Analysis of contention tree-algorithms

A. J. E. M. Janssen and M. J. M. de Jong

Abstract— **The Capetanakis-Tsybakov-Mikhailov contention tree-algorithm provides an efficient scheme for multi-accessing a broadcast-communication channel. This article studies statistical properties of multiple-access contention tree-algorithms with ternary feedback for arbitrary degree of node. The particular quantities under investigation are the number of levels required for a random contender to have successful access, as well as the number of levels and the number of contention frames required to provide access for all contenders. Through classical Fourier analysis approximations to both the average and the variance are calculated as a function of the number of contenders n. It is demonstrated that in the limit of large n these quantities do not converge to a fixed mode, but contain an oscillating term as well.**

Keywords— **Broadcast communication, random multiple-access, collision resolution, contention tree-algorithms.**

I. Introduction

The allocation of a single broadcast-communication channel among a large number of independent transmitters usually requires more advanced medium-access protocols than time-division multiple access (TDMA). The reason is that TDMA provides a notoriously low performance with respect to channel utilization, unless all transmitters are continuously transmitting, and with respect to access delay, unless the number of users is low.

Introduced in the early 1970s as a solution to the problem sketched above, the ALOHA protocol yields an elegant scheme to provide immediate random access to the channel [1]. The concept of random access implies that two or more transmitters may be active at the same time, prohibiting error-free reception. If such a collision occurs, the transmitters try again later, each one after a randomly chosen time. However, the performance of the ALOHA protocol becomes very poor, if the channel occupancy increases beyond a certain level.

Basically, there are two strategies to improve the performance of random multiple-access protocols: carrier-sense multiple-access [2] and collision-resolution algorithms [3], [4]. This article studies statistical properties of the basic collision-resolution algorithm: the contention tree-algorithm. The outline is as follows: Section II overviews the development and explains the operation of the contention tree-algorithm. In Secs. III and IV, we investigate the number of levels and the number of contention frames, respectively, required to complete the tree algorithm. We present conclusions in Sec. V. Appendix A and B provide the details of our mathematical analysis. A brief account of this work will be presented in Ref. [5].

A. J. E. M. Janssen and M. J. M. de Jong are with Philips Research Laboratories, 5656 AA Eindhoven, The Netherlands. Correspondence should be addressed to M. J. M. de Jong, e-mail: jongmjm@natlab.research.philips.com. This paper will be presented in part at the IEEE Information Theory Workshop, Killarney, Ireland, June 1998 [5].

II. The contention tree-algorithm

Let us now describe the multiple-access contention tree-algorithm as first reported by Capetanakis [3] and by Tsybakov and Mikhailov [4]. A large number of transmitters (stations, terminals, sources, etc.) share a single, slotted broadcast channel. The transmitters that contend for channel access are able to acquire ternary feedback on what happened during a contention slot, i.e. whether zero transmitters (an *empty* slot), one transmitter (a *success*ful transmission), or more than one transmitter (a *collision*) has been broadcasting during the particular slot. The ternary feedback can either be detected by the stations themselves or by a central controller and is not required to be immediate, i.e. there may be a certain delay between the transmission during the contention slot and the reception of the feedback. Furthermore, the tree has nodal degree $m \geq 2$, and as a consequence (see below) m consecutive contention slots are grouped into a contention frame.

The contention tree-algorithm utilizes the ternary feedback as follows. Let us assume that there are n contending transmitters at the start of a new tree algorithm, i.e. n transmitters want to broadcast a data packet. During the first contention frame, i.e. the frame at the root of the tree, each of the n transmitters picks at random a number (say k) between 1 and m — with equal probabilities — and transmits its packet during the kth contention slot. If after completion of the contention frame the ternary feedback becomes available, each transmitter knows whether its packet has been successfully broadcast or not. If not, a new contention frame is assigned to all transmitters that caused the collision during the particular slot. Therefore, if there were collisions in all contention slots, m new contention frames would become available. This leads to the formation of a tree with nodal degree m. The expansion of the tree stops at either empty or successful slots. Upon completion of the tree algorithm, all the n contenders have successfully broadcast their data. Thereafter a new tree algorithm may start again. To exemplify the contention tree-algorithm, Fig. 1 depicts a possible contention tree for $n = 13$ contenders and $m = 3$ slots per frame. Note that the formation of the tree is a stochastic process, because in each frame each contender picks a slot at random. The contention tree depicted in Fig. 1 is just one realization out of an infinite number of possible trees for $n = 13$ and $m = 3$. The development of the algorithm is thus the result of an interplay between the exponential growth of the contention tree and the random choices made by the contenders.

Capetanakis [3] has shown that in the case of Poisson generated data packets the maximum throughput of the binary ($m = 2$) contention tree-algorithm equals 0.347 packets/slot. Capetanakis also demonstrated that if the number of contention slots in the root frame is allowed to be

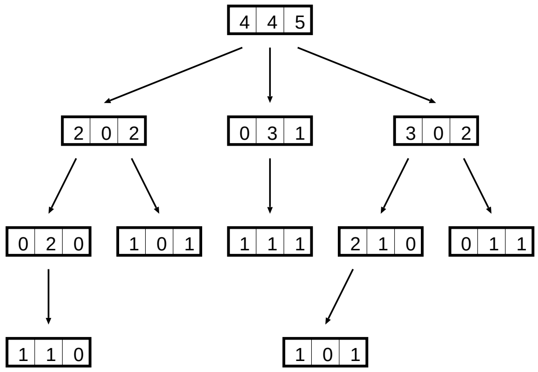

Fig. 1. An example contention tree with $m = 3$ slots per contention frame and $n = 13$ contending transmitters involved. The numbers in each contention slot denote the number of packets being transmitted. If this number is larger than 1 (a collision) a new frame is assigned to the contenders involved. If this number is 0 (an empty slot) or 1 (a successful transmission) the tree expansion stops.

variable, but all other frames still contain 2 slots, the maximum throughput equals 0.430 packets/slot. The most efficient contention tree-algorithm demonstrated so far leads to a maximum capacity of 0.487 packets/slot, again in the case of Poisson generated data packets [6], [7], [8], [9], [10].

A variant of the contention tree-algorithm, the contention stack-algorithm, has been introduced by Tsybakov and Vvedenskaya [11], [12], [13]. In the stack algorithm a transmitter does not need to wait until the tree is completed, but is allowed to contend in the first available contention frame. An advantage is that this eliminates the requirement that all transmitters continuously monitor the status of the channel.

In the above described algorithms the contention process is being executed with the data packets themselves, in other words a successful contention implies that the data is transmitted as well. Alternatively, one can make use of the contention tree-algorithm to make reservations or requests for data transmission. As generally the length of a reservation packet is smaller than the length of a data packet, the reservation contention tree-algorithm may lead to even larger channel utilization [14], [15]. This mechanism is currently being proposed in several network standards [16], [17].

Apart from the channel utilization, the access delay, i.e. the time it takes before the data packet has successfully been transmitted, is also an important performance parameter. In order to minimize the access delay, one must take into account the round-trip delay between the transmission of a packet and the reception of the feedback [3], [18]. If the round-trip delay is negligibly small, it is most advantageous to perform a serial search, in which each branch is fully completed before returning to the root. However, in communication channels with a large round-trip delay, such as metropolitan cable networks and satellite networks, it may be advantageous to perform a parallel tree search, in which all the contention frames at a certain level are executed before proceeding to the next level [3].

In the present article, we investigate statistical properties of the contention tree-algorithm. In particular we study as a function of the number of contenders n and the number of slots in a contention frame (or nodal degree) m the following statistical quantities:

- The number of levels d_n required for a random contender to have successful contention. This number is of importance to calculate the mean access delay in systems with a large round-trip delay. In Fig. 1, d_{13} is either 2, 3, or 4, depending on the contender. The average equals $\frac{42}{13}$.
- The number of levels D_n required to complete the tree. This number is of importance to calculate the duration of the algorithm in systems with a large round-trip delay. In Fig. 1, $D_{13} = 4$.
- The number of contention frames L_n required to complete the tree algorithm. In Fig. 1, $L_{13} = 11$. This quantity determines how much of the channel capacity is needed for the tree algorithm. In the case of a negligible round-trip delay, it determines the duration of the algorithm, as well.

As far as we know, the quantities d_n and D_n have not been studied in detail before. The quantity L_n has been thoroughly investigated from the moment of its introduction [3], [4], [6], [7], [8], [9], [19], [20], [21], [22]. The reason for reinvestigating L_n is that the techniques taken at hand to calculate various statistical properties of d_n and D_n can be readily applied to the quantity L_n. This allows us to confirm and state precisely various results and conjectures presented earlier [8], [20], [21].

The aim of our calculations is to analyze both the expectation values $\overline{d_n}$, $\overline{D_n}$, and $\overline{L_n}$ as well as the variances $\mathrm{var}(d_n)$, $\mathrm{var}(D_n)$, and $\mathrm{var}(L_n)$. Through classical Fourier analysis we have found analytical expressions for these quantities in the limit of large n. In a nutshell, the results can be summarized as follows:

$$\overline{d_n} \simeq \log_m(n-1)\,, \tag{1}$$
$$\overline{D_n} \simeq 2\log_m n\,, \tag{2}$$
$$\overline{L_n} \simeq \frac{n}{\log m}\,, \tag{3}$$

where the logarithm base m is given by $\log_m n \equiv \log n/\log m$ and $\log n \equiv \ln n$. More precise results are presented in Secs. III and IV, with the mathematical details given in Appendix A and B. From comparison with the exact results it follows that the expressions obtained are already quite accurate for rather small values of n. Furthermore, we demonstrate that the averages and variances obtained do not converge for large n to the laws (1), (2), and (3), but contain oscillating terms as well, reflecting the discrete-level nature of the contention tree.

This article considers the statistical properties of the contention tree-algorithm, only. However, our results can be combined with arbitrary traffic models in order to make predictions on the performance of various cases. In many situations, Eqs. (1)–(3) suffice to make back-of-the-envelope estimates on the performance.

III. The number of tree levels

The probability distribution of the number of levels d_n required for a random transmitter to have successful con-

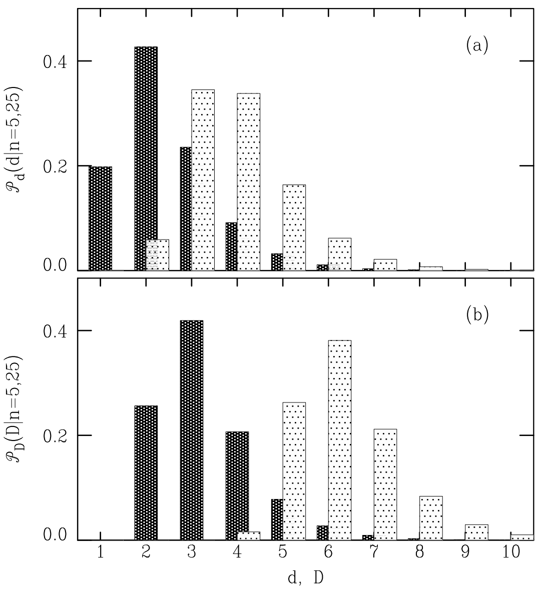

Fig. 2. Probability distribution $\mathcal{P}_d(d|n)$ for d_n, the number of levels required for a random contender to have successful transmission (a) and $\mathcal{P}_D(D|n)$ for D_n, the number of levels required to complete the tree algorithm (b). The heavily shaded bars are for $n = 5$ and the lightly shaded bars for $n = 25$ contenders, both for $m = 3$ slots per contention frame.

tention when n transmitters contend in a tree algorithm with m contention slots per frame reads

$$\mathcal{P}_d(d|n) = \begin{cases} \delta_{1,d} , & n = 1 , \\ a_{n-1}(m^d) - a_{n-1}(m^{d-1}) , & n \geq 2 , \end{cases} \qquad (4)$$

with the Kronecker delta function $\delta_{1,d} = 1$ if $d = 1$ and $\delta_{1,d} = 0$ otherwise. The function $a_n(M)$ is given by

$$a_n(M) = (1 - 1/M)^n , \qquad (5)$$

if $M > 1$, $a_n(M) = 0$ otherwise. Equations (4) and (5) can be understood as follows, where we borrow an argument from Refs. [21], [23]: We consider an infinite, complete tree of nodal degree m. The number of slots in level d amounts to m^d. In the first level, each of the n contenders picks at random one of the m slots. This process is repeated for each subsequent level. As a result, the n contenders in level d are independently and identically, randomly distributed with equal probabilities over the m^d slots. Therefore, the probability that a slot in level d occupied by a random contender is not occupied by any of the $n-1$ other contenders equals $a_{n-1}(m^d)$. The difference between $a_{n-1}(m^d)$ and $a_{n-1}(m^{d-1})$ equals the probability that the random contender requires precisely d levels to be the single occupant of a contention slot. The fact that in the implementation of the contention tree-algorithm the tree is not expanded upon an empty or successful slot, does not change this argument.

Similarly, we have for the probability distribution of the number of levels D_n required to complete the tree,

$$\mathcal{P}_D(D|n) = \begin{cases} \delta_{1,D} , & n = 1 , \\ b_n(m^D) - b_n(m^{D-1}) , & n \geq 2 , \end{cases} \qquad (6)$$

where $b_n(m^D)$ denotes the probability that all n contenders in level D occupy different slots:

$$b_n(M) = \frac{M!}{(M-n)!M^n} . \qquad (7)$$

if $M \geq n$, $b_n(M) = 0$ otherwise. Figure 2 provides a histogram of $\mathcal{P}_d(d|n)$ and $\mathcal{P}_D(D|n)$ for $n = 5, 25$ and $m = 3$. Indeed, it takes a few levels more to have successful transmission for all contenders than for a random contender.

Given the probability distribution $\mathcal{P}_d(d|n)$ one can readily calculate the average,

$$\overline{d_n} \equiv \sum_{d=1}^{\infty} d\, \mathcal{P}_d(d|n) , \qquad (8)$$

the second moment,

$$\overline{d_n^2} \equiv \sum_{d=1}^{\infty} d^2\, \mathcal{P}_d(d|n) , \qquad (9)$$

and the variance,

$$\mathrm{var}(d_n) \equiv \overline{d_n^2} - \overline{d_n}^2 . \qquad (10)$$

Similarly, $\overline{D_n}$, $\overline{D_n^2}$, and $\mathrm{var}(D_n)$ can be evaluated.

The moments and variance of d_n and D_n as a function of n and m can be computed up to arbitrary precision. However, much more insight in the tree algorithm can be obtained if the general behavior of $\overline{d_n}$, $\mathrm{var}(d_n)$, $\overline{D_n}$, and $\mathrm{var}(D_n)$ is known as a function of n and m. In Appendix A, we derive accurate analytical approximations for these values in the limit of large n. In short, our derivation for d_n proceeds as follows. Firstly, we expand the function $a_n(m^p)$ in a series with terms $n^{-k} f_k(n/m^p)$. Secondly, we note that $f_k(n/m^p) - f_k(n/m^{p-1}) \simeq 0$, except when $m^p \simeq n$, so that the summation in Eqs. (8) and (9) can be replaced by a summation from $-\infty$ to ∞. Thirdly, we utilize classical Fourier analysis to approximate the summation up to the aimed accuracy. This approach can be directly applied to D, as well. As expected, we find a logarithmic dependence on n for both $\overline{d_n}$ and $\overline{D_n}$. However, around this "DC-value" there is a Fourier series of oscillations. For $m \lesssim 20$ only the first oscillation is significant, the others being exponentially small.

Let us now quote our results. From Eqs. (64), (65), and (76) we have

$$\overline{d_n} = \tilde{d}_n + \delta d_n + \mathcal{O}\left(n^{-2}\right) . \qquad (11)$$

The first term on the right-hand side (r.h.s.) denotes the "DC-value," which is given by

$$\tilde{d}_n = \log_m(n-1) + \left(\frac{1}{2} + \frac{\gamma}{\log m}\right) + \frac{1}{2n \log m} , \qquad (12)$$

with Euler's constant $\gamma \approx 0.5772$. We find an approximate logarithmic dependence of $\overline{d_n}$ as a function of n, as is not unexpected for the tree algorithm. Multiplying the

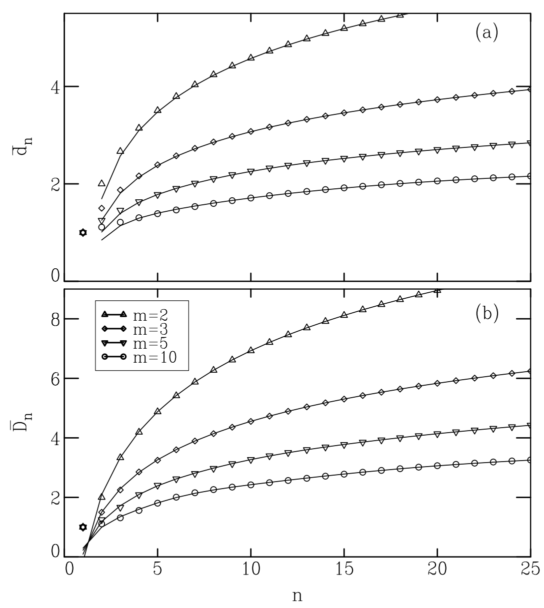

Fig. 3. The average $\overline{d_n}$ of the number of levels required for a random contender to have successful transmission (a) and the average $\overline{D_n}$ of the number of levels required to complete the tree algorithm (b) as a function of the number of contenders n. The symbols denote the exact value $\overline{d_n}$ and $\overline{D_n}$ calculated from Eqs. (4), (8) and (6), the lines the approximation $\widetilde{d}_n + \delta d_n$ and $\widetilde{D}_n + \delta D_n$ according to Eqs. (12), (13) and (15), (16), respectively.

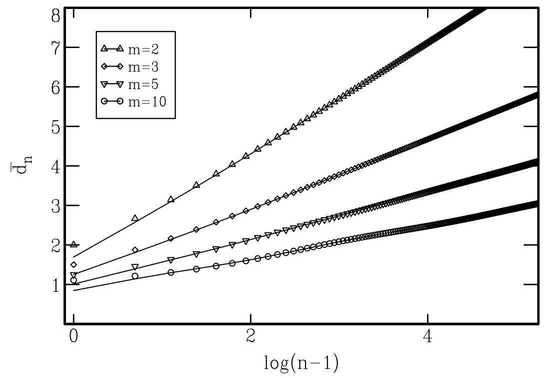

Fig. 4. Same as Fig. 3a, but now on a logarithmic scale.

number of contenders with a factor of m, leads to an increase of 1 required level. Around this "DC-value" there are oscillations given by

$$\delta d_n = \kappa_1 \sin[2\pi \log_m (n-1) - \phi_1] \,, \tag{13}$$

where κ_1 and ϕ_1 are defined according to Eq. (65). The values of κ_1 and ϕ_1 are enlisted in Table I for several values of m. The magnitude of the oscillation increases with m.

Figure 3a compares the approximation $\widetilde{d}_n + \delta d_n$ with the exact value $\overline{d_n}$. We note that our large-n approximation is remarkably accurate. Already, for $n = 5$ the deviation is below 1%. Figure 4 displays the same data but now versus $\log(n-1)$. Indeed, one notices the logarithmic behavior, but there is a deviation from this behavior as can be clearly observed. This is due to the $1/n$ term in Eq. (12) and the oscillation (13).

The oscillation in $\overline{d_n}$ can be studied in more detail in

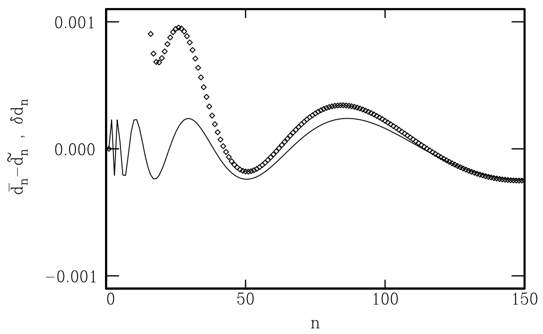

Fig. 5. The oscillations in $\overline{d_n}$, the average number of levels required for a random contender to have successful transmission, as a function of the number of contenders n for $m = 3$ contention slots per frame. The symbols denote $\overline{d_n} - \widetilde{d}_n$ according to Eqs. (8) and (12) and the line denotes δd_n according to Eq. (13).

Fig. 5, which compares $\overline{d_n} - \widetilde{d}_n$ with δd_n. Note, that for $m = 3$ the oscillation is at least three orders of magnitude smaller than the "DC-value," and is therefore barely visible in Figs. 3a and 4. The approximation becomes excellent when n increases. The existence of these oscillations is a consequence of the discreteness of the number of levels in the contention tree, whereas the exponential increase of the period with n reflects the nature of the tree expansion.

Similarly to $\overline{d_n}$, we find for the approximation of $\overline{D_n}$ using Eq. (78) instead of Eq. (76)

$$\overline{D_n} = \widetilde{D}_n + \delta D_n + \mathcal{O}\left(n^{-2}\right) \,, \tag{14}$$

where the "DC-value" is given by

$$\widetilde{D}_n = 2\log_m n + \left(\frac{1}{2} + \frac{\gamma - \log 2}{\log m}\right) - \frac{1}{3n \log m} \,. \tag{15}$$

It takes approximately twice as much levels to have successful transmission for all contenders than for a random contender. The oscillation δD_n around $\widetilde{D}_n$ has the same amplitude as δd_n, at — approximately — a doubled frequency

$$\delta D_n = \kappa_1 \sin[2\pi \log_m (n^2/2) - \phi_1] \,. \tag{16}$$

Figure 3b plots $\overline{D_n}$ as a function of n.

The variance of d_n and D_n follows from Eqs. (66) and (67), in combination with Eq. (77)and (79), respectively,

$$\begin{aligned} \mathrm{var}(d_n) &= \widetilde{v}_{d_n} + \delta v_{d_n} + \mathcal{O}\left(n^{-2}\right) \,, & (17)\\ \mathrm{var}(D_n) &= \widetilde{v}_{D_n} + \delta v_{D_n} + \mathcal{O}\left(n^{-2}\right) \,. & (18)\end{aligned}$$

Again, we have found a "DC-value" of magnitude

$$\begin{aligned} \widetilde{v}_{d_n} &= \frac{1}{12} + \frac{\pi^2}{6\log^2 m} - \frac{1}{n \log^2 m} \,, & (19)\\ \widetilde{v}_{D_n} &= \frac{1}{12} + \frac{\pi^2}{6\log^2 m} - \frac{4}{3n \log^2 m} \,, & (20)\end{aligned}$$

and an oscillation around this value according to

$$\delta v_{d_n, D_n} = -\kappa_2 \sin(2\pi z - \phi_2) \,, \tag{21}$$

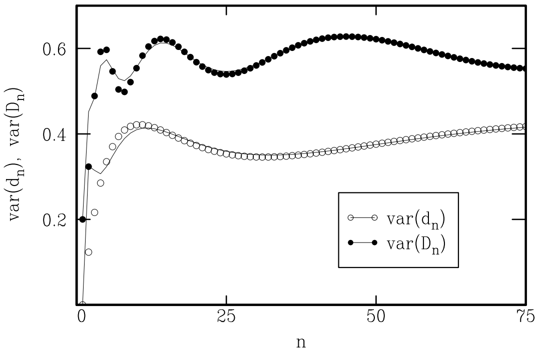

Fig. 6. The variances var(d_n) and var(D_n) as a function of n for $m = 10$. Symbols denote the exact value calculated from Eqs. (4)–(10), lines the approximation according to Eqs. (19)–(21). For clarity, var(D_n) has been offset by a value of 0.2.

where $z = \log_m(n-1)$ for δv_{d_n} and $z = \log_m(n^2/2)$ for δv_{D_n}. The parameters κ_2 and ϕ_2 are defined in Eq. (67), see also Table I.

We find that var$(d_n) \simeq$ var(D_n). This is not unexpected, since D_n is in fact the largest value of the n values of d_n, for each realization of the tree algorithm. For large n, we have

$$\lim_{n\to\infty} \widetilde{v}_{d_n} = \lim_{n\to\infty} \widetilde{v}_{D_n} \equiv \widetilde{v}_{d,D} = \frac{1}{12} + \frac{\pi^2}{6\log^2 m} . \quad (22)$$

Values are displayed in Table I. The variances var(d_n) and var(D_n) are plotted in Fig. 6. for $m = 10$ (the oscillations are less prominent for smaller m). Note, the difference in the period of the oscillation.

IV. The number of contention frames

Let us now address the number of contention frames L_n required to complete the contention tree-algorithm with n contenders. This quantity has been studied extensively upon the introduction of the contention tree-algorithm itself [3], [4], [6], [7], [8], [9], [14], [19], [20], [21], [22]. We note that the exact definition of L_n varies a bit from author to author. The differences are due to whether the root frame consists of 1 or m contention slots and to whether the number of contention slots are counted instead of the number of contention frames. We will follow the definition as given in Ref. [21], which corresponds to counting the number of contention frames. For the tree of Fig. 1 this implies $L_{13} = 11$. The expectation value of L_n can be expressed recursively according to [8], [22]

$$\overline{L_n} = \left(1 - m^{1-n}\right)^{-1} \left[1 + \sum_{k=2}^{n-1} \binom{n}{k} \frac{(m-1)^{n-k}}{m^{n-1}} \overline{L_k}\right] , \quad (23)$$

for $n \geq 3$. The first 2 values are $\overline{L_1} = 1$, $\overline{L_2} = m/(m-1)$. It has been demonstrated in Refs. [8], [21] that $\overline{L_n}$ increases proportionally with n. It has been suggested by Massey [8] that for a binary tree ($m = 2$) the constant of proportionality equals $1/\log 2$. However, this suggestion was rebutted by the observation that $\overline{L_n}/n$ does not really converge to a fixed value, but rather oscillates weakly around some value [8], [21]. Below we reinvestigate this issue and obtain exact expressions for the "DC-value" as well as the magnitude and the phase of the oscillation.

Equation (23) allows easy calculation of the values of $\overline{L_n}$. However for further analysis it is more convenient to start from the expression by Kaplan and Gulko [21]:

$$\overline{L_n} = 1 + \sum_{p=1}^{\infty} c_n(m^p) , \quad (24)$$

where the function $c_n(M)$ is given by

$$c_n(M) = M\left[1 - (1 - 1/M)^n\right] - n(1 - 1/M)^{n-1} . \quad (25)$$

The term $c_n(m^p)$ in the summation of Eq. (24) equals the expected number of contention slots with collisions in level p and thus the expected number of contention frames in the next level.

In Appendix B, we analyze and approximate the infinite series (24) in a similar fashion as used for the evaluation of d_n and D_n. The result can be written as

$$\overline{L_n} = \widetilde{L}_n + \delta L_n + \mathcal{O}\left(n^{-1}\right) , \quad (26)$$

where $\widetilde{L}_n$ denotes the "DC-value," given by

$$\widetilde{L}_n = \frac{n}{\log m} - \frac{1}{m-1} . \quad (27)$$

Note, that $\widetilde{L}_n$ increases linearly with n. Indeed, this result confirms and generalizes the conjecture by Massey [8] and the results by Mathys and Flajolet [20] that the constant of proportionality equals $1/\log m$. The fluctuation around this linear behavior can be approximated with excellent accuracy by

$$\delta L_n = n\lambda_1 \cos\left(2\pi \log_m n + \theta_1\right) + \lambda_2 \sin\left(2\pi \log_m n + \theta_2\right) , \quad (28)$$

where the parameters are given by

$$\lambda_1 = 2\sqrt{\frac{2\pi^2/\log m}{(4\pi^2 + \log^2 m)\sinh(2\pi^2/\log m)}} , \quad (29)$$

$$\theta_1 = \arg\left[\frac{\Gamma(1 - 2\pi i/\log m)}{1 + 2\pi i/\log m}\right] , \quad (30)$$

$$\lambda_2 = \frac{2\pi}{\log^2 m}\sqrt{\frac{2\pi^2/\log m}{\sinh(2\pi^2/\log m)}} , \quad (31)$$

$$\theta_2 = \arg\left[\Gamma(1 - 2\pi i/\log m)\right] . \quad (32)$$

Note that $\theta_2 = -\phi_1$. In Table II we display the numerical value of these parameters for selected values of m. We remark that, similar to $\widetilde{L}_n$, the quantity δL_n contains a part which rises linearly with n as well. However, for not too large values of m the amplitude of the fluctuation λ_1 is at least an order of magnitude smaller than the leading coefficient $1/\log m$. The limiting behavior of $\overline{L_n}/n$ for $m = 2$ is in complete agreement with the numerical values found by Kaplan and Gulko [21].

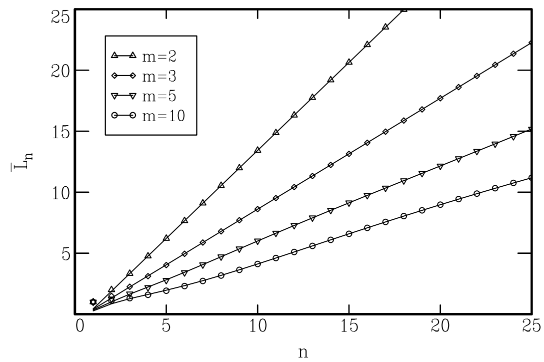

Fig. 7. The average number of contention frames versus the number of contenders n for $m = 2, 3, 5, 10$ contention slots per frame. The symbols denote the exact value $\overline{L_n}$ according to Eq. (23), the lines the approximation $\widetilde{L}_n + \delta L_n$ according to Eqs. (27) and (28).

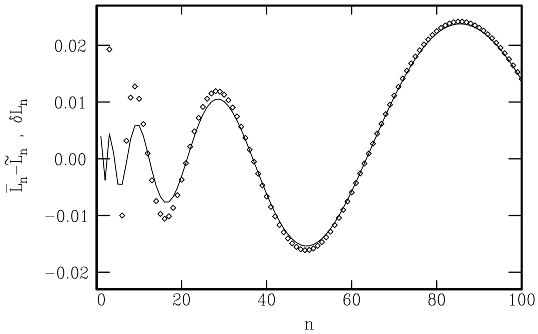

Fig. 8. The oscillations in the average number of contention frames $\overline{L_n}$ as a function of the number of contenders n for $m = 3$ contention slots per frame. The symbols denote $\overline{L_n} - \widetilde{L}_n$ according to Eqs. (23) and (27) and the line denotes δL_n according to Eqs. (28).

Figure 7 compares the exact values of $\overline{L_n}$ as calculated from Eq. (23) with our approximation $\widetilde{L}_n + \delta L_n$ calculated from Eqs. (27) and (28). Deviations are only observable for a small number of contenders $n \leq 2$ and are smaller than 1% if $n \geq 3, 3, 6, 7$ for the values of $m = 2, 3, 5, 10$, respectively. Note further that the oscillating behavior is barely visible on this plot. The oscillation is depicted in more detail by Fig. 8, which compares the exact value minus the "DC-value" $\overline{L_n} - \widetilde{L}_n$ with the "AC-approximation" δL_n for $m = 3$. Note, that the vertical scale is 3 orders of magnitude smaller than in Fig. 7. Clearly, the approximation becomes better upon increasing n. More details upon the accuracy of Eqs. (25)–(28) can be found in Appendix B.

Finally, we address the variance of the number of contention frames. In Ref. [21] an expression is given for $\mathrm{var}(L_n)/n$ in the limit of large n. Again, it was found from numerical evaluation that this value does not converge to a fixed value, but oscillates with small amplitude around a "DC-value." In Appendix B, we derive from the expression in Ref. [21] an analytical approximation for the variance, again using classical Fourier analysis. The result can be written as

$$\lim_{j\to\infty,\, n=lm^j} \frac{\mathrm{var}(L_n)}{n} = \widetilde{v}_L + \delta v_L(l)\,, \tag{33}$$

where l is fixed. The second term $\delta v_L(l)$ oscillates as a function of l. We have only explicitly evaluated the "DC-value" $\widetilde{v}_L$ since the exact magnitude of the oscillation, though computable, is not relevant and very small. The result is:

$$\widetilde{v}_L = \frac{2}{\log m} \sum_{r=1}^{\infty} \frac{1}{1+m^r} + \frac{1}{2\log m} - \frac{1}{(\log m)^2}\,. \tag{34}$$

The summation can be easily carried out numerically. The magnitude of $\widetilde{v}_L$ agrees with the numerical values given in Refs. [8], [21]. In Table II the number $\widetilde{v}_L$ is given for some values of m. As expected, the variance decreases upon increasing number of contention slots per frame.

V. Conclusions

We have analyzed properties of the contention tree-algorithm for multi-accessing a broadcast-communication channel as a function of the number of contenders n and the number of contention slots per frame m. The quantities under study are the number of levels d_n required for a random contender to have successful access, as well as the number of levels D_n and the number of contention frames L_n required to complete the tree algorithm. These quantities are of importance for the evaluation of the performance of the contention-tree protocol in communication channels with both low and high round-trip delays.

We have presented the probability distribution of d_n and D_n, which enables us to determine various statistical quantities, such as the average $\overline{d_n}, \overline{D_n}$ and the variance $\mathrm{var}(d_n)$, $\mathrm{var}(D_n)$. Through classical Fourier analysis we have derived accurate, analytical approximations for these quantities. Both $\overline{d_n}$ and $\overline{D_n}$ increase logarithmically with n. Around this increase there is a small oscillation with exponentially increasing period which reflects the discrete-level nature of the contention tree. The amplitude increases with m. In addition, it is found that $\mathrm{var}(d_n) \approx \mathrm{var}(D_n)$ apart from similar oscillations.

Starting from expressions given by Kaplan and Gulko [21], the average $\overline{L_n}$ and variance $\mathrm{var}(L_n)$ have been evaluated as well. This has allowed us to confirm the conjecture by Massey [8] and the results by Mathys and Flajolet [20] that $\overline{L_n}$ increases linearly with n with constant of proportionality equal to $1/\log m$. This surmise was under debate because it had been found numerically that $\overline{L_n}/n$ does not converge to a fixed value, but rather oscillates. We have identified this oscillation as well.

Appendix A: Details for Section III

In this Appendix we derive the results given in Section III for the expectation value and the variance of the number levels in the tree algorithm required for a random contender and for all the contenders. The results follow from the approximation and the (approximate) Fourier analysis of the quantities

$$\overline{d_n^k} \equiv \sum_{p=1}^{\infty} p^k \left[a_{n-1}(m^p) - a_{n-1}(m^{p-1})\right] , \quad (35)$$

$$\overline{D_n^k} \equiv \sum_{p=1}^{\infty} p^k \left[b_n(m^p) - b_n(m^{p-1})\right] , \quad (36)$$

for $k = 1, 2$, as well as of the quantities $\mathrm{var}(d_n)$ and $\mathrm{var}(D_n)$, when $n \to \infty$. The functions a_n and b_n are defined in Eqs. (5) and (7).

A1. Approximation and relevant summation ranges

To obtain convenient expressions for mean and variance of d, D we expand a_n, b_n as

$$\begin{aligned} a_n(M) &= e^{-n/M} - \frac{1}{2n}\left(\frac{n}{M}\right)^2 e^{-n/M} \\ &+ \frac{1}{n^2}\left[\frac{1}{8}\left(\frac{n}{M}\right)^4 - \frac{1}{3}\left(\frac{n}{M}\right)^3\right] e^{-n/M} \\ &+ \cdots , \quad M \geq 1 , \end{aligned} \quad (37)$$

$$\begin{aligned} b_n(M) &= e^{-n^2/2M} + \frac{1}{n}\left[\frac{n^2}{2M} - \frac{2}{3}\left(\frac{n^2}{2M}\right)^2\right] e^{-n^2/2M} \\ &+ \frac{1}{n^2}\left[\frac{3}{2}\left(\frac{n^2}{2M}\right)^2 - \frac{4}{3}\left(\frac{n^2}{2M}\right)^3 + \frac{2}{9}\left(\frac{n^2}{2M}\right)^4\right] \\ &\times e^{-n^2/2M} + \cdots , \quad M \geq n . \end{aligned} \quad (38)$$

Here the errors caused by truncating the series are of the same order as the first deleted term.

Therefore $a_n(M)$ is either close to 0 or close to 1, unless M is confined to a region $(\varepsilon n, \varepsilon^{-1} n)$. It follows that in the series (35) only those p contribute that satisfy

$$p \approx \frac{\log n}{\log m} = \log_m n . \quad (39)$$

Similarly, in the series (36) only those p contribute that satisfy

$$p \approx \frac{\log(n^2/2)}{\log m} = 2\log_m n - \log_m 2 . \quad (40)$$

The proof of Eq. (37) with truncation error assessment follows easily from the Taylor expansion of $\log(1 - 1/M) \equiv \log(1-t)$ around $t = 0$ and the inequality

$$(1-t)^n \leq e^{-nt} , \quad 0 < t \leq 1 , \quad (41)$$

so that the t-regime $(0, 1]$ for which (37) has to be established can be split up conveniently in $(0, n^{-1/2}]$ and $(n^{-1/2}, 1]$. The proof of Eq. (38) uses the approximation

$$\log\Gamma(z) = (z - \frac{1}{2})\log z - z + \frac{1}{2}\log(2\pi) + \mathcal{O}\left(z^{-1}\right) , \quad z \to \infty, \quad (42)$$

together with the inequality

$$\frac{M!}{(M-n)!M^n} \leq \exp\left[-\frac{n(n-1)}{2M}\right] , \quad M \geq n , \quad (43)$$

for conveniently splitting up the range for M in $[n, n^{3/2})$ and $(n^{3/2}, \infty)$, and a lengthy but elementary computation.

A2. Fourier analysis of leading approximations

We replace in Eqs. (35) and (36)

$$a_n(m^p) \quad \text{by} \quad \exp\left(-n/m^p\right) , \quad (44)$$
$$b_n(m^p) \quad \text{by} \quad \exp\left(-n^2/2m^p\right) , \quad (45)$$

and we extend the summation range of p to all integers, at the expense of errors of order $\exp(-n)$. We thus arrive at the leading approximation

$$\mu_{k,d}(n+1) = \sum_{p=-\infty}^{\infty} p^k \left[\exp(-n/m^p) - \exp(-n/m^{p-1})\right] , \quad (46)$$

$$\mu_{k,D}(n) = \sum_{p=-\infty}^{\infty} p^k \left[\exp(-n^2/2m^p) - \exp(-n^2/2m^{p-1})\right] , \quad (47)$$

for the k-th moment of d, D, respectively. Observe that

$$\mu_{k,D}(n) = \mu_{k,d}(\frac{1}{2}n^2 + 1) , \quad (48)$$

so that we can restrict ourselves in the remainder of this subsection to the evaluation of $\mu_{k,d}$.

We introduce the notation $\alpha = \log m$, $z = \log_m(n-1)$, i.e. $n = e^{\alpha z} + 1$, and we define

$$f(z) = \exp\left(-e^{\alpha z}\right) - \exp\left(-e^{\alpha(z+1)}\right) . \quad (49)$$

Then there holds

$$\mu_{k,d}(n) = \sum_{p=-\infty}^{\infty} p^k f(z-p) . \quad (50)$$

For $k = 1, 2$ we have

$$\mu_{1,d}(n) = z g_0(z) - g_1(z) , \quad (51)$$
$$\mu_{2,d}(n) = z^2 g_0(z) - 2z g_1(z) + g_2(z) . \quad (52)$$

Here we have set for $k = 0, 1, 2$

$$g_k(z) = \sum_{p=-\infty}^{\infty} (z-p)^k f(z-p) , \quad (53)$$

which are 1-periodic, bounded, smooth functions of z. Noting that $g_0(z) \equiv 1$ [as $\exp(-e^{\alpha z})$ decreases from 1 to 0 as z increases from $-\infty$ to ∞], we get

$$\mu_{1,d}(n) = z - g_1(z) , \quad (54)$$
$$\sigma_d^2(n) \equiv \mu_{2,d}(n) - \mu_{1,d}^2(n) = g_2(z) - g_1^2(z) , \quad (55)$$

for the leading approximations of mean and variance of d, respectively. Hence, $\mu_{1,d}(n)$ grows like $\log_m(n-1)$ with oscillations due to the term $g_1(z)$, and $\sigma_d^2(n)$ is a bounded function of n, which is 1-periodic in $z = \log_m(n-1)$.

Let us now analyze the functions $g_k(z)$ a bit further. From the Poisson summation formula and some elementary properties of the Fourier transform it follows that

$$g_k(z) = \left(\frac{-1}{2\pi i}\right)^k \sum_{q=-\infty}^{\infty} F^{(k)}(q) e^{2\pi i q z} , \quad (56)$$

where $F^{(k)}(\nu) \equiv d^k F(\nu)/d\nu^k$ is the kth derivative of the Fourier transform

$$F(\nu) = \int_{-\infty}^{\infty} dz e^{-2\pi i\nu z} f(z)\,, \quad \nu \in \mathbb{R}\,, \tag{57}$$

of f in Eq. (49). The function $F(\nu)$ can be expressed in terms of the Γ-function as

$$F(\nu) = \frac{e^{2\pi i\nu} - 1}{2\pi i\nu}\Gamma(1 - 2\pi i\nu/\alpha)\,. \tag{58}$$

It thus follows that

$$\begin{aligned} g_1(z) &= -\left(\frac{1}{2} + \frac{\gamma}{\alpha}\right) \\ &- \frac{1}{2\pi}\sum_{q\neq 0}\frac{1}{iq}\Gamma\left(1 - 2\pi iq/\alpha\right)e^{2\pi iqz}\,, \quad (59) \\ g_2(z) &= \frac{1}{3} + \frac{\gamma}{\alpha} + \frac{1}{\alpha^2}\left(\frac{\pi^2}{6} + \gamma^2\right) \\ &- \frac{1}{2\pi^2}\sum_{q\neq 0}\frac{1}{q}\Gamma\left(1 - 2\pi iq/\alpha\right)e^{2\pi iqz} \\ &\times\left[\pi i - 1/q - (2\pi i/\alpha)\psi\left(1 - 2\pi iq/\alpha\right)\right]\,, \quad (60) \end{aligned}$$

with the digamma function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}\,. \tag{61}$$

and Euler's constant $\gamma = \psi(1) \approx 0.5772$.

We now argue that the terms in the two series at the r.h.s. of Eqs. (59) and (60) can be largely ignored. To this end we quote the formulas, see Ref. [24],

$$|\Gamma(1 + iy)|^2 = \frac{\pi y}{\sinh \pi y}\,, \quad y \in \mathbb{R}\,, \tag{62}$$

$$\psi(z) = \log z + \mathcal{O}(z^{-1})\,, \quad z \to \infty\,,\ \Re(z) \geq 0\,. \tag{63}$$

It thus follows that the two series in Eqs. (59) and (60) are negligibly small for not too large values of α ($m = e^\alpha = 2, 3, 4$), while it is sufficient to include only the terms with $q = \pm 1$ for moderately larger values of α ($m = 5, \ldots, 20$).

Returning to Eqs. (59) and (60) for the leading approximations, we see that $\mu_{1,d}(n)$ has, within a negligible error, a sinusoidal oscillation around a monotonously increasing term,

$$\begin{aligned} \mu_{1,d}(n) &= z + \left(\frac{1}{2} + \frac{\gamma}{\alpha}\right) + \kappa_1 \sin(2\pi z - \phi_1)\,, \quad (64) \\ \kappa_1 e^{i\phi_1} &= \Gamma(1 + 2\pi i/\alpha)/\pi\,. \quad (65) \end{aligned}$$

Similarly, we find for the variance

$$\begin{aligned} \sigma_d^2(n) &= \frac{1}{12} + \frac{\pi^2}{6\alpha^2} - \kappa_2 \sin(2\pi z - \phi_2)\,, \quad (66) \\ \kappa_2 e^{i\phi_2} &= \Gamma(1 + 2\pi i/\alpha) \\ &\times\{2\pi[\psi(1 + 2\pi i/\alpha) + \gamma]/\alpha + i\}/\pi^2\,. \quad (67) \end{aligned}$$

A3. Error analysis for leading approximations

We now briefly indicate how the leading approximations change when instead of per Eq. (44) $a_n(M)$ is replaced by [see Eq. (37)]

$$e^{-n/M} - \frac{1}{2n}\left(\frac{n}{M}\right)^2 e^{-n/M}\,. \tag{68}$$

An analysis similar to the one given in Section A2 shows that the r.h.s. of Eq. (51) has to be changed into

$$zg_0(z) - g_1(z) - \frac{1}{2n}\left[zj_0(z) - j_1(z)\right]\,, \tag{69}$$

and the r.h.s. of Eq. (52) into

$$z^2 g_0(z) - 2zg_1(z) - g_2(z) - \frac{1}{2n}\left[z^2 j_0(z) - 2zj_1(z) + j_2(z)\right]\,, \tag{70}$$

where

$$j_k(z) = \sum_{p=-\infty}^{\infty}(z - p)^k h(z - p)\,, \tag{71}$$

with

$$h(z) = e^{2\alpha z}\exp\left(-e^{\alpha z}\right) - e^{2\alpha(z+1)}\exp\left[-e^{\alpha(z+1)}\right]\,. \tag{72}$$

Since $j_0(z) = 0$, it follows that

$$\mu_{1,d}(n) = z - g_1(z) + \frac{1}{2n}j_1(z)\,, \tag{73}$$

$$\begin{aligned} \sigma_d^2(n) &= g_2(z) - g_1^2(z) \\ &- \frac{1}{2n}\left[j_2(z) - 2g_1(z)j_1(z) + \frac{1}{2n}j_1^2(z)\right]\,. \quad (74) \end{aligned}$$

Hence the corrected values differ by $\mathcal{O}(n^{-1})$ from Eqs. (54) and (55).

One can make calculations for j_1 and j_2 in the same manner as was done in Section A2 for g_1 and g_2. It thus turns out that the function $j_1(z)$, required for the average, and the function between the square brackets in Eq. (74)], required for the variance, both oscillate with period 1 and small amplitude around the values $1/\alpha$ and

$$\frac{2}{\alpha^2}\left(1 + \frac{1}{4n}\right) - \frac{2\pi^2}{\alpha^3}\sum_{q\neq 0}\frac{[2 - (1 + 4\pi^2 q^2/\alpha^2)/(2n)]q}{\sinh(2\pi^2 q/\alpha)}\,, \tag{75}$$

respectively. The series in Eq. (75) is very small and will be neglected together with the $1/(4n)$. The first order corrections to Eqs. (54) and (55) are thus given by

$$\begin{aligned} \mu_{1,d}(n) &= z - g_1(z) + \frac{1}{2n\alpha}\,, \quad (76) \\ \sigma_d^2(n) &= g_2(z) - g_1^2(z) - \frac{1}{n\alpha^2}\,. \quad (77) \end{aligned}$$

In principle one can continue the process of adding terms, see Eqs. (37) and (68), so as to obtain higher order corrections. However, the expressions (76) and (77) approximate $\overline{d_n}$ and $\mathrm{var}(d_n)$ already up to sufficient accuracy. Inclusion of the third term in Eq. (37) only has the modest effect on $\overline{d_n}$ of magnitude $1/(12\alpha n^2)$. We also note that the so-obtained series of approximations is asymptotic in nature, in the sense that the n-range where inclusion of the kth term of the the r.h.s. of Eqs. (37) and (38) yields a better approximation shifts towards ∞ with increasing k.

An evaluation of the corrections on mean and variance of D from Eq. (38) is almost similar. Note, that the leading approximation of $\overline{D_n}$ and $\mathrm{var}(D_n)$ can be obtained from the relation (48). Including the constant terms of the next order of approximation we have

$$\mu_{1,D}(n) = z - g_1(z) - \frac{1}{3n\alpha}, \qquad (78)$$

$$\sigma_D^2(n) = g_2(z) - g_1^2(z) - \frac{4}{3n\alpha^2}, \qquad (79)$$

where now $z = \log_m(n^2/2)$.

Appendix B: Details for Section IV

In this Appendix we derive the results presented in Section IV on the approximation of the average and the variance of the number of contention frames required for the tree algorithm in the limit of a large number of contenders n. We start from the expressions given in Ref. [21] for the average

$$\overline{L_n} = \sum_{p=0}^{\infty} c_n(m^p) \qquad (80)$$

where $c_n(M)$ is defined in Eq. (25) and for the variance

$$\begin{aligned}\lim_{j\to\infty, n=lm^j} \frac{\mathrm{var}(L_n)}{n} &= 2\sum_{p=-\infty}^{\infty} \rho\left(\frac{l}{m^p}\right)\sum_{r>p}\chi\left(\frac{l}{m^r}\right) \\ &+ \sum_{p=-\infty}^{\infty} \rho\left(\frac{l}{m^p}\right)\chi\left(\frac{l}{m^p}\right) \\ &- \left[\sum_{p=-\infty}^{\infty}\tau\left(\frac{l}{m^p}\right)\right]^2, \end{aligned} \qquad (81)$$

for fixed integer l, where

$$\rho(x) = (1+x)e^{-x}, \qquad (82)$$

$$\chi(x) = \frac{1-e^{-x}}{x} - e^{-x}, \qquad (83)$$

$$\tau(x) = xe^{-x}, \qquad (84)$$

for $x \geq 0$. Our large-n analysis makes use of the same type of approximations and Fourier analysis as the analysis in Appendix A.

B1. Approximation of $\overline{L_n}$

There holds the following approximation for Eq. (25)

$$c_n(M) = n\chi\left(\frac{n}{M}\right) + \phi\left(\frac{n}{M}\right) + \frac{1}{n}\theta\left(\frac{n}{M}\right) + \frac{1}{n^2}\eta\left(\frac{n}{M}\right) + \cdots, \qquad (85)$$

where χ is given in Eq. (83) and

$$\phi(x) = \frac{1}{2}x(x-1)e^{-x}, \qquad (86)$$

$$\theta(x) = \left(-\frac{2}{3}x^2 + \frac{17}{24}x^3 - \frac{1}{8}x^4\right)e^{-x}, \qquad (87)$$

$$\eta(x) = \left(-x^3 + \frac{11}{12}x^4 - \frac{13}{48}x^5 + \frac{1}{48}x^6\right)e^{-x}, \qquad (88)$$

for $x \geq 0$. The proof of Eq. (85) follows straightforwardly from Eq. (37).

Substitution of Eq. (85) into Eq. (80) yields

$$\overline{L_n} = n\sum_{p=0}^{\infty}\chi\left(\frac{n}{m^p}\right) + \sum_{p=0}^{\infty}\phi\left(\frac{n}{m^p}\right) + E, \qquad (89)$$

where E is the error due to the last two terms at the r.h.s. of Eq. (85). This $E = \mathcal{O}(n^{-1})$, since, for instance,

$$\sum_{p=0}^{\infty}\frac{n}{m^{2p}}e^{-n/m^p} \leq \frac{1}{n}\sum_{p=-\infty}^{\infty}\left(\frac{n}{m^p}\right)^2 e^{-n/m^p}, \qquad (90)$$

the r.h.s.-series being periodic (and whence bounded) in $\log_m n$. Furthermore, we extend the lower summation limit in Eq. (89) to minus infinity at the expense of an exponentially small error so that we obtain

$$\overline{L_n} = \mu_{1,L} + \mathcal{O}\left(n^{-1}\right), \qquad (91)$$

where

$$\mu_{1,L} = n\sum_{p=-\infty}^{\infty}\chi\left(\frac{n}{m^p}\right) + \sum_{p=-\infty}^{\infty}\phi\left(\frac{n}{m^p}\right) - \frac{1}{m-1}, \qquad (92)$$

where the last term on the r.h.s. is due to the $1/x$ in Eq. (83). The highest order term of our result is equal to the function given in Ref. [21]. Below, it is demonstrated how from this term and the other series in Eq. (92) an analytical approximation to $\overline{L_n}$ can be obtained.

B2. Fourier analysis of $\mu_{1,L}$

Similarly to Appendix A, we use the notation $\alpha = \log m$ and $z = \log_m n$, i.e. $n = e^{\alpha z}$. From the following Fourier transforms

$$\int_{-\infty}^{\infty} dz e^{-2\pi i\nu z}\chi\left(e^{\alpha z}\right) = \frac{1}{\alpha}\frac{\Gamma(1-2\pi i\nu/\alpha)}{1+2\pi i\nu/\alpha}, \qquad (93)$$

$$\int_{-\infty}^{\infty} dz e^{-2\pi i\nu z}\phi\left(e^{\alpha z}\right) = -\frac{\pi i\nu}{\alpha^2}\Gamma(1-2\pi i\nu/\alpha), \qquad (94)$$

and from the Poisson summation formula it follows that

$$\sum_{p=-\infty}^{\infty}\chi\left(\frac{n}{m^p}\right) = \frac{1}{\alpha} + \frac{1}{\alpha}\sum_{q\neq 0}\frac{\Gamma(1-2\pi iq/\alpha)}{1+2\pi iq/\alpha}e^{2\pi iqz}, \qquad (95)$$

$$\sum_{p=-\infty}^{\infty}\phi\left(\frac{n}{m^p}\right) = -\frac{\pi i}{\alpha^2}\sum_{q\neq 0} q\Gamma(1-2\pi iq/\alpha)e^{2\pi iqz}. \qquad (96)$$

The result (95) has already been obtained by Mathys and Flajolet [20] on the basis of an asymptotic analysis. The assessment under what conditions and which terms in the r.h.s. of Eqs. (95) and (96) with $q \neq 0$ are significant is the same as in Appendix A2. For not too large values of m, including only the $q = \pm 1$ terms lead to sufficient accuracy. From Eqs. (91), (92), (95), and (96) one can easily derive the results (26)–(32) presented in the main text.

In a similar fashion as in Appendix A3, we can give corrections to the approximation just found by incorporating the higher-order terms of Eq. (85) in the analysis. The term $\theta(n/M)/n$ yields a correction to $\overline{L_n}$ which oscillates with small amplitude around "DC-value" 0, while the term $\eta(n/M)/n^2$ yields a correction term of magnitude $-1/(2\alpha n^2)$. Hence, it is sufficient to consider only the first two terms in Eq. (85).

B3. Fourier analysis of Eq. (81)

We briefly outline how the r.h.s. of Eq. (81) can be evaluated, the method being similar to the derivations above. We only explicitly calculate the "DC-value," but we have checked that the oscillations around this value are indeed at least an order of magnitude smaller.

Let us consider for $\beta > 0$ the quantities

$$s(l;\beta) \equiv \sum_{p=-\infty}^{\infty} \rho\left(\frac{l}{m^p}\right)\chi\left(\beta\frac{l}{m^p}\right) . \tag{97}$$

In the r.h.s. of Eq. (81), the first term equals $2\sum_{r\geq 1} s(l, m^{-r})$ and the second term equals $s(l,1)$. Setting $z = \log_m l$, we obtain for the Fourier transform of $s(e^{\alpha z};\beta)$

$$S(\nu;\beta) = \frac{\Gamma(1-2\pi i\nu/\alpha)}{\beta(\alpha+2\pi i\nu)}\left[\frac{1+(1-2\pi i\nu/\alpha)\beta+\beta^2}{(1+\beta)^{1-2\pi i\nu/\alpha}} - 1\right] . \tag{98}$$

Note, that $S(\nu;\beta) = \mathcal{O}(\beta)$ as $\beta \to 0$. This ensures that the summation $s(l, m^{-r})$ over r converges rapidly. The "DC-term" of $s(l;\beta)$ is given by

$$S(0;\beta) = \frac{1}{\alpha}\frac{\beta}{1+\beta} . \tag{99}$$

For the third therm on the r.h.s. of Eq. (81) we note that, as before,

$$\sum_{p=-\infty}^{\infty} \tau\left(\frac{l}{m^p}\right) = \frac{1}{\alpha} + \frac{1}{\alpha}\sum_{q\neq 0}\Gamma(1-2\pi iq/\alpha)e^{2\pi iqz} , \tag{100}$$

so that its contribution to the "DC-value" of Eq. (81) is given by

$$-\int_0^1 dz \left[\sum_p \tau\left(\frac{l}{m^p}\right)\right]^2 = -\frac{1}{\alpha^2} - \frac{2}{\alpha^2}\sum_{q=1}^{\infty}\frac{2\pi q^2/\alpha}{\sinh(2\pi^2 q/\alpha)} . \tag{101}$$

The second term on the r.h.s. is very small and will be neglected. Collecting the results, we arrive at our expression (34).

We conclude this Appendix with the remark that the techniques used here can be applied to the series $d(\xi)$ given in Theorem 7 of Ref. [21] (this quantity is of importance for contention tree-algorithms where the number of slots in the root is variable). We immediately give the result

$$d(\xi) = \sum_{p=-\infty}^{\infty}\frac{\xi m^p}{(1+\xi m^p)^2} = \frac{1}{\alpha} + \frac{4\pi^2}{\alpha^2}\sum_{q=1}^{\infty}\frac{q\cos(2\pi qz)}{\sinh(2\pi^2 q/\alpha)} , \tag{102}$$

where $z = 1/\log_m \xi$. As remarked in Ref. [21], the oscillations around the "DC-value" are indeed substantially smaller than those in Eq. (95).

References

[1] N. Abramson, "Development of the ALOHANET," *IEEE Trans. Inform. Theory*, vol. 31, pp. 119–123, 1985.

[2] L. Kleinrock and F. A. Tobagi, "Packet switching in radio channels: Part 1: CSMA modes and their throughput-delay characteristics," *IEEE Trans. Commun.*, vol. 23, pp. 1400–1416, 1975.

[3] J. I. Capetanakis, "Tree algorithms for packet broadcast channels," *IEEE Trans. Inform. Theory*, vol. 25, pp. 505–515, 1979.

[4] B. S. Tsybakov and V. A. Mikhailov, "Free synchronous packet access in a broadcast channel with feedback," *Probl. Peredachi Inf.*, vol. 14, no. 4, pp. 32–59, 1978 [*Probl. Inform. Trans.*, vol. 14, pp. 259–280, 1978].

[5] M. J. M. de Jong and A. J. E. M. Janssen, "Analytic properties of contention tree-algorithms," in *Proc. Inform. Theory Workshop* 1998, Killarney, Ireland.

[6] B. S. Tsybakov and V. A. Mikhailov, "Random multiple access of packets: Part and try algorithm," *Probl. Peredachi Inf.*, vol. 16, pp. no. 4, 65–79, 1980 [*Probl. Inform. Trans.*, vol. 16, pp. 305–317, 1980].

[7] T. Berger, "The Poisson multiple-access conflict resolution problem," in *Multi-user communication systems*, G. Longo, Ed., CISM Course and Lecture Notes 265 (Springer Verlag, New York, 1981), pp. 1–27.

[8] J. L. Massey, "Collision resolution algorithms and random access algorithms," in *Multi-user communication systems*, G. Longo, Ed., CISM Course and Lecture Notes 265 (Springer Verlag, New York, 1981), pp. 73–137.

[9] G. Ruget, "Some tools for the study of channel sharing algorithms," in *Multi-user communication systems*, G. Longo, Ed., CISM Course and Lecture Notes 265 (Springer Verlag, New York, 1981), pp. 201–231.

[10] R. G. Gallager, "A perspective on multiple access channels," *IEEE Trans. Inform. Theory*, vol. 31, pp. 124–142, 1985.

[11] B. S. Tsybakov and N. D. Vvedenskaya, "Random multiple-access stack algorithms," *Probl. Peredachi Inf.*, vol. 16, no. 3, pp. 80–94, 1980 [*Probl. Inform. Trans.*, vol. 16, pp. 230–243, 1980].

[12] N. D. Vvedenskaya and B. S. Tsybakov, "Packet delay in the case of a multiple-access stack algorithm," *Probl. Peredachi Inf.*, vol. 20, no. 2, pp. 77–97, 1984 [*Probl. Inform. Trans.*, vol. 20, pp. 137–153, 1984].

[13] G. Fayolle, P. Flajolet, M. Hofri, and P. Jacquet, "Analysis of a stack algorithm for random multiple-access communication," *IEEE Trans. Inform. Theory*, vol. 31, pp. 244–254, 1985.

[14] B. S. Tsybakov and M. A. Burkovskii, "Multiple access with reservation," *Probl. Peredachi Inf.*, vol. 16, no. 1, pp. 50–76, 1980 [*Probl. Inform. Trans.*, vol. 16, pp. 35–54, 1980].

[15] W. Xu and G. Campbell, "A distributed queueing random access protocol for a broadcast protocol," SIGCOMM, *Computer Communication Review*, vol. 23, no. 4, pp. 270–278, 1993.

[16] IEEE Project 802.14 Draft2 Revision 2, January 1998.

[17] DAVIC 1.3 specification Part 8, September 1997.

[18] B. Hajek, N. B. Likhanov, and B. S. Tsybakov, "On the delay in a multiple-access system with large propagation delay," *IEEE Trans. Inform. Theory*, vol. 40, pp. 1158–1166, 1994.

[19] B. S. Tsybakov and N. B. Likhanov, "An upper bound on packet delay in a multiple access channel," *Probl. Peredachi Inf.*, vol. 18, no. 4, pp. 76–84, 1982 [*Probl. Inform. Trans.*, vol. 18, pp. 279–285, 1982].

[20] P. Mathys and P. Flajolet, "Q-ary collision resolution algorithms in random-access systems with free or blocked channel access," *IEEE Trans. Inform. Theory*, vol. 31, pp. 217–243, 1985.
[21] M. A. Kaplan and E. Gulko, "Analytic properties of multiple-access trees," *IEEE Trans. Inform. Theory*, vol. 31, pp. 255–263, 1985.
[22] J.-C. Huang and T. Berger, "Delay analysis of interval searching contention resolution algorithms," *IEEE Trans. Inform. Theory*, vol. 31, pp. 264–273, 1985.
[23] J. I. Capetanakis, "Generalized TDMA," *IEEE Trans. Commun.*, vol. 27, pp. 1476–1484, 1979.
[24] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions* (National Bureau of Standards, Washington, 1964), pp. 257–259.

TABLE I

THE PARAMETERS REQUIRED TO CALCULATE THE AVERAGE AND THE VARIANCE OF THE NUMBER OF LEVELS d_n AND D_n FOR VARIOUS NUMBER m OF CONTENTION SLOTS PER FRAME. THE OSCILLATIONS IN THE AVERAGES δd_n AND δD_n HAVE AMPLITUDE κ_1 AND PHASE ϕ_1, SEE EQS. (13) AND (16). THE LARGE-n LIMIT OF THE "DC-VALUE" OF THE VARIANCES $\widetilde{v}_{d_n}$ AND $\widetilde{v}_{d_n}$ IS GIVEN BY $\widetilde{v}_{d,D}$, SEE EQ. (22). THE OSCILLATIONS AROUND THIS VALUE δv_{d_n} AND $\delta v D_n$ CAN BE DESCRIBED BY κ_2 AND ϕ_2, SEE EQ. (21).

m	κ_1	ϕ_1	κ_2	ϕ_2	$\widetilde{v}_{d,D}$
2	$1.573 \cdot 10^{-6}$	-0.873	$1.463 \cdot 10^{-5}$	2.798	3.5071
3	$2.394 \cdot 10^{-4}$	-1.258	$1.244 \cdot 10^{-3}$	2.503	1.4462
5	$3.423 \cdot 10^{-3}$	2.177	$1.099 \cdot 10^{-2}$	-0.246	0.7184
10	$1.813 \cdot 10^{-2}$	0.765	$3.731 \cdot 10^{-2}$	-1.543	0.3936

TABLE II

THE PARAMETERS REQUIRED FOR THE CALCULATION OF THE AVERAGE AND THE VARIANCE OF THE NUMBER OF CONTENTION FRAMES L_n ENLISTED FOR VARIOUS VALUES OF m, THE NUMBER OF CONTENTION SLOTS PER FRAME. THE CONSTANT OF PROPORTIONALITY FOR THE "DC-VALUE" OF $\overline{L_n}$ EQUALS $1/\log m$, SEE EQ. (27). THE QUANTITIES $\lambda_{1,2}$ AND $\theta_{1,2}$ DESCRIBE THE FLUCTUATION AROUND THIS VALUE, SEE EQS. (28)–(32). THE LAST COLUMN ENLISTS THE NUMBER $\widetilde{v}_L$, WHICH APPROXIMATES $\mathrm{var}(L_n)/n$ FOR LARGE n, ACCORDING TO EQS. (33) AND (34).

m	$1/\log m$	λ_1	θ_1	λ_2	θ_2	$\widetilde{v}_L$
2	1.4427	$1.564 \cdot 10^{-6}$	-0.588	$6.463 \cdot 10^{-5}$	0.873	0.8459
3	0.9102	$2.358 \cdot 10^{-4}$	-0.139	$3.915 \cdot 10^{-3}$	1.258	0.3622
5	0.6213	$3.316 \cdot 10^{-3}$	2.786	$2.609 \cdot 10^{-2}$	-2.177	0.1919
10	0.4343	$1.702 \cdot 10^{-2}$	-1.985	$6.750 \cdot 10^{-2}$	-0.765	0.1171