On analytic design of loudspeaker arrays with uniform radiation characteristics

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Abstract

Some notes on analytical derived loudspeaker arrays with uniform radiation characteristics are presented. The array coefficients are derived via analytical means and compared with so-called maximal flat sequences known from telecommunications and information theory. It appears that the newly derived array, the quadratic phase array, has a higher efficiency than the Bessel array and a flatter response than the Barker array. Our method admits generalization to the design of arrays with desired non-uniform radiating characteristics.

1 Introduction

There is a vast amount of literature on loudspeaker arrays radiating sound in a particular direction, see e.g. [6] and the classical paper [7]. The directional characteristics of such an array— when depicted graphically— assume the form of a major or principal lobe and several minor or secondary lobes. Instead of making the array directive, one may desire an array having a directional response proportional to that of a single loudspeaker with a gain factor as high as possible. The application of such array could be to address an audience, where a single loudspeaker does not radiate sufficient power while it is desired that the perception of the listeners is independent of their position to the array. The calculation of the array coefficients is the topic of the present paper. In [8] another approach is followed where the problem is treated as an approximation to a continuous distributed sensor.

The directional response of a linear loudspeaker array depends on the coefficients assigned to the individual loudspeakers. The sound pressure of a loudspeaker array with N identical equally spaced loudspeakers is given by

$$p(\Omega,\theta,r) = A(\omega,\theta)R(\omega,r)\sum_{\ell=-M}^{M} x_\ell \mathrm{e}^{i\ell\Omega}, \tag{1}$$

where θ is the angle of observation,

$A(\omega,\theta)$ the directional response of a single loudspeaker,

d is the distance between the loudspeakers,

$\Omega = \omega\ d\sin(\theta)/c$,

ω is the radial frequency of the sound,

r is the common distance to the array,

c is the velocity of sound,

$R(\omega, r) = r^{-1}\mathrm{e}^{-i\omega r/c}$,

x_ℓ is the coefficient for the ℓ^{th} loudspeaker,

$N = 2M + 1$ is the number of loudspeakers.

We assume here that the observation point is in the far field so that, $r \gg c/\omega$ and $r \gg 2Md$.

In general terms the paper discusses various loudspeaker array coefficients: one based on Bessel functions, a newly derived array called 'the Quadratic Phase array', and coefficients with optimal autocorrelation properties known from telecommunications and information theory (such as Barker, Huffman, and binary maximally-flat (MF) sequences) [2]. The paper is organized as follows.

In Sec. 2 we consider arrays in which we take as the array coefficients $x_\ell = J_\ell(z)/\sigma$, with appropriately chosen z and σ which we call 'Bessel array'. Using the asymptotics of the Bessel functions we indicate values for z (depending on their length N) such that the resulting array has a good trade-off between spectral flatness and efficiency (see Eq. 15 for the definition of efficiency).

In Sec. 3 we present some considerations that lead to new arrays which we call 'Quadratic Phase array'. These arrays turn out to have efficiencies much better than the Bessel arrays, and a better spectral flatness than the Barker and other binary arrays ($|x_\ell| = 1$).

Contrary to Barker arrays (that have at most 13 elements), there are no length limitations for Quadratic Phase arrays. Moreover, the coefficients of the Quadratic Phase arrays are easy to compute since they are given in analytical form. Such a thing does not hold for Barker, binary (MF) sequences and (non-binary) Huffman arrays, for which an exhaustive search must be done to find the coefficients.

2 Bessel array

It was suggested by N.V. Franssen and elaborated by W. Kitzen [3] to use $x_\ell = J_\ell(z)/\sigma$. Here J_ℓ is the Bessel function of the first kind of order ℓ, the argument z is to be chosen appropriately, and σ is a normalization constant such that $\max|x_\ell| = 1$. Using the generating function of $J_\ell(z)$ [1, Eq. 9.1.41],

$$\mathrm{e}^{z(t-1/t)/2} = \sum_{\ell=-\infty}^{\infty} t^\ell J_\ell(z), \tag{2}$$

with $t = \mathrm{e}^{i\Omega}$, Eq. 1 can for $M \to \infty$ be written as

$$p(\Omega, \theta, r)_{M\to\infty} = A(\omega, \theta) R \mathrm{e}^{iz\sin\Omega}. \tag{3}$$

Eq. 3 shows that apart from a phase factor, the array exhibits a directional response proportional to that of a single loudspeaker, i.e.

$$|p(\Omega, \theta, r)_{M\to\infty}|_{Bessel} = |A(\omega, \theta) R|. \tag{4}$$

It appears from Eq. 4 that the amplitude of the sound pressure does not depend on z. However, for practical Bessel arrays, where M is finite, a judicious choice of z is necessary. Since the right-hand side of Eq. 3 has an absolute value independent of Ω, the array exhibits as an acoustical all-pass filter. It is known from psycho-acoustic theory that the human ear is not very sensitive to phase distortion, whence ignoring the phase factor does not cause serious problems.

To obtain a finite sum as in Eq. 1, the infinite series of Eq. 2 must be truncated at both sides to a finite length M, the coefficients x_ℓ must be normalized by a suitable factor σ and an appropriate fixed value of z must be chosen, depending on M and such that the modulus of the sound pressure is, to a good approximation, independent of Ω. These topics, including the influence of the truncation, are discussed below.

2.1 Calculation of appropriate fixed value of z

The error introduced due to the truncation of the infinite sum in Eq. 2 is equal to

$$\Delta = \mathrm{e}^{iz\sin\Omega} - \sum_{\ell=-M}^{M} J_\ell(z)\mathrm{e}^{i\ell\Omega}. \tag{5}$$

This error will influence the array behavior when we use the truncated version of Eq. 2 to implement Eq. 1. Clearly Δ depends on Ω, and we are interested in particular in the maximal error $\hat{\Delta}$. For fixed $z = z_F \approx M$, and Ω such that the error is maximal, it can be shown from the asymptotics of the Bessel functions ([1,

§ 9.3.]) that

$$|\hat{\Delta}| \approx \int_{\gamma}^{\infty} \mathrm{Ai}(s)ds, \tag{6}$$

where

$$\gamma = (\frac{2}{M+1})^{1/3}(M+1-z_F), \tag{7}$$

and Ai(s) is an Airy function [1, § 10.4.]. As to the choice of γ there is a trade-off between taking large $\gamma \geq 0$ so that $|\hat{\Delta}|$ is small and taking small $\gamma \geq 0$ so that the efficiency of the array is large.

A value suggested by N.V. Franssen was $z_F = M - 1$, which yields $0.5 \leq \gamma \leq 1.4$ for $100 \geq M \geq 5$. Using the asymptotic expansion [1, Eq. 10.4.82]

$$\int_0^x \mathrm{Ai}(s)ds \sim \frac{1}{3} - \frac{1}{2}\pi^{-\frac{1}{2}}x^{-\frac{3}{4}}\mathrm{e}^{-\frac{2}{3}x^{\frac{3}{2}}}, \tag{8}$$

we obtain

$$|\hat{\Delta}| \approx \frac{1}{2}\pi^{-\frac{1}{2}}\gamma^{-\frac{3}{4}}\mathrm{e}^{-\frac{2}{3}\gamma^{\frac{3}{2}}}; \tag{9}$$

the right-hand side of Eq. 9 is plotted in Fig. 1.

Using $\gamma = 2^{\frac{1}{3}}$ gives $\hat{\Delta} \approx 0.1$ and $z_F = M+1-(M+1)^{1/3}$. The advantage of this 'rule of thumb' value of z_F over Franssen's rule is that it yields an error $\hat{\Delta}$ independent of M.

2.2 Calculation of the scaling factor

To compare the performance of the array with that of a single loudspeaker, we normalize the coefficients by a scalar σ such that the largest coefficient in the summation of Eq. 2 is equal to unity. Therefore the value of ν where the maximum value of $J_\nu(z_F)$ occurs, must be calculated.

To that end we use the asymptotic expansion [1, Eq. 9.3.23] of J_ν in the transition region where J_ν takes its largest value (see Fig. 2).

Thus

$$J_\nu(\nu + a\nu^{1/3}) \approx (2/\nu)^{1/3}\mathrm{Ai}(-2^{1/3}a). \tag{10}$$

It appears that $\mathrm{Ai}(-2^{1/3}a)$ has maximum value ≈ 0.5357 at $a \approx 0.8086$. Solving $z_F = \nu + a\nu^{1/3}$ for ν, keeping in mind that $\nu \approx z_F$, gives

$$\nu \approx z_F - a{z_F}^{1/3}. \tag{11}$$

Using Eq. 10, with ν as in Eq. 11 and $z_F \approx M = (N-1)/2$ yields

$$J_\nu(z_F) \approx 0.85N^{-1/3}. \tag{12}$$

When we truncate the infinite sum in Eq. 2 to $\pm M$ and use

$$\sigma_M = J_\nu(z_F) \tag{13}$$

as a normalization factor, the maximum coefficient is equal to one. Finally, the summation of Eq. 1 becomes

$$|p(\Omega,\theta,r)|/\sigma_M \approx |A(\omega,\theta)R| \sum_{\ell=-M}^{M} J_\ell(z_F)/\sigma_M\ . \tag{14}$$

2.3 The efficiency of Bessel arrays

The efficiency of an array is defined as

$$\eta = E/(N\ \max|x_i|^2), \tag{15}$$

with normalization factor $E = \sum_{\ell=-M}^{M} |x_\ell|^2$. Using the addition theorem for the Bessel functions [1, Eq. 9.1.76], the efficiency for a Bessel array is

$$\eta_{Bessel} \approx 1/(N\sigma_M^2) \tag{16}$$

so that for large N (using Eqs. 12–13)

$$\eta_{Bessel} \approx 1.4N^{-\frac{1}{3}}. \tag{17}$$

Clearly, for large Bessel arrays the efficiency decreases.

2.4 Comparison with Barker arrays

A different approach for array beam forming was given in [4, 5] where instead of Bessel coefficients, Barker sequences were used, where $x_\ell = \pm 1$. The largest Barker sequence known is for $N = 13$. A comparison between a uniform, a Barker and a Bessel array, using $z = z_F$ as presented at the end of Section 2.1, with five loudspeakers, is shown in Fig. 3.

One can state in general that for loudspeaker array applications, sequences with good autocorrelation properties (that is having a high efficiency and a maximally flat amplitude spectrum) are superior to Bessel arrays with respect to efficiency. This can be shown easily as follows.

The spectrum X_j of such sequences x_ℓ is approximately flat, or $\forall j\ |X_j| \approx C$. Using Parseval's theorem we have

$$E = \sum_\ell |x_\ell|^2 \approx C^2, \tag{18}$$

and with Eq. 15 we can write

$$C \approx \sqrt{N\eta}. \tag{19}$$

For a flat binary sequence ($\eta = 1$) we get

$$C_F = \sqrt{N} \tag{20}$$

and for a Bessel sequence (using Eq. 17)

$$C_B \approx 1.18 N^{1/3}. \tag{21}$$

Using the above and considering C as the gain factor of using N loudspeakers instead of one, we can write for an array with coefficients equal to a flat binary sequence

$$|p(\Omega, \theta, r)|_{binary} = \sqrt{N}|A(\omega, \theta)R|, \tag{22}$$

while for Bessel coefficients we get

$$|p(\Omega, \theta, r)|_{Bessel} = 1.18 N^{1/3}|A(\omega, \theta)R|. \tag{23}$$

Here we see that to obtain an array with a directional response proportional to that of one single loudspeaker, the increase in sound pressure level is, at best, proportional to $\sqrt{N}$. To obtain a level of 10 times a single loudspeaker we need approximately 600 loudspeakers in a Bessel configuration.

3 Quadratic Phase array

In the previous section it was shown that the gain of Bessel arrays is of the order of $N^{1/3}$ (Eq. 23). This power of 1/3 is due to the relatively large values of the Bessel coefficients (for fixed z) in the transition region. In order to improve the efficiency, the following idea was developed. The reason for using the Bessel array was that the generating function of $J_\ell(z)$ (see Eq. 2) is the Fourier transform of the sequence $(J_\ell(z))_\ell$. Substitution of $t = \mathrm{e}^{i\theta}$ in Eq. 2 gives

$$\mathrm{e}^{iz\sin\theta} = \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{i\theta\ell} J_\ell(z). \tag{24}$$

Applying the inverse Fourier transform to both sides of Eq. 24 gives the integral representation of the Bessel function ([1, Eq. 9.1.21])

$$J_\ell(z) = 1/(2\pi)\int_{-\pi}^{\pi} \mathrm{e}^{i(z\sin\theta - \ell\theta)} d\theta. \tag{25}$$

From Eq. 25 one can derive the behavior of $J_\ell(z)$ for $|\ell| \leq z$ by employing the stationary phase method [9]. It can thus be seen that for values of ℓ with $|\ell| \approx z$ one must expect $|J_\ell(z)|$ to be relatively large since $z\sin\theta - \ell\theta$ has a nearly vanishing second derivative with respect to θ at $\theta = 0$ or π for these values of ℓ. Such a thing can be avoided by replacing the $\sin\theta$ in the exponential at the left-hand side of Eq. 24 by a phase function $\phi(\theta)$ for which the second derivative of $z\phi(\theta)$ always stays away from 0. We thus propose to choose $C_\ell(z)$ such that

$$c(\theta; z) := \mathrm{e}^{iz\phi(\theta)} = \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{i\ell\theta} C_\ell(z), \tag{26}$$

whence

$$C_\ell(z) = 1/(2\pi) \int_{-\pi}^{\pi} \mathrm{e}^{i(z\phi(\theta)-\ell\theta)} d\theta, \tag{27}$$

where

$$\phi(\theta) = (1 - |\theta|/\pi)\theta/\pi, \tag{28}$$

with $|\theta| \leq \pi$. Thus $c(\theta; z)$ and $C_\ell(z)$ are a Fourier pair, and since $|c(\theta; z)| = 1$ it is guaranteed that the sequence C_ℓ has a flat spectrum.

We call the sequence C_ℓ the 'Quadratic Phase array'. It is not easy to calculate the values of C_ℓ directly, but by evaluating Eq. 26, and applying a discrete Fourier transform to this result, the C_ℓ can be computed and are plotted in Fig. 4 (dashed curve). Instead of using a discrete Fourier transform, C_ℓ can be approximated directly as discussed in the appendix.

As the figure shows, there is no peaking of the sequence C_ℓ in the transition region, as occurred in the Bessel coefficients case. The sequence C_ℓ has a nice compact burst-like behavior. This is an attractive feature because the efficiency will be higher than for the Bessel coefficients.

Using the method of stationary phase [9], the integral in Eq. 27 when $|\ell| \leq z/\pi$ can be approximated. The stationary points follow from

$$z\phi'(\theta) = \ell, \tag{29}$$

and using Eq. 28 we obtain

$$\theta_\ell = \pi^2(1/\pi - \ell/z)/2. \tag{30}$$

Now we get from the stationary phase method

$$C_\ell(z) \approx 2\Re\{1/(2\pi)\mathrm{e}^{i[z\phi(\theta_\ell)-\ell\theta_\ell]}\int_{-\infty}^{\infty}\mathrm{e}^{iz\phi''(\theta_\ell)(\theta-\theta_\ell)^2/2}d\theta\}. \tag{31}$$

Then for $0 \le \ell \le z/\pi$ we have $\phi''(\theta) = -2/\pi^2$, and using

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\gamma(\theta-\theta_\ell)^2}d\theta = \sqrt{\pi/\gamma}, \tag{32}$$

with $\gamma = -iz\phi''/2$ we finally obtain

$$C_\ell(z) \approx \sqrt{\pi/z}\cos(z(1-\ell\pi/z)^2/4 - \pi/4). \tag{33}$$

For $-z/\pi \le \ell \le 0$ we use that $C_{-\ell} = (-1)^\ell C_\ell$.

The approximated sequence C_ℓ (with Eq. 33) is plotted in Fig. 4 (dotted curve). The difference between the exact and approximated version of C_ℓ is plotted in Fig. 5. The Fourier transform of the approximated C_ℓ (using Eq. 33 without the leading term $\sqrt{\pi/z}$) is plotted in Fig. 6 for $z = 70$, together with two Fourier transforms of sequences known from telecommunications and information theory as maximal-efficient Huffman sequence and a maximal-flat binary sequence (MF) [2, Table III], respectively, and finally a Bessel array ($z = 20.2$). Since the maximal length of a Barker sequence corresponds to $M = 6$, the Barker array is not included in the comparison, where we have 45 loudspeakers ($M = 22$).

Fig. 6 shows that— due to the approximation and truncation of the C_ℓ— there appears a ripple in the response. However, it has a good efficiency. If one desires a flatter response, the C_ℓ can be obtained by using Eqs. 26, 28 an FFT, and a smaller value for z_F, however this is at the expense of a smaller efficiency. The Huffman array has a good performance, but there are neither analytical methods available to calculate the coefficients nor their efficiency, and the efficiency behaves somewhat as an erratic function of array length.

The aim of the proposed quadratic phase array was, like in the case of the Bessel array, to radiate uniformly over the entire Ω range. To meet this end with an efficiency that compares favorably with that of a Bessel array, we have chosen $\phi(\theta)$ such that the envelope of the $C_\ell(z)$ is approximately constant in the range $|\ell| \leq z$ of interest (see Eq. 33). When one prefers a different envelope, this can be achieved by choosing $\phi(\theta)$ in such a way that the desired envelope results in terms of ϕ'' in the same manner as this occurred in Eqs. 31–33 for the case of a uniform envelope. This makes the method very flexible and can be implemented even as an adaptive array. We finally observe that our method generalizes in a straightforward way when the desired radiation characteristic is a slowly varying function of Ω, rather than a constant. We intend further investigation on this point.

4 Design example

To gain some insight into the coefficients of the various arrays, a design example is given in Table 1 for an array with 13 loudspeakers. The table shows that the Quadratic Phase array, has a higher efficiency than the Bessel array. The maximally flat sequence (MF) has— because it is a binary array— the maximally attainable efficiency, but does not allow to trade flatness against efficiency as opposed to Quadratic Phase arrays by varying the value of the parameter z. Using the coefficients of Table 1 the array responses are calculated and are shown in Fig. 7–9, together with an uniform array Fig. 10 (all coefficients equal to one). The frequency axis in the plots are normalized as

$$\omega_n = \omega \ d/c. \tag{34}$$

It appears that the newly derived array, the Quadratic Phase array, has a higher efficiency than the Bessel array and a flatter response than the Barker array. The ripple in the MF array (Fig. 7) looks similar as that of the Quadratic Phase array (Fig. 9). However, for the MF array there is a rather strong increase in output for values of Ω in the neighborhood of integer multiples of π (e.g. small values of ω_n or θ). This can be seen more clearly in the plot for the smaller array of Fig. 3. For the larger array ($N = 45$) there is for the MF array not such a ridge (see Fig. 6), but the response is a somewhat erratic function of Ω.

Table 1: Design example of various array types of length $N = 13$, with their efficiency η. The first coefficients are given, the others are found by using the skew-symmetry property $C_{-\ell} = (-1)^{\ell} C_{\ell}$. MF is the best possible Maximally Flat sequence, the so called Barker sequence. For the Bessel array $z = 5.0$, for the QPA (using Eq. 33) $z = 18.0$.

ℓ	MF	Bessel	QPA
0	-1	-.454	-.864
1	1	-.837	-.670
2	1	.119	.447
3	-1	.933	1.000
4	1	1.000	.957
5	-1	.667	.778
6	1	.335	.735
η	1	.499	.628

5 Conclusions

It appeared that the newly derived array, the Quadratic Phase array, has a higher efficiency than the Bessel array and a flatter response than the Barker array. The Quadratic Phase array and the Bessel array allow for trading flatness against efficiency by varying the value of the parameter z. The proposed method allows generalization of the design of arrays with desired non-uniform radiating characteristics.

A Expression of $C_\ell(z)$ in terms of the Fresnel auxiliary function

The integral in Eq. 27 can be expressed in terms of the Fresnel integrals and, using the auxiliary function $g(x)$ (see [1, Eq. 7.3.6]), we have

$$C_\ell(z) = \sqrt{\pi/(2z)}(g(-\omega_\ell) - (-1)^\ell g(\omega_{-\ell})), \tag{35}$$

where $\omega_\ell = \sqrt{z/\pi}(1 - \ell\pi/z)$. Accurate asymptotic approximations of $g(x)$ are given in [1, Eq. 7.3.28]. A rational approximation of $g(x)$ for $x \geq 0$ is given in [1, Eq. 7.3.33]; for $x < 0$ we can write (using [1, Eqs. 7.3.6, 7.3.17])

$$g(-x) = \cos(\pi x^2/2) + \sin(\pi x^2/2) - g(x). \tag{36}$$

The function $g(x)$ is plotted in Fig. A1.

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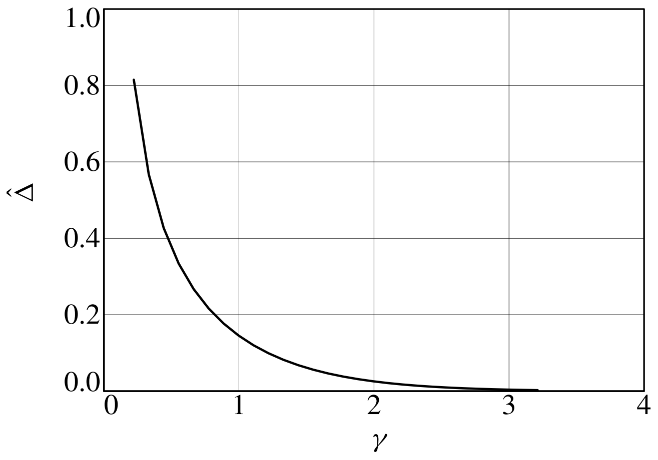

Figure 1: The approximate truncation error $\hat{\Delta}$ vs. γ, see Eq. 9.

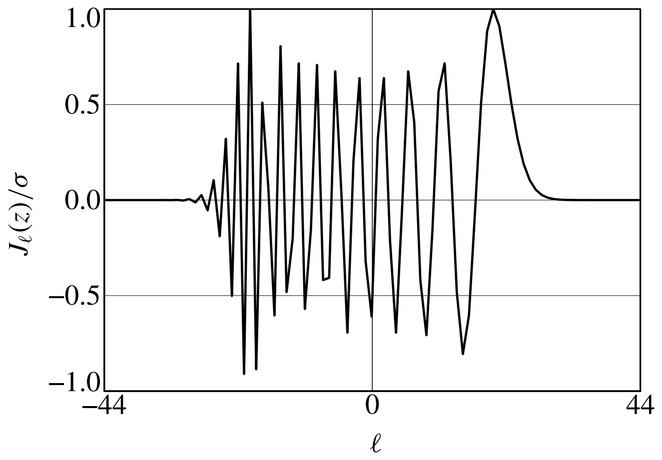

Figure 2: $J_\ell(z)/\sigma$ vs. ℓ, with $\sigma = 0.243$ and $z = 70/\pi$.

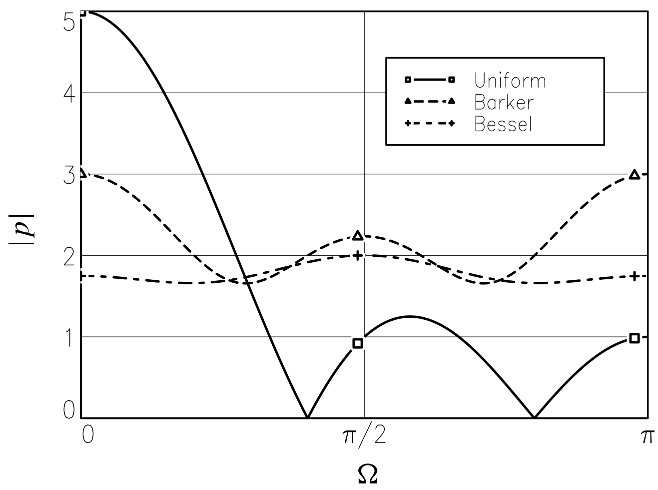

Figure 3: Comparison of the directional dependency ($|p/R|$ vs. Ω) of a uniform, a Barker and a Bessel array, with $N = 5$ loudspeakers ($M = 2$).

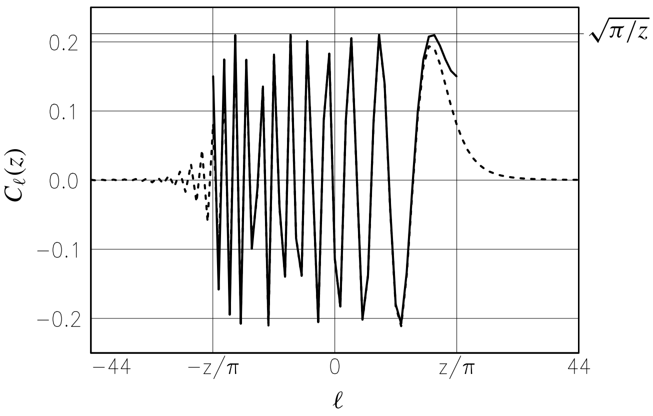

Figure 4: Exact and approximated sequence C_ℓ for z=70. Dashed curve: C_ℓ (exact via a 1024 point FFT). Solid curve: (with Eq. 33) approximated C_ℓ for $|\ell| \leq z/\pi$.

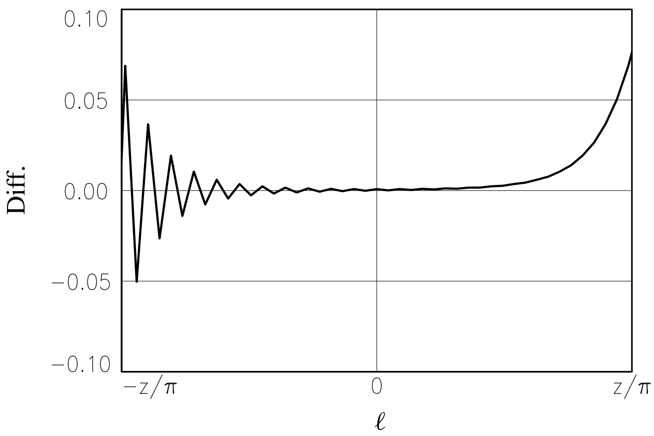

Figure 5: The difference between the exact and approximated version of C_ℓ. (for $z = 70$).

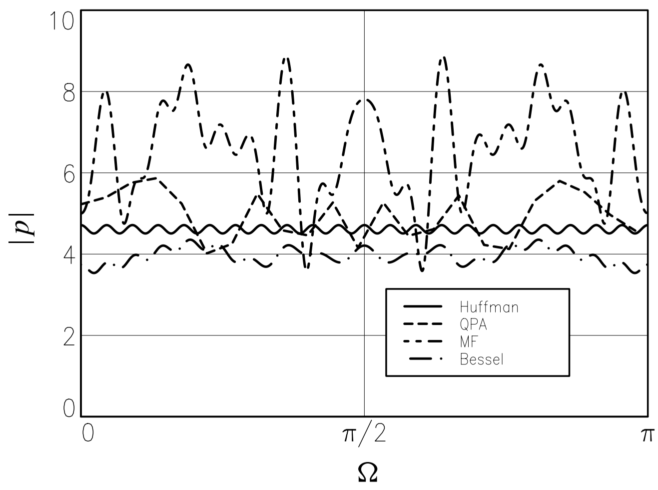

Figure 6: Comparison of the direction dependency ($|p|$ vs. Ω) of a maximal-efficient Huffman sequence, a Quadratic Phase array (QPA, $z = 70$) a maximal-flat binary sequence (MF, [2, Table III]), and a Bessel array ($\gamma = 2^{1/3}$, $z = 20.2$), each with 45 loudspeakers ($M = 22$).

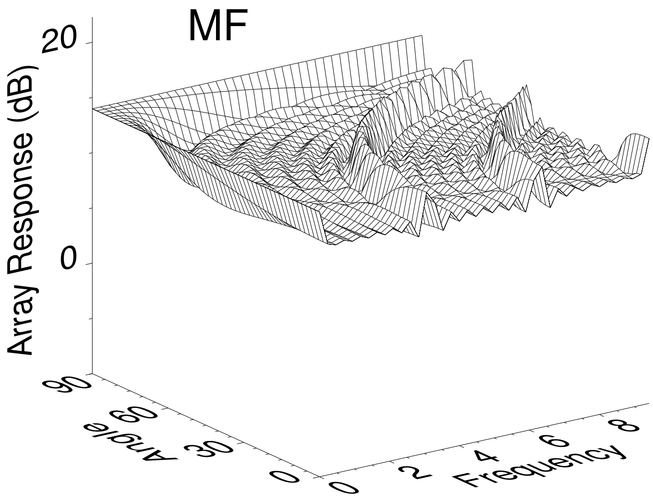

Figure 7: Design example of an MF array ($N = 13$), the best possible Maximally Flat sequence (Barker sequence). Frequencies have been normalized and are expressed in terms of ω_n.

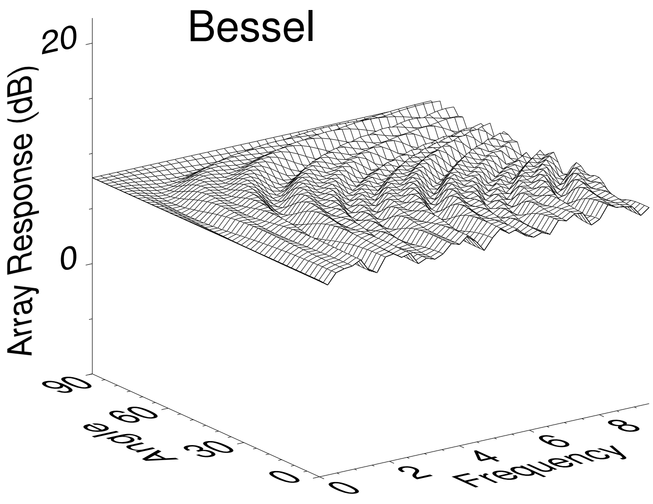

Figure 8: Design example of a Bessel array ($N = 13$, $z = 5.0$). Frequencies have been normalized and are expressed in terms of ω_n.

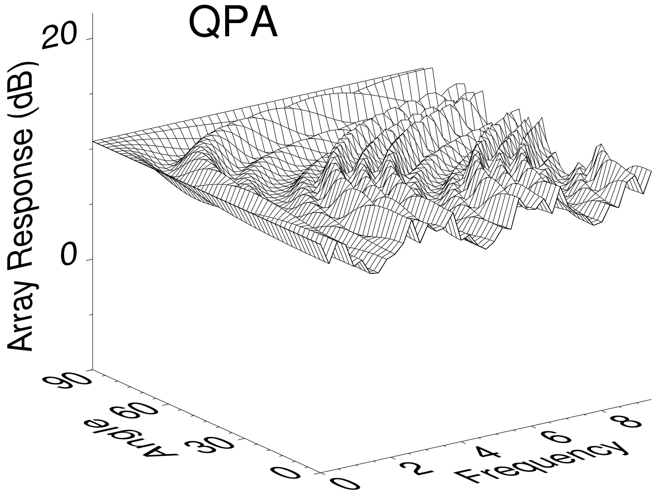

Figure 9: Design example of a QPA array (using Eq. 33, $N = 13$, $z = 18.0$). Frequencies have been normalized and are expressed in terms of ω_n.

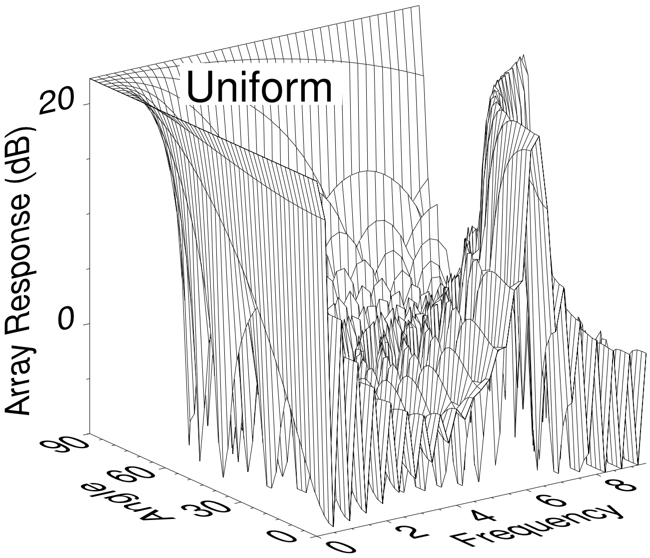

Figure 10: Design example of an uniform array (all coefficients equal to one, $N = 13$). Frequencies have been normalized and are expressed in terms of ω_n.

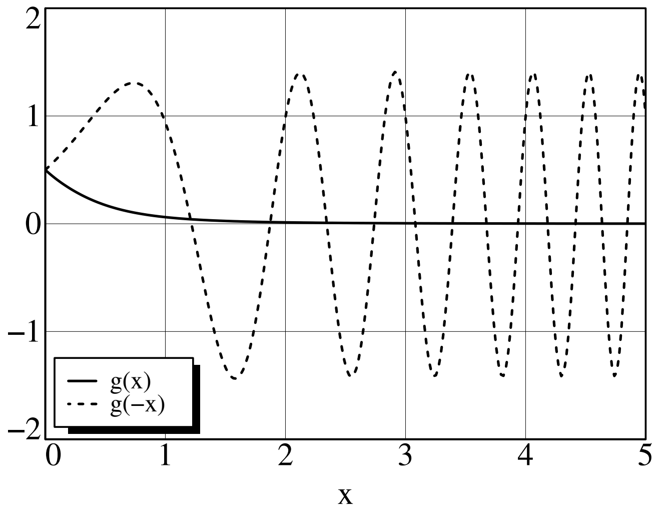

Figure A1: Auxiliary function $g(x)$ (solid) and $g(-x)$ (dotted).