Box-functions and mismatched log-likelihood ratios

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Extended Abstract

Abstract — **The input-output relationship of an a-posteriori probability decoder often can not be made explicit. When the outputs of such an decoder are represented as log-likelihood ratios, the input-output relationship can be regarded as a log-likelihood function. The box-function is an example of an explicit log-likelihood function. We verify the log-likelihood ratio property and study the effects of mismatched inputs to that function.**

I. Introduction

In an a-posteriori probability (APP) decoder, for each information and/or code bit b, the a-posteriori probability $P(b=1|\mathbf{Y}=\mathbf{y})$ is calculated. A hard-decision decoder can make a firm decision based on this probability, i.e. $\hat{b}=1$ if $P(b=1|\mathbf{Y}=\mathbf{y}) > P(b=0|\mathbf{Y}=\mathbf{y})$. In iterative decoding schemes, such as used in Turbo decoders, a soft-decision is preferred. Often these soft-decisions are represented as log-likelihood ratios, which are defined by

$$L(b) = \log \frac{P(b=0|\mathbf{Y}=\mathbf{y})}{P(b=1|\mathbf{Y}=\mathbf{y})},$$

where log is the natural logarithm. Log-likelihood ratios can therefore be generated by APP decoders. For calculating log-likelihood ratios, the decoder has to know the signal-to-noise ratio. In general, the explicit relation between the input sequence $\mathbf{y}$ and the log-likelihood ratio $L(b)$ is unknown and should be expected to be complicated. Therefore studying effects due to erroneous SNR estimations has to be done via simulations. However, in some cases an explicit function that produces log-likelihood ratios as function of the input $\mathbf{y}$ can be determined. In Section II we introduce the box-function, which is an example of a relation between received symbols and log-likelihood ratios. In Section III the log-likelihood behaviour of the box-function is verified. The effect of erroneous SNR estimations is shown in Section IV. Finally, some conclusions are given.

II. The box-function

Hagenauer et al. [1] introduced the box-function in some elementary log-likelihood ratio calculations. The box-function is given by

$$z = x \boxplus y \doteq \log \frac{1+e^{x+y}}{e^x + e^y}.$$

When u_1 and u_2 are two independent binary values with log-likelihood ratios $L(u_1)$ and $L(u_2)$, then the sum $u_1 \oplus u_2$ has log-likelihood ratio $L(u_1) \boxplus L(u_2)$. For the [3,2,2] SPC code, one can show that the two information bits b_0 and b_1 have log-likelihood ratios $L(b_0) = L_c y_0 + (L_c y_1) \boxplus (L_c y_2)$ and $L(b_1) = L_c y_1 + (L_c y_0) \boxplus (L_c y_2)$, where (y_0, y_1, y_2) is the received word and $L_c = 4\sqrt{E_s}/N_0$. Another example in which the box-function plays a role is the symbol-by-symbol APP decoding of the $R=1$ recursive encoder with generator $G(D) = (1+D)^{-1}$ (binary integrator or DBPSK modulator). In that case we have that $L(b_i) = (L_c y_i) \boxplus (L_c y_{i-1})$. These examples show that the box-function is a non-trivial example of a relation between received symbols and log-likelihood ratios. Therefore, this function is a suitable subject for further analysis and can act as a model for more complicated functions.

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III. Log-likelihood property of the box-function

We assume that $\mathbf{x}$ and $\mathbf{y}$ are uncorrelated Gaussian distributed random variables with mean μ and variance σ^2 and we want to study the random variable $\mathbf{z}$ which is given by $\mathbf{z} = \mathbf{x} \boxplus \mathbf{y}$. We say that the random variable $\mathbf{z}$ has the log-likelihood ratio property if and only if

$$\log \frac{p_{\mathbf{z}}(z)}{p_{\mathbf{z}}(-z)} = z, \quad \forall z. \tag{1}$$

It can be shown by verification that the probability density function of a Gaussian distributed random variable with mean μ and variance σ^2 exhibits this property if and only if $\alpha \doteq \frac{2\mu}{\sigma^2} = 1$. The interpretation of this setting is that we encode the all-zero information sequence with the binary integrator encoder, such that the all-zero code sequence is transmitted. BPSK transmission is assumed and the noise is assumed to be AWGN. The received sequence is scaled with $L_c = 4\sqrt{E_s}/N_0$ so that after scaling the received values exhibit the property that $\frac{2\mu}{\sigma^2} = 1$. An APP decoder should generate log-likelihood ratios by combining two succesive received and scaled values in the box-function.

The probability density function of $\mathbf{z}$ can be determined as :

$$p_{\mathbf{z}}(z) = e^{-\frac{\mu^2}{\sigma^2}} \int_{-\infty}^{\infty} f_\alpha(e^t) p_{\mathbf{w}}(t) p_{\mathbf{w}}(t-z) \mathrm{d}t,$$

where

$$\begin{aligned} p_{\mathbf{w}}(w) &= \begin{cases} 0, & w<0 \\ \frac{2}{\sigma\sqrt{\pi}} \frac{\exp\left(-\frac{1}{\sigma^2}\log^2(e^w + \sqrt{e^{2w}-1})\right)}{\sqrt{1-e^{-2w}}}, & w>0 \end{cases} \\ f_\alpha(x) &= \cosh(\alpha \operatorname{arccosh}(x)), \quad x \geq 1. \end{aligned}$$

It can be easily verified that

$$p_{\mathbf{z}}(-z) = e^{-\frac{\mu^2}{\sigma^2}} \int_{-\infty}^{\infty} f_\alpha(e^{t-z}) p_{\mathbf{w}}(t) p_{\mathbf{w}}(t-z) \mathrm{d}t.$$

Hence, when $\alpha = 1$ so that $f_\alpha(x) = x$, we obtain that $p_{\mathbf{z}}(z) = e^z p_{\mathbf{z}}(-z)$, whence the box-function exhibits the log-likelihood ratio property in that case. It should be noted that $p_{\mathbf{z}}(z)$ has a logarithmic singularity at $z=0$. We thank our colleague William Rey, who showed by simulations that due to this singularity, experimental results are prone to be unreliable, especially for z close to zero. This fact emphasizes that for further study, mathematical analysis is required.

IV. Mismatched inputs

In the previous section we showed that the box-function has the log-likelihood ratio property when $\alpha = 1$. However, in practice $\sqrt{E_s}$ and N_0 are not perfectly known at the decoder. This results in having $L_c \neq 4\sqrt{E_s}/N_0$ so that $\alpha \neq 1$. We show the effect of a mismatch of the inputs on the output of the decoder. Let us define $L(z)$,

$$L(z) \doteq \log \frac{p_{\mathbf{z}}(z)}{p_{\mathbf{z}}(-z)}.$$

It can be shown that for $\alpha > 1$ and $y \geq 1$:

$$y^\alpha f_\alpha(x) \leq f_\alpha(xy) \leq f_\alpha(x) f_\alpha(y), \quad x \geq 1,$$

while for $\alpha < 1$ the inequality signs should be reversed. Applying this inequality with $f_\alpha(xy) = f_\alpha(e^t) = f_\alpha(e^{t-z}e^z)$ results for $\alpha > 1$ and $z > 0$ in the following bounds for $L(z)$:

$$\alpha z \leq L(z) \leq \log f_\alpha(e^z).$$

For $\alpha < 1$ the inequalities should be reversed. In accordance with the result given in the previous section, both upper and lower bounds coincide for $\alpha = 1$ so that $L(z) = z$. Furthermore, it should be noted that the above inequality for $L(z)$ is sharp in the sense that $L(z) \to \alpha z$ when α is kept fixed while $\sigma \to \infty$ and $L(z) \to \log f_\alpha(e^z)$ when α is kept fixed while $\sigma \downarrow 0$. Experiments show that for σ not too small and fixed α, the bound αz is close to being attained.

The derived bounds of $L(z)$ are illustrated for $\alpha = 2$ and for $\alpha = 1/2$ in Figure 1. The bounds in Figure 1 show that for $\alpha > 1$, the mismatched output of the box-function is smaller than the (*true*) log-likelihood ratio. The case $\alpha > 1$ means that the inputs of the box-function are not matched to the signal-to-noise ratio on the transmission channel and that the true signal-to-noise ratio is higher than assumed. This underestimation leads to outputs of the box-function that are also smaller than the corresponding log-likelihood ratios (even the lower bound on $L(z)$ is larger than the mismatched output). Similarly, for $\alpha < 1$ the outputs of the box-function are larger than the corresponding log-likelihood ratios.

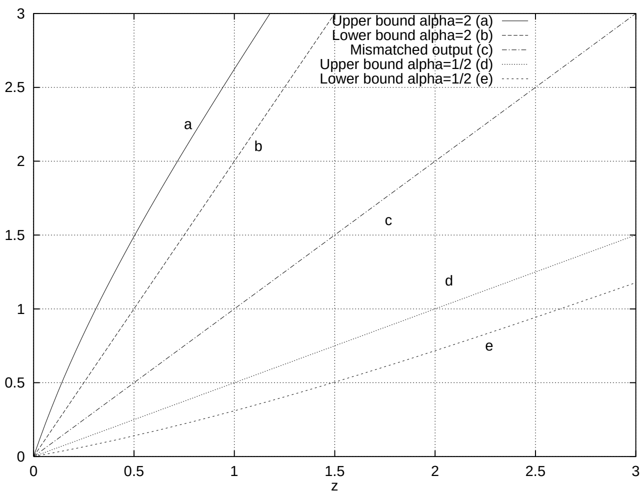

Figure 1: Bounds on the LLR of box-function with mismatched inputs.

Both the upper bound for $\alpha > 1$ and the lower bound for $\alpha < 1$ of $L(z)$ exhibit a linear behaviour for small z and for large z. This linear behaviour of these bounds can be made explicit by applying an upper bound on $f_\alpha(y)$ for $\alpha > 1$ and $y \geq 1$:

$$f_\alpha(y) \leq \begin{cases} y^{\alpha^2}, \\ 2^{\alpha-1} y^\alpha. \end{cases}$$

For $\alpha < 1$ the inequality signs should be reversed and then we obtain a lower bound on $f_\alpha(y)$. When we apply these bounds with $y = e^z$ we obtain

$$L(z) \leq \log f_\alpha(e^z) \leq \begin{cases} \alpha^2 z, \\ \alpha z + (\alpha - 1)\log 2, \end{cases}$$

where the inequality signs should be reversed for $\alpha < 1$. It can be shown that for fixed α it holds that

$$\lim_{z \downarrow 0} \frac{L(z)}{z} = \alpha^2, \quad \lim_{z \to \infty} (L(z) - \alpha z) = (\alpha - 1)\log 2,$$

Hence, for large z the upper bound and the lower bound on $L(z)$ have the same slope α and the bounds have a vertical offset of $(\alpha - 1)\log 2$.

V. Conclusions

The box-function is a non-trivial example of a relation between received symbols and log-likelihood ratios. Provided that the inputs are properly scaled ($\alpha = 1$), the box-function exhibits the log-likelihood property. Incorrectly scaled inputs lead to mismatched outputs of the box-function, i.e. the outputs are no log-likelihood ratios anymore. We give upper and lower bounds on the log-likelihood ratio at the output of the box-function. An underestimation of the signal-to-noise ratio ($\alpha > 1$) leads to pessimistic outputs of the box-function and overestimation of the signal-to-noise ratio ($\alpha < 1$) leads to optimistic outputs of the box-function. For not too small σ and fixed α, the bound αz is tight.

References

[1] J. Hagenauer, E. Offer and L. Papke, "Iterative Decoding of Binary and Convolutional Codes," *IEEE trans. on Information Theory*, Vol. 42, No. 2, pp. 429–445, March 1996.