Proof of a conjecture on the supports of Wigner distributions

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Abstract.
In this note we prove that the Wigner distribution of an $f \in L^2(\mathbb{R}^n)$ cannot be supported by a set of finite measure in $\mathbb{R}^{2n}$ unless $f = 0$. We prove a corresponding statement for cross-ambiguity functions. As a strengthening of the conjecture we show that for an $f \in L^2(\mathbb{R}^n)$ its Wigner distribution has a support of measure 0 or ∞ in any half-space of $\mathbb{R}^{2n}$.

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1. Introduction

Benedicks [1] has shown that when $f \in L^1(\mathbb{R}^n)$ satisfies

(1) $$|\Sigma(f)|\,|\Sigma(\hat{f})| < \infty \ ,$$

then $f = 0$. Here $\Sigma(f)$ is the set $\{x \mid f(x) \neq 0\}$, $\Sigma(\hat{f})$ is the set $\{y \mid \hat{f}(y) \neq 0\}$ with

(2) $$\hat{f}(y) = (\mathcal{F}f)(y) = \int e^{-2\pi i x \cdot y} f(x)\, dx \ , \qquad y \in \mathbb{R}^n \ ,$$

the Fourier transform of f, and $|\ |$ denotes Lebesgue measure. We refer to [2], Section 7, for historical notes and further comments on this theorem.

It is conjectured in [2], Section 7, that when $f \in L^2(\mathbb{R}^n)$ and $|\Sigma\,(W(f,f))| < \infty$, then $f = 0$. Here

(3) $$\Sigma\,(W(f,f)) = \{(t,\nu) \in \mathbb{R}^{2n} \mid W(f,f)(t,\nu) \neq 0\} \ ,$$

and $W(f,f)$ is the Wigner distribution of f. When $f, g \in L^2(\mathbb{R}^n)$, we define the Wigner transform of f and g as

(4) $$W(f,g)(t,\nu) = \int e^{-2\pi i\nu\cdot x} \, f(t+\tfrac{1}{2}x)\, g^*(t-\tfrac{1}{2}x)\, dx \;, \qquad t,\nu \in \mathbb{R}^n \;,$$

and when $f = g$ in (4), we speak of the Wigner distribution of f. We refer to [2], Section 6, and [3], Ch. 1, Secs. 4, 8, for more information on Wigner distributions, and to [2], Sec. 7, for some partial results regarding the conjecture. The author was kindly informed by G.B. Folland that the conjecture occurred during discussions between D. Mustard and A. Sitaram who noticed, for instance, that one must have $|\sum (W(f,f))| = \infty$ when $f \in L^2(\mathbb{R}^n)$ is even or odd. It was furthermore observed by Mustard, see [2], Sec. 7, that (as $W(f,f) \in L^1(\mathbb{R}^{2n})$ when $f \in L^2(\mathbb{R}^n)$ and $|\Sigma(W(f,f))| < \infty$) one could use Benedicks' theorem with $f \in L^1(\mathbb{R}^n)$ replaced by $W(f,f) \in L^1(\mathbb{R}^{2n})$ once one would know that $|\Sigma(\mathcal{F}(W(f,f)))| < \infty$ as well.

2. Proof of the conjecture

We shall now present a proof of the conjecture. It is based on the following formula,

(5) $$\iint W(f_1,f_2)(t,\nu)\, W^*(g_1,g_2)(s-t,\mu-\nu)\, e^{-2\pi i a\cdot t + 2\pi i b\cdot\nu}\, dt\, d\nu =$$
$$= \frac{1}{4^n}\, e^{-\pi i a\cdot s + \pi i b\cdot\mu}\, W(f_1,g_1)(\tfrac{1}{2}s + \tfrac{1}{4}b, \tfrac{1}{2}\mu + \tfrac{1}{4}a)\, W^*(f_2,g_2)(\tfrac{1}{2}s - \tfrac{1}{4}b, \tfrac{1}{2}\mu - \tfrac{1}{4}a) \;,$$

valid for $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^n)$ and $s,\mu,a,b \in \mathbb{R}^n$. This formula, see Sec. 3, is a straightforward consequence of Moyal's formula, see [3], Ch. 1, Sec. 8, (1.93), and a special case, viz. where $f_1 = f_2 = g_1 = g_2$ and $a = b = 0$, has been used already in [4], Sec. 6, (75), to study the interference phenomena present in Wigner distributions. Also, formula (5) can be regarded as a generalization of Siebert's self-transform property [5] for ambiguity functions, see (8) and (9) below, for which the choice $f_1 = g_2 = f$, $f_2 = g_1 = \tilde{f}$, $s = \mu = 0$ must be made. Formula (5) was found by F. Hlawatsch in 1986, see [6], [7], formula (7.77–78) and also [8], and, independently, by A.H. Nuttall in 1989 (private communication to the author, [9]).

To prove the conjecture, we let $s,\mu \in \mathbb{R}^n$ be fixed, and we choose $f_1 = f_2 = g_1 = g_2 = f \in L^2(\mathbb{R}^n)$ in (5), so that

(6) $$\iint W(f,f)(t,\nu)\, W^*(f,f)(s-t,\mu-\nu)\, e^{-2\pi i a\cdot t + 2\pi i b\cdot\nu}\, dt\, d\nu =$$
$$= \frac{1}{4^n}\, e^{-\pi i a\cdot s + \pi i b\cdot\mu}\, W(f,f)(\tfrac{1}{2}s + \tfrac{1}{4}b, \tfrac{1}{2}\mu + \tfrac{1}{4}a)\, W^*(f,f)(\tfrac{1}{2}s - \tfrac{1}{4}b, \tfrac{1}{2}\mu - \tfrac{1}{4}a)$$

for all $a, b \in \mathbb{R}^n$. The function

(7) $$\chi(t,\nu) = W(f,f)(t,\nu)\, W^*(f,f)(s-t,\mu-\nu) \;, \qquad t,\nu \in \mathbb{R}^n \;,$$

which is in $L^1(\mathbb{R}^{2n})$ since $W(f,f) \in L^2(\mathbb{R}^{2n})$, has a support of finite measure when $|\Sigma(W(f,f))| < \infty$, and so has its Fourier transform as we see from (6). Hence Benedicks' theorem yields that $\chi = 0$. Since $s, \mu \in \mathbb{R}^n$ are arbitrary, it follows easily that $W(f,f) = 0$, i.e. $f = 0$.

When we take $f, g \in L^2(\mathbb{R}^n)$, and we define the cross-ambiguity function $A(f,g)$ of f and g by

$$A(f,g)(p,q) = \int e^{2\pi i q\cdot y}\, f(y + \tfrac{1}{2}\,p)\, g^*(y - \tfrac{1}{2}\,p)\, dy\ , \qquad p,q \in \mathbb{R}^n\ , \tag{8}$$

then it follows in a similar fashion that $f = 0$ or $g = 0$ whenever $|\Sigma\,(A(f,g))| < \infty$. For this we use the fact

$$W(f,\tilde{g})(u,v) = 2^n\, A(f,g)(2u,2v) = W^*(\tilde{g},f)(u,v)\ , \tag{9}$$

valid for all $u, v \in \mathbb{R}^n$, where we have set $\tilde{g}(x) = g(-x)$, $x \in \mathbb{R}^n$, together with formula (5) in which $f_1 = g_2 = f$, $f_2 = g_1 = \tilde{g}$ is taken, to conclude that $A(f,g) = 0$ whenever $|\Sigma\,(A(f,g))| < \infty$. Since $A(f,g) = 0$ if and only if $f = 0$ or $g = 0$ we get the result. Obviously, we now also have that $f \in L^2(\mathbb{R})$ is the null function whenever its short-time Fourier transform $S_g\, f = |A(f,g)|^2$, using a window $0 \neq g \in L^2(\mathbb{R})$, vanishes outside a set of finite measure.

The argument used to prove the conjecture gives somewhat more. The extension of the conjecture given below is significant since it shows, for instance, that supporting sets of Wigner distributions cannot have finite-measure protrusions.

Corollary. Assume that $f \in L^2(\mathbb{R}^n)$, and let H be any half-space in $\mathbb{R}^{2n}$. Then

$$|\{(t,\nu) \in H \,|\, W(f,f)(t,\nu) \neq 0\}| = 0 \ \text{ or } \ \infty\ . \tag{10}$$

Proof. We start by noting that we can write (6) as

$$(\mathcal{F}_t\, \mathcal{F}_\nu\, \chi)(a,-b) = \frac{1}{4^n}\, e^{-\pi i a\cdot s + \pi i b\cdot \mu}\, \chi(\tfrac{1}{2}\,s + \tfrac{1}{4}\,b, \tfrac{1}{2}\,\mu + \tfrac{1}{4}\,a) \tag{11}$$

with χ given for $s, \mu \in \mathbb{R}^n$ by (7) and $\mathcal{F}_t$, $\mathcal{F}_\nu$ denoting Fourier transforms with respect to the respective variables.
By using symplectic transformations, see [3], Ch. 4, Sec. 2, we can assume that H is given by

$$H = \{(t,\nu) \,|\, t = (t_1, ..., t_n) \in \mathbb{R}^n,\ t_1 < 0,\ \nu \in \mathbb{R}^n\}\ . \tag{12}$$

Suppose that the set in (10) with H in (12) has finite measure; we must show that it has measure 0. For any $s = (s_1, ..., s_n) \in \mathbb{R}^n$, $s_1 < 0$, $\mu \in \mathbb{R}^n$ we have that χ of (7) has support of finite measure since at least one of t_1 and $s_1 - t_1$ is negative when $t_1 \in \mathbb{R}$. Hence (11) and Benedicks' theorem yields that

$$W(f,f)(t,\nu)\, W^*(f,f)(s-t,\mu-\nu) = 0\ , \qquad t,\nu \in \mathbb{R}^n\ , \tag{13}$$

with arbitrary $s = (s_1, ..., s_n) \in \mathbb{R}^n$, $s_1 < 0$, $\mu \in \mathbb{R}^n$. Now when $(t, \nu) \in H$, we use (13) with $s = 2t$, $\mu = 2\nu$ to conclude that $W(f, f)(t, \nu) = 0$. Hence $W(f, f)$ vanishes on H, as required.

We finally note that the Corollary remains valid, with essentially the same proof using (9), when we replace $W(f, f)$ by $A(f, g)$ in (10) with $g \in L^2(\mathbb{R}^n)$.

3. Proof of formula (5)

For $f \in L^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ we let

(14) $$f_{x,y}(t) = e^{2\pi i y\cdot t} f(t - x) \ , \qquad t \in \mathbb{R}^n \ .$$

Now when $s, t, \mu, \nu, a, b \in \mathbb{R}^n$ and $g_1, g_2 \in L^2(\mathbb{R}^n)$, we have

(15) $$W(g_1, g_2)(s - t, \mu - \nu)\, e^{2\pi i a\cdot t - 2\pi i b\cdot \nu} = W(h_1, h_2)(t, \nu) \ ,$$

where

(16) $$h_1 = ((\tilde{g}_1)_{s,\mu})_{\frac{1}{2}b, \frac{1}{2}a} \ , \qquad h_2 = ((\tilde{g}_2)_{s,\mu})_{-\frac{1}{2}b, -\frac{1}{2}a} \ .$$

Hence, Moyal's formula, see [3], Ch. 1, Sec. 8, (1.93),

(17) $$\iint W(f_1, f_2)(t, \nu)\, W^*(h_1, h_2)(t, \nu)\, dt\, d\nu = \\ = \left(\int f_1(x)\, h_1^*(x)\, dx\right) \left(\int f_2(y)\, h_2^*(y)\, dy\right)^* \ ,$$

yields formula (5) on using

(18) $$\int f_1(x)\, h_1^*(x)\, dx = \\ = (\tfrac{1}{2})^n\, e^{\frac{1}{2}\pi i \mu\cdot b - \pi i \mu\cdot s - \frac{1}{4}\pi i a\cdot b - \frac{1}{2}\pi i a\cdot s}\, W(f_1, g_1)(\tfrac{1}{2}\, s + \tfrac{1}{4}\, b, \tfrac{1}{2}\, \mu + \tfrac{1}{4}\, a)$$

and a similar formula for the second factor at the right-hand side of (17).

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