A note on Unser-Zerubia generalized sampling theory applied to the linear interpolator

A.J.E.M. Janssen, T. Kalker

Abstract

In this letter we calculate the condition number of the linear operator that maps sequences of samples $f(2k)$, $f(2k+a)$, $k \in \mathbb{Z}$, of an unknown continuous $f \in L^2(\mathbb{R})$ consistently (in the senses of the Unser-Zerubia generalized sampling theory) onto the set of continuous, piecewise linear functions in $L^2(\mathbb{R})$ with nodes at the integers as a function of $a \in (0,2)$. It turns out that the minimum condition numbers occur at $a = \sqrt{2/3}$ and $a = 2 - \sqrt{2/3}$ and not at $a = 1$ as one might have expected.

Keywords

Sampling, interpolation, error analysis, video, de-interlacing, motion estimation.

I. Introduction and Announcements of Results

Generalized sampling theory can successfully be applied to de-interlacing of television images. The interlaced video format entails that odd and even television lines, referred to as the odd and even *fields*, be sampled at different times. To create a progressive television image, even and odd lines, collectively referred to as a *frame*, are sampled at the same moment. In order to de-interlace a field it is therefore necessary to reconstruct the missing even or odd lines of a progressive frame from preceding or following fields. If there is no motion in the scene, this reconstruction process is trivial. In general, however, the scene is not static. A common approach to de-interlacing in this non-static case is the assumption that successive frames are locally related by uniform motion. The image samples in successive fields can then be viewed as a non-uniform sampling of the underlying progressive frames. Viewed in this way, de-interlacing can be mathematically modeled as the reconstruction of uniformly spaced sampling values $f(k)$, $k \in \mathbb{Z}$ of an unknown function $f(x)$, sampled at positions $2k$ and $2k+a$, $k \in \mathbb{Z}$. The velocity v of the scene and the parameter a are related by $a = 1 - v$. For reasons of symmetry we may assume without loss of generality that $0 \leq a < 1$. For an example of a successful application of this approach, see [1].

The parameter a is directly related to the assumed motion in the scene. However, this motion model is usually only an approximation, and even if the model were exact, the parameter a could only be estimated. Moreover, the sampling values of the interlaced fields usually have a large noise component. It is therefore relevant to study, as in [2], the stability of the sequence of reconstructed sampling values. Intuitively one would expect that the most stable reconstruction is obtained if there is no motion in the scene, i.e. if $a = 1$. In this case, reconstruction amounts to copying of sampling values. For $a \neq 1$, a simple and practical solution for consumer devices is the use of a linear interpolator function. However, directly applying the stability theory of [2] to this situation leads to the somewhat surprising result that the most stable situation occurs at $a = \sqrt{2/3}$. The derivation of this result is presented in Section III.

So what is wrong? As it turns out, nothing is wrong. Both our intuition and the theory of [2] are correct. The explanation of these of counter-intuitive results is based on the fact that the theory of [2] does not directly apply to the case of de-interlacing. The de-interlacing problem can be described as the study of the discrete operator $Q_a : l_2^2 \to l_2$, mapping non-uniform sampling values to uniform sampling values $c(k)$, $k \in \mathbb{Z}$. The theory of [2], on the other hand, studies the construction of the piecewise linear function

$$\tilde{f}(x) = E(\{c(k)\}) = \sum_{k \in \mathbb{Z}} c(k)\varphi(x-k) \tag{1}$$

i.e. an operator $\tilde{Q}_a = E \circ Q_a : l_2^2 \to L_2$. The function $\varphi(x)$ in (1) above is the interpolator for piecewise linear functions and is equal to the centered linear spline $\beta^{(1)}(x)$.

In the case of linear interpolation, the operator Q_a is represented by the matrix (see (10))

$$\mathbf{Q_a(z)} = \frac{\mathbf{1}}{\mathbf{a}} \begin{bmatrix} a & 0 \\ a-1 & 1 \end{bmatrix}, \tag{2}$$

where we have used the isomorphism $\iota : l_2 \to l_2^2$, $\{c(k)\} \to (\{c(2k)\}, \{c(2k+1)\})$ to view Q_a as an operator $l_2^2 \to l_2^2$. Following the analysis of [3], the stability of the reconstruction $\alpha_a^{(l_2)}$ is measured by the condition number of the matrix $\mathbf{A_a^{(l_2)}(z)} = \mathbf{Q^T(z^{-1})Q(z)}$. It is easy to verify that maximal stability (i.e. a minimal condition number) is obtained for

Philips Research Eindhoven, Prof. Holstlaan 4, 5656 AA, Eindhoven, 31-40-2743839 (tel), 31-40-2744675 (fax), {janssena,kalker}@natlab.research.philips.com

$a = 1$. Figure 1 gives a graphical presentation of the dependency of the condition number as a function of the shift parameter a.

In [2] it is shown that the condition for the operator $\tilde{Q}_a : l_2^2 \to L_2$ not only involves the the matrix Q_a, but also the autocorrelation properties of the linear interpolator function $\varphi(x)$. In particular, all computations are based upon the matrix (see [2, (18)]

$$\mathbf{A_a(z) = Q_a^T(z^{-1})A_\varphi Q_a(z)}, \tag{3}$$

where the entries of $\mathbf{A}_\varphi$ record the autocorrelation properties of the interpolator function $\varphi(x)$. Stability is measured in this case by the square root of the condition number α_a of the matrix $\mathbf{A_a(z)}$. As opposed to the discrete case, the computation of the minimal α_a is quite involved. However, the computations of Section III show that an analytical solution is possible. In particular, it follows from those computations that maximal stability is reached for $a = \sqrt{2/3}$ (see also Figure 2). This is conclusive evidence that linear interpolation in the discrete domain and continuous domain are really different from the point of view of stability .

A similar counter-intuitive result as above does not hold for some higher order interpolators. For example, as shown in [2], the cardinal cubic spline and sinc interpolator achieve maximal stability in the continuous domain at $a = 1$. The corresponding discrete domain operators Q_a have of course the same property. The authors of this correspondence are not aware of any general condition on interpolator functions that imply stability at $a = 1$.

The organization of this letter is as follows. In Section II we give a short overview of the most important ingredients of the Unser-Zerubia theory. In Section III we compute analytically the value a for which stability is maximal. This letter ends with a short summary of the achieved results.

II. De-interlacing in the Unser-Zerubia theory

For the sake of simplicity we will assume that we are dealing with 1 dimensional digital video. We will also assume that the video captures a uniformly moving scene $f(x,t) = f(x - vt)$, where v is the motion of the scene. This scene is sampled at time instances $t \in \mathbb{Z}$, but it is sampled differently for t odd and even. If t is even, the scene is sampled at $x \in 2\mathbb{Z}$ (the even field), and if t is odd, the scene is sampled at $2\mathbb{Z} + \nVdash$ (the odd field). If we only consider the sampling moments $t = 0$ and $t = 1$, we obtain two different samplings of the function $f(x)$, viz. the sequences $\{f(2k)\}$ and $f(2k+1-v)$, where $k \in \mathbb{Z}$. We may assume without loss of generality that $0 <= v <= 1$: if necessary the function $f(x)$ is reversed to $f(-x)$ and/or an integer shift is applied to the odd field.

The problem of de-interlacing can now be formulated concisely either as the reconstruction of the function $f(x)$ or, in a more restrictive formulation, as the construction of the *progressive frame* $f(k)$, $k \in \mathbb{Z}$. As de-interlacing is particularly relevant for consumer devices, only simple interpolation functions can be used, in order to limit costs. A common choice is to use bi-linear interpolation, using the first order spline $\beta^{(1)}$ as interpolator. Setting $a = 1 - v$, we are interested in the stability of the reconstruction process. As shown in the previous section, the reconstruction of the discrete progressive frame is most stable if $a = 1$, that is, if there is no motion. In order to analyze the stability for the continuous case, we formulate the de-interlacing problem in terms of the Unser-Zerubia theory.

The sampling of the odd and even fields is modeled by sampling functions $\phi_1(x) = \delta(x)$ and $\phi_2(x) = \delta(x - a)$ (see [2, III.C]). The interpolator function is the centered linear spline $\beta^{(1)}(x)$. The first step in the stability analysis is the construction of the matrix $\mathbf{Q(z)}$ as the inverse of the matrix $\mathbf{A}_{\phi,\varphi}(\mathbf{z})$,

$$[\mathbf{A}_{\phi,\varphi}]_{\mathbf{i,j}}(\mathbf{k}) = \langle \varphi(\mathbf{x}), \phi_\mathbf{i}(\mathbf{x - 2k + (j - 1)}) \rangle, \quad \mathbf{i,j} \in \{\mathbf{0,1}\}. \tag{4}$$

Secondly, the autocorrelation matrix $\mathbf{A}_\varphi(\mathbf{z})$ needs to be computed, where

$$[\mathbf{A}_\varphi]_{\mathbf{i,j}}(\mathbf{k}) = \langle \varphi(\mathbf{x - i - 2k}), \varphi(\mathbf{x - j}) \rangle, \quad \mathbf{i,j} \in \{\mathbf{0,1}\}. \tag{5}$$

Thirdly, the dual autocorrelation matrix $\mathbf{A}_{\tilde{\phi}}(\mathbf{z})$ is computed as

$$\mathbf{A}_{\tilde{\phi}}(\mathbf{z}) = \mathbf{Q^T(z^{-1})A}_\varphi\mathbf{(z)Q(z)}. \tag{6}$$

Fourthly, the largest and smallest eigenvalues, λ_+ and λ_- respectively, of $\mathbf{A}_{\tilde{\phi}}(\mathbf{z})$ over the unit circle have to be computed. Note that $\mathbf{A}_{\tilde{\phi}}(\mathbf{e^{j\omega}})$ is Hermitian and positive definite by construction, and that therefore $\lambda_\pm$ is non-negative. The condition number α is then computed as

$$\alpha = \sqrt{\frac{\lambda_+}{\lambda_-}}. \tag{7}$$

In Sections III-B and III-C this condition number α is analyzed as a function $\alpha(a)$ of the parameter a. In particular it is proven that the minimum condition number is obtained for $a = \sqrt{2/3}$.

III. Derivations

A. Computation of eigenvalues

We have

$$(\mathbf{A}_{\phi,\varphi}(\mathbf{k}))_{\mathbf{i},\mathbf{j}} = \varphi(\mathbf{2k} + (\mathbf{i}-\mathbf{1})\mathbf{a} - (\mathbf{j}-\mathbf{1})), \quad \mathbf{i},\mathbf{j} = \mathbf{1},\mathbf{2}, \tag{8}$$

so that

$$\mathbf{A}_{\phi,\varphi}(\mathbf{z}) = \begin{bmatrix} 1 & 0 \\ 1-a & a \end{bmatrix}, \tag{9}$$

and

$$\mathbf{Q}(\mathbf{z}) = \mathbf{A}^{-\mathbf{1}}_{\phi,\varphi}(\mathbf{z}) = \frac{\mathbf{1}}{\mathbf{a}} \begin{bmatrix} a & 0 \\ a-1 & 1 \end{bmatrix}. \tag{10}$$

Now, using Kronecker's delta $\delta_{p,q}$, we have

$$\mathbf{A}_{\varphi}(\mathbf{k}) = \begin{bmatrix} \frac{2}{3}\delta_{k,0} & \frac{1}{6}\delta_{k,0} + \frac{1}{6}\delta_{k,1} \\ \frac{1}{6}\delta_{k,0} + \frac{1}{6}\delta_{k,-1} & \frac{2}{3}\delta_{k,0} \end{bmatrix}, \quad \mathbf{k} \in \mathbb{Z}, \tag{11}$$

so that

$$\mathbf{A}_{\varphi}(\mathbf{z}) = \frac{\mathbf{1}}{\mathbf{6}} \begin{bmatrix} 4 & 1+z^{-1} \\ 1+z & 4 \end{bmatrix}. \tag{12}$$

Then (6), (10) and (12) yield

$$\mathbf{A}_{\tilde{\phi}}(\mathbf{z}) = \frac{\mathbf{1}}{\mathbf{6a^2}} \begin{bmatrix} 10a^2 - 10a + 4 + a(a-1)(z+z^{-1}) & 5a-4+az^{-1} \\ 5a-4+az & 4 \end{bmatrix}. \tag{13}$$

With

$$x = \frac{1}{2}(z+z^{-1}) = \cos(\omega) \in [-1,1], \tag{14}$$

we compute the two eigenvalues $\lambda_{\pm}(x)$ of $\mathbf{A}_{\tilde{\phi}}(\mathbf{z})$ as

$$6a^2\lambda_{\pm}(x) = B_1 \pm \sqrt{B_2}, \tag{15}$$

where

$$B_1 = 5a^2 - 5a + 4 + a(a-1)x, \tag{16}$$

and

$$B_2 = 36(a-2/3)^2 + a^2(1-a)^2(5+x)^2 - 2a(5a-4)(1-x). \tag{17}$$

It is easily seen that B_2 is a smooth and strictly positive function of $(a,x) \in [0,1]\times[-1,1]\setminus(1,-1)$ and that it vanishes at $(a,x) = (1,-1)$.

B. Analysis of eigenvalues

With $y = 5 + x \in [4,6]$ we can write the right-hand side of (15) as

$$\psi_{\pm}(y) := 4 - a(1-a)y \pm \sqrt{E}, \tag{18}$$

where

$$E = 16 - 24a^2 + 2a(5a-4)y + a^2(1-a)^2y^2. \tag{19}$$

Differentiating with respect to y we find

$$\psi'_{\pm}(y) = \frac{1}{\sqrt{E}}\left(-a(1-a)\sqrt{E} \pm (a(5a-4) + a^2(1-a)^2 y)\right). \tag{20}$$

Now observe that

$$(a(1-a)\sqrt{E})^2 - (a(5a-4) + a^2(1-a)^2 y)^2 = a^3(8 - 33a + 48a^2 - 24a^3) \tag{21}$$

is independent of y. Thus when we take an $a \in (0,1)$ such that

$$P(a) := 8 - 33a + 48a^2 - 24a^3 \neq 0, \tag{22}$$

we see that $\psi'_{\pm}(y)$, being the sum/difference of two smooth, real expressions of unequal modulus, is of one sign. The unique root $a_0 \in (0,+1]$ of $P(a) = 0$ can be computed as

$$a_0 = \frac{2}{3} + \frac{1}{6}\left((\frac{3}{2\sqrt{2}} + 1)^{1/3} - (\frac{3}{2\sqrt{2}} - 1)^{1/3} \right) = 0.813269147. \tag{23}$$

Hence, when $a \neq a_0$, the extrema of $\psi_{\pm}(y)$ occur at $y = 4, 6$. By smoothness of $\psi_{\pm}(y)$ (as a function of $(a,y) \in (0,1) \times [4,6]$) we have that $\arg\max_y(\psi_+(y))$ and $\arg\max_y(\psi_-(y))$ are constant as a function of a in the intervals $(0,a_0)$ and $(a_0,1)$ with a possible jump at $a = a_0$. An analysis for small $a > 0$ and small $1 - a > 0$ then shows that

$$\psi_+(y)\,\text{is maximal at}\,y = 4 \quad (x = -1), \quad 0 < a < a_0, \tag{24}$$

$$\psi_+(y)\,\text{is maximal at}\,y = 6 \quad (x = +1), \quad a_0 < a < 1, \tag{25}$$

$$\psi_-(y)\,\text{is mimimal at}\,y = 6 \quad (x = +1), \quad 0 < a < 1. \tag{26}$$

C. Computation of extreme eigenvalues and condition numbers

It follows from (15), (18), (24), (25), (26) and some simplifications that

$$6a^2\lambda_{\max} = \psi_+(4) = 4\left(a^2 - a + 1 + \sqrt{(a^2-a+1)^2 - a^2}\right), \quad 0 < a < a_0, \tag{27}$$

$$6a^2\lambda_{\max} = \psi_+(6) = 6\left(a^2 - a + \frac{2}{3} + \sqrt{(a^2 - a + \frac{2}{3})^2 - \frac{1}{3}a^2}\right), \quad a_0 < a < 1, \tag{28}$$

$$6a^2\lambda_{\min} = \psi_-(6) = 6\left(a^2 - a + \frac{2}{3} - \sqrt{(a^2 - a + \frac{2}{3})^2 - \frac{1}{3}a^2}\right), \quad 0 < a < 1. \tag{29}$$

We thus obtain that

$$\frac{\lambda_{\max}}{\lambda_{\min}} = 2\left(\frac{a^2-a+1}{a} + \sqrt{\left(\frac{a^2-a+1}{a}\right)^2 - 1}\right)\left(\frac{a^2-a+2/3}{a} + \sqrt{\left(\frac{a^2-a+2/3}{a}\right)^2 - \frac{1}{3}}\right) \tag{30}$$

for $0 < a < a_0$, and

$$\frac{\lambda_{\max}}{\lambda_{\min}} = 3\left(\frac{a^2-a+2/3}{a} + \sqrt{\left(\frac{a^2-a+2/3}{a}\right)^2 - \frac{1}{3}}\right)^2 \tag{31}$$

for $a_0 < a < 1$. The function $(a^2 - a + 1)/a$ decreases for $0 < a \leq 1$, and the function $(a^2 - a + 2/3)/a$ decreases for $0 < a \leq \sqrt{2/3} = 0.816496581$ and increases for $a \geq \sqrt{2/3}$. Note that $\sqrt{2/3} > a_0$. Hence (30) decreases for $0 < a \leq a_0$ and (31) decreases for $a_0 < a < \sqrt{2/3}$ and increases for $a \geq \sqrt{2/3}$. It follows that the minimum of $\lambda_{\max}/\lambda_{\min}$ is assumed at $a = \sqrt{2/3}$, yielding the minimum condition number

$$\begin{aligned} \alpha_{\min} &= \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \\ &= \sqrt{3}\left(2\sqrt{\frac{2}{3}} - 1 + \sqrt{(2\sqrt{\frac{2}{3}} - 1)^2 - \frac{1}{3}}\right) \\ &= 1.54586606. \end{aligned} \tag{32}$$

For comparison we have from (15) at $a = 1$

$$6\lambda_{\pm}(x) = 4 \pm \sqrt{2 + 2x}, \tag{33}$$

so that

$$\lambda_{\max} = 6, \quad \lambda_{\min} = 2; \tag{34}$$

$$\alpha = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} = \sqrt{3} = 1.73205081.$$

See Figure 2 for a graphical representation of the condition number as a function of the interlace parameter α.

IV. Conclusions

In this letter we have argued that from the point of view of stability, reconstructions in the discrete and continuous domains are essentially different. This has been verified through the example of de-interlacing using a bilinear interpolator.

Acknowledgments

The authors would like to thank Erwin Bellers for pointing out that the minimum condition number for linear interpolation does not occur at $a = 1$.

References

[1] E. Bellers and G. de Haan, "New algorithm for motion estimation on interlaced video," in *Proceedings of SPIE – Visual Communications and Image Processing '98*, vol. 3309 (part 1), pp. 111 – 121, Jan. 1998.

[2] M. Unser and J. Zerubia, "Generalized sampling: Stability and performance analysis," *IEEE Transactions on Signal Processing*, vol. 45, pp. 2941–2950, Dec. 1997.

[3] M. Unser and J. Zerubia, "A generalized sampling theory without bandlimiting constraints," *IEEE Transactions on Circuits and Systems–II: Analog and Digital Signal Processing*, pp. 959 – 969, Aug. 1998.

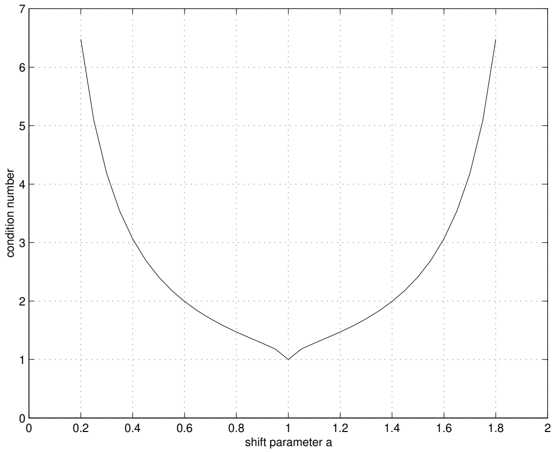

Fig. 1. Condition number for discrete reconstruction as a function of the interlace parameter a.

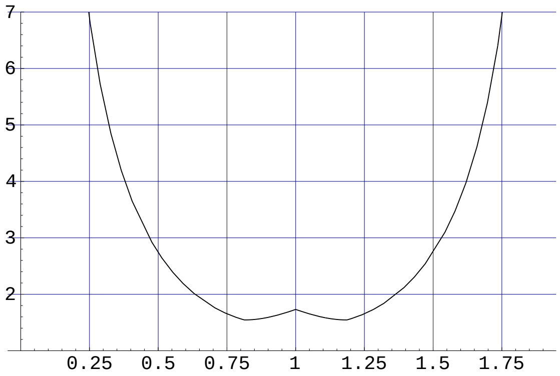

Fig. 2. Condition number for continuous reconstruction as a function of the interlace parameter a.