounded and supported by $(0, 1)$, for which such numbers a_c, b_c do not exist. In the first one of these examples, the frame bound equals 0 for all $a > 0$, $b > 0$, $b < 1$. In the second example, the frame lower bound equals 0 for all a of the form $l \cdot 3^{-k}$ with $l, k \in \mathbb{N}$ and all b, $0 < b < 1$, while the frame lower bound is at least 1 for all a of the form $(2m)^{-1}$ with $m \in \mathbb{N}$ and all b, $0 < b < 1$.

Index terms: Weyl-Heisenberg frame, Zak transform, frame bounds.

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1. Introduction and notations

Let $x, y \in \mathbb{R}$, and let $f \in L^2(\mathbb{R})$. We define the time-frequency translate $f_{x,y}$ of f by

$$f_{x,y}(t) = e^{2\pi i y t} f(t-x) \ , \qquad t \in \mathbb{R} \ . \tag{1}$$

Next let $a > 0$, $b > 0$ and $g \in L^2(\mathbb{R})$. We say that g has a finite (Weyl-Heisenberg) frame upper bound for the shift parameters a, b when there is a $B = B(a,b) < \infty$ such that

$$\sum_{n,m \in \mathbb{Z}} |(f, g_{na,mb})|^2 \le B(a,b) \, \|f\|^2 \ , \qquad f \in L^2(\mathbb{R}) \ . \tag{2}$$

Actually, it is sufficient to know that (2) is valid for all f in some dense subspace of $L^2(\mathbb{R})$, e.g. Schwartz $\mathcal{S}$. We refer to [3], Secs. 3.2, 3.4, 4.1, 4.22 for generalities about (Weyl-Heisenberg) frames, and to [5], Ch. 3, for a detailed discussion of universal sufficient conditions which imply that g has a finite frame upper bound for any $a > 0$, $b > 0$.

There are many functions $g \in L^2(\mathbb{R})$ such that g does not have a finite frame upper bound for any choice of $a > 0$, $b > 0$; one can use (12) below to generate examples. Simple natural examples are thus essentially unbounded $g \in L^2(\mathbb{R})$. Alternatively, and more directly, when $f(t) = g(t) = (1+|t|)^{-s}$ in (2) with $\frac{1}{2} < s < \frac{3}{4}$, the decay of $(f, g_{x,0})$, which behaves like $|x|^{1-2s}$ as $|x| \to \infty$, is insufficient for the left-hand side of (2) to be finite.

It is rather straightforward to show (see Sec. 2) that a $g \in L^2(\mathbb{R})$ has a finite frame upper bound for the parameters $a > 0$, $b > 0$ if and only if it has such a bound for the parameters $a\,p/q$, $b\,r/s$, where p/q and r/s are arbitrary positive rational numbers. Thus the question naturally arises, see [5], whether g has a finite frame upper bound for any real $a > 0$, $b > 0$ whenever g has such a bound for the parameters $a = b = 1$. Let $\alpha > 0$ be any irrational. We shall present a $g \in L^2(\mathbb{R})$ such that g has a finite frame upper bound for any rational pair $a > 0$, $b > 0$, while g has no such bound for any pair $p\,\alpha/q$, β with p/q any positive rational and β any positive real.

The frame coefficient mapping

$$f \to C(f\,;\, g, a, b) = ((f, g_{na,mb}))_{n,m \in \mathbb{Z}} \tag{3}$$

is well-defined for this $g \in L^2(\mathbb{R})$ as (i) a mapping from Schwartz space $\mathcal{S}$ into $l^2(\mathbb{Z}^2)$ for any $a > 0$, $b > 0$, and as (ii) a bounded linear mapping from $L^2(\mathbb{R})$ into $l^2(\mathbb{Z}^2)$ with operator norm $\le B^{1/2}(a,b)$ for any rational $a > 0$, $b > 0$. Hence when $\beta > 0$ is rational and $a_k > 0$ are rationals with $a_k \to \alpha$ as $k \to \infty$, then one can derive that $B(a_k, \beta) \to \infty$ as $k \to \infty$, despite the fact that $C(f, g\,;\, a_k, \beta) \to C(f, g\,;\, \alpha, \beta)$ in $l^2(\mathbb{Z}^2)$-sense as $k \to \infty$ for the dense set of all $f \in \mathcal{S} \subset L^2(\mathbb{R})$.

It is quite natural to ask now whether there also exist examples as the one just described for positive frame *lower* bounds. We say that $g \in L^2(\mathbb{R})$ has a positive frame lower bound for the parameters a, b when there is an $A = A(a,b) > 0$ such that

$$\sum_{n,m\in\mathbb{Z}} |(f, g_{na,mb})|^2 \geq A(a,b)\, \|f\|^2 \ , \qquad f \in L^2(\mathbb{R}) \ . \tag{4}$$

When g has both a finite frame upper bound and a positive frame lower bound, we say that the system $(g_{na,mb})_{n,m\in\mathbb{Z}}$ is a frame. It follows from [2], Thms. 2.5–6 that there are $a_c > 0$, $b_c > 0$ such that $(g_{na,mb})_{n,m\in\mathbb{Z}}$ is a frame whenever $g \neq 0$ is sufficiently well-behaved and $0 < a < a_c$, $0 < b < b_c$; such a result occurs in greater generality in [4], Thm. 6.1. When we restrict to bounded g's supported by $(0, 1)$, the finiteness of frame upper bounds is no issue by [1], Thm. 3.13. For these g's the interesting question to ask is whether one can find one with zero lower frame bound for arbitrarily small $a > 0$, $b > 0$. We shall present in Sec. 5 two such examples. The first example is a $g \neq 0$ such that g has frame lower bound $= 0$ for any a, b with $0 < a < 1$, $0 < b < 1$. The second one is a $g \neq 0$ such that g has frame lower bound $= 0$ for all a of the form $a = l \cdot 3^{-k}$, $l, k \in \mathbb{N}$, and all b, $0 < b < 1$, while g has frame lower bound ≥ 1 for all a of the form $a = (2m)^{-1}$, $m \in \mathbb{Z}$, and all b, $0 < b < 1$.

2. Rationally related lattices

Let $a > 0$, $b > 0$, and let p, q, r, s be positive integers with $\gcd(p, q) = 1 = \gcd(r, s)$. We shall show that when g has a finite frame upper bound for one of the shift parameter pairs (a, b) or $(a\,p/q, b\,r/s)$, then g has such a bound for the other pair, and the frame upper bounds satisfy

$$B(a\,p/q, b\,r/s) \leq qs\, B(a, b) \leq pqrs\, B(a\,p/q, b\,r/s) \ . \tag{5}$$

To show this we take for convenience $a = 1 = b$, the proof for the general case being the same. We have for $f \in L^2(\mathbb{R})$

$$\begin{aligned}
\sum_{n,m\in\mathbb{Z}} |(f, g_{np/q,mr/s})|^2 &= \sum_{n,m\in\mathbb{Z}} \sum_{l=0}^{q-1} \sum_{j=0}^{s-1} |(f, g_{(nq+l)p/q,(ms+j)r/s})|^2 = \\
&= \sum_{l=0}^{q-1} \sum_{j=0}^{s-1} \sum_{n,m\in\mathbb{Z}} |(f_{-lp/q,-jr/s}, g_{np,mr})|^2 \leq \\
&\leq \sum_{l=0}^{q-1} \sum_{j=0}^{s-1} \sum_{n,m\in\mathbb{Z}} |(f_{-lp/q,-jr/s}, g_{n,m})|^2 \ ,
\end{aligned} \tag{6}$$

and similarly

$$\begin{aligned}
\sum_{n,m\in\mathbb{Z}} |(f, g_{n,m})|^2 &\leq \sum_{n,m\in\mathbb{Z}} |(f, g_{n/q,m/s})|^2 = \\
&= \sum_{n,m\in\mathbb{Z}} \sum_{k=0}^{p-1} \sum_{l=0}^{r-1} |(f, g_{(np+k)/q,(mr+l)/s})|^2 =
\end{aligned}$$

$$= \sum_{k=0}^{p-1} \sum_{l=0}^{r-1} \sum_{n,m\in\mathbb{Z}} |(f_{-k/q,-l/s}, g_{np/q,mr/s})|^2 \; . \tag{7}$$

Since time-frequency shift operators are unitary, it follows from (6) that

$$\sum_{n,m\in\mathbb{Z}} |(f, g_{np/q,mr/s})|^2 \leq qs\, B(1,1)\, \|f\|^2 \tag{8}$$

whenever g has a finite frame upper bound $B(1,1)$ for the shift parameters $(1,1)$, and it follows from (7) that

$$\sum_{n,m\in\mathbb{Z}} |(f, g_{n,m})|^2 \leq pr\, B(p/q, r/s)\, \|f\|^2 \tag{9}$$

whenever g has the finite frame upper bound $B(p/q, r/s)$ for the shift parameters $(p/q, r/s)$. This proves the result.

3. The example for frame upper bounds

Let $\alpha > 0$, irrational be given. We shall construct a smooth $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ such that g has a finite frame upper bound for the parameters $(1,1)$, while it has no such bound for the parameters (α, β) for any real $\beta > 0$. In view of the result in Section 2, this implies that g has a finite frame upper bound for any pair of rationals $a > 0$, $b > 0$, while it has no such bound for any pair $c\alpha$, β with $c > 0$ rational and any $\beta > 0$.

For the construction of g we use the fact, see [2], p. 981, that g has a finite frame upper bound for the parameters $(1,1)$ if and only if

$$\operatorname*{ess\,sup}_{(t,s)\in[0,1]^2} |(U_z\, g)(t,s)|^2 < \infty \; . \tag{10}$$

Here

$$(U_z\, g)(t,s) = \sum_{n\in\mathbb{Z}} e^{2\pi i n s}\, g(t-n) \tag{11}$$

is the Zak transform of g. We also use the fact that

$$\sum_{n\in\mathbb{Z}} |g(t-n\alpha)|^2 \leq \beta\, B(\alpha,\beta) \; , \qquad \text{a.e. } t \in \mathbb{R} \; , \tag{12}$$

whenever g has the finite frame upper bound $B(\alpha,\beta)$ for the parameters α, β. The latter result can be distilled from the proof of Thm. 2.5 in [2], Appendix C; we also refer to [1], Thm. 3.12.

We conclude from (10–12) that it is sufficient to construct a smooth $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$

such that

$$\operatorname*{ess\,sup}_{t\in[0,1]} \sum_{n\in\mathbb{Z}} |g(t-n)| < \infty \ , \tag{13}$$

while

$$\operatorname*{ess\,sup}_{|t|\le 1} \sum_{n\in\mathbb{Z}} |g(t-n\alpha)|^2 = \infty \ . \tag{14}$$

Denote by $\lfloor x \rfloor$ the largest integer $\le\ x \in \mathbb{R}$. Since the set $\{n\alpha - \lfloor n\alpha \rfloor \mid n \in \mathbb{N}\}$ is dense in $(0,1)$ we can find $n_1, n_2, ...,$ in $\mathbb{N}$ and positive numbers $\varepsilon_1, \varepsilon_2, ...$ such that the intervals

$$I_k = (n_k\alpha - \lfloor n_k\alpha \rfloor - \varepsilon_k, n_k\alpha - \lfloor n_k\alpha \rfloor + \varepsilon_k) \ , \qquad k = 1, 2, ... \ , \tag{15}$$

are pairwise disjoint and contained in $(0,1)$. We choose smooth functions g_k, supported by I_k and satisfying

$$0 \le g_k(t) \le 1 \ , \ \ t \in I_k \ ; \qquad g_k(t) = 1 \ , \ \ t \in J_k \ , \tag{16}$$

where J_k is the middle third part of the interval I_k for $k = 1, 2, ...$ Now we define

$$g(t) = \sum_{k=1}^{\infty} g_k(t - \lfloor n_k\alpha \rfloor) \ , \qquad t \in \mathbb{R} \ . \tag{17}$$

It is easy to see that g is smooth and belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, for the g_k have disjoint supports, and $|g|$ is bounded by 1 and supported by a set of measure less than 1. Also, for any $t \in [0,1]$, there is at most one $k = 1, 2, ...$ such that $t \in I_k$, whence $\sum_{n\in\mathbb{Z}} |g(t-n)| \le 1$. Thus (13) holds. On the other hand, for any $K = 1, 2, ...,$ the set of $t \in \mathbb{R}$ such that $\sum_{n\in\mathbb{Z}} |g(t-n\alpha)|^2 \ge K$ contains the intersection of the intervals $(-\varepsilon_k/3, \varepsilon_k/3)$, $k = 1, ..., K$, which has positive measure. Therefore (14) holds as well, and the construction is complete.

Remark. The rationals do not play a specific role in the previous example. Indeed, it only matters that two lattices are not rationally related to one another. Hence one can produce another function which has no finite frame upper bound for any rational pair (a, b) while it has such bounds for all rational multiples of (α, β), where α and β are both irrational.

4. Local unboundedness of frame upper bounds

The estimate in (5) indicates that one has to expect that the frame upper bounds $B(a_k, b_k)$ may be unbounded when g has a finite frame upper bound $B(1,1)$ for the parameters $(1,1)$ and (a_k, b_k) any pair of rationals approaching a pair (α, β) of which at least one of α, β is irrational. The example g given in the previous section can be used to show that the frame bounds $B(a,b)$ are indeed unbounded in any neighbourhood of (α, β) with α as above.

To prove the statement just made, suppose that there is some $B < \infty$ such that $B(a,b) \le B$ for all pairs of rationals (a,b) sufficiently close to (α,β). It is not very difficult to see that for any test function $f \in \mathcal{S}$ the mapping $(a,b) \to C(f\,;\,g,a,b)$, see (3), is continuous from $\mathbb{R}_+^2$ to $l^2(\mathbb{Z}^2)$. (Indeed, when $f \in \mathcal{S}$, there is a uniform frame upper bound for f when a,b ranges over a compact subset of $\mathbb{R}_+^2$. Next, the continuity property certainly holds when g is replaced by an $h \in \mathcal{S}$. And, finally, approximating $g \in L^2(\mathbb{R})$ by $h \in \mathcal{S}$ and applying the triangle inequality, we get the continuity of $(a,b) \to C(f\,;\,g,a,b)$ as well.) Taking a sequence (a_k,b_k) of pairs of rationals with $\lim_{k\to\infty}(a_k,b_k) = (\alpha,\beta)$, we thus see that the mappings $f \to C(f\,;\,g,a_k,b_k)$ constitute a bounded sequence of bounded linear operators from the Banach space $L^2(\mathbb{R})$ to $l^2(\mathbb{R}^2)$, convergent for a dense subspace of $L^2(\mathbb{R})$ to the limit operator $f \to C(f\,;\,g,\alpha,\beta)$. However, by a simple approximation argument this implies that this sequence is convergent for all $f \in L^2(\mathbb{R})$ and, furthermore that the limit operator is itself bounded. Contradiction.

<u>Remark</u>. The argument of Sec. 2 can be used to show that $g \in L^2(\mathbb{R})$ has a finite frame upper bound for all pairs (a,b) in $\mathbb{R}_+^2$ if and only if $B(a,b) \le B < \infty$ for all pairs (a,b) in an arbitrary small disk in $\mathbb{R}_+^2$.

5. The examples for frame lower bounds

Let $0 < \varepsilon < 1$. We shall first present a g, bounded and supported by $(0,1)$, such that the measure of the set of all $t \in (0,1)$ with $|g(t)| \ge 1$ exceeds $1-\varepsilon$ while nevertheless g has frame lower bound 0 for all $a > 0$, $0 < b < 1$. This example boils down to pushing to the extreme the example in [2], Remark 2 after Thm. 2.5.

We start with the observation that, when $b < 1$, a g supported by $(0,1)$ has a positive frame lower bound for the parameters a, b if and only if

$$\operatorname*{ess\,inf}_{t\in I} \sum_{n\in\mathbb{Z}} |g(t+na)|^2 > 0\,, \tag{18}$$

where I is any interval of length a. This is implicit in the proof of Thm. 2.5 in [2], Appendix C; also see [1], Thm. 3.13.

We let

$$O = \bigcup_{k=1}^{\infty} \bigcup_{l\in\mathbb{Z}} \Big(\frac{l}{3^k} - \frac{\varepsilon}{3^{2k}}\,,\,\frac{l}{3^k} + \frac{\varepsilon}{3^{2k}}\Big)\,, \tag{19}$$

and we put

$$g = \chi_{(0,1)\backslash O}\,. \tag{20}$$

It is easy to see that $O \cap (0,1)$ has Lebesgue measure $< \varepsilon$, whence the measure of the set of all t with $g(t) = 1$ exceeds $1-\varepsilon$. Also, the set O is open and dense in $\mathbb{R}$.

Now let $a > 0$, and take a non-empty interval $I_0 = (c,d) \subset (0,1) \cap O$ with $d < a$. Next, let I_1 be a non-empty open interval contained in $(a+I_0)\cap O$. Next, let I_2 be a non-empty open interval contained in $(a+I_1)\cap O$, etc. We continue this process until we find the first open interval, I_{m+1}, which is entirely contained in $(1,\infty)$. Then we set

$$K=(I_m\cap(0,1))-ma\,. \tag{21}$$

This K is a non-empty open interval while $t+na\in O$ for $n=0,1,...,m\,,\ t\in K$, and $t+na\notin(0,1)$ when $n\neq 0,1,...,m\,,\, t\in K$. Hence

$$\sum_{n\in\mathbb{Z}}|g(t+na)|^2=0\,,\qquad t\in K\,, \tag{22}$$

so that g has frame lower bound 0 for the parameters a, b by (18), as required.

We next present a g, bounded and supported by $(0,1)$, such that g has frame lower bound 0 for all $a=l\cdot 3^{-k}$, $l,k\in\mathbb{N}$, and all b, $0<b<1$, while g has frame lower bound $\geq\ 1$ for all $a=(2m)^{-1}$, $m\in\mathbb{N}$, and all b, $0<b<1$. Again, we use (18). Hence is it sufficient to check for the g we give below that

(i) all points $l\cdot 3^{-k}$ are Lebesgue points of g with $g(l\cdot 3^{-k})=0$,
(ii) there holds

$$|g(t)|^2+|g(t+\tfrac{1}{2})|^2\geq 1\,,\qquad t\in(0,\tfrac{1}{2})\,. \tag{23}$$

We let O as in (19) with $\varepsilon=\frac{1}{2}$, we put

$$S=O\cap(0,1)\,,\qquad S^*=(0,1)\backslash O\,, \tag{24}$$

and we set $g=\chi_T$, where

$$T=S^*\cup[(\tfrac{1}{2}+S)\cap S^*]\cup\Big[\Big((\tfrac{1}{2}+S)\cap S\Big)-\tfrac{1}{2}\Big]\,. \tag{25}$$

The rationale behind choosing this g is as follows. We start with the g of the previous example so that (i) certainly holds. In order to achieve that (23) holds, we would like to add to S^* all points of the form $t+\frac{1}{2}$ with $t\in(0,\frac{1}{2})\cap S$. However, this destroys (i) for all $l\cdot 3^{-k}$ that are in $(\frac{1}{2}+S)\cap(0,1)$, and so we add $(\frac{1}{2}+S)\cap S^*$, rather than $\frac{1}{2}+S$, to S^*. However, then for the g thus obtained, (23) is violated for all $t\in S\cap(0,\frac{1}{2})$ for which $t+\frac{1}{2}\in S$ as well. Hence we finally add to $S^*\cup[(\frac{1}{2}+S)\cap S^*]$ the set $\Big((\frac{1}{2}+S)\cap S\Big)-\frac{1}{2}$, to obtain the set T in (25). We shall now check (i) and (ii).

As to (i) we start by noting that $\frac{1}{2}$ has ternary representation

$$\tfrac{1}{2}=\sum_{k=1}^{\infty}3^{-k}\,. \tag{26}$$

We have depicted the set S with two detail pictures around $\frac{1}{2}$, displaying the intervals $J_{l,k}$ that occur at the right-hand side of (19) of order $k = 1$, $k = 1, 2$, $k = 2, 3$ in Fig. 1.a, b, c, respectively. We first observe that for any $l \cdot 3^{-k} \in (\frac{1}{2}, 1)$ the interval $J_{l,k} \subset (\frac{1}{2}, 1)$ has empty intersection with T. This is so since $\Big((\frac{1}{2} + S) \cap S\Big) - \frac{1}{2}$ is contained in $(0, \frac{1}{2})$. As to the other intervals, let us consider the point $\frac{1}{3}$ contained in the interval $J = J_{1,1} = (\frac{5}{18}, \frac{7}{18})$ as an example. (The reasoning that follows is valid for the other intervals $J_{l,k} \subset (0, \frac{1}{2})$ with no essential changes.) We have pictured in Fig. 2 the situation for this J, displaying the interval J in Fig. 2.a, the set $(\frac{1}{2} + J) \cap S$ as far as subintervals of S of order $k = 2, 3, 4$ are concerned in Fig. 2.b, and the corresponding portion $J' = J \backslash \Big(\Big((\frac{1}{2} + J) \cap S\Big) - \frac{1}{2}\Big)$ of the interval J that lies outside T. It is a simple matter to show that $(\frac{1}{2} + J) \cap S$ has density 0 at $\frac{5}{6}$ (i.e. $(2\varepsilon)^{-1}$ meas $[(\frac{5}{6} - \varepsilon, \frac{5}{6} + \varepsilon) \cap (\frac{1}{2} + J) \cap S] \to 0$ as $\varepsilon \downarrow 0$), the ratio of the length of the subintervals of S and the distance of the midpoint of such an interval tending to 0 exponentially fast, and also that $\frac{5}{6} \notin (\frac{1}{2} + J) \cap S$. Hence $\frac{1}{3}$ is a point of $S \backslash T$ of density 1. Hence we conclude, the point $\frac{1}{3}$ being typical, that all $l \cdot 3^{-k}$, $l, k \in \mathbb{N}$, are points of $S \backslash T$ of density 1. This shows (i).

To show (ii), we shall prove that for any $t \in (0, \frac{1}{2})$ at least one of t and $t + \frac{1}{2}$ is in T. Indeed, take $t \in (0, \frac{1}{2})$. When $t \in S^*$, we are done, so suppose that $t \in S$. When $t + \frac{1}{2} \in S^*$, we are done as well, so we assume that $t \in S, t + \frac{1}{2} \in S$. Then $t = (\frac{1}{2} + t) - \frac{1}{2} \in \Big((\frac{1}{2} + S) \cap S\Big) - \frac{1}{2} \subset T$, so we are done too. This completes the proof.

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Figure captions

Fig. 1. The set S as given in (24) with O of (19) with $\varepsilon = \frac{1}{2}$ and a) intervals $J_{l,k} \subset S$ of order $k = 1$, b) intervals $J_{l,k} \subset S$ of order $k = 1, 2$ as far as they lie in $(\frac{1}{3}\,,\,\frac{2}{3})$, c) intervals $J_{l,k} \subset S$ of order $k = 2, 3$ as far as they lie in $(\frac{4}{9}\,,\,\frac{5}{9})$.

Fig. 2. a) The set $J = J_{1,1} \subset S$, b) the set $(\frac{1}{2} + J) \cap S$ as far as intervals $J_{l,k} \subset S$ of order $k = 2, 3, 4$ are concerned, c) the corresponding portion $J' = J \backslash \left(\left((\frac{1}{2} + J) \cap S \right) - \frac{1}{2} \right)$ of the interval J that lies outside T.

Figure 1.

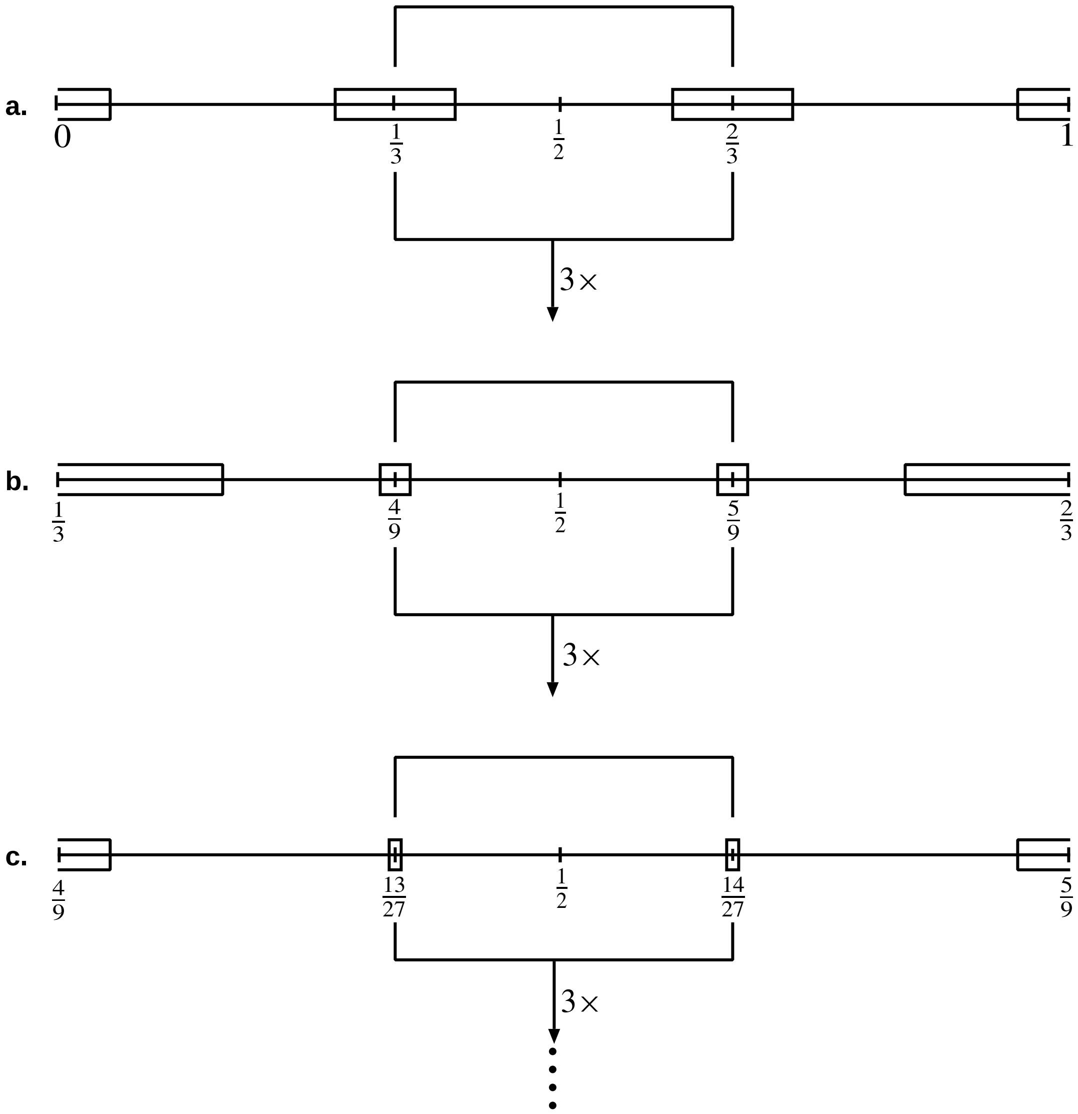

Figure 2.