Gabor frames, Unimodularity, and Window Decay*

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Abstract—We study time-continuous Gabor frame generating window functions g satisfying decay properties in time and/or frequency with particular emphasis on rational time-frequency lattices. Specifically, we show under what conditions these decay properties of g are inherited by its minimal dual γ^0 and by generalized duals γ. We consider compactly supported, exponentially decaying, and faster than exponentially decaying (i.e. decay like $|g(t)| \leq C\, e^{-\alpha|t|^{1/\alpha}}$ for some $\frac{1}{2} \leq \alpha < 1$) window functions. Particularly, we find that g and γ^0 have better than exponential decay in both domains if and only if the associated Zibulski-Zeevi matrix is unimodular, i.e., its determinant is a constant. In the case of integer oversampling unimodularity of the Zibulski-Zeevi matrix is equivalent to tightness of the underlying Gabor frame. For arbitrary oversampling, we furthermore consider tight Gabor frames canonically associated to window functions g satisfying certain decay properties. Here, we show under what conditions and to what extent the canonically associated tight frame inherits decay properties of g. Our proofs rely on the Zak transform, on the Zibulski-Zeevi representation of the Gabor frame operator, on a result by S. Jaffard, on a functional calculus for Gabor frame operators, on results from the theory of entire functions, and on the theory of polynomial matrices.

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1 Introduction, Announcement of Results, and Outline

1.1 Introduction and Motivation

Gabor frames. Let $a > 0$, $b > 0$, and let $g \in L^2(\mathbb{R})$. For $x, y \in \mathbb{R}$ we define

$$f_{x,y}(t) = e^{2\pi i y t} f(t-x), \qquad t \in \mathbb{R}.$$

We say that g generates a Gabor frame for $L^2(\mathbb{R})$ for the shift parameters a, b when there exist constants $A > 0$ and $B < \infty$ such that

$$A\|f\|^2 \leq \sum_{n,m} |(f, g_{na,mb})|^2 \leq B\|f\|^2, \qquad f \in L^2(\mathbb{R}). \tag{1}$$

The constants A and B are called a lower and upper frame bound for g, respectively. When (1) holds for $A = B$, we say that g generates a tight frame. It is well known that for g to generate a Gabor frame for $L^2(\mathbb{R})$ it is necessary that $ab \leq 1$ [8], Sec. 4.1, [21, 16]. The cases $ab = 1$ and $ab < 1$ are referred to as critical sampling and oversampling, respectively. The frame condition (1) can be equivalently written as

$$A\boldsymbol{I} \leq \boldsymbol{S} \leq B\boldsymbol{I},$$

where $\boldsymbol{I}$ is the identity operator of $L^2(\mathbb{R})$, and $\boldsymbol{S}$ is the frame operator defined by

$$\boldsymbol{S}f = \sum_{n,m} (f, g_{na,mb}) g_{na,mb}, \qquad f \in L^2(\mathbb{R}).$$

When g generates a frame and $f \in L^2(\mathbb{R})$, f can be represented as

$$f = \sum_{n,m} a_{nm} g_{na,mb} \tag{2}$$

with $\underline{a} \in l^2(\mathbb{Z}^2)$. In the critically sampled case the a_{nm} are unique for given f and g. In the oversampled case, one possible choice for the coefficients a_{nm} in (2) is

$$a_{nm} = (f, \gamma^0_{na,mb}), \tag{3}$$

where $\gamma^0 = \boldsymbol{S}^{-1}g$ is the minimal dual (or Wexler-Raz dual) of g. This γ^0 is minimal in the sense that for any $f \in L^2(\mathbb{R})$ and any $\underline{a} \in l^2(\mathbb{Z}^2)$ such that (2) holds, we have

$$\sum_{n,m} |(f, \gamma^0_{na,mb})|^2 \leq \sum_{n,m} |a_{nm}|^2$$

with equality if and only if a_{nm} is given by (3). Note that the minimal dual γ^0 generates a Gabor frame with frame operator $\boldsymbol{S}^{-1}$, i.e.,

$$\boldsymbol{S}^{-1}f = \sum_{n,m} (f, \gamma^0_{na,mb}) \gamma^0_{na,mb}, \qquad f \in L^2(\mathbb{R}).$$

In the oversampled case, we say that two functions g and γ having a finite upper frame bound are dual if

$$f = \sum_{n,m} (f, \gamma_{na,mb}) g_{na,mb} \tag{4}$$

holds for all $f \in L^2(\mathbb{R})$.

The minimal dual γ^0 can be computed from g and $\boldsymbol{S}$ as follows. With

$$\boldsymbol{V} = \boldsymbol{I} - \frac{2}{B+A}\boldsymbol{S},$$

we have

$$\|\boldsymbol{V}f\| \leq \frac{B-A}{B+A}\|f\|, \qquad f \in L^2(\mathbb{R}).$$

The Neumann series expansion for γ^0 is then given by

$$\gamma^0 = \boldsymbol{S}^{-1}g = \frac{2}{B+A}(\boldsymbol{I}-\boldsymbol{V})^{-1}g = \frac{2}{B+A}\sum_{n=0}^{\infty}\boldsymbol{V}^n g.$$

To every $g \in L^2(\mathbb{R})$ generating a Gabor frame for $L^2(\mathbb{R})$ we can associate a tight frame generated by

$$h = \boldsymbol{S}^{-1/2}g = \left(\frac{2}{B+A}\right)^{1/2}\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}\boldsymbol{V}^n g.$$

We say that h is canonically associated to g. For further generalities about Gabor frames the interested reader is referred to [9, 8, 12, 2].

The Zak transform. In the cases of rational oversampling, i.e., $ab = p/q, p \in \mathbb{Z}, q \in \mathbb{Z}, p < q, (p,q) = 1$ and critical sampling, the frame operator can conveniently be expressed using the Zak transform (ZT) [15, 5, 27, 28]. For $\lambda > 0$ the ZT of a signal $f \in L^2(\mathbb{R})$ is defined as

$$(Z_\lambda f)(t,\nu) = \lambda^{1/2}\sum_{k=-\infty}^{\infty} f\left(\lambda(t+k)\right) e^{-2\pi i k\nu}, \qquad a.e. \quad t,\nu \in \mathbb{R}. \tag{5}$$

Important properties of the Zak transform include

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\, h^*(t)\, dt &= \int_0^1\int_0^1 (Z_\lambda f)(t,\nu)(Z_\lambda h)^*(t,\nu)\, dt\, d\nu, \\
\lambda^{1/2} f(\lambda t) &= \int_0^1 (Z_\lambda f)(t,\nu)\, d\nu, \qquad a.e. \quad t \in \mathbb{R} \qquad (6)\\
(Z_\lambda f)(t+1,\nu) &= e^{2\pi i\nu}\,(Z_\lambda f)(t,\nu), \qquad a.e. \quad t,\nu \in \mathbb{R} \\
(Z_\lambda f)(t,\nu+1) &= (Z_\lambda f)(t,\nu), \qquad a.e. \quad t,\nu \in \mathbb{R} \\
(Z_\lambda \hat{f})(t,\nu) &= e^{2\pi i\nu t}(Z_{\frac{1}{\lambda}} f)(-\nu,t), \qquad a.e. \quad t,\nu \in \mathbb{R}, \qquad (7)
\end{aligned}$$

where $f, h \in L^2(\mathbb{R})$ and $\hat{f}(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i\nu t}\, dt$ denotes the Fourier transform of f. Moreover, for any $F \in L^2_{\text{loc}}(\mathbb{R}^2)$ such that

$$\begin{aligned}
F(t+1,\nu) &= e^{2\pi i\nu}F(t,\nu), \qquad a.e. \quad t,\nu \in \mathbb{R} \\
F(t,\nu+1) &= F(t,\nu), \qquad a.e. \quad t,\nu \in \mathbb{R}
\end{aligned}$$

there is a unique $f \in L^2(\mathbb{R})$ such that $F = Z_\lambda f$.

Representations of the Gabor frame operator. We shall next provide time, frequency, and time-frequency representations of the Gabor frame operator $\boldsymbol{S}$. These representations constitute the basis for the results presented in what follows. For $f \in L^2(\mathbb{R})$, the Walnut representation [25] of the Gabor frame operator is given by

$$\boldsymbol{S}f(t) = \frac{1}{b}\sum_{l=-\infty}^{\infty} f\left(t - \frac{l}{b}\right) \sum_{k=-\infty}^{\infty} g(t-ka)\, g^*\left(t - ka - \frac{l}{b}\right) \tag{8}$$

with absolute convergence for a.e. $t \in \mathbb{R}$. Similarly, for $f \in L^2(\mathbb{R})$ we have

$$\widehat{\boldsymbol{S}f}(\nu) = \frac{1}{a}\sum_{l=-\infty}^{\infty} \hat{f}\left(\nu - \frac{l}{a}\right) \sum_{k=-\infty}^{\infty} \hat{g}(\nu-kb)\, \hat{g}^*\left(\nu - kb - \frac{l}{a}\right) \tag{9}$$

with absolute convergence for a.e. $\nu \in \mathbb{R}$. For a rigorous proof of the convergence properties of (8) and (9) the reader is referred to [19, 6].

Following the presentation in [19], Sec. 1.5, we shall next provide a time-frequency representation of the Gabor frame operator in terms of the ZT. With $\boldsymbol{S}f = y$, the Zibulski-Zeevi representation [27, 28, 19] of $\boldsymbol{S}$ for rational oversampling ($ab = p/q$) and $\lambda = a$ reads [19]

$$(\mathbf{z}_a y)(t,\nu) = \mathbf{S}^{(a)}(t,\nu)\,(\mathbf{z}_a f)(t,\nu), \tag{10}$$

where

$$\begin{aligned}
(\mathbf{z}_a y)(t,\nu) &= \left[(Z_a y)(t,\nu),\ (Z_a y)\left(t - \frac{q}{p}, \nu\right),\ \ldots,\ (Z_a y)\left(t - (p-1)\frac{q}{p}, \nu\right)\right]^T, \\
(\mathbf{z}_a f)(t,\nu) &= \left[(Z_a f)(t,\nu),\ (Z_a f)\left(t - \frac{q}{p}, \nu\right),\ \ldots,\ (Z_a f)\left(t - (p-1)\frac{q}{p}, \nu\right)\right]^T
\end{aligned}$$

and $\mathbf{S}^{(a)}(t,\nu)$ is a $p \times p$ matrix with elements ($k = 0, 1, \ldots, p-1$, $l = 0, 1, \ldots, p-1$)

$$[\mathbf{S}^{(a)}(t,\nu)]_{k,l} = \frac{1}{p}\sum_{i=0}^{q-1} (Z_a g)\left(t - k\frac{q}{p}, \nu - \frac{i}{q}\right)\,(Z_a g)^*\left(t - l\frac{q}{p}, \nu - \frac{i}{q}\right). \tag{11}$$

The matrix $\mathbf{S}^{(a)}(t,\nu)$ is 1-periodic in t, i.e., $\mathbf{S}^{(a)}(t+1,\nu) = \mathbf{S}^{(a)}(t,\nu)$ and $\frac{1}{q}$-periodic in ν, i.e., $\mathbf{S}^{(a)}\left(t, \nu + \frac{1}{q}\right) = \mathbf{S}^{(a)}(t,\nu)$ [19]. Furthermore, it is easily seen from (11) that $\mathbf{S}^{(a)}(t,\nu)$ is hermitian, i.e., $\mathbf{S}^{(a)}(t,\nu) = \mathbf{S}^{(a)^*}(t,\nu)$.

We shall also need the Zibulski-Zeevi representation of $\boldsymbol{S}$ in terms of the ZT with $\lambda = \frac{1}{b}$ given by [27, 28, 19]

$$\left(\mathbf{z}_{\frac{1}{b}} y\right)(t,\nu) = \mathbf{S}^{(\frac{1}{b})}(t,\nu)\,(\mathbf{z}_{\frac{1}{b}} f)(t,\nu), \tag{12}$$

where

$$\begin{aligned}
\left(\mathbf{z}_{\frac{1}{b}} y\right)(t,\nu) &= \left[(Z_{\frac{1}{b}} y)(t,\nu),\ (Z_{\frac{1}{b}} y)\left(t, \nu + \frac{1}{p}\right),\ \ldots,\ (Z_{\frac{1}{b}} y)\left(t, \nu + \frac{p-1}{p}\right)\right]^T, \\
\left(\mathbf{z}_{\frac{1}{b}} f\right)(t,\nu) &= \left[(Z_{\frac{1}{b}} f)(t,\nu),\ (Z_{\frac{1}{b}} f)\left(t, \nu + \frac{1}{p}\right),\ \ldots,\ (Z_{\frac{1}{b}} f)\left(t, \nu + \frac{p-1}{p}\right)\right]^T
\end{aligned}$$

and $\mathbf{S}^{(\frac{1}{b})}(t,\nu)$ is a $p \times p$ matrix with elements $(k = 0,1,...,p-1,\ l = 0,1,...,p-1)$

$$[\mathbf{S}^{(\frac{1}{b})}(t,\nu)]_{k,l} = \frac{1}{p}\sum_{i=0}^{q-1}(Z_{\frac{1}{b}}g)\left(t - i\frac{p}{q}, \nu + \frac{k}{p}\right)\,(Z_{\frac{1}{b}}g)^*\left(t - i\frac{p}{q}, \nu + \frac{l}{p}\right). \tag{13}$$

The matrix $\mathbf{S}^{(\frac{1}{b})}(t,\nu)$ is $\frac{p}{q}$-periodic in t, i.e., $\mathbf{S}^{(\frac{1}{b})}\left(t + \frac{p}{q}, \nu\right) = \mathbf{S}^{(\frac{1}{b})}(t,\nu)$, and 1-periodic in ν, i.e., $\mathbf{S}^{(\frac{1}{b})}(t,\nu+1) = \mathbf{S}^{(\frac{1}{b})}(t,\nu)$ [17, 19]. Furthermore, it is easily seen from (13) that $\mathbf{S}^{(\frac{1}{b})}(t,\nu)$ is hermitian, i.e., $\mathbf{S}^{(\frac{1}{b})}(t,\nu) = \mathbf{S}^{(\frac{1}{b})^*}(t,\nu)$.

Zibulski-Zeevi representation of the analysis-synthesis relation. We shall furthermore need the Zibulski-Zeevi representation for expressing the duality condition (4). Defining the $p \times q$ matrices $(k = 0,1,...,p-1, l = 0,1,...,q-1)$

$$[\mathbf{G}^{(a)}(t,\nu)]_{k,l} = \sqrt{\frac{1}{p}}(Z_a g)\left(t - k\frac{q}{p}, \nu - \frac{l}{q}\right) \qquad \text{and} \qquad [\mathbf{\Gamma}^{(a)}(t,\nu)]_{k,l} = \sqrt{\frac{1}{p}}(Z_a \gamma)\left(t - k\frac{q}{p}, \nu - \frac{l}{q}\right),$$

it is shown in [17, 19] that g and γ are dual if and only if

$$\mathbf{\Gamma}^{(a)}(t,\nu)\mathbf{G}^{(a)^*}(t,\nu) = \mathbf{I}_p, \tag{14}$$

where $\mathbf{I}_p$ denotes the $p \times p$ identity matrix. Note that the Zibulski-Zeevi representation of the frame operator $\boldsymbol{S}$ can be expressed in terms of the matrix $\mathbf{G}^{(a)}(t,\nu)$ as

$$\mathbf{S}^{(a)}(t,\nu) = \mathbf{G}^{(a)}(t,\nu)\mathbf{G}^{(a)^*}(t,\nu).$$

Denoting the Zibulski-Zeevi representation (with $\lambda = a$) of the frame operator corresponding to the minimal dual γ^0 as $\mathbf{S}_{\gamma^0}^{(a)}(t,\nu)$ and setting $\mathbf{S}_g^{(a)}(t,\nu) = \mathbf{S}^{(a)}(t,\nu)$, we furthermore have [19]

$$\mathbf{S}_{\gamma^0}^{(a)}(t,\nu)\mathbf{S}_g^{(a)}(t,\nu) = \mathbf{I}_p. \tag{15}$$

The duality condition can be rephrased using the ZT with $\lambda = \frac{1}{b}$ as

$$\mathbf{\Gamma}^{(\frac{1}{b})}(t,\nu)\mathbf{G}^{(\frac{1}{b})^*}(t,\nu) = \mathbf{I}_p \tag{16}$$

with the $p \times q$ matrices

$$[\mathbf{G}^{(\frac{1}{b})}(t,\nu)]_{k,l} = \sqrt{\frac{1}{p}}(Z_{\frac{1}{b}} g)\left(t - l\frac{p}{q}, \nu + \frac{k}{p}\right) \qquad \text{and} \qquad [\mathbf{\Gamma}^{(\frac{1}{b})}(t,\nu)]_{k,l} = \sqrt{\frac{1}{p}}(Z_{\frac{1}{b}} \gamma)\left(t - l\frac{p}{q}, \nu + \frac{k}{p}\right).$$

The Zibulski-Zeevi representation of the Gabor frame operator $\boldsymbol{S}$ with respect to the ZT with $\lambda = \frac{1}{b}$ is given by

$$\mathbf{S}^{(\frac{1}{b})}(t,\nu) = \mathbf{G}^{(\frac{1}{b})}(t,\nu)\mathbf{G}^{(\frac{1}{b})^*}(t,\nu).$$

Denoting the Zibulski-Zeevi representation (with $\lambda = \frac{1}{b}$) of the frame operator corresponding to γ^0 as $\mathbf{S}_{\gamma^0}^{(\frac{1}{b})}(t,\nu)$ and setting $\mathbf{S}_g^{(\frac{1}{b})}(t,\nu) = \mathbf{S}^{(\frac{1}{b})}(t,\nu)$, we have [19]

$$\mathbf{S}_{\gamma^0}^{(\frac{1}{b})}(t,\nu)\mathbf{S}_g^{(\frac{1}{b})}(t,\nu) = \mathbf{I}_p. \tag{17}$$

Unimodularity of the Zibulski-Zeevi matrices. A square polynomial matrix $\mathbf{P}(z)$ with $z \in \mathbb{C}$ is unimodular [11, 24] if its determinant equals a nonzero constant, i.e.,

$$\det[\mathbf{P}(z)] = c \qquad \text{with} \qquad c \in \mathbb{C},\, c \neq 0.$$

The inverse of a unimodular matrix is again a unimodular polynomial matrix. Conversely, if $\mathbf{P}(z)$ is a polynomial matrix with polynomial inverse, $\mathbf{P}(z)$ is unimodular. In the following we use the terminology unimodular for arbitrary (not only polynomial) matrices $\mathbf{P}(z)$ with constant nonzero determinant. The matrix $\mathbf{S}^{(\lambda)}(t,\nu)$ (with $\lambda = a$ or $\lambda = \frac{1}{b}$) is said to be unimodular in ν for all t if

$$\det[\mathbf{S}^{(\lambda)}(t,\nu)] = c(t) \qquad \text{with} \quad c(t) \neq 0 \quad \text{for} \quad t \in \mathbb{R},$$

i.e., if its determinant is a zero-free function of t only. Likewise, $\mathbf{S}^{(\lambda)}(t,\nu)$ is said to be unimodular in t for all ν if

$$\det[\mathbf{S}^{(\lambda)}(t,\nu)] = d(\nu) \qquad \text{with} \quad d(\nu) \neq 0 \quad \text{for} \quad \nu \in \mathbb{R},$$

i.e., if its determinant is a zero-free function of ν only. The matrix $\mathbf{S}^{(\lambda)}(t,\nu)$ is unimodular if and only if its determinant is independent of t and ν, i.e.,

$$\det[\mathbf{S}^{(\lambda)}(t,\nu)] = c \qquad \text{with} \qquad c \in \mathbb{C}, c \neq 0.$$

In the case of integer oversampling ($p = 1$) we have

$$\begin{aligned}
\mathbf{S}^{(a)}(t,\nu) &= \sum_{i=0}^{q-1} \left| (Z_a g)\left(t, \nu - \frac{i}{q}\right) \right|^2 \\
\mathbf{S}^{(\frac{1}{b})}(t,\nu) &= \sum_{i=0}^{q-1} \left| (Z_{\frac{1}{b}} g)\left(t - i\frac{p}{q}, \nu\right) \right|^2,
\end{aligned}$$

which implies that unimodularity of $\mathbf{S}^{(\lambda)}(t,\nu)$ is equivalent to tightness of the Gabor frame generated by g.

Functional calculus for frame operators. Since we are interested in how decay etc. is inherited by $\gamma^0 = \boldsymbol{S}^{-1}g$, $h = \boldsymbol{S}^{-1/2}g$ from g, it would be pleasant when we could express the operators $\boldsymbol{S}^{-1}, \boldsymbol{S}^{-1/2}$ in a Walnut-type form or in the Zak transform domain, just as we have this for the operator $\boldsymbol{S}$ itself. This can be done indeed. The basic observations are as follows. Let $A > 0, B < \infty$. The operator $\boldsymbol{S}$ has frame bounds A, B if and only if the corresponding matrices

$$\mathbf{S}(t) = \left(\frac{1}{b} \sum_{r=-\infty}^{\infty} g\left(t - ra - \frac{k}{b}\right) g^*\left(t - ra - \frac{l}{b}\right) \right)_{k,l \in \mathbb{Z}}, \qquad a.e. \quad t \in \mathbb{R} \tag{18}$$

satisfy

$$A\mathbf{I} \leq \mathbf{S}(t) \leq B\mathbf{I}, \qquad a.e. \quad t \in \mathbb{R} \tag{19}$$

(with $\mathbf{I}$ the identity matrix of infinite size). Similarly, the operator $\boldsymbol{S}$ has frame bounds A, B if and only if the corresponding matrices

$$\hat{\mathbf{S}}(\nu) = \left(\frac{1}{a}\sum_{r=-\infty}^{\infty} \hat{g}\left(\nu - rb - \frac{k}{a}\right)\hat{g}^*\left(\nu - rb - \frac{l}{a}\right)\right)_{k,l\,\in\,\mathbb{Z}}, \qquad a.e. \quad \nu \in \mathbb{R} \tag{20}$$

satisfy

$$A\mathbf{I} \,\leq\, \hat{\mathbf{S}}(\nu) \,\leq\, B\mathbf{I}, \qquad a.e. \quad \nu \in \mathbb{R} \tag{21}$$

(see [19], Sec. 1.2). Furthermore, when $ab = p/q$ with $p, q \in \mathbb{Z}$, $(p, q) = 1$, we have that the operator $\boldsymbol{S}$ has frame bounds A, B if and only if

$$A\mathbf{I}_p \,\leq\, \mathbf{S}^{(a)}(t,\nu) \,\leq\, B\mathbf{I}_p, \qquad a.e. \quad t,\nu \in \mathbb{R}, \tag{22}$$

if and only if

$$A\mathbf{I}_p \,\leq\, \mathbf{S}^{(\frac{1}{b})}(t,\nu) \,\leq\, B\mathbf{I}_p, \qquad a.e. \quad t,\nu \in \mathbb{R} \tag{23}$$

(see [27, 17], and [19], Sec. 1.5).

The matrices $\mathbf{S}(t), \hat{\mathbf{S}}(\nu)$ and $\mathbf{S}^{(a)}(t,\nu), \mathbf{S}^{(\frac{1}{b})}(t,\nu)$ represent the frame operator $\boldsymbol{S}$ according to

$$\begin{aligned}
((\boldsymbol{S}f)(t - j/b))_{j\,\in\,\mathbb{Z}} &= \mathbf{S}(t)\,(f\,(t - l/b))_{l\,\in\,\mathbb{Z}}, && a.e. \quad t \in \mathbb{R} && (24)\\
((\widehat{\boldsymbol{S}f})(\nu - j/a))_{j\,\in\,\mathbb{Z}} &= \hat{\mathbf{S}}(\nu)\left(\hat{f}\,(\nu - l/a)\right)_{l\,\in\,\mathbb{Z}}, && a.e. \quad \nu \in \mathbb{R} && (25)\\
(\mathbf{z}_a\boldsymbol{S}f)(t,\nu) &= \mathbf{S}^{(a)}(t,\nu)(\mathbf{z}_a f)(t,\nu), && a.e. \quad t,\nu \in \mathbb{R} && (26)\\
(\mathbf{z}_{\frac{1}{b}}\boldsymbol{S}f)(t,\nu) &= \mathbf{S}^{(\frac{1}{b})}(t,\nu)(\mathbf{z}_{\frac{1}{b}} f)(t,\nu), && a.e. \quad t,\nu \in \mathbb{R}, && (27)
\end{aligned}$$

respectively, where $f \in L^2(\mathbb{R})$. Eqs. (24),(25) follow on rewriting (8) and (9), where we use that $[\mathbf{S}(t - j/b)]_{0,l} = [\mathbf{S}(t)]_{j,l+j}$, a.e. $t \in \mathbb{R}$ and $[\hat{\mathbf{S}}(\nu - j/a)]_{0,l} = [\hat{\mathbf{S}}(\nu)]_{j,l+j}$, a.e. $\nu \in \mathbb{R}$, respectively, and (26) and (27) are (10) and (12), respectively. Observing the matrix-vector multiplications at the right-hand sides of (24)-(27), we get for any polynomial φ that the operator $\varphi(\boldsymbol{S})$ is represented as

$$\begin{aligned}
((\varphi(\boldsymbol{S})f)(t - j/b))_{j\,\in\,\mathbb{Z}} &= \varphi(\mathbf{S}(t))\,(f\,(t - l/b))_{l\,\in\,\mathbb{Z}}, && a.e. \quad t \in \mathbb{R} && (28)\\
((\widehat{\varphi(\boldsymbol{S})}f)(\nu - j/a))_{j\,\in\,\mathbb{Z}} &= \varphi(\hat{\mathbf{S}}(\nu))(\hat{f}\,(\nu - l/a))_{l\,\in\,\mathbb{Z}}, && a.e. \quad \nu \in \mathbb{R} && (29)\\
(\mathbf{z}_a\varphi(\boldsymbol{S})f)(t,\nu) &= \varphi(\mathbf{S}^{(a)}(t,\nu))(\mathbf{z}_a f)(t,\nu), && a.e. \quad t,\nu \in \mathbb{R} && (30)\\
(\mathbf{z}_{\frac{1}{b}}\varphi(\boldsymbol{S})f)(t,\nu) &= \varphi(\mathbf{S}^{(\frac{1}{b})}(t,\nu))(\mathbf{z}_{\frac{1}{b}} f)(t,\nu), && a.e. \quad t,\nu \in \mathbb{R}, && (31)
\end{aligned}$$

where $f \in L^2(\mathbb{R})$. By (18)-(23) the correspondences (28)-(31) extend to all functions φ analytic in an open set containing the closed segment $[A, B]$ by basic facts about functional calculus of operators. For functions $\varphi(x)$ of the form x^α it is convenient in the sequel to have this in the explicit form

$$x^\alpha = \left(\frac{B+A}{2}(1-v)\right)^\alpha = \frac{B+A}{2}\sum_{n=0}^{\infty}\left(\begin{array}{c}\alpha\\ n\end{array}\right)(-1)^n v^n$$

with

$$v = 1 - \frac{2x}{B+A} \quad \in \quad \left[-\frac{B-A}{B+A}, \frac{B-A}{B+A}\right].$$

As a particular case we have that the inverse frame operator $\boldsymbol{S}^{-1}$ is represented by the inverses of the matrices that occur at the right-hand sides of (24)-(27). Since $\boldsymbol{S}^{-1}$ is the frame operator corresponding to γ^0 it thus follows that the $\mathbf{S}$-matrices in (24)-(27) corresponding to γ^0 are the inverses of these $\mathbf{S}$-matrices corresponding to g.

Motivation. In practical applications one is often interested in Gabor frames with compactly supported synthesis functions $g_{na,mb}$ and compactly supported analysis functions $\gamma^0_{na,mb}$ or $\gamma_{na,mb}$. Compact support of both g and γ^0 is hard to achieve [4]. It therefore makes sense to relax this constraint, and to study Gabor window functions satisfying certain decay conditions or to consider generalized duals γ. Good time-frequency localization of the Gabor window functions allows an interpretation of the Gabor coefficients a_{nm} as the local energy content of the signal f around the lattice point (na, mb).

Previous results. A question of practical relevance is whether a given compactly supported function g generating a Gabor frame for $L^2(\mathbb{R})$ has a compactly supported minimal dual γ^0. It is well-known [2] that a compactly supported g having a frame operator $\boldsymbol{S}$ which is a multiplication operator in the time-domain has a compactly supported minimal dual γ^0 (in fact, in this case γ^0 has the same support as g). Recently, a necessary and sufficient condition on a compactly supported g generating a Gabor frame for $L^2(\mathbb{R})$ for its minimal dual γ^0 to be compactly supported has been presented in [4].

1.2 Announcement of Results

This subsection is meant to provide a brief overview of the results presented in the paper.[1]

Compactly supported and strictly band-limited Gabor frames. Extending the result reported in [4], we provide a necessary and sufficient condition on a compactly supported g to have a compactly supported dual γ. A similar result will be provided for strictly band-limited Gabor frames. We note that this result uses the Zibulski-Zeevi representation of the Gabor frame operator and therefore holds only for rational oversampling. For rational oversampling we furthermore provide a parameterization of all Gabor frame generating compactly supported (strictly band-limited) window functions g having a compactly supported (strictly band-limited) dual γ.

Exponential decay in time and/or frequency. We prove that exponential decay of g implies exponential decay of γ^0. By time-frequency symmetry we find that exponential decay of $\hat{g}$ implies exponential decay of $\widehat{\gamma^0}$. Consequently, exponential decay of g in both domains implies exponential decay of γ^0 in both domains. These results use the functional calculus for Gabor frame operators and therefore hold for arbitrary

[1] In the following, by strictly band-limited we mean that the Fourier transform is compactly supported.

$ab < 1$. As an important special case it follows that the minimal dual of a compactly supported (strictly band-limited) g has exponential decay in time (frequency).

Gabor frames with faster-than-exponential decay. For rational oversampling, we show that Gabor frames with window g and minimal dual γ^0 having faster-than-exponential decay in both domains (by faster-than-exponential we mean decay like $e^{-\alpha|t|^{1/\alpha}}$ for some $\frac{1}{2} \le \alpha < 1$) have unimodular matrices $\mathbf{S}^{(\lambda)}(t,\nu)$ with $\lambda = a$ or $\lambda = \frac{1}{b}$. *We note, however, that whether such frames exist is an open problem.* Specifically, we show that in the case of integer oversampling, faster-than-exponential decay of g and γ^0 in both domains is possible if and only if g generates a tight frame. Finally, we prove that in the case of integer oversampling faster-than-exponential decay in the time-domain (frequency-domain) only is possible if and only if the frame operator is a multiplication operator in the time-domain (frequency-domain).

Construction of tight Gabor frames satisfying decay conditions. We furthermore study the decay properties of tight Gabor frames canonically associated to Gabor frame generating windows g satisfying certain decay conditions. Specifically, we show for arbitrary ab that a window function g having exponential decay in both domains can be modified using $\boldsymbol{S}^{-1/2}$ into a tight frame generating window function having exponential decay in both domains. This result can be specialized to exponential decay in time or frequency only. Moreover, we show for integer oversampling that applying $\boldsymbol{S}^{-1/2}$ to the Gaussian function yields a tight window function which decays no faster than exponentially in both domains.

2 Gabor frames with compact support or exponential decay

The proof of the following lemma can be found in [4].

> **Lemma 2.1:** For a compactly supported g the ZT $(Z_a g)(t,\nu)$ is a polynomial in $e^{2\pi i\nu}$ for all t. If g is supported in $t \in [0, T_0)$, the maximum degree of $(Z_a g)(t,\nu)$ for $t \in [0,1)$ is given by $\left\lfloor \frac{T_0}{a} \right\rfloor$. Conversely, if $(Z_a g)(t,\nu)$ has a finite maximum degree for $t \in [0,1)$ the function g is compactly supported.

Straightforward calculations reveal that

$$[\mathbf{S}^{(a)}(t,\nu)]_{k,l} = \frac{aq}{p} \sum_{u=-\infty}^{\infty} e^{-2\pi i\nu uq} \sum_{s=-\infty}^{\infty} g\left(at - k\frac{q}{p}a - sa\right) g^*\left(at - l\frac{q}{p}a - sa - uqa\right). \tag{32}$$

Thus, for a compactly supported g the matrix $\mathbf{S}^{(a)}(t,\nu)$ is a polynomial matrix in $e^{2\pi i\nu}$ for all t.

> **Lemma 2.2:** For a strictly band-limited g the function $(\widetilde{Z_{\frac{1}{b}} g})(t,\nu) = e^{-2\pi i t\nu}(Z_{\frac{1}{b}} g)(t,\nu) = \sqrt{b}\sum_{k=-\infty}^{\infty} \hat{g}(b(\nu+k))\, e^{2\pi i k t}$ is a polynomial in $e^{2\pi i t}$ for all ν. If $\hat{g}$ is supported in $\nu \in [0, F_0)$ the maximum degree of $(\widetilde{Z_{\frac{1}{b}} g})(t,\nu)$ for $\nu \in [0,1)$ is given by $\left\lfloor \frac{F_0}{b} \right\rfloor$. Conversely, if $(\widetilde{Z_{\frac{1}{b}} g})(t,\nu)$ has a finite maximum degree for $\nu \in [0,1)$, the function g is strictly band-limited.

The proof of Lemma 2.2 can be found in [4].

In the following, we shall also need the Smith form decomposition of polynomial matrices [11, 24], which essentially allows to decompose polynomial matrices into simpler forms such as triangular and diagonal forms. This decomposition will prove useful in what follows. Since for a compactly supported g the $q \times p$ matrix $\mathbf{G}^{(a)^*}(t,\nu)$ is a polynomial matrix in $e^{2\pi i\nu}$ for all t, it can be decomposed into its Smith form according to

$$\mathbf{G}^{(a)^*}(t,\nu) = \mathbf{A}^{(a)}(t,\nu)\,\mathbf{D}^{(a)}(t,\nu)\,\mathbf{B}^{(a)}(t,\nu)$$

with the matrix polynomials $\mathbf{A}^{(a)}(t,\nu)$ and $\mathbf{B}^{(a)}(t,\nu)$ of size $q \times q$ and $p \times p$, respectively. Furthermore, $\mathbf{A}^{(a)}(t,\nu)$ and $\mathbf{B}^{(a)}(t,\nu)$ are unimodular in ν for all t. The $q \times p$ matrix $\mathbf{D}^{(a)}(t,\nu)$, called the Smith form of $\mathbf{G}^{(a)^*}(t,\nu)$, is given by

$$\mathbf{D}^{(a)}(t,\nu) = \begin{bmatrix} d_0^{(a)}(t,\nu) & 0 & \ldots & 0 \\ 0 & d_1^{(a)}(t,\nu) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & d_{p-1}^{(a)}(t,\nu) \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \tag{33}$$

where we made use of the fact that the frame property implies that the normal rank of $\mathbf{G}^{(a)^*}(t,\nu)$ is p for all t [28]. Since $\mathbf{A}^{(a)}(t,\nu)$ and $\mathbf{B}^{(a)}(t,\nu)$ are unimodular in ν for all t, we have $\det\left[\mathbf{A}^{(a)}(t,\nu)\right] = c_A(t)$ and $\det\left[\mathbf{B}^{(a)}(t,\nu)\right] = c_B(t)$, where both $c_A(t)$ and $c_B(t)$ are zero-free. We furthermore note that the matrices $\mathbf{A}^{(a)}(t,\nu)$ and $\mathbf{B}^{(a)}(t,\nu)$ are not defined uniquely, but $\mathbf{D}^{(a)}(t,\nu)$ is unique. The latter statement follows from the fact that the entries of $\mathbf{D}^{(a)}(t,\nu)$ can be expressed in terms of the matrix $\mathbf{G}^{(a)^*}(t,\nu)$ itself [11].

We are now ready to formulate

> **Lemma 2.3:** A compactly supported g generates a Gabor frame for $L^2(\mathbb{R})$ if and only if the $d_i^{(a)}(t,\nu)$ $(i = 0,1,...,p-1)$ do not have zeros for $(t,\nu) \in ([0,1)^2)$.

Proof: According to Lemma 2.1 compact support of g implies that $(Z_a g)(t,\nu)$ is a polynomial in $e^{2\pi i\nu}$ for all t. Thus, $(Z_a g)(t,\nu)$ is bounded on $[0,1)^2$, which by Lemma 2 in [28] implies the existence of an upper frame bound $B < \infty$. We shall next show that for such a g a full rank $\mathbf{G}^{(a)^*}(t,\nu)$ and hence a full rank $\mathbf{D}^{(a)}(t,\nu)$ is necessary and sufficient for the existence of a lower frame bound A. From $\mathbf{G}^{(a)}(t,\nu+1) = \mathbf{G}^{(a)}(t,\nu)$ and $\mathbf{G}^{(a)}(t+1,\nu) = \mathbf{G}^{(a)}(t,\nu)\,\mathrm{diag}\{e^{2\pi i(\nu-l/q)}\}_{l=0}^{q-1}$ it follows that it suffices to consider $\mathbf{G}^{(a)}(t,\nu)$ on $[0,1)^2$ only. If $\mathbf{G}^{(a)^*}(t,\nu)$ has full rank on $[0,1)^2$, then $\mathbf{S}^{(a)}(t,\nu) = \mathbf{G}^{(a)}(t,\nu)\mathbf{G}^{(a)^*}(t,\nu)$ has full rank on $[0,1) \times \left[0,\frac{1}{q}\right)$, which means that the eigenvalues of $\mathbf{S}^{(a)}(t,\nu)$ satisfy $\lambda_i^{(a)}(t,\nu) > 0$ for $(t,\nu) \in \left([0,1) \times \left[0,\frac{1}{q}\right)\right)$ and

$i = 0, 1, ..., p-1$. Since $(Zg)(t,\nu)$ is a polynomial in $e^{2\pi i\nu}$ for all t, it follows that the $\lambda_i^{(a)}(t,\nu)$ are continuous functions of ν for all t, and therefore we can conclude that $A = \operatorname{ess\,inf}_{(t,\nu)\in\left([0,1)\times\left[0,\frac{1}{q}\right)\right),\, i=0,1,...,p-1}\lambda_i^{(a)}(t,\nu) > 0$. We next prove that, conversely, a full rank $\mathbf{G}^{(a)^*}(t,\nu)$ is necessary for the existence of an $A > 0$. Suppose that $\mathbf{G}^{(a)^*}(t,\nu)$ does not have full rank on $[0,1)^2$. It follows that $\mathbf{S}^{(a)}(t,\nu)$ does not have full rank on $[0,1) \times \left[0, \frac{1}{q}\right)$. This implies that there is at least one eigenvalue with $\lambda_i^{(a)}(t_0,\nu) = 0$ for some $t_0 \in [0,1)$ on a measurable set with positive measure. Hence, using [28]

$$A = \operatorname{ess\,inf}_{(t,\nu)\in\left([0,1)\times\left[0,\frac{1}{q}\right)\right),\, i=0,1,...,p-1}\lambda_i^{(a)}(t,\nu)$$

we conclude that $A = 0$. We have thus shown that a compactly supported g generates a Gabor frame for $L^2(\mathbb{R})$ if and only if $\mathbf{D}^{(a)}(t,\nu)$ has full rank for $(t,\nu) \in ([0,1)^2)$. From (33) it therefore follows that this is satisfied if and only if the $d_i^{(a)}(t,\nu)$ $(i = 0, 1, ..., p-1)$ do not have zeros for $(t,\nu) \in ([0,1)^2)$. $\square$

Multiplying the duality relation (16) by the $p \times p$ unitary matrix

$$\mathbf{U}(t,\nu) = \operatorname{diag}\left\{e^{-2\pi i t\left(\nu+\frac{l}{p}\right)}\right\}_{l=0}^{p-1}$$

from the left and multiplying it by $\mathbf{U}^*(t,\nu)$ from the right we obtain

$$\mathbf{\Gamma}^{\left(\frac{1}{b}\right)'}(t,\nu)\mathbf{G}^{\left(\frac{1}{b}\right)'^*}(t,\nu) = \mathbf{I}_p,$$

where

$$\mathbf{G}^{\left(\frac{1}{b}\right)'}(t,\nu) = \mathbf{U}(t,\nu)\mathbf{G}^{\left(\frac{1}{b}\right)}(t,\nu) \qquad \text{and} \qquad \mathbf{\Gamma}^{\left(\frac{1}{b}\right)'}(t,\nu) = \mathbf{U}(t,\nu)\mathbf{\Gamma}^{\left(\frac{1}{b}\right)}(t,\nu).$$

It follows from

$$[\mathbf{G}^{\left(\frac{1}{b}\right)'}(t,\nu)]_{k,l} = \sqrt{\frac{1}{p}}e^{-2\pi i l\frac{p}{q}\left(\nu+\frac{k}{p}\right)}(\widetilde{Z_{\frac{1}{b}}g})\left(t - l\frac{p}{q}, \nu + \frac{k}{p}\right)$$

and

$$[\mathbf{\Gamma}^{\left(\frac{1}{b}\right)'}(t,\nu)]_{k,l} = \sqrt{\frac{1}{p}}e^{-2\pi i l\frac{p}{q}\left(\nu+\frac{k}{p}\right)}(\widetilde{Z_{\frac{1}{b}}\gamma})\left(t - l\frac{p}{q}, \nu + \frac{k}{p}\right)$$

and Lemma 2.2 that for strictly band-limited g and γ the matrices $\mathbf{G}^{\left(\frac{1}{b}\right)'}(t,\nu)$ and $\mathbf{\Gamma}^{\left(\frac{1}{b}\right)'}(t,\nu)$ are polynomial in $e^{2\pi i t}$ for all ν. We can therefore decompose $\mathbf{G}^{\left(\frac{1}{b}\right)'^*}(t,\nu)$ into its Smith form according to

$$\mathbf{G}^{\left(\frac{1}{b}\right)'^*}(t,\nu) = \mathbf{A}^{\left(\frac{1}{b}\right)}(t,\nu)\,\mathbf{D}^{\left(\frac{1}{b}\right)}(t,\nu)\,\mathbf{B}^{\left(\frac{1}{b}\right)}(t,\nu)$$

with the matrix polynomials $\mathbf{A}^{\left(\frac{1}{b}\right)}(t,\nu)$ and $\mathbf{B}^{\left(\frac{1}{b}\right)}(t,\nu)$ of size $q \times q$ and $p \times p$, respectively. Furthermore, $\mathbf{A}^{\left(\frac{1}{b}\right)}(t,\nu)$ and $\mathbf{B}^{\left(\frac{1}{b}\right)}(t,\nu)$ are unimodular in t for all ν. The $q \times p$ matrix $\mathbf{D}^{\left(\frac{1}{b}\right)}(t,\nu)$ is given by

$$\mathbf{D}^{(\frac{1}{b})}(t,\nu) = \begin{bmatrix} d_0^{(\frac{1}{b})}(t,\nu) & 0 & ... & 0 \\ 0 & d_1^{(\frac{1}{b})}(t,\nu) & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & d_{p-1}^{(\frac{1}{b})}(t,\nu) \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & 0 \end{bmatrix},$$

where we made use of the fact that the frame property implies that the normal rank of $\mathbf{G}^{(\frac{1}{b})'^*}(t,\nu)$ is p for all ν [28]. Since $\mathbf{A}^{(\frac{1}{b})}(t,\nu)$ and $\mathbf{B}^{(\frac{1}{b})}(t,\nu)$ are unimodular in t for all ν, we have $\det[\mathbf{A}^{(\frac{1}{b})}(t,\nu)] = d_A(\nu)$ and $\det[\mathbf{B}^{(\frac{1}{b})}(t,\nu)] = d_B(\nu)$, where both $d_A(\nu)$ and $d_B(\nu)$ are zero-free functions. Repeating the arguments in the proof of Lemma 2.3 it is now easily seen that a strictly band-limited g generates a Gabor frame for $L^2(\mathbb{R})$ if and only if the $d_i^{(\frac{1}{b})}(t,\nu)$ $(i = 0, 1, ..., p-1)$ do not have zeros for $(t,\nu) \in ([0,1)^2)$.

We furthermore need the following proposition, which is a special case of Proposition 2 in [13]

Proposition 2.4: Let $\mathbf{A}$ be a doubly infinite matrix satisfying

$$|[\mathbf{A}]_{k,l}| \leq \alpha e^{-\beta|k-l|}$$

with some constants α and β. Then, if $\mathbf{A}$ is invertible its inverse $\mathbf{A}^{-1}$ satisfies

$$\left|[\mathbf{A}^{-1}]_{k,l}\right| \leq \alpha' e^{-\beta'|k-l|}$$

with some constants α' and β'. These constants depend only on $\inf_{\|x\|=1} \|\mathbf{A}\mathbf{x}\|^2$ and $\sup_{\|x\|=1} \|\mathbf{A}\mathbf{x}\|^2$.

Proof: For a proof of this proposition in a more general context see [13]. $\square$

Theorem 2.5:[2] Let g generate a Gabor frame for $L^2(\mathbb{R})$ and let $a, b > 0$, $ab < 1$. Assume that g has exponential decay in time, i.e., g satisfies $|g(t)| \leq \alpha\, e^{-\beta|t|}$ with some constants α and β. Then, the minimal dual γ^0 associated to g has exponential decay, i.e., γ^0 satisfies $|\gamma^0(t)| \leq \alpha'\, e^{-\beta'|t|}$ with some constants α' and β'.

Proof: Note that

$$[\mathbf{S}(t)]_{k,l} = \frac{1}{b} \sum_{r=-\infty}^{\infty} g\left(t - ra - \frac{k}{b}\right) g^*\left(t - ra - \frac{l}{b}\right)$$

is periodic in t with period a and that there is uniform (in t) exponential decay of $[\mathbf{S}(t)]_{k,l}$ in $|k-l| \to \infty$. The frame property implies

$$A\mathbf{I} \leq \mathbf{S}(t) \leq B\mathbf{I},$$

[2]The authors are grateful to K. Gröchenig for pointing out the proof of this theorem to them.

where $A > 0$ and $B < \infty$ are frame bounds for the system $\{g_{na,mb}\}_{n,m\in\mathbb{Z}}$. Using $\gamma^0 = \boldsymbol{S}^{-1}g$ and (28) with $\varphi(x) = x^{-1}$, we get

$$\gamma^0(t) = \sum_{k=-\infty}^{\infty} [\mathbf{S}^{-1}(t)]_{0,k}\, g\left(t - \frac{k}{b}\right). \tag{34}$$

Now, Proposition 2.4 implies that $\mathbf{S}^{-1}(t)$ has uniform (in t) exponential decay in $|k-l| \to \infty$. Using (34) and the fact that g has exponential decay itself, it finally follows that γ^0 has exponential decay. □

The result provided in Theorem 2.5 has first been reported in [23] for Gabor frames on $l^2(\mathbb{Z})$.

Corollary 2.6: Let g generate a Gabor frame for $L^2(\mathbb{R})$ and let $a, b > 0$, $ab < 1$. Assume that g has exponential decay in frequency, i.e., $\hat{g}$ satisfies $|\hat{g}(\nu)| \leq \alpha\, e^{-\beta|\nu|}$ with some constants α and β. Then, the minimal dual associated to g has exponential decay in frequency, i.e., $|\widehat{\gamma^0}(\nu)| \leq \alpha'\, e^{-\beta'|\nu|}$ with some constants α' and β'.

Proof: The proof proceeds along the same lines as the proof of Theorem 2.5 and will therefore be omitted. □

As an immediate consequence of Theorem 2.5 and Corollary 2.6, we get that exponential decay of g in both domains implies exponential decay of γ^0 in both domains. We can furthermore conclude that the minimal dual of a compactly supported (strictly band-limited) g has exponential decay in time (frequency). For rational oversampling this has previously been shown in [22] using the Zak transform.

We shall next provide a necessary and sufficient condition on a compactly supported g to have *a* compactly supported dual γ.

Theorem 2.7: A compactly supported[3] g generating a Gabor frame for $L^2(\mathbb{R})$ and rational $ab < 1$ has a compactly supported dual γ if and only if $\mathbf{G}^{(a)}(t,z)$ has full rank for $t \in \mathbb{R}$, $z \in \mathbb{C}$, i.e.,

$$\operatorname{rank}\left\{\mathbf{G}^{(a)}(t,z)\right\} = p \qquad \text{for} \quad t \in \mathbb{R},\, z \in \mathbb{C}.$$

Proof: The proof follows from a result on polynomial matrices, which says that the $q \times p$ ($q > p$) polynomial matrix $\mathbf{A}(z)$ has a polynomial left-inverse $\mathbf{B}(z)$, i.e.,

$$\mathbf{B}(z)\mathbf{A}(z) = \mathbf{I}_p \qquad \forall \quad z \in \mathbb{C}$$

if and only if $\operatorname{rank}\{\mathbf{A}(z)\} = p$ for all $z \in \mathbb{C}$ [20, 7]. Assuming that γ is such that $(Z_a\gamma)(t,z)$ is well-defined for $z \in \mathbb{C}$, the duality relation (14) extends to $z \in \mathbb{C}$ as

$$\mathbf{\Gamma}^{(a)}(t,z)\mathbf{G}^{(a)^*}\left(t, \frac{1}{z^*}\right) = \mathbf{I}_p.$$

[3]Note that since g is compactly supported its ZT $(Z_a g)(t,z) = \sqrt{a}\sum_{k=-\infty}^{\infty} g(a(t+k))\, z^{-k}$ is well-defined for $z \in \mathbb{C}$. This constitutes a slight abuse of notation, since we actually have $(Z_a g)(t,\nu) = (Z_a g)(t,z)$ with $z = e^{2\pi i\nu}$.

A compactly supported g has a polynomial $\mathbf{G}^{(a)^*}\left(t, \frac{1}{z^*}\right)$. Hence, it follows that a polynomial $\mathbf{\Gamma}^{(a)}(t,z)$ and hence a compactly supported γ exists if and only if $\mathbf{G}^{(a)^*}\left(t, \frac{1}{z^*}\right)$ and hence $\mathbf{G}^{(a)}(t,z)$ has full rank for $t \in \mathbb{R}$, $z \in \mathbb{C}$. $\square$

In the case of integer oversampling (i.e. $p = 1$), the condition in Theorem 2.7 allows an interesting interpretation. For $p = 1$ we have

$$\mathbf{G}^{(a)}(t,z) = [(Z_a g)(t,z),\ \ (Z_a g)(t,zW_q),\ \ \ldots,\ \ (Z_a g)(t,zW_q^{(q-1)})],$$

where $W_q = e^{-2\pi i/q}$. In this case the conditions in Theorem 2.7 are satisfied if the $(Z_a g)(t, zW_q^l)$ $(l = 0,1,...,q-1)$ do not have common zeros for $t \in \mathbb{R}$. By inspection it follows that this is the case if there is no $t \in \mathbb{R}$ such that $Z_g(t,z)$ has q zeros equally spaced on a circle in the z-plane.

Corollary 2.8: A strictly band-limited[4] g generating a Gabor frame for $L^2(\mathbb{R})$ and rational $ab < 1$ has a strictly band-limited dual γ if and only if $\mathbf{G}^{(\frac{1}{b})'}(z,\nu)$ has full rank for $\nu \in \mathbb{R}, z \in \mathbb{C}$, i.e.,

$$\text{rank}\left\{\mathbf{G}^{(\frac{1}{b})'}(z,\nu)\right\} = p \qquad \text{for} \quad \nu \in \mathbb{R},\, z \in \mathbb{C}.$$

Proof: The proof proceeds along the same lines as the proof of Theorem 2.7 and will therefore be omitted. $\square$

From Theorem 2.7, Corollary 2.8, and the results presented in [4] it follows that there do exist situations where a compactly supported (strictly band-limited) g does not have a compactly supported (strictly band-limited) minimal dual γ^0 but possibly some dual γ that is compactly supported (strictly band-limited). We shall next provide an alternative condition on a compactly supported g to have a compactly supported dual γ.

Corollary 2.9: A compactly supported g generating a Gabor frame for $L^2(\mathbb{R})$ and rational $ab < 1$ has a compactly supported dual γ if and only if the $d_i^{(a)}(t,z)$ are independent of z, i.e., $d_i^{(a)}(t,z) = d_i^{(a)}(t)$ with $d_i^{(a)}(t) \neq 0$ for all $t \in \mathbb{R}$.

Proof: Using $\text{rank}\,[\mathbf{G}^{(a)}(t,z)] = \text{rank}\,[\mathbf{D}^{(a)}(t,z)]$ for $t \in \mathbb{R}$, $z \in \mathbb{C}$, and Theorem 2.7 it follows that $d_i^{(a)}(t,z) \neq 0$ for $t \in \mathbb{R}$, $z \in \mathbb{C}$. Since the $d_i^{(a)}(t,z)$ are polynomials in z for all $t \in \mathbb{R}$, this is possible only if they are independent of z, i.e., $d_i^{(a)}(t,z) = d_i^{(a)}(t)$ for $t \in \mathbb{R}$. Moreover, the frame property requires that $d_i^{(a)}(t) \neq 0$ for all t. $\square$

Equivalently, it can be shown that a strictly band-limited g has a strictly band-limited dual if and only if the $d_i^{(\frac{1}{b})}(z,\nu)$ $(i = 0,1,...,p-1)$ are independent of z, i.e., $d_i^{(\frac{1}{b})}(z,\nu) = d_i^{(\frac{1}{b})}(\nu) \neq 0$ for $\nu \in \mathbb{R}$. As a

[4]Note that since $\hat{g}$ is compactly supported $(\widetilde{Z_{\frac{1}{b}} g})(z,\nu) = \sqrt{b}\sum_{k=-\infty}^{\infty} \hat{g}(b(\nu+k))\, z^k$ is well-defined for $z \in \mathbb{C}$. This constitutes a slight abuse of notation, since we actually have $(\widetilde{Z_{\frac{1}{b}} g})(t,\nu) = (\widetilde{Z_{\frac{1}{b}} g})(z,\nu)$ with $z = e^{2\pi i t}$.

consequence of Corollary 2.9, we can provide a parameterization of all compactly supported (strictly band-limited) g generating a Gabor frame for $L^2(\mathbb{R})$ and having a compactly supported (strictly band-limited) dual γ.

Corollary 2.10: Every $\mathbf{G}^{(a)^*}\left(t, \frac{1}{z^*}\right)$ corresponding to a compactly supported Gabor frame generating g having a compactly supported dual γ can be written as

$$\mathbf{G}^{(a)^*}\left(t, \frac{1}{z^*}\right) = \mathbf{A}^{(a)}(t,z)\mathbf{D}^{(a)}(t)\mathbf{B}^{(a)}(t,z),$$

with the matrix polynomials $\mathbf{A}^{(a)}(t,z)$ and $\mathbf{B}^{(a)}(t,z)$ being unimodular in z for all $t \in \mathbb{R}$, i.e., $\det[\mathbf{A}^{(a)}(t,z)] = c_A(t)$, and $\det[\mathbf{B}^{(a)}(t,z)] = c_B(t)$, where $c_A(t)$ and $c_B(t)$ are zero-free functions, and

$$\mathbf{D}^{(a)}(t) = \begin{bmatrix} d_0^{(a)}(t) & 0 & \dots & 0 \\ 0 & d_1^{(a)}(t) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & d_{p-1}^{(a)}(t) \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

with $d_i^{(a)}(t) \neq 0$ for $t \in \mathbb{R}$.

Since every unimodular matrix can be expressed as a product of a finite number of elementary matrices[5] [11, 24], Corollary 2.10 states that $\mathbf{G}^{(a)^*}\left(t, \frac{1}{z^*}\right)$ can be expressed as the product of a finite number of matrices of a special form.

Corollary 2.11: Every $\mathbf{G}^{(\frac{1}{b})'^*}\left(\frac{1}{z^*}, \nu\right)$ corresponding to a strictly band-limited Gabor frame generating g having a strictly band-limited dual γ can be written as

$$\mathbf{G}^{(\frac{1}{b})'^*}\left(\frac{1}{z^*}, \nu\right) = \mathbf{A}^{(\frac{1}{b})}(z,\nu)\mathbf{D}^{(\frac{1}{b})}(\nu)\mathbf{B}^{(\frac{1}{b})}(z,\nu),$$

with the matrix polynomials $\mathbf{A}^{(\frac{1}{b})}(z,\nu)$ and $\mathbf{B}^{(\frac{1}{b})}(z,\nu)$ being unimodular in z for all $\nu \in \mathbb{R}$, i.e., $\det[\mathbf{A}^{(\frac{1}{b})}(z,\nu)] = d_A(\nu)$ and $\det[\mathbf{B}^{(\frac{1}{b})}(z,\nu)] = d_B(\nu)$, where $d_A(\nu)$ and $d_B(\nu)$ are zero-free

[5] Premultiplying a polynomial matrix with an elementary matrix provides elementary row and column operations, such as interchanging two rows (columns), multiplying a row (column) with a nonzero constant, or adding a polynomial multiple of a row (column) to another row (coulmn).

functions, and

$$\mathbf{D}^{(\frac{1}{b})}(\nu) = \begin{bmatrix} d_0^{(\frac{1}{b})}(\nu) & 0 & \dots & 0 \\ 0 & d_1^{(\frac{1}{b})}(\nu) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & d_{p-1}^{(\frac{1}{b})}(\nu) \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

with $d_i^{(\frac{1}{b})}(\nu) \neq 0$ for $\nu \in \mathbb{R}$.

These parameterizations provide a systematic way of constructing compactly supported (strictly band-limited) Gabor frame generating window functions having a compactly supported (strictly band-limited) dual γ.

3 Gabor frames with faster-than-exponential decay

In this section we consider classes of functions f that decay faster than exponentially in time and/or frequency and we ask the question what it means for a Gabor frame to have a window g with minimal dual γ^0 having this sort of decay in terms of the Zibulski-Zeevi matrices. To that end we introduce spaces related to the Gel'fand-Shilov spaces $\mathcal{W}$, see [10], Ch. 1. We let $M : [0, \infty) \to [0, \infty)$ be twice differentiable, and with $m = M'$, we require

$$m(0) = M(0) = 0, \qquad m'(x) > 0, \qquad \lim_{x \to \infty} m(x) = \infty. \tag{35}$$

Hence M is a convex function on $[0, \infty)$ that increases to ∞ faster than linearly. We also let

$$M^{\times}(x) = \int_0^x m^{\leftarrow}(t)\, dt, \qquad x \geq 0,$$

where $m^{\leftarrow}$ is the inverse function of m. The function $M^{\times}$ is called the Young dual of M and shares the properties (35) with M. We have $(M^{\times})^{\times} = M$, and there is Young's inequality

$$xy \leq M(x) + M^{\times}(y), \qquad x, y \geq 0, \tag{36}$$

with equality if and only if $y = m(x)$.

Definition 3.1: With M as above, we let $\mathcal{V}_{M,\mu}$ for $\mu > 0$ be the set of all $f \in L^2(\mathbb{R})$ such that for any $\mu_1 < \mu$ there is a $C > 0$ such that

$$|f(t)| \leq Ce^{-2\pi M(\mu_1 |t|)}, \qquad t \in \mathbb{R}.$$

The spaces $\mathcal{V}_{M,\mu}$ are related to the spaces $\mathcal{W}_{M,a}$ in [10], Sec. 1.1, but they differ from these spaces since there is no requirement on the behavior of the derivatives of the members of $\mathcal{V}_{M,\mu}$.

We start by presenting a characterization of the spaces $\mathcal{V}_{M,\mu}$ in terms of the Zak transforms of their elements. Here we use the standard Zak transform $Z = Z_1$, ($\lambda = 1$ in (5)). Since $Z_\lambda f = Z_1 f_\lambda$, with $f_\lambda(t) = \lambda^{1/2} f(\lambda t)$, this is no real limitation of generality.

Proposition 3.2: Let $f \in L^2(\mathbb{R})$. Then we have that $f \in \mathcal{V}_{M,\mu}$ if and only if $(Zf)(t,\nu)$ can be extended for any $t \in \mathbb{R}$ to an entire function of ν while for any $\theta > 1/\mu$ there is a C independent of t such that

$$|(Zf)(t,\nu_1+i\nu_2)| \le Ce^{-2\pi t\nu_2+2\pi M^\times(\theta|\nu_2|)}, \qquad t,\nu_1,\nu_2 \in \mathbb{R}. \tag{37}$$

Proof: Assume that $f \in \mathcal{V}_{M,\mu}$, take $\mu_1 < \mu$, and let $K > 0$ be such that

$$|f(t)| \le Ke^{-2\pi M(\mu_1|t|)}, \qquad t \in \mathbb{R}.$$

We estimate for $t,\nu_1,\nu_2 \in \mathbb{R}$

$$\left|\sum_k f(t+k)\,e^{-2\pi ik(\nu_1+i\nu_2)}\right| \le K\sum_k e^{-2\pi M(\mu_1|t+k|)+2\pi k\nu_2} = Ke^{-2\pi t\nu_2}\sum_k e^{-2\pi M(\mu_1|t+k|)+2\pi(t+k)\nu_2}.$$

Take ϵ with $0 < \epsilon < \mu_1$. By (36) we have

$$|t+k||\nu_2| = (\mu_1-\epsilon)|t+k|\frac{|\nu_2|}{\mu_1-\epsilon} \le M((\mu_1-\epsilon)|t+k|) + M^\times\left(\frac{|\nu_2|}{\mu_1-\epsilon}\right).$$

It follows that

$$\left|\sum_k f(t+k)e^{-2\pi ik(\nu_1+i\nu_2)}\right| \le Ke^{-2\pi t\nu_2+2\pi M^\times\left(\frac{|\nu_2|}{\mu_1-\epsilon}\right)}\sum_k e^{-2\pi M(\mu_1|t+k|)+2\pi M((\mu_1-\epsilon)|t+k|)}.$$

By convexity of M we have

$$M(\mu_1|t+k|) - M((\mu_1-\epsilon)|t+k|) \ge M(\epsilon|t+k|).$$

Therefore, since M increases faster than linearly to ∞, we find

$$\left|\sum_k f(t+k)e^{-2\pi ik(\nu_1+i\nu_2)}\right| \le Ce^{-2\pi t\nu_2+2\pi M^\times\left(\frac{|\nu_2|}{\mu_1-\epsilon}\right)},$$

where

$$C = K\sup_{t\in[0,1)}\sum_k e^{-2\pi M(\epsilon|t+k|)} < \infty.$$

Hence, when we extend $(Zf)(t,\nu)$ to an entire function of ν according to

$$(Zf)(t,\nu) = \sum_k f(t+k)\,e^{-2\pi ik\nu}, \qquad \nu \in \mathbb{C},$$

we get a bound (37) for any $\theta > 1/\mu$ with a C independent of t. This is so since in the above we can allow any $\mu_1 < \mu$ and any $\epsilon > 0$.

Conversely, assume that we have an $f \in L^2(\mathbb{R})$ such that for any $t \in \mathbb{R}$ its Zak transform $(Zf)(t,\nu)$ can be extended to an entire function of $\nu \in \mathbb{C}$ while for any $\theta > 1/\mu$ there is a $C > 0$ independent of t such that (37) holds. The inversion relation (6) can be written by analyticity and $1-$periodicity of $(Zf)(t,\nu)$ in ν as

$$f(t) = \int_0^1 (Zf)(t, \nu_1 + i\nu_2)\, d\nu_1,$$

where we can take any $\nu_2 \in \mathbb{R}$. Using (37) we thus get for any $\nu_2 \in \mathbb{R}$

$$|f(t)| \leq Ce^{-2\pi t\nu_2 + 2\pi M^\times(\theta|\nu_2|)}.$$

By Young's inequality, see (36), we have that

$$-t\nu_2 + M^\times(\theta|\nu_2|) = -\frac{t}{\theta}\theta\nu_2 + M^\times(\theta|\nu_2|) = -M\left(\frac{1}{\theta}|t|\right)$$

when $\nu_2 = \theta^{-1}m(\theta^{-1}|t|)\operatorname{sgn}(t)$. Hence we get

$$|f(t)| \leq Ce^{-2\pi M(\frac{1}{\theta}|t|)}$$

with C independent of t, and since we can allow here any $\theta > 1/\mu$, the converse is proved also. □

Proposition 3.3: Let $f \in L^2(\mathbb{R})$. Then we have that $\hat{f} \in \mathcal{V}_{M,\mu}$ if and only if $(Zf)(t,\nu)$ can be extended for any $\nu \in \mathbb{R}$ to an entire function of t while for any $\theta > 1/\mu$ there is a C independent of ν such that

$$|(Zf)(t_1 + it_2, \nu)| \leq Ce^{2\pi M^\times(\theta|t_2|)}, \quad t_1, t_2, \nu \in \mathbb{R}. \tag{38}$$

Proof: This is a consequence of formula (7) and Proposition 3.2; indeed, there are no factors $e^{\pm 2\pi\nu t_2}$ at the right-hand side of (38) since these drop out. □

We also need a result from complex function theory that we prove here since we were unable to find an appropriate reference for it in the literature.

Lemma 3.4: Assume that $P(z)$ is a β-periodic, zero-free, entire function. Also assume that for some $c > 0$, $A > 0$

$$|P(z)| \leq Ae^{ce^{\alpha|z|}}, \qquad z \in \mathbb{C}, \tag{39}$$

where $\alpha \in (0, \pi\beta)$. Then, there is a $k \in \mathbb{Z}$ such that

$$P(z) = P(0)e^{2\pi ikz/\beta}, \qquad z \in \mathbb{C}.$$

Proof: Clearly we can restrict to the case $\beta = 1$. There is an entire function $Q(z)$ such that $P(z) = e^{Q(z)}$. By 1-periodicity of $P(z)$, for every $z \in \mathbb{C}$ there is an integer k such that $Q(z+1) = Q(z) + 2\pi ik$; by continuity of Q this integer k is independent of z. Now consider

$$R(z) = Q(z) - 2\pi ikz - Q(0), \qquad z \in \mathbb{C}.$$

Then R is 1-periodic , and

$$\mathrm{Re}(R(z)) = \log|P(z)| + 2\pi ik \mathrm{Im}(z) - \mathrm{Re}(Q(0)), \qquad z \in \mathbb{C}.$$

We thus see that for some $d > 0$ and α as in (39)

$$\mathrm{Re}(R(z)) \le d\, e^{\alpha|z|}, \qquad z \in \mathbb{C}.$$

By Caratheodory's theorem, see [3], Sec. 1.3, we get that

$$|R(z)| \le De^{\alpha|z|}, \qquad z \in \mathbb{C},$$

for some $D > 0$ and α as in (39). Then by Carlson's theorem, see [3], Sec. 9.2, we have $R(z) = R(0) = 0$, as required. $\square$

In the following theorem we consider certain dual windows g, γ^0 that decay just a little bit faster than exponentially; in particular, the theorem applies to g, γ^0 that decay like $e^{-\alpha|t|^{1/\alpha}}$ for some $\frac{1}{2} \le \alpha < 1$.

Theorem 3.5: Assume that $g \in L^2(\mathbb{R})$ generates a Gabor frame for the parameters a, b with $ab = p/q$ and $p, q \in \mathbb{Z}$, $(p,q) = 1$. Also assume that $g, \gamma^0 \in \mathcal{V}_{M,c}$, where

$$M(x) = (1+x)\log(1+x) - x, \qquad x \ge 0, \tag{40}$$

and $c > (\pi qa)^{-1}$. Then $\det[\mathbf{S}^{(a)}(t,\nu)]$ does not depend on ν.

Proof: The function M in (40) satisfies the requirements given in the beginning of this section, see [10], Example 2 in Sec. 1.1 and Sec. 3.1, and we have

$$M^{\times}(x) = e^x - 1 - x, \qquad x \ge 0. \tag{41}$$

The minimal duality relation between g and γ^0 is expressed by the formula, see (15),

$$[\mathbf{S}^{(a)}(t,\nu)]^{-1}\mathbf{S}^{(a)}(t,\nu) = \mathbf{I}_p, \qquad t, \nu \in \mathbb{R}, \tag{42}$$

where

$$[\mathbf{S}^{(a)}(t,\nu)]^{\pm 1}_{k,l} = \frac{1}{p}\sum_{i=0}^{q-1}(Z_a f)\left(t - k\frac{q}{p}, \nu - \frac{i}{q}\right)(Z_a f)^*\left(t - l\frac{q}{p}, \nu - \frac{i}{q}\right) \tag{43}$$

for $f = g, \gamma^0$ corresponding to the $+1$ and the -1 at the left of (43), respectively. Since the f in (43) is in $\mathcal{V}_{M,c}$, we can extend for given t the right-hand side of (43) to an entire function of $\nu \in \mathbb{C}$. We thus let

$$[\mathbf{S}^{(a)}(t,\nu)]_{k,l}^{\pm 1} = \frac{1}{p} \sum_{i=0}^{q-1} (Z_a f) \left(t - k\frac{q}{p}, \nu - \frac{i}{q} \right) (Z_a f)^* \left(t - l\frac{q}{p}, \nu^* - \frac{i}{q} \right)$$

for $t \in \mathbb{R}$, $\nu \in \mathbb{C}$. When doing so, the relation (42) remains valid for $t \in \mathbb{R}, \nu \in \mathbb{C}$. In particular, we have for fixed t that

$$F_t(\nu) = \det[\mathbf{S}^{(a)}(t,\nu)], \qquad \nu \in \mathbb{C},$$

is zero-free.

By Proposition 3.2 (adapted to the Z_a-case), we can find for every $\theta > \frac{1}{ac}$ a $C > 0$ such that for $t, \nu_1, \nu_2 \in \mathbb{R}$

$$\left| \left[\mathbf{S}^{(a)}(t, \nu_1 + i\nu_2) \right]_{k,l} \right| \le C^2 \frac{q}{p} e^{4\pi M^{\times}(\theta|\nu_2|) + 2\pi(k-l)\frac{q}{p}\nu_2}.$$

From the representation

$$F_t(\nu) = \sum_{\pi \in \Pi_p} (-1)^{\sigma(\pi)} \prod_{k=0}^{p-1} [\mathbf{S}^{(a)}(t,\nu)]_{k,\pi(k)},$$

where Π_p denotes the set of all permutations of $\{0, 1, ..., p-1\}$, we see

$$|F_t(\nu_1 + i\nu_2)| \le p! \left(C^2 \frac{q}{p} \right)^p e^{4\pi p M^{\times}(\theta|\nu_2|)}, \qquad \nu_1, \nu_2 \in \mathbb{R}. \tag{44}$$

Furthermore, $F_t(\nu)$ is $\frac{1}{q}$-periodic in $\nu \in \mathbb{C}$ since all entries of $\mathbf{S}^{(a)}(t,\nu)$ are $\frac{1}{q}$-periodic in ν. Since $c > (\pi q a)^{-1}$ and since we can allow any $\theta > \frac{1}{ac}$ in (44), it follows that there is an $A > 0$ and an $\alpha < \pi q$ such that

$$|F_t(\nu_1 + i\nu_2)| \le A e^{4\pi p M^{\times}(\alpha|\nu_2|)}, \qquad \nu_2 \in \mathbb{C}.$$

By Lemma 3.4 and (41) it follows that for some $k \in \mathbb{Z}$

$$F_t(\nu) = F_t(0) e^{2\pi i k q \nu}, \qquad \nu \in \mathbb{C}.$$

Since $F_t(\nu) > 0$ for $\nu \in \mathbb{R}$, it follows that $k = 0$, i.e., $F_t(\nu)$ does not depend on ν, as required. □

Note 3.6: It does not pay to repeat the above arguments with the matrices $[\mathbf{S}^{(\frac{1}{b})}(t,\nu)]^{\pm 1}$, see (17). By [19], (1.5.17) we have that $\det[\mathbf{S}^{(\frac{1}{b})}(t,\nu)]$ is $\frac{1}{p}$-periodic in ν. Arguing as in the proof of Theorem 3.5 we then find the condition $c > \frac{b}{\pi p}$ for constantness of this determinant. Since $ab = p/q$, this condition on c is the same as the one in Theorem 3.5.

Note 3.7: It is evident that a result like Theorem 3.5 also holds when we have decay of g, γ^0 in the frequency-domain. Thus one finds that $\det[\mathbf{S}^{(a)}(t,\nu)]$, $\det[\mathbf{S}^{(\frac{1}{b})}(t,\nu)]$ do not depend on t when $\hat{g}, \widehat{\gamma^0} \in \mathcal{V}_{M,c}$ with M and c as in Theorem 3.5.

Note 3.8: When g generates a Gabor frame with $g, \gamma^0, \hat{g}, \widehat{\gamma^0} \in \mathcal{V}_{M,c}$, where M and c are as in Theorem 3.5, we find that $\det[\mathbf{S}^{(a)}(t,\nu)]$, $\det[\mathbf{S}^{(\frac{1}{b})}(t,\nu)]$ are constant, i.e., do not depend on t, ν. When, moreover $(ab)^{-1} = q$ is integer ($p = 1$), we thus have that the Gabor frame is tight.

Theorem 3.9: Assume that $g \in L^2(\mathbb{R})$ generates a Gabor frame for the parameters a, b with $(ab)^{-1} = q$ and $q \in \mathbb{Z}$. Also assume that $g, \gamma^0 \in \mathcal{V}_{M,c}$ with M and c as in Theorem 3.5. Then, the Gabor frame operator $\boldsymbol{S}$ is a multiplication operator in the time-domain, i.e.,

$$\boldsymbol{S}f(t) = aqf(t) \sum_{r=-\infty}^{\infty} |g(t-ra)|^2$$

and hence g and γ^0 are related by

$$\gamma^0(t) = \frac{g(t)}{aq\sum_{r=-\infty}^{\infty} |g(t-ra)|^2}.$$

Proof: Using Theorem 3.5 it follows that $\det[\mathbf{S}^{(a)}(t,\nu)]$ does not depend on ν. From (32) we conclude that this is the case if and only if the function $S^{(a)}(t,\nu) = aq\sum_{u=-\infty}^{\infty} e^{-2\pi i\nu uq} \sum_{s=-\infty}^{\infty} g(at-sa)g^*(at-sa-uqa)$ does not depend on ν, which is equivalent to requiring that $\sum_{s=-\infty}^{\infty} g(at-sa)g^*(at-sa-uqa) = \sum_{s=-\infty}^{\infty} |g(at-sa)|^2 \,\delta_u$. Consequently, we have that $\sum_{s=-\infty}^{\infty} g(t-sa)g^*(t-sa-uqa) = \sum_{s=-\infty}^{\infty} |g(t-sa)|^2 \,\delta_u$. Inserting this into (8), which in the case of integer oversampling reads

$$\boldsymbol{S}f(t) = aq \sum_{l=-\infty}^{\infty} f(t-lqa) \sum_{k=-\infty}^{\infty} g(t-ka)\, g^*(t-ka-lqa),$$

it follows that $\boldsymbol{S}f(t) = aq\, f(t)\, \sum_{k=-\infty}^{\infty} |g(t-ka)|^2$, which concludes the proof. $\square$

It is evident that a similar result holds when $\hat{g}, \widehat{\gamma^0} \in \mathcal{V}_{M,c}$.

Example 3.10: Consider the case that

$$g(t) = 2^{1/4} e^{-\pi t^2}, \qquad t \in \mathbb{R}. \tag{45}$$

It is well known that g generates a Gabor frame when $ab < 1$. In [18], Sec. 6 the minimal dual γ^0 is computed for the case that $(ab)^{-1} = q$ is an even integer. It turns out that $\gamma^0(t)$ decays not faster than $e^{-\frac{\pi|t|}{2b}}$ and that $\gamma^0(t)$ does not extend to an entire function (which would have been the case when $\widehat{\gamma^0}$ were in a space $\mathcal{V}_{M,\mu}$).

There do exist (non-minimal) dual functions γ for the g in (45) that have Gaussian decay in time and frequency. The condition of duality reads, see [16], Proposition A

$$(\gamma_{\frac{k}{b},\frac{l}{a}}, g) = ab\, \delta_k \delta_l, \qquad k, l \in \mathbb{Z}. \tag{46}$$

Take for convenience $a = b < 1$. The condition (46) means that the Bargmann transform $(\mathcal{B}\gamma)(z)$ should vanish for all $z = (k + il)/a$ with $k, l \in \mathbb{Z}$, $(k, l) \neq (0, 0)$, see [14], Sec. 2. In [14], Subsec. 4.4, there is analyzed a $\gamma_1 \in L^\infty(\mathbb{R})$ (Bastiaans' singular function, [1]) such that

$$(\mathcal{B}\gamma_1)(z) = 0, \qquad z = k + il, \quad k, l \in \mathbb{Z}, (k, l) \neq (0, 0).$$

Setting $a = e^{-\alpha}$ with $\alpha > 0$, we see that $(\mathcal{B}\gamma_1)(e^{-\alpha} z)$ vanishes for all $z = (k + il)/a$ with $k, l \in \mathbb{Z}$, $(k, l) \neq (0, 0)$. Now in the notations of [14], Sec. 2, we have

$$(\mathcal{B}\gamma_1)(e^{-\alpha} z) = e^{\frac{\alpha}{2}} (\mathcal{B} N_\alpha \gamma_1)(z).$$

The function $N_\alpha \gamma_1$ has Gaussian decay in both time and frequency, and (46) is satisfied for $\gamma = c N_\alpha \gamma_1$ with c an appropriate constant.

4 Construction of tight Gabor frames satisfying decay properties

In this section we shall study tight Gabor frames canonically associated to Gabor frames satisfying certain decay conditions. We shall first show that Gabor frame generating window functions having exponential decay in time and/or frequency can be modified into tight Gabor window functions having the same decay properties.

Theorem 4.1: Let g generate a Gabor frame for $L^2(\mathbb{R})$ and let $a, b > 0$ and $ab < 1$. Assume that g has exponential decay, i.e., $|g(t)| \leq \alpha\, e^{-\beta|t|}$ with some constants α and β. Then, the tight frame generating function $h = \boldsymbol{S}^{-1/2} g$ has exponential decay in time, i.e., $|h(t)| \leq \alpha'\, e^{-\beta'|t|}$ with some constants α' and β'.

Proof: The proof is based on the functional calculus presented in Sec. 1. The operator $\boldsymbol{S}^{-1/2}$ has the matrix representation

$$\mathbf{S}^{-1/2}(t) = \left(\frac{2}{B + A}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} \left(\mathbf{I} - \frac{2}{B + A} \mathbf{S}(t)\right)^n . \tag{47}$$

As in the proof of Proposition 2 in [13] we see that the matrices in (47) have uniform exponential decay in $|k - l| \to \infty$ (uniform with respect to t) whenever $\mathbf{S}(t)$ has. Now, using (28) it follows that

$$h(t) = \boldsymbol{S}^{-1/2} g(t) = \left(\frac{2}{B + A}\right)^{1/2} \sum_{k=-\infty}^{\infty} \left[\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} \left(\mathbf{I} - \frac{2}{B + A} \mathbf{S}(t)\right)^n\right]_{0,k} g\left(t - \frac{k}{b}\right),$$

which implies exponential decay of h. $\square$

For Gabor frames on $l^2(\mathbb{Z})$ the result in Theorem 4.1 has previously been stated in [23]. Using (29) and the same arguments as in the proof of Theorem 4.1, it can be shown that exponential decay of $\hat{g}$ implies exponential decay of $\hat{h}$. We can therefore conclude that exponential decay of g in both domains implies exponential decay of $\boldsymbol{S}^{-1/2}g$ in both domains.

Example 4.2: Consider $a = 1, b = 1/M$ with integer $M \geq 2$ and let $g(t)$ be the Gaussian $2^{1/4}e^{-\pi t^2}$. With $Z = Z_1$ as usual, we define h by

$$(Zh)(t,\nu) = \frac{(Zg)(t,\nu)}{(\sum_{m=0}^{M-1} |(Zg)(t,\nu+\frac{m}{M})|^2)^{1/2}}, \qquad t,\nu \in \mathbb{R}. \tag{48}$$

Then h generates a tight Gabor frame, and we ask the question whether h can be an element of a space $\mathcal{V}_{M,\mu}$, see Sec. 3. Note, by Fourier invariance of g, we have $\hat{h} = h$ as well. Hence, h and $\hat{h}$ are or are not in $\mathcal{V}_{M,\mu}$ at the same time.

Suppose that $h \in \mathcal{V}_{M,\mu}$. We take squares in (48) and extend the resulting relation as in the proof of Theorem 3.5 to all $t,\nu \in \mathbb{C}$ to get

$$(Zh)^2(t,\nu) \sum_{m=0}^{M-1} (Zg)\left(t,\nu+\frac{m}{M}\right)(Zg)^*\left(t^*,\nu^*+\frac{m}{M}\right) = (Zg)^2(t,\nu). \tag{49}$$

Define

$$K(t,\nu) = \sum_{m=0}^{M-1} (Zg)\left(t,\nu+\frac{m}{M}\right)(Zg)^*\left(t^*,\nu^*+\frac{m}{M}\right), \qquad t,\nu \in \mathbb{C}.$$

Suppose that $K(t_0,\nu_0) = 0$ for some $t_0,\nu_0 \in \mathbb{C}$. Then by $\frac{1}{M}$-periodicity of $K(t_0,\nu)$ in ν we get by (49) that

$$(Zg)\left(t_0,\nu_0+\frac{m}{M}\right) = 0, \qquad m \in \mathbb{Z}. \tag{50}$$

The zero-set of Zg is precisely known. There holds

$$(Zg)(t,\nu) = 2^{1/4}e^{-\pi t^2}\theta_3(\pi(\nu - it)), \qquad t,\nu \in \mathbb{C},$$

see [15], Subsec. 3.3, with θ_3 the third order theta function, see [26], Ch. 21. Since $\theta_3(\pi z) = 0$ if and only if z is of the form $n+\frac{1}{2}+i(m+\frac{1}{2})$ with $n,m \in \mathbb{Z}$, it is easily seen that (50) cannot hold. Hence $K(t_0,\nu_0) \neq 0$ for $t_0,\nu_0 \in \mathbb{C}$. Observing that K is an entire function of its both variables, 1-periodic in its first and $\frac{1}{M}$-periodic in its second, and that for some $A > 0$, $B > 0$ we have

$$|K(t,\nu)| \leq Ae^{\pi B(|t|^2+|\nu|^2)}, \qquad t,\nu \in \mathbb{C}$$

we easily conclude that $K(t,\nu)$ is constant, see Lemma 3.4. However, this implies that g generates a tight Gabor frame which is definitely not the case. Therefore, h does not belong to any of the spaces $\mathcal{V}_{M,\mu}$.

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