# Equivalence of two methods for constructing tight Gabor frames*

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**Abstract**–Recently, in the context of Orthogonal Frequency Division Multiplexing (OFDM) a new method (FAB-method) [11] for constructing tight Gabor frames (with redundancy 2) from a (non-tight) gaussian $g$ was proposed. In this letter, we prove that the FAB-method yields the tight window function canonically associated to the gaussian. We furthermore provide a necessary and sufficient condition on the initial window function $g$ in the Zak transform domain for the FAB-method to yield a tight Gabor frame. This yields a characterization of all initial window functions $g$ for which the FAB-method works.

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# 1 Introduction

Recently, in the context of Orthogonal Frequency Division Multiplexing (OFDM) LeFloch, Alard, and Berrou [11] proposed a new method for constructing tight Gabor frames for oversampling factor 2. In the following, this method will be referred to as FAB-method. The essence of the FAB-method is to start from a (non-tight) gaussian $g$ and to "tighten" it using a two-step orthogonalization procedure.

In this letter, we show that the FAB-method is equivalent to applying the positive definite inverse square root of the Gabor frame operator to the initial window function. As a consequence the FAB-method yields the tight Gabor frame canonically associated to $g$. We provide a necessary and sufficient condition on the initial window function $g$ in the Zak transform domain for the FAB-method to yield a tight Gabor frame. This condition characterizes all functions for which the FAB-method works (i.e. yields a tight Gabor frame).

The letter is organized as follows. In Section 2, we provide a brief review of Gabor theory and Zak transforms, and we describe the FAB-method. In Section 3, we consider the FAB-method in the Zak transform domain, and in Section 4 we show that the FAB-method yields the canonical tight frame for gaussian $g$.

# 2 Gabor frames, Zak transforms, and the FAB-Method

**Gabor frames.** Let $a > 0$, $b > 0$, and denote for $x, y \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$

$$f_{x,y}(t) = e^{2\pi i y t} f(t-x), \qquad t \in \mathbb{R}. \tag{1}$$

We say that $g$ generates a Gabor frame for $L^2(\mathbb{R})$ for the shift parameters $a, b$ when there exist constants $A > 0$ and $B < \infty$ such that

$$A\|f\|^2 \le \sum_{n,m} |(f, g_{na,mb})|^2 \le B\|f\|^2, \qquad f \in L^2(\mathbb{R}). \tag{2}$$

The constants $A$ and $B$ are called a lower and upper frame bound for $g$, respectively. When (2) holds for $A = B$, we say that $g$ generates a tight frame. It is well known that for $g$ to generate a Gabor frame for $L^2(\mathbb{R})$ it is necessary that $ab \le 1$ [3], Sec. 4.1, [10, 7]. The cases $ab = 1$ and $ab < 1$ are referred to as critical sampling and oversampling, respectively. The frame condition (2) can be equivalently written as

$$A\boldsymbol{I} \le \boldsymbol{S} \le B\boldsymbol{I}, \tag{3}$$

where $\boldsymbol{I}$ is the identity operator of $L^2(\mathbb{R})$, and $\boldsymbol{S}$ is the frame operator defined by

$$\boldsymbol{S}f = \sum_{n,m} (f, g_{na,mb})\, g_{na,mb}, \qquad f \in L^2(\mathbb{R}). \tag{4}$$

When $g$ generates a frame and $f \in L^2(\mathbb{R})$, $f$ can be represented as

$$f = \sum_{n,m} (f, \gamma^0_{na,mb}) g_{na,mb}, \tag{5}$$

where $\gamma^0 = \boldsymbol{S}^{-1} g$ is the minimal (or Wexler-Raz) dual of $g$. The importance of tight Gabor frames is derived from the fact that $\gamma^0 = \frac{1}{A} g$ [3].

To every $g \in L^2(\mathbb{R})$ generating a Gabor frame for $L^2(\mathbb{R})$ we can associate a tight frame generated by [3]

$$h = \boldsymbol{S}^{-1/2} g = \left(\frac{2}{B+A}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \left(\boldsymbol{I} - \frac{2}{B+A}\boldsymbol{S}\right)^n g, \tag{6}$$

where $\boldsymbol{S}^{-1/2}$ denotes the positive definite operator square root of the inverse Gabor frame operator. We say that $h$ is canonically associated to $g$. For further generalities about Gabor frames the interested reader is referred to [4, 3, 5, 1].

**The Zak transform.** In the cases of rational oversampling, i.e., $ab = p/q$, $p \in \mathbb{Z}, q \in \mathbb{Z}, p < q, (p,q) = 1$, and critical sampling, the Gabor frame operator can conveniently be expressed using the Zak transform (ZT) [6, 2, 14, 15]. For $\lambda > 0$ the ZT of a signal $f \in L^2(\mathbb{R})$ is defined as

$$(Z_\lambda f)(t,\nu) = \lambda^{1/2} \sum_{k=-\infty}^{\infty} f\left(\lambda(t+k)\right) e^{-2\pi i k\nu}. \tag{7}$$

In this letter, we will be only concerned with the case $a = 1/2, b = 1$, and $\lambda = 1/b = 1$. The Zibulski-Zeevi representation [14, 15, 9] of $\boldsymbol{S}$ reads

$$(Z\boldsymbol{S}f)(t,\nu) = \left(|(Zg)(t,\nu)|^2 + |(Zg)(t+1/2,\nu)|^2\right)(Zf)(t,\nu), \tag{8}$$

where $(Zf)$ is the ZT of $f$ with $\lambda = 1$. A $g$ generates a tight frame if and only if

$$|(Zg)(t,\nu)|^2 + |(Zg)(t+1/2,\nu)|^2 = c \tag{9}$$

for some constant $c$. Also, the canonical tight frame generating $h$ associated to $g$ is given in the ZT domain by

$$(Zh)(t,\nu) = \frac{(Zg)(t,\nu)}{(|(Zg)(t,\nu)|^2 + |(Zg)(t+1/2,\nu)|^2)^{1/2}}. \tag{10}$$

We shall need the following properties of the ZT. When $f \in L^2(\mathbb{R})$, $H$ is 1-periodic and $\Psi$ is any function defined on $[0,1]$, we have under mild smoothness and decay properties of $f, H$ and $\Psi$ that

$$\begin{aligned} f(t) &= \int_0^1 (Zf)(t,\nu)\, d\nu, \quad t \in \mathbb{R}, && (11)\\ (Z\mathbf{F}f)(\nu,-t) &= e^{-2\pi i t\nu}(Zf)(t,\nu), \quad t,\nu \in \mathbb{R}, && (12)\\ (Zf)(t-n,\nu) &= e^{-2\pi i n\nu}(Zf)(t,\nu), \quad t,\nu \in \mathbb{R}, n \in \mathbb{Z}, && (13)\\ (Zf)(t,\nu-m) &= (Zf)(t,\nu), \quad t,\nu \in \mathbb{R}, m \in \mathbb{Z}, && (14) \end{aligned}$$

$$
\begin{aligned}
|(Zf)(t,\nu)|^2 &= \sum_{n,m}(f,f_{n,m})e^{2\pi imt-2\pi in\nu}, \quad t,\nu\in\mathbb{R}, &(15)\\
Z(Hf)(t,\nu) &= H(t)(Zf)(t,\nu), \quad t,\nu\in\mathbb{R}, &(16)\\
Z\left[\int_0^1 \Psi(\mu)(Zf)(\cdot,\mu)\,d\mu\right](t,\nu) &= \Psi(\nu)(Zf)(t,\nu), \quad t,\nu\in\mathbb{R}, &(17)\\
\int_0^1 |(Zf)(t,\nu)|^2\,dt &= \sum_k |(\mathbf{F}f)(\nu-k)|^2, \quad \nu\in\mathbb{R}. &(18)
\end{aligned}
$$

In (12) we have written $\mathbf{F}f$ for the Fourier transform of $f$. The properties (11)-(16) are well known in ZT theory. Property (17) follows from (11) and the fact that $Z$ is an isometry of $L^2(\mathbb{R})$ onto $L^2([0,1]^2)$. Indeed, either side of (17) is a ZT of an $L^2$-function. These functions are the same as can be seen from integrating either side of (17) over $\nu\in[0,1]$ and using (11). Hence their ZTs are the same and this is (17). Finally, property (18) follows from (12) and Parseval's theorem for Fourier series.

**The FAB-method.** In the FAB-method for computing a tight Gabor frame for oversampling factor 2 one starts with $g_\alpha(t)=(2\alpha)^{1/4}e^{-\pi\alpha t^2}$ and computes

$$h^{(FAB)}(t)=\mathbf{O}_{1/2}\mathbf{F}^{-1}\mathbf{O}_1\mathbf{F}g_\alpha, \tag{19}$$

where

$$(\mathbf{O}_\lambda f)(t)=\frac{f(t)}{(\lambda\sum_{k=-\infty}^{\infty}|f(t-k\lambda)|^2)^{1/2}} \tag{20}$$

and $\mathbf{F}$ denotes the Fourier transform operator as above. This approach for computing tight Gabor frames with redundancy 2 recently occurred in certain work on OFDM [11, 12, 13]. That the FAB-method works is a non-trivial fact for which we could not find a proof in the literature.

## 3 The FAB-method in the Zak transform domain

In this section, we consider the FAB-method in the ZT domain. More specifically, we shall show that the FAB-method, starting from a general well-behaved $g$, indeed yields a tight frame if and only if we have a factorization

$$|(Zg)(t,\nu)|^2+|(Zg)(t+1/2,\nu)|^2=\Psi(\nu)K(t), \tag{21}$$

where $\Psi$ is 1-periodic in $\nu$ and $K$ is $\frac{1}{2}$-periodic in $t$, and in that case we have that $h^{(FAB)}$ in (19) equals the canonical tight frame generating $h$ of (6). The following lemma is basic.

**Lemma.** For general well-behaved $g$, we have

$$Z(\mathbf{O}_{1/2}\mathbf{F}^{-1}\mathbf{O}_1\mathbf{F}g)(t,\nu)=\frac{(Zg)(t,\nu)\Phi^{-1/2}(\nu)}{\left(\frac{1}{2}\int_0^1\left(|(Zg)(t,\mu)|^2+|(Zg)(t+1/2,\mu)|^2\right)\Phi^{-1}(\mu)\,d\mu\right)^{1/2}}, \tag{22}$$

where $\Phi(\nu)=\sum_{k=-\infty}^{\infty}|(\mathbf{F}g)(\nu-k)|^2$ is assumed to be strictly positive.

*Proof:* We compute, using (12),

$$\begin{aligned}(\mathbf{F}^{-1}\mathbf{O}_1\mathbf{F}g)(t) &= \int_{-\infty}^{\infty} e^{2\pi i\nu t}(\mathbf{O}_1\mathbf{F}g)(\nu)\,d\nu = \int_{-\infty}^{\infty} e^{2\pi i\nu t}\frac{(\mathbf{F}g)(\nu)}{(\sum_{k=-\infty}^{\infty}|(\mathbf{F}g)(\nu-k)|^2)^{1/2}}\,d\nu \\ &= \int_0^1 \Phi^{-1/2}(\nu)\sum_{k=-\infty}^{\infty}(\mathbf{F}g)(\nu+k)e^{2\pi i(\nu+k)t}\,d\nu = \int_0^1 \Phi^{-1/2}(\nu)e^{2\pi i\nu t}(Z\mathbf{F}g)(\nu,-t)\,d\nu \\ &= \int_0^1 \Phi^{-1/2}(\nu)(Zg)(t,\nu)\,d\nu. \end{aligned} \tag{23}$$

Then from (13) and Parseval's theorem

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\left|(\mathbf{F}^{-1}\mathbf{O}_1\mathbf{F}g)(t-k/2)\right|^2 \\ &= \sum_{n=-\infty}^{\infty}\left|\int_0^1 \Phi^{-1/2}(\nu)(Zg)(t-n,\nu)\,d\nu\right|^2 + \sum_{n=-\infty}^{\infty}\left|\int_0^1 \Phi^{-1/2}(\nu)(Zg)(t+1/2-n,\nu)\,d\nu\right|^2 \\ &= \int_0^1 \Phi^{-1}(\nu)(|(Zg)(t,\nu)|^2+|(Zg)(t+1/2,\nu)|^2)\,d\nu =: H(t). \end{aligned} \tag{24}$$

By $\frac{1}{2}$- (and thus 1-) periodicity of $H$ and (16) we then get

$$Z(\mathbf{O}_{1/2}\mathbf{F}^{-1}\mathbf{O}_1\mathbf{F}g)(t,\nu) = Z\left[\frac{(\mathbf{F}^{-1}\mathbf{O}_1\mathbf{F}g)}{(\frac{1}{2}H)^{1/2}}\right](t,\nu) = \left(\frac{1}{2}H(t)\right)^{-1/2} Z(\mathbf{F}^{-1}\mathbf{O}_1\mathbf{F}g)(t,\nu). \tag{25}$$

Finally, an application of (17) using (23) yields the desired result. □

We now show the results stated in the beginning of this section. Assume that we have a $g$ such that $h^{(FAB)}$ generates a tight frame for $a = 1/2, b = 1$. Then we have that

$$|(Zh^{(FAB)})(t,\nu)|^2 + |(Zh^{(FAB)})(t+1/2,\nu)|^2 = c \tag{26}$$

for some constant $c$. From the Lemma we thus see that

$$|(Zg)(t,\nu)|^2 + |(Zg)(t+1/2,\nu)|^2 = \frac{c}{2}\,\Phi(\nu)\int_0^1 (|(Zg)(t,\mu)|^2 + |(Zg)(t+1/2,\mu)|^2)\Phi^{-1}(\mu)\,d\mu \tag{27}$$

showing that the left-hand side of (27) has a factorization as in (21).

Conversely, assume that (21) holds with a 1-periodic $\Psi$ and a $\frac{1}{2}$-periodic $K$. Then we have from (18) that

$$\Psi(\nu)\int_0^{1/2} K(t)\,dt = \int_0^1 |(Zg)(t,\nu)|^2\,dt = \sum_{k=-\infty}^{\infty}|(\mathbf{F}g)(\nu-k)|^2 = \Phi(\nu), \tag{28}$$

with $\Phi$ as in the Lemma. We thus get from the Lemma that

$$|(Zh^{(FAB)})(t,\nu)|^2 + |(Zh^{(FAB)})(t+1/2,\nu)|^2 = \frac{\Psi(\nu)K(t)\Phi^{-1}(\nu)}{\frac{1}{2}\int_0^1 \Psi(\mu)K(t)\Phi^{-1}(\mu)\,d\mu} = 2. \tag{29}$$

Hence, $h^{(FAB)}$ generates a tight frame for $a = 1/2, b = 1$.

Finally, when $h^{(FAB)}$ generates a tight frame, we see from (27) and the Lemma that

$$\begin{aligned}(Zh^{(FAB)})(t,\nu) &= \frac{(Zg)(t,\nu)\Phi^{-1/2}(\nu)}{(\frac{1}{c}\Phi^{-1}(\nu)(|(Zg)(t,\nu)|^2+|(Zg)(t+1/2,\nu)|^2))^{1/2}} \\ &= c^{1/2}\frac{(Zg)(t,\nu)}{(|(Zg)(t,\nu)|^2+|(Zg)(t+1/2,\nu)|^2)^{1/2}}. \qquad (30)\end{aligned}$$

This shows that $h^{(FAB)}$ coincides, within a constant (see (10)), with the canonical tight frame generating $h$.

We finally note that the formulation of the FAB-method, as well as the results of this section, admit straightforward generalization to the case that the oversampling factor $a^{-1}$ is an integer $\geq 2$.

## 4 Application to gaussian $g$

In this section, we show that the FAB-method applied to a gaussian $g_\alpha(t) = (2\alpha)^{1/4}e^{-\pi\alpha t^2}$ indeed yields a tight frame generating $h^{(FAB)}$ which coincides (up to a factor) with the canonical tight frame generating $h$. It is for all this enough to show that (21) holds with $\Psi$ and $K$ periodic functions of period 1 and 1/2, respectively. It follows from (15) for general $g$ that

$$|(Zg)(t,\nu)|^2+|(Zg)(t+1/2,\nu)|^2 = 2\sum_{k,l=-\infty}^{\infty}(g,g_{k,2l})e^{4\pi ilt-2\pi ik\nu}. \qquad (31)$$

For $g = g_\alpha$ and $x, y \in \mathbb{R}$ we have

$$(g,g_{x,y}) = e^{-\frac{1}{2}\pi\alpha x^2-\frac{1}{2}\pi\alpha^{-1}y^2}e^{-\pi ixy}. \qquad (32)$$

Hence, when $x = k, y = 2l$ with $k, l \in \mathbb{Z}$, the phase factor $e^{-\pi ixy}$ in the right-hand side expression in (32) equals unity, and we get

$$\sum_{k,l=-\infty}^{\infty}(g,g_{k,2l})e^{4\pi ilt-2\pi ik\nu} = \sum_{k=-\infty}^{\infty}e^{-\frac{1}{2}\pi\alpha k^2-2\pi ik\nu}\sum_{l=-\infty}^{\infty}e^{-2\pi\alpha^{-1}l^2+4\pi ilt}, \qquad (33)$$

which is the desired factorization.

We finally note that gaussians $g$ are among the very few functions for which a factorization as in (21) holds, see [8] for some more examples.

## 5 Conclusion

We proved that the tight Gabor frame generated by the FAB-method applied to the gaussian is equal to the canonical tight Gabor frame associated to the gaussian. Hence, the FAB-method is equivalent to applying the positive definite inverse square root of the Gabor frame operator to the initial window function. We furthermore showed that for general windows $g$ the FAB-method yields a tight Gabor frame (and, indeed, the canonical tight frame) if and only if $|(Zg)(t,\nu)|^2+|(Zg)(t+1/2,\nu)|^2$ is separable, and thereby characterized all window functions for which the FAB-method works (i.e. yields a tight Gabor frame). It appears that the Zak transform based method is more general since it does not impose any restrictions on the initial window function.

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