CLASSIFYING TIGHT WEYL-HEISENBERG FRAMES

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Abstract. A Weyl-Heisenberg frame for $L^2(R)$ is a frame consisting of translates and modulates of a fixed function in $L^2(R)$, i.e. $(E_{mb}T_{na}g)_{m,n\in Z}$, with $a,b>0$, and $g\in L^2(R)$. A central question is to write down exact conditions on g,a,b so that $(E_{mb}T_{na}g)_{m,n\in Z}$ forms a frame. In this paper we will give necessary and sufficient conditions for this family to form a tight WH-frame. This allows us to write down explicitly all functions $g\in L^2(R)$ for which the corresponding WH-frame is an orthonormal basis for $L^2(R)$. We also give a classification for alternate dual frames to Weyl-Heisenberg frames. Although many of these results have appeared before, our proofs are new and follow directly from the definition. Also, by integrating the material into the existing literature, we hope this will also provide a good survey of the subject.

1. Introduction

In 1946 D. Gabor [9] introduced a technique for signal processing which soon became a paradigm for the spectral analysis associated with time-frequency methods. His introduction of the short-time (windowed) Fourier Transform led eventually to Wavelet theory. Gabor received the Nobel prize in 1971 for his fundamental work in signal analysis. In 1952, Duffin and Schaeffer [6] introduced frame theory which put Gabor's technique into the framework of a larger model - that of a frame for a Hilbert space.

A sequence $(f_i)_{i\in I}$ of elements of a Hilbert space H is called a **frame** for H if there are constants $A,B>0$ so that

$$A\|f\|^2 \leq \sum_{i\in I} |<f,f_i>|^2 \leq B\|f\|^2, \quad \text{for all } \ f\in H. \tag{1.1}$$

The numbers A,B are called the **lower** and **upper** frame bounds. The frame is a **tight frame** if $A=B$ and a **normalized tight frame** if $A=B=1$. Two frames

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(or just sequences) $(f_i), (g_i)$ are said to be **equivalent** if the map $Tf_i = g_i$ extends to a well-defined bounded linear invertible operator. A bounded unconditional basis for a Hilbert space H is called a **Riesz basis**. A frame is said to be **exact** if it ceases to be a frame when any one of its elements is removed. It is known that a frame is exact if and only if it is a Riesz basis for H. A non-exact frame is called **overcomplete** in that at least one vector can be removed from the frame and the remaining set of vectors will still form a frame for H. For the basic results on frames we refer to [4,7,10,11,19]. The Introduction of [7] provides a good historical development of the ideas in this paper.

Gabor's work gave rise to Gabor frames (with Gaussian frame generator g), also called Weyl-Heisenberg frames. We fix $a, b \in R$ and a function $g \in L^2(R)$. If the family $(E_{mb}T_{na}g)_{m,n\in Z}$ is a frame for $L^2(R)$(see Section 2 for notation) we call this frame a **Weyl-Heisenberg** frame or a **WH-frame**. A central question is to give a concrete representation of the (g, a, b) which yield WH-frames. In this paper we will give a complete classification of the normalized tight WH-frames. The general conditions required for (g, a, b) to give a normalized tight WH-frame are given in Theorem 3.2. In Section 4, we will give a concrete representation of the functions satisfying these conditions for a few cases. In Section 5 we will give similar conditions which characterize all alternate dual WH-frames for a given WH-frame. Parts of this were done earlier by Ron and Shen [18] and Janssen [12]. Also, Gabardo and Han [8] have already generalized some of the results of this paper to the setting of invariant subspaces for Weyl-Heisenberg unitary systems.

2. Preliminaries

If we replace f in (1.1) by f_j, we see that

$$\|f_j\|^4 + \sum_{i\in I-\{j\}} |<f_j, f_i>|^2 \leq B\|f_j\|^2.$$

This yields immediately the following well-known remark:

Remark 2.1. *For all $j \in Z$, $\|f_j\|^2 \leq B$ and $\|f_j\|^2 = B$ implies $f_j \perp span_{i\neq j} f_i$. In particular, if $(f_i)_{i\in I}$ is a normalized tight frame, then $\|f_j\| \leq 1$ and $\|f_j\| = 1$ if and only if $f_j \perp span_{i\neq j} f_i$.*

If $(e_i)_{i\in I}$ is an orthonormal basis for H and $(f_i)_{i\in I}$ is a sequence of vectors in H, we will call the operator $T : \ell_2 \to H$ given by: $T(e_i) = f_i$ the **pre-frame operator** associated with $(f_i)_{i\in I}$. A direct calculation shows that

$$T^*f = \sum_{i\in I} < f, f_i > e_i, \quad \text{for all} \quad f \in H.$$

It follows that the pre-frame operator is bounded if and only if there is a constant $B > 0$ satisfying:

$$\sum_{i\in I} |< f, f_i >|^2 \leq B\|f\|^2, \quad \text{for all} \quad f \in H.$$

In that case $(f_i)_{i\in I}$ is called a **Bessel sequence**. Moreover, $(f_i)_{i\in I}$ is a frame if and only if the pre-frame operator is a bounded, linear, onto map. The dimension of the kernel of T is called the **excess** of the frame. It follows that $S = TT^*$ is an invertible operator on H, called the **frame operator**, which has the form:

$$Sf = \sum_{i\in I} < f, f_i > f_i, \quad \text{for all} \quad f \in H. \tag{2.2}$$

Now, for all $f \in H$ we have,

$$< Sf, f >= \sum_{i\in I} |< f, f_i >|^2. \tag{2.3}$$

So, the frame operator is a positive, self-adjoint invertible operator on H. One can check that $(f_i)_{i\in I}$ is a normalized tight frame if and only if $S = I$. It is a straightforward calculation that for all $f \in H$,

$$f = \sum_{i\in I} < S^{-1}f, f_i > f_i = \sum_{i\in I} < f, S^{-1/2}f_i > S^{-1/2}f_i. \tag{2.4}$$

One interpretation of (2.4) is that $(S^{-1/2}f_i)_{i\in I}$ is a normalized tight frame for H which is equivalent to the frame $(f_i)_{i\in I}$.

The particular frames of interest to us will be the Weyl-Heisenberg frames. To define these frames, let $a, b \in R$ and define the operators of **modulation** E_b, and **translation** T_a for functions $f \in L^2(R)$ by:

$$E_bf(x) = e^{2\pi ibx}f(x),$$

and

$$T_af(x) = f(x-a).$$

Given $g \in L^2(R)$, and $a, b > 0$, we say that (g, a, b) **generates a WH-frame for** $L^2(R)$ if $(E_{mb}T_{na}g)_{m,n\in Z}$ is a frame for $L^2(R)$. The function g is referred to as the **window function** while in other areas it is also called the **mother wavelet**. The numbers a, b are the **frame parameters**, with a being the **shift parameter** and b the **modulation parameter**.

The Zak Transform has been a major tool in this area. For a good survey of the properties of the Zak Transform, we refer the reader to [15]. Most of the time a,b will be arbitrary real numbers, while examples etc. with the Zak Transform will be done for $a = b = 1$. So we will give a special form of the Zak Transform for this case.

Definition 2.2. *The* **Zak Transform of a function** $f \in L^2(R)$ **is**

$$(Zf)(t,\nu) = \sum_{k\in Z} f(t-k)e^{2\pi ik\nu}, \quad \textbf{a.e.} \quad t,\nu \in R. \tag{2.5}$$

Zf satisfies the quasi-periodicity relations for a.e. $t, \nu \in R$ and all $k, \ell \in Z$

$$(Zf)(t+k,\nu) = e^{2\pi ik\nu} \tag{2.6}$$

and

$$(Zf)(t,\nu+\ell) = (Zf)(t,\nu). \tag{2.7}$$

So the values of Zf are completely determined by its values on the unit square $[0,1] \times [0,1]$, and in fact, the Zak Transform is a unitary map of $L^2(R)$ onto $L^2([0,1]^2)$. Consequently, if $F \in L^2_{loc}(R^2)$ satisfies

$$F(t+k,\nu) = e^{2\pi ik\nu}F(t,\nu) \quad \text{and} \quad F(t,\nu+\ell) = F(t,\nu), \tag{2.8}$$

for a.e. $t, \nu \in R$ and $k, \ell \in Z$, then there is a unique $f \in L^2(R)$ such that $F = Zf$. Moreover, a function can be recovered from its Zak Transform by the formula

$$f(t) = \int_0^1 (Zf)(t,\nu)\, d\nu. \tag{2.9}$$

Also, since we are working with the case $a = b = 1$ we have for all $m, n \in Z$

$$Z(E_mT_ng)(t,\nu) = e^{-2\pi in\nu+2\pi imt}(Zg)(t,\nu) \;\text{ a.e. }\; t,\nu, \tag{2.10}$$

and

$$(2.11) \qquad \sum_{m,n\in Z} |< f, E_m T_n g >|^2 = \int_0^1 \int_0^1 |(Zf)(t,\nu)|^2 |(Zg)(t,\nu)|^2 \, dt \, d\nu.$$

For well-behaved g, we have that

$$(2.12) \qquad |(Zg)(t,\nu)|^2 = \sum_k \left(\sum_n g(t-n)\overline{g(t-n-k)} \right) e^{-2\pi i k\nu}.$$

Finally, we see from (2.11) and the fact that the Zak Transform is an isometry, that $(E_m T_n g)$ has a finite upper frame bound if and only if Zg is essentially bounded, the upper frame bound being given by

$$B = \text{ess sup}_{t,\nu\in[0,1]} |(Zg)(t,\nu)|^2.$$

The class of functions we will be working with are those $g \in L^2(R)$ for which (g, a, b) has a finite upper frame bound. We call these functions the **pre-frame functions** and denote this class of functions by **PF**.

It is known [11] that if $g \in$ PF, then g is bounded. The class PF has not been identified yet but our earlier discussion on frame properties yields:

Proposition 2.3. *The following are equivalent:*

(1) $g \in PF$.

(2) The pre-frame operator $T : \ell_2 \to L^2(R)$ *given by:* $T(e_{m,n}) = E_{mb}T_{na}g$ *is a bounded linear operator, where* $(e_{m,n})_{m,n\in Z}$ *is any orthonormal basis for* ℓ_2.

(3) There is a constant $B > 0$ *satisfying for all* $f \in L^2(R)$:

$$\sum_{m,n} |< f, E_{mb}T_{na}g >|^2 \le B\|f\|^2.$$

Moreover, if $a = b = 1$, *(1)-(3) are equivalent to*

(4) Zg *is essentially bounded.*

Theorem 2.1 of Casazza and Christensen [1] yields a sufficient condition for $g \in$ PF:

Proposition (Casazza/Christensen [1]). *If* $g \in L^2(R)$, $a, b \in R$ *and*

$$(2.13) \qquad \sum_{k\in Z} \Big| \sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)} \Big| \le B \quad a.e.,$$

then $f \in PF$.

Condition (2.13) is usually easy to check by hand. Furthermore it is automatically satisfied for all functions g belonging to the **Wiener Amalgam space**

$$W(L^\infty,\ell_1) := \{g \in L^2(R) \mid \sum_{m\in Z} ||g\cdot 1_{[m,m+1[}||_\infty < \infty\}.$$

For a proof we refer to [11], Theorem 4.1.8. It can be shown that $W(L^\infty,\ell_1)$ is contained in the **Feichtinger algebra** S_0, which is a dense subspace of $L^2(R)$. For more information about $W(L^\infty,\ell_1), S_0$ and related spaces we refer to Chapter 3 in the research monograph [7].

It can be shown that a variation of (2.13) produces a necessary condition for $f \in$ PF. That is, if $f \in$ PF then,

$$\sum_{k\in Z} \Big|\sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)}\Big|^2 \le B \quad \text{a.e.}$$

We will not show that this condition is necessary since it is clearly not strong enough to classify PF. Also, the sufficient condition for $g \in$ PF given in (2.13) is not necessary as we show next. This condition and its relationship to the Walnut representation of the frame operator for general $a, b \in R$ are examined in detail by Casazza, Christensen, and Janssen [2].

Proposition 2.4. *There is a function* $g \in PF$ *which fails to satisfy condition (2.13).*

Proof. Condition (2.13) and the properties of the Zak Transform indicate we should be looking for a Zg (whence a g) so that Zg is essentially bounded but $(Zg)(t,\cdot)$ is not continuous a.e. t. So let

$$F(t,\nu) = \begin{cases} 1: & 0 \le t \le 1,\ 0 \le \nu \le \dfrac{1}{2} \\ \dfrac{1}{2}: & 0 \le t < 1,\ \dfrac{1}{2} < \nu < 1 \end{cases}$$

and extend F quasi-periodically according to (2.6) and (2.7). Then $F = Zg$ for some unique $g \in L^2(R)$, and by Proposition 2.3, $g \in$ PF. We can explicitly compute the function g. For $0 \le t < 1$, $k \in Z$ we have that

$$(Zg)(t+k,\nu) = e^{2\pi i k\nu}(Zg)(t,\nu), \quad 0 \le \nu < 1.$$

Thus,

$$g(t+k) = \int_0^1 (Zg)(t+k,\nu)\ d\nu = \int_0^1 e^{2\pi i k\nu} F(t,\nu)\ d\nu$$

$$= \int_0^{1/2} e^{2\pi ik\nu}\ d\nu + \frac{1}{2}\int_{1/2}^{1} e^{2\pi ik\nu}\ d\nu$$

$$= \begin{cases} \frac{3}{4} : \ k = 0 \\ \frac{1}{4\pi ik}((-1)^k - 1) : \ 0 \neq k \in Z. \end{cases}$$

Now, for $k \in Z$ and $t \in [0,1]$ we have

$$\sum_{n\in Z} g(t-n)\overline{g(t-n-k)} = \tag{2.14}$$

$$\frac{3((-1)^k - 1)}{8\pi ik} + \frac{1+(-1)^k}{8\pi^2}\left[\sum_{n\ odd;\ n\neq 0,-k} \frac{1}{n(n+k)}\right].$$

If $k \in Z$ is odd, then $1 + (-1)^k = 0$, so the series on the right hand side of (2.14) only needs to be considered for k even. In this case,

$$\sum_{n\ odd;\ n\neq 0,-k} \frac{1}{n(n+k)} = \sum_{n\ odd;\ n\neq 0,-k} \frac{1}{k}\left(\frac{1}{n} - \frac{1}{n+k}\right) = 0 \ \text{ if } \ k \neq 0,$$

and

$$\sum_{n\ odd} \frac{1}{n^2} = 2\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} = \frac{\pi^2}{4}.$$

Thus, for all $t \in [0,1)$ we get,

$$\sum_{n\in Z} g(t-n)\overline{g(t-n-k)} = \begin{cases} (\frac{3}{4})^2 + \frac{1}{4\pi^2}\sum_{n\ \ odd} \frac{1}{n^2} = \frac{5}{8} : \ k = 0 \\ \frac{3((-1)^k - 1)}{8\pi ik} : \ k \neq 0. \end{cases}$$

This yields immediately that

$$\sum_{k\in Z} \left|\sum_{n\in Z} g(t-n)\overline{g(t-n-k)}\right| = \infty,$$

which completes the proof of Proposition 2.4.

Throughout the paper, we will be working with infinite sums of the form:

$$\sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)}.$$

So let us discuss the question of convergence here, and ignore it for the rest of the paper.

Proposition 2.5. *If $g \in L^2(R)$ then for all $k \in Z$ the series*

$$\sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)}$$

converges absolutely a.e. $x \in R$.

Proof. Since $g, T_{k/b}g \in L^2(R)$ we have that $g\overline{T_{k/b}g} \in L^1(R)$. Also,

$$\|g\overline{T_{k/b}g}\|_{L^1} = \int_R |g(x)T_{k/b}(x)|\;dx = \int_0^a \sum_{n\in Z} |g(x-na)g(x-na-k/b)|\;dx < \infty.$$

It follows that

$$\sum_{n\in Z} |g(x-na)\overline{g(x-na-k/b)}| < \infty \;\text{ a.e.}$$

The main result we need throughout the paper is a basic calculation due to Daubechies [3] (see also [11]). Careful examination of the proof of this result in [11], Theorem 4.1.5, shows that we do not need the assumption of $g \in$ PF used there. By checking functions which are bounded and supported just on an interval of length 1/b, one can see that the proof goes through without change for any $g \in L^2(R)$ with the property that $\sum_n |g(x-na)|^2 \le B$ a.e. It is this stronger result we now state.

WH-Frame Identity. *Let $g \in L^2(R)$ and assume there is a constant $B > 0$ such that $\sum_n |g(x-na)|^2 \le B$ a.e. If $f \in L^2(R)$ is bounded and compactly supported, then*

$$\sum_{n\in Z}\sum_{m\in Z} |<f, E_{mb}T_{na}g>|^2 = F_1(f) + F_2(f) \tag{2.15}$$

where

$$F_1(f) = b^{-1}\int_R |f(x)|^2 \sum_n |g(x-na)|^2\;dx, \tag{2.16}$$

$$F_2(f) = b^{-1}\sum_{k\neq 0}\int_R \overline{f(x)}f(x-k/b)\sum_n g(x-na)\overline{g(x-na-k/b)}\;dx = \tag{2.17}$$

$$b^{-1}\sum_{k\geq 1} 2Re\int_R \overline{f(x)}f(x-k/b)\sum_n g(x-na)\overline{g(x-na-k/b)}\;dx.$$

To simplify the notation a little we introduce the following auxiliary functions:

$$G(x) = \sum_{n \in Z} |g(x-na)|^2, \tag{2.18}$$

and for all $0 \neq k \in Z$,

$$G_k(x) = \sum_{n \in Z} g(x-na)\overline{g(x-na-k/b)}, \tag{2.19}$$

It follows that $G_0 = G$, and G_k are periodic functions on R of period a.

We mention one interesting consequence of the WH-frame Identity.

Theorem 2.6. *If $g \in L^2(R)$ and $G_k(x) = 0$ a.e. for all $k \neq 0$, then the following are equivalent:*

(1) $(E_{mb}T_{na}g)_{m,n \in Z}$ is a Weyl-Heisenberg frame with frame bounds A, B.

(2) We have

$$A \leq G_0(x) \leq B, \quad a.e.$$

Proof. Our hypotheses and the WH-frame Identity imply that for all bounded, compactly supported $f \in L^2(R)$ we have

$$\sum_{m,n \in Z} | < f, E_{mb}T_{na}g > |^2 = b^{-1} \int_R |f(x)|^2 G_o(x)\ dx.$$

The theorem is now obvious.

3. Classifying Tight WH-Frames

Now we will classify the tight WH-frames both abstractly in terms of the behavior of related families of vectors and concretely in terms of the behavior of the functions $G_k(x)$. In the next section we will write down explicitly the functions which satisfy the conditions of our theorem for some cases. We start with a basic fact which will simplify parts of the theorem and will be used in Section 5 to classify the alternate dual frames for a WH-frame. The proof consists of standard arguments.

Proposition 3.1. *Let $g, h \in L^2(R)$ and $a, b \in R$.*

(1) $h \perp E_{mb}g$, for all $m \neq 0$ if and only if there is a constant C so that

$$\sum_{n \in Z} h(x-n/b)\overline{g(x-n/b)} = C, \quad a.e.$$

(2) If $n \neq 0$, then $h \perp E_{mb}T_{na}g$ for all $m \in Z$ if and only if

$$\sum_k h(x-k/b)\overline{g(x-k/b-na)} = 0, \quad a.e.$$

Proof. (2): We just calculate:

$$< h, E_{mb}T_{na}g >= \int_R h(x)\overline{E_{mb}g(x-na)}\ dx = \int_R h(x)\overline{g(x-na)}e^{-2\pi imbx}\ dx =$$

$$\int_0^{1/b} \sum_{k\in Z} h(x-k/b)\overline{g(x-k/b-na)}e^{-2\pi imbx}\ dx.$$

(2) follows immediately from here and (1) is just the case of n=0 in (2).

We are now ready to prove the classification theorem for tight WH-frames. The equivalence of (1) and (2) below can be found in [18], Corollary 2.19, while it also follows from the developments in [12], Subsection 1.3, and the equivalence of (1) and (3) can be deduced from [18], Corollary 6.8 or from [12], Theorem 1.4.1.

Theorem 3.2. *Let $g \in L^2(R)$ and $a, b \in R$. The following are equivalent:*

(1) $(E_{mb}T_{na}g)_{n,m\in Z}$ is a normalized tight Weyl-Heisenberg frame for $L^2(R)$.

(2) We have:
(a) $G(x) = \sum_{n\in Z} |g(x-na)|^2 = b$ a.e.,
(b) $G_k(x) = \sum_{n\in Z} g(x-na)\overline{g(x-na-k/b)} = 0$ a.e. for all $k \neq 0$.

(3) We have, $g \perp E_{n/a}T_{m/b}g$, for all $(n,m) \neq (0,0)$, and $\|g\|^2 = ab$.

(4) $(E_{n/a}T_{m/b}g)_{n,m\in Z}$ is an orthogonal sequence in $L^2(R)$ and $\|g\|^2 = ab$.

(5) $(E_{mb}T_{na}g)_{n,m\in Z}$ is a Weyl-Heisenberg frame for $L^2(R)$ with frame operator S and $Sg = g$.

Moreover, when at least one of (1)-(5) holds, $(E_{mb}T_{na}g)_{m,n\in Z}$ is an orthonormal basis for $L^2(R)$ if and only if $\|g\| = 1$.

Proof. (1) $\Rightarrow$ (2): Assume $(E_{mb}T_{na}g)_{n,m\in Z}$ is a normalized tight frame for $L^2(R)$. For any function $f \in L^2(R)$ which is supported on an interval of length $< 1/b$ we have for all $x \in R$ and all $0 \neq k \in Z$ that $\overline{f(x)}f(x-k/b) = 0$. That is, $F_2(f) = 0$ in the WH-frame identity. Now, again by the WH-frame Identity, we have

$$\|f\|^2 = \int_R |f(x)|^2\ dx = \sum_n \sum_m | < f, E_{mb}T_{na}g > |^2 = F_1(f) + F_2(f)$$

$$= b^{-1}\int_R |f(x)|^2 \sum_n |g(x-na)|^2\ dx = b^{-1}\int_R |f(x)|^2 G(x)\ dx.$$

Since this equality holds for all $f \in L^2(I)$, for any interval I of length $< 1/b$, it follows easily that $G(x) = b$, a.e. Hence, for all $f \in L^2(R)$, $F_1(f) = \|f\|^2$. But now, again by the WH-frame Identity, we have for all bounded, compactly supported $f \in L^2(R)$,

$$\|f\|^2 = F_1(f) + F_2(f) = \|f\|^2 + F_2(f).$$

That is, $F_2(f) = 0$, for all bounded compactly supported $f \in L^2(R)$. Now fix $k_0 \geq 1$ and let I be any interval in R of length $\leq 1/b$. Define a function $f \in L^2(R)$ by:

$$f(x) = e^{i\arg G_{k_0}(x)}, \quad \text{for all} \ \ x \in I,$$

and $f(x - k_0/b) = 1$ for all $x \in I$ and $f(x) = 0$, otherwise. Then, by the WH-frame Identity,

$$0 = F_2(f) = b^{-1}\sum_{k\geq 1} 2\mathrm{Re} \int_R \overline{f(x)} f(x-k/b) \sum_n g(x-na)\overline{g(x-na-k/b)}\ dx =$$

$$b^{-1} 2\mathrm{Re}\int_R \overline{f(x)} f(x-k_0/b) G_{k_0}(x)\ dx = b^{-1} 2 \int_I |G_{k_0}(x)|\ dx.$$

It follows that $G_{k_0}(x) = 0$, a.e. on I. Since k_0 and I were arbitrary, we have $(2b)$.

(2) $\Rightarrow$ (1): By assumption $(2b)$ and the WH-frame Identity, we have $F_2(f) = 0$ for all bounded, compactly supported $f \in L^2(R)$. Hence, applying assumption $(2a)$ and the WH-frame Identity:

$$\sum_n \sum_m |< f, E_{mb}T_{na}g >|^2 = F_1(f) = b^{-1}\int_R |f(x)|^2 \sum_n |g(x-na)|^2\ dx =$$

$$\int_R |f(x)|^2\ dx = \|f\|^2.$$

Since this equality holds on a dense subset of $L^2(R)$, it holds for all $f \in L^2(R)$. So $(E_{mb}T_{na}g)_{n,m\in Z}$ is a normalized tight frame for $L^2(R)$.

(2) $\Leftrightarrow$ (3): By (2) of Proposition 3.1, (2b) is equivalent to $g \perp E_{n/a}T_{m/b}g$, for all $m \neq 0$. Also, we have,

$$\|g\|^2 = \int_R |g(x)|^2\ dx = \int_0^a \sum_{n\in Z} |g(x-na)|^2\ dx = \int_0^a G(x)\ dx.$$

So the rest of the equivalence follows from (1) of Proposition 3.1.

(3) $\Leftrightarrow$ (4): This is immediate from the observation that for all $m, n, \ell, k \in Z$ we have:

$$< E_{n/a}T_{m/b}g, E_{k/a}T_{\ell/b}g >= e^{2\pi i \frac{n-k}{a}\frac{m}{b}} < g, E_{\frac{k-n}{a}}T_{\frac{\ell-m}{b}}g > .$$

(1) $\Leftrightarrow$ (5): Since S commutes with $E_{mb}T_{na}$, we have that $S = I$ if and only if $Sg = g$.

For the moreover part of the theorem, we just observe that for all $m, n \in Z$, $\|g\| = \|E_{mb}T_{na}g\|$. Hence, if $\|g\| = 1$, then our tight frame consists of norm 1 elements which now form an orthonormal basis by Remark 2.1.

Now we can give one simple class of non-compactly supported functions which produce normalized tight WH-frames.

Corollary 3.3. *Let $a, b \in R$, and assume that g is supported on an interval I with $|I| \leq \frac{1}{a}$. If*

$$\sum_{m \in Z} |g(x - mb)|^2 = a \quad a.e.$$

then $(E_{mb}T_{na}\hat{g})$ is a normalized tight WH-frame consisting of non-compactly supported functions.

Proof. By Theorem 3.2, the conditions stated in Corollary 3.3 guarantee that $(E_{na}T_{mb}g)$ is a normalized tight WH-frame for $L^2(R)$. Taking the Fourier Transform of this yields that $(E_{mb}T_{na}\hat{g})$ is a normalized tight WH-frame. Evidently, $\hat{g}$ is not compactly supported.

Since $ab \leq 1$ for a WH-frame (see below), it follows that $b \leq 1/a$. So it is easy to produce functions satisfying the conditions in Corollary 3.3.

With the characterization in Theorem 3.2, we can now recover easily and directly from the definition the basic properties of WH-frames which formerly required some more work [3,4,11]. Recently, Janssen [13,14] has derived some of these results also in an easier fashion. Our proof however, is simple and follows directly from the definitions. For any WH-frame $(E_{mb}T_{na}g)_{m,n \in Z}$ with frame operator S, a direct computation shows that

$$S(E_{mb}T_{na}g) = E_{mb}T_{na}Sg, \quad \text{for all} \quad m, n \in Z.$$

It follows that $S^{-1/2}$ also commutes with $E_{mb}T_{na}$ and so $(E_{mb}T_{na}S^{-1/2}g)_{m,n \in Z}$ is a normalized tight WH-frame which is equivalent to $(E_{mb}T_{na}g)_{m,n \in Z}$ and hence must satisfy the conditions of Theorem 3.2. In particular, for all $(m,n) \neq (0,0)$ we have:

$$< S^{-1}g, E_{n/a}T_{m/b}g >=< S^{-1/2}g, E_{n/a}T_{m/b}S^{-1/2}g >= 0.$$

This leads to:

Corollary 3.4. *Let $(E_{mb}T_{na}g)_{m,n\in Z}$ be a WH-frame for $L^2(R)$.*

(1) $S^{-1}g \perp E_{n/a}T_{m/b}g$, for all $(m,n) \neq (0,0)$ and

$$< S^{-1}g, g >=< S^{-1/2}g, S^{-1/2}g >= \|S^{-1/2}g\|^2 = ab \leq 1.$$

(2) If $ab < 1$ then the WH-frame is overcomplete.

(3) If $ab = 1$ then the WH-frame is a Riesz basis.

Proof. (1): All of this was observed before we stated the theorem except the last inequality $ab \leq 1$ which follows immediately from Remark 2.1.

(2): If our WH-frame is exact then $(E_{mb}T_{na}S^{-1/2}g)$ is an orthonormal basis. Hence,

$$1 = \|S^{-1/2}g\|^2 = ab.$$

(3): If $ab = 1$ then $(E_{mb}T_{na}S^{-1/2}g)$ is a tight WH-frame and hence $(E_{n/a}T_{m/b}S^{-1/2}g)$ is an orthogonal sequence by Theorem 3.2 (4). But, $ab = 1$ implies that $n/a = nb$ and $m/b = ma$, so $(E_{nb}T_{ma}S^{-1/2}g)$ is an orthogonal basis for $L^2(R)$. Since $S^{-1/2}$ is an invertible operator on $L^2(R)$, it follows that $(E_{mb}T_{na}g)$ is a Riesz basis for $L^2(R)$.

Actually, there is a stronger statement than (1) of Corollary 3.4. That is, if $ab > 1$ then for any $g \in L^2(R)$ there is an $f \in L^2(R)$ with $f \perp \text{span}_{m,n\in Z}(E_{mb}T_{na}g)$. See Daubechies [3], page 978 for a discussion of this result.

Finally, we point out what appears to be a surprising consequence of Theorem 3.2 (which can also be gleaned out of sections 1.2-1.4 of [12]) which indicate a relationship between the values of a function and its Fourier Transform.

Corollary 3.5. *If $g \in PF$ and $ab \leq 1$, the following are equivalent:*

(1) The function g satisfies:

$$\sum_n g(x-na)\overline{g(x-na-k/b)} = 0 \quad a.e. \text{ for all } \; k \neq 0$$

and

$$\sum_n |g(x-na)|^2 = b \quad a.e.$$

(2) The Fourier Transform of the function g satisfies:

$$\sum_n \hat{g}(x-nb)\overline{\hat{g}(x-nb-k/a)} = 0 \quad a.e. \text{ for all } \; k \neq 0$$

and

$$\sum_n |\hat{g}(x-nb)|^2 = a \quad a.e.$$

Proof. Corollary 3.5 comes from Theorem 3.2 applied to the families $(E_{mb}T_{na}g)$ and $(E_{ma}T_{nb}\hat{g})$ which are normalized tight WH-frames together.

4. The Functions g Giving Tight WH-Frames

In this section we will give an explicit representation for the functions giving tight WH-frames given in Theorem 3.2 (2) for two specific cases. If $f(x,y)$ is any function of two variables, we denote by $f_y(x)$ the function:

$$f_y(x) = f(x,y).$$

We start with a simple proposition which contains the basic notions which will be used in our characterization. This is essentially a triviality, but because of its central role in all the examples, we give a short proof.

Proposition 4.1. *The following are equivalent:*

(1) The sequence $z = (c_n)_{n\in Z} \in \ell_2(Z)$ is orthogonal to all of its proper shifts and $\|z\|^2 = b$.

(2) The unique function $h : [0,1] \to C$ with Fourier series coefficients $\hat{h}(n) = c_n$, for all $n \in Z$ has $|h(x)|^2 = b$ a.e.

(3) There is a measurable function $f : [0,1] \to R$ so that the function h in (2) is of the form

$$h(x) = \sqrt{b}\; e^{2\pi i f(x)}.$$

Proof. (1) $\Rightarrow$ (2) If $(\hat{h}(n)) = (c_n)$ is orthogonal to all its shifts then $h \perp e^{2\pi i m\cdot}h$, for all $m \neq 0$. That is, for all $m \neq 0$ we have

$$0 =< h, e^{2\pi i m\cdot}h >=< |h|^2, e^{2\pi i m\cdot} > .$$

Hence, $|h|^2 = C$ a.e. Also, $\|h\|^2 = \|(c_n)\|^2 = b$.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): Given h as in (3), for any $m \in Z$ we have:

$$< h, E_m h >= \int_0^1 h(x)\overline{h(x)e^{2\pi i m x}}\, dx = \int_0^1 |h(x)|^2 e^{-2\pi i m x} dx =$$

$$\int_0^1 b\ e^{-2\pi i m x} dx = b\delta_m.$$

So h is orthogonal to $e^{2\pi i m \cdot} h$ for all $m \neq 0$. Hence, $(\hat{h}(n))$ is orthogonal to all its proper shifts and $\|(\hat{h}(n))\|^2 = b$.

In the case $ab = 1$ we are now ready to give an explicit representation of the functions g for which $(E_{mb}T_{na}g)$ is an orthonormal basis for $L^2(R)$.

Theorem 4.2. *Let $ab = 1$ and $g \in L^2(R)$. The following are equivalent:*
(1) $(E_{mb}T_{na}g)$ is a normalized tight WH-frame for $L^2(R)$.
(2) $(E_{mb}T_{na}g)$ is an orthonormal basis for $L^2(R)$.
(3) There is a measurable function $f : [0,1] \times [0,a) \to R$ and

$$h(x,y) = \sqrt{b}\ e^{2\pi i f(x,y)}$$

so that

$$g(y + na) = \hat{h_y}(n), \quad \textit{for all} \quad y \in [0,a).$$

Proof. This is essentially immediate from our assumptions and Theorems 3.2 and 4.1. Since $ab = 1$, we have $a = 1/b$ so Theorem 3.2 (2) becomes:

(a) $G(x) = \sum_{n\in Z} |g(x - na)|^2 = b$, a.e.,

(b) $G_k(x) = \sum_{n\in Z} g(x - na)\overline{g(x - (n - k)a)} = 0$, a.e. for all $k \neq 0$.

But condition (b) is equivalent to: $z_y = (g(y - na))_{n\in Z}$ is orthogonal to all of its proper shifts and (a) is equivalent to $\|z_y\|^2 = b$. By Theorem 4.1 these conditions are equivalent to: For each $y \in [0,a)$ there is a function $f_y : [0,1] \to R$ and $h_y : [0,a] \to C$ with

$$h_y(x) = e^{2\pi i f_y(x)},$$

and

$$\hat{h_y}(n) = g(y - na).$$

So defining $f(x,y) : [0,1] \times [0,a) \to R$ and $h(x,y)$ by:

$$f(x,y) = f_y(x), \qquad h(x,y) = h_y(x),$$

yields the theorem modulo the measurability conditions which are obvious.

Note that the h in Theorem 4.2 (3) is essentially the Zak transform of g.

We now show how the conditions in Theorem 4.2 quickly and easily give the standard examples from the literature, as well as some new ones, on tight WH-frames.

Example 4.3. *Letting $f(x,y) = C$, for all $y \in [0,a]$ gives the function*

$$g(x) = \sqrt{b}\ \chi_{[0,a)}$$

which yields a normalized tight frame (in fact, an orthonormal basis for $L^2(R)$). Letting

$$f(x,y) = C_y, \quad \text{for all} \;\; x \in [0,1],$$

gives a function g of the form:

$$g(x) = \sqrt{b}e^{ih(x)}\chi_{[0,a)}$$

where $h : [0,1] \to R$.

Example 4.4. *Letting $f(x,y) = f(x,y')$, for all $y, y' \in [0,a]$ gives the function*

$$g(x) = \sum_{n \in Z} c_n \chi_{[na,(n+1)a)}$$

where (c_n) is an ℓ_2 sequence which is orthogonal to all its proper shifts. This now yields a normalized tight WH-frame.

Example 4.5. *If we partition $[0,a)$ into disjoint measurable sets $(A_n)_{n \in Z}$, let $B_n = A_n + \{n\}$ and define*

$$f(x,y) = d_n + nx, \quad \text{for all} \;\; y \in A_n.$$

Then

$$g(x) = \sum_{n \in Z} c_n \chi_{B_n},$$

yields a normalized tight frame.

We leave it to the reader to check other variations for the functions $f(x,y)$ to yield old and new examples of tight WH-frames.

We proceed to the next level of the representations of the functions g giving WH-frames. See Daubechies [3] (page 981) for a formulation of this result in terms of the Zak Transform.

Proposition 4.6. *Let $ab = 1/q$, $q \in Z$ and $g \in L^2(R)$. The following are equivalent:*

(1) $(E_{mb}T_{na}g)$ is a normalized tight WH-frame for $L^2(R)$.

(2) There are functions $f_j(x,y) : [0,1] \times [0,a) \to C$, $1 \le j \le q$ satisfying:

(a)
$$\sum_{j=1}^{q} |f_{j,y}(x)|^2 = b, \quad \text{for all } \ x \in [0,1].$$

(b)
$$g(y + ja + n/b) = \hat{f_{j,y}}(n), \quad \text{for all } \ 1 \le j \le q, \quad 0 \le y < \frac{1}{qb} = a.$$

Proof. We leave it to the reader to check that g is a well-defined function on all of R. For $0 \le y < \frac{1}{qb} = a$ we have:

(4.1)
$$\sum_{n \in Z} g(y - na)\overline{g(x - na - k/b)} = \sum_{j=1}^{q} \sum_{n \in Z} g(y + ja + nqa)\overline{g(y + ja + nqa + k/b)} =$$

$$\sum_{j=1}^{q} \sum_{n \in Z} g(y + ja + n/b)\overline{g(y + ja + (n+k)/b)} = \sum_{j=1}^{q} \sum_{n \in Z} \hat{f_{j,y}}(n)\overline{\hat{f_{j,y}}(n+k)}.$$

Now, we have the equality in (4.1) equal to 0 for all $k \ne 0$ if and only if:

$$0 = \sum_{j=1}^{q} < f_{j,y}, E_k f_{j,y} >= \int_0^1 \sum_{j=1}^{q} |f_{j,y}(x)|^2 e^{-2\pi i k x} \, dx, \quad \text{for all } \ k \ne 0.$$

But, this is equivalent to:

$$\sum_{j=1}^{q} |f_{j,y}(x)|^2 = C_y, \quad \text{for all } \ x \in [0,1]$$

for some constant C_y. However, for $k = 0$ and all $0 \le y < 1/qb$ we have:

(4.2)
$$\sum_{n \in Z} |g(y - na)|^2 = \sum_{j=1}^{q} \sum_{n \in Z} |\hat{f_{j,y}}(n)|^2 = \sum_{j=1}^{q} \|f_{j,y}\|^2 =$$

$$\int_0^1 \sum_{j=1}^{q} |f_{j,y}(x)|^2 \, dx = C_y.$$

Since we have a normalized tight WH-frame if and only if the sums in (4.2) equal b a.e., i.e. if and only if $C_y = b$ for all $y \in [0,a)$.

We leave it to the reader to examine the special cases of Proposition 4.6. Common examples are when all the functions f_j are equal or when they are each a constant function.

One can find a representation similar to that of Proposition 4.6 for the case where ab = p/q is a rational number [21,22,23] and [12], Section 1.5. Consider the case that $ab = p/q \leq 1$ with $p, q \in Z$, and gcd (p,q) = 1. We let for $\lambda > 0$ and $f \in L^2(R)$

$$(Z_\lambda f)(t,\nu) = \lambda^{1/2} \sum_{k \in Z} f(\lambda(t-k)) e^{2\pi i k \nu}, \quad \text{a.e. } t,\nu \in R. \tag{4.3}$$

The generalized Zak transform Z_λ has similar properties as the standard Zak transform Z $(\lambda = 1)$ of Section 2. We define for $f \in L^2(R)$ and a.e. $t, \nu \in R$ the $(p \times q)-$matrices

$$\Phi^f(t,\nu) = \left(p^{-1/2} (Z_{1/b} f)(t - \ell \frac{p}{q}, \nu + \frac{k}{p}) \right)_{k=0,1,\cdots,p-1;\ \ell=0,1,\cdots,q-1},$$

$$\Psi^f(t,\nu) = \left(p^{-1/2} (Z_a f)(t - k \frac{q}{p}, \nu - \frac{\ell}{q}) \right)_{k=0,1,\cdots,p-1;\ \ell=0,1,\cdots,q-1},$$

where we have chosen $\lambda = b^{-1}$, $\lambda = a$, respectively, in (4.3). Then we have that $g \in L^2(R)$ generates a tight WH-frame for the parameters a,b if and only if there is a $c > 0$ such that for a.e. $t, \nu \in R$

$$\Phi^g(t,\nu) \left(\Phi^g(t,\nu) \right)^* = c I_{p \times p},$$

if and only if there is a $d > 0$ such that for a.e. $t, \nu \in R$

$$\Psi^g(t,\nu) \left(\Psi^g(t,\nu) \right)^* = d I_{p \times p},$$

where $I_{p \times p}$ denotes the $p \times p$ identity matrix and * represents the conjugate transpose.

The case where ab is irrational seems to be very difficult and we have not found any good general examples what-so-ever for this case. Yet, we know that every WH-frame is equivalent to a normalized tight WH-frame and so there must be a huge number of such functions even for irrational ab. It would be very interesting to have an exact representation of this class of functions.

5. Alternate Dual Frames

If $(f_i)_{i\in I}$ is a frame for a Hilbert space H, a frame $(h_i)_{i\in I}$ for H is called an **alternate dual frame** or a **pseudo-dual** for $(f_i)_{i\in I}$ if

$$f = \sum_{i\in I} < f, h_i > f_i, \quad \text{for all} \quad f \in H. \tag{5.1}$$

We already know one sequence $(h_i)_{i\in I}$ satisfying (5.1). Namely, the sequence $(S^{-1}f_i)_{i\in I}$. That is, for all $f \in H$,

$$f = S(S^{-1}f) = \sum_{i\in I} < S^{-1}f, f_i > f_i = \sum_{i\in I} < f, S^{-1}f_i > f_i.$$

We call $(S^{-1}f_i)_{i\in I}$ the **canonical dual** of (f_i). If $(f_i)_{i\in I}$ is a normalized tight frame, then $S = I$, so the frame equals its canonical dual frame. The converse of this clearly holds also. But in general, there may be many alternate dual frames for a given frame. A simple example would be to let our frame be $\{e_1, e_1, e_2, e_2, \dots\}$ where (e_i) is an orthonormal basis for H, and observe that $\{e_1, 0, e_2, 0, \dots\}$ is an alternate dual frame while $\{e_1/2, e_1/2, e_2/2, e_2/2, \dots\}$ is the canonical dual frame. In the definition we assumed that the sequence $(h_i)_{i\in I}$ is a frame for H. For a given frame $(f_i)_{i\in I}$ there might exist sequences $(h_i)_{i\in I}$which are not frames, but such that (5.1) holds. For example,

$$\{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \dots\}$$

is a normalized tight frame for H and the sequence $(h_i)_{i\in I}$ given by,

$$\{e_1, \sqrt{2}e_2, 0, \sqrt{3}e_3, 0, 0, \dots\}$$

is not a frame for H but satisfies (5.1). But if $(f_i)_{i\in I}$ and $(h_i)_{i\in I}$ are Bessel sequences and they satisfy (5.1), then $(h_i)_{i\in I}$ and $(f_i)_{i\in I}$ are alternate dual frames. For the basic properties of alternate dual frames we refer to [10,12,16,18]. In particular, Li [16] has given a characterization of all alternate duals for a given frame.

It is also known that a frame has a unique alternate dual frame if and only if it is a Riesz basis. No two distinct alternate dual frames are equivalent to one another [10], so the canonical dual frame is the only alternate dual frame which is equivalent to the original frame. Finally, if $(h_i)_{i\in I}$ is an alternate dual frame to $(f_i)_{i\in I}$, then $(f_i)_{i\in I}$ is an alternate dual frame to $(h_i)_{i\in I}$. Now we will use the techniques developed in Section 3 to characterize the Weyl-Heisenberg alternate dual frames for a given Weyl-Heisenberg frame. We need a beautiful result of Wexler-Raz [19]. The rigorous form of the result we state is due to Janssen [13].

Theorem(Wexler-Raz). *Let $g, h \in PF$. Then $(E_{mb}T_{na}h)$ and $(E_{mb}T_{na}g)$ are alternate dual frames if and only if $h \perp E_{n/a}T_{m/b}g$, for all $(m,n) \neq (0,0)$, and $< h, g >= ab$.*

We proceed with the corresponding result to Theorem 3.2 for alternate dual frames. Parts of this were done in the more general setting of shift invariant systems by Ron and Shen [17] and Janssen [12]. Again, our proof is simple, and follows directly from the defintions.

Theorem 5.1. *For $g, h, f \in PF$ the following are equivalent:*

(1) $(E_{mb}T_{na}h)_{m,n\in Z}$ is an alternate dual frame for $(E_{mb}T_{na}g)_{m,n\in Z}$.

(2) We have:

(a) $\sum_{n\in Z} h(x-na)\overline{g(x-na-k/b)} = 0$ a.e. for all $k \neq 0$.

(b) $\sum_{n\in Z} h(x-na)\overline{g(x-na)} = b$, a.e.

(3) $h = S^{-1}g + f$, where $f \in L^2(R)$ and $f \perp span_{n,m\in Z} E_{n/a}T_{m/b}g$.

Proof. (1) $\Leftrightarrow$ (2): This is Proposition 3.1 combined with the theorem of Wexler-Raz.

(1) $\Rightarrow$ (3): By the Wexler-Raz Theorem, $h \perp E_{n/a}T_{m/b}g$, for all $(m,n) \neq (0,0)$. By Corollary 3.3, we also have that $S^{-1}g \perp E_{n/a}T_{m/b}g$. Hence, $f = h - S^{-1}g \perp E_{n/a}T_{m/b}g$. Again by the Wexler-Raz Theorem, $< h, g >= ab$ and applying Corollary 3.3 again,

$$< h - S^{-1}g, g >=< h, g > + < S^{-1}g, g >= ab - ab = 0.$$

It follows that $h = S^{-1}g + (h - S^{-1}g) = S^{-1}g + f$ and $f \perp E_{n/a}T_{m/b}g$, for all $n, m \in Z$.

(3) $\Rightarrow$ (1): Fix $(m,n) \neq (0,0)$. We compute using Corollary 3.3:

$$< h, E_{n/a}T_{m/b}g >=< S^{-1}g + f, E_{n/a}T_{m/b}g >=$$

$$< S^{-1}g, E_{n/a}T_{m/b}g > + < f, E_{n/a}T_{m/b}g >= 0 + 0 = 0.$$

Also, using Corollary 3.3,

$$< h, g >=< S^{-1}g + f, g >=< S^{-1}g, g > + < f, g >= ab + 0 = ab.$$

So this implication follows from the Wexler-Raz Theorem.

It can be shown that the canonical dual $S^{-1}g$ has the least L^2-norm among all $h \in L^2(R)$ satisfying $h \perp E_{n/a}T_{m/b}g, \quad (n,m) \neq (0,0)$, and $< h, g >= ab$. For proofs of this fact we refer to [14], Proposition 3.3 and [5], Proposition 4.2.

Note that $(E_{mb}T_{na}g)_{m,n\in Z}$ is a normalized tight frame if and only if we can replace h in Theorem 5.2 by the function g and in this case Theorem 5.1 reduces to Theorem 3.2. Also in this case, $S = I$ so part (3) of the theorem becomes: $h = g+f$ where $f \perp \text{span}_{n,m\in Z} E_{n/a}T_{m/b}g$.

The canonical dual to a WH-frame is again a WH-frame, but in general there exists alternate duals not having the WH-structure [16].

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