Zak transforms with few zeros and the tie

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Abstract.
We consider the difficult problem of deciding whether a triple (g, a, b), with window $g \in L^2(\mathbb{R})$ and time shift parameter a and frequency shift parameter b, is a Gabor frame from two different points of view. We first identify two classes of non-negative windows $g \in L^2(\mathbb{R})$ such that their Zak transforms have no and just one zero per unit square, respectively. The first class consists of all integrable, non-negative windows g that are supported by $[0, \infty)$. The second class consists of all even, non-negative, continuous, integrable windows g that satisfy on $[0, \infty)$ a condition slightly stronger than strict convexity (super convexity). Accordingly, the members of these two classes generate Gabor frames for integer oversampling factor $(ab)^{-1} \geq 1$ and ≥ 2, respectively. When we weaken the condition of super convexity, the Zak transforms Zg may have as many zeros as one wants, but in all cases (g, a, b) is still a Gabor frame when $(ab)^{-1}$ is an integer ≥ 2. As a second issue we consider the question for which $a, b > 0$ the triple (g, a, b) is a Gabor frame, where g is the characteristic function of an interval $[0, c_0)$ with $c_0 > 0$ fixed. It turns out that the answer to the latter question is quite complicated, where irrationality or rationality of ab gives rise to quite different situations. A pictorial display, in which the various cases are indicated in the positive (a, b)-quadrant, shows a remarkable resemblance with the design of a low-budget tie.

Index terms. Zak transform, Gabor frame, integer oversampling, super convexity, characteristic function.

1 Introduction and announcements of results

We consider in this contribution time-continuous Gabor systems (g, a, b) with window $g \in L^2(\mathbb{R})$ and time shift parameter a and frequency shift parameter b. To be more specific, let $a > 0$, $b > 0$, and let $g \in L^2(\mathbb{R})$. A Gabor system (g, a, b) consists of all time and frequency shifted functions $g_{na,mb}$, where for $x, y \in \mathbb{R}$ we denote by

$$g_{x,y}(t) = e^{2\pi i y t}\, g(t - x) \ , \qquad t \in \mathbb{R} \ , \tag{1}$$

the window g shifted by x in time and by y in frequency. We say that the triple (g, a, b) is a Gabor frame when there are $A > 0$, $B < \infty$ such that for all $f \in L^2(\mathbb{R})$ we have

$$A \, \|f\|^2 \le \sum_{n,m} |\langle f, g_{na,mb} \rangle|^2 \le B \, \|f\|^2 \ , \tag{2}$$

where $\| \ \|$ and $\langle \ , \ \rangle$ denote the standard norm and inner product of $L^2(\mathbb{R})$. We refer to [1]–[3] for generalities about frames for a Hilbert space and Gabor frames in particular, and how they give rise to methods of efficient signal representation.

It is a basic and highly non-trivial problem in the theory of Gabor frames to identify those triples (g, a, b) that are indeed Gabor frames. It is well known that a triple (g, a, b) can be a Gabor frame only when $(ab)^{-1} \ge 1$, see [1], p. 978, [2], Sec. 4.1, [4] and the references therein for the history and proof of this result. On the other hand, when $(ab)^{-1} > 1$, one can construct a smooth g supported by an interval of length $\le 1/b$ such that (g, a, b) is a (tight) frame (tightness means that $A = B$ in (2)), see [2], 3.4.4.A. When $(ab)^{-1} = 1$ there still do exist $g \in L^2(\mathbb{R})$ such that (g, a, b) is a (tight) frame, but these g cannot decay rapidly in both time and frequency, see [2], 4.1.1. Hence, the matter of existence of (well-behaved) windows g such that (g, a, b) is a frame when $ab \le 1$ is not an issue. This does, however, not resolve the question of determining for a given $g \in L^2(\mathbb{R})$ the set of all $a > 0$, $b > 0$ such that (g, a, b) is a frame. Even for windows g as “elementary” as Gaussians, exponentials (one-sided or two-sided), hyperbolic secants, characteristic functions of an interval, this problem has proved to be quite hard. We refer to [1], pp. 980–983, [2], 3.4.4.B, [4]–[5] for getting an idea of the non-triviality of this basic problem as well as for some analytic solutions of it (in many cases only when $(ab)^{-1}$ is an integer).

In Sec. 2 of this contribution we consider the cases of critical sampling and integer oversampling, so that $(ab)^{-1} = N = 1, 2, \ldots$. Given $a > 0$, $b > 0$, $c > 0$ it is a basic fact that (g, a, b) is a Gabor frame if and only if $(D_c g, a/c, bc)$ is a Gabor frame and the frame bounds for both systems are the same. Here D_c is the dilation operator $(D_c f)(t) = c^{1/2} f(ct)$, $t \in \mathbb{R}$, defined for $f \in L^2(\mathbb{R})$. We shall therefore restrict attention to the cases that $b = 1$, $a^{-1} = N$ and $a = 1$, $b^{-1} = N$. Then the optimal values $A_{\max}$, $B_{\min}$ in (2) for A, B can be read off from the Zak transform

$$(Zg)(t,\nu) = \sum_{k=-\infty}^{\infty} g(t+k)\, e^{-2\pi i k\nu}\ , \qquad \text{a.e. } t,\nu \in \mathbb{R}\ , \tag{3}$$

of g. Indeed, when $b = 1$, $a^{-1} = N$ we have that

$$A_{\max} = \operatorname*{ess\,inf}_{(t,\nu)\in[0,1)^2} \sum_{l=0}^{N-1} \Big|(Zg)\Big(t+\frac{l}{N}\,,\nu\Big)\Big|^2\ , \tag{4}$$

$$B_{\min} = \operatorname*{ess\,sup}_{(t,\nu)\in[0,1)^2} \sum_{l=0}^{N-1} \Big|(Zg)\Big(t+\frac{l}{N}\,,\nu\Big)\Big|^2\ , \tag{5}$$

and when $a = 1$, $b^{-1} = N$ we have that

$$A_{\max} = \operatorname*{ess\,inf}_{(t,\nu)\in[0,1)^2} \sum_{l=0}^{N-1} \Big|(Zg)\Big(t,\nu+\frac{l}{N}\Big)\Big|^2\ , \tag{6}$$

$$B_{\min} = \operatorname*{ess\,sup}_{(t,\nu)\in[0,1)^2} \sum_{l=0}^{N-1} \Big|(Zg)\Big(t,\nu+\frac{l}{N}\Big)\Big|^2\ . \tag{7}$$

Hence $(g, N^{-1}, 1)$ is a Gabor frame if and only if $A_{\max}$ in (4) is positive and $B_{\max}$ in (5) is finite, and $(g, 1, N^{-1})$ is a Gabor frame if and only if $A_{\max}$ in (6) is positive and $B_{\min}$ in (7) is finite.

The finiteness of $B_{\min}$ in (5) or (7) is satisfied under mild boundedness and decay conditions on g (for instance, when g is in the Wiener amalgam space $W(L^\infty, l')$, see [3], Sec. 3.2.2). The positivity of $A_{\max}$ in (4) or (6) poses a much harder problem. It is well known, see [6], Sec. 5, that when $g \in L^2(\mathbb{R})$ is such that Zg is continuous then Zg has a zero in $[0,1)^2$. Hence, when g is continuous and decays sufficiently rapidly (say, like $(1+|t|)^{-\alpha}$ with some $\alpha > 1$), the triple (g, a, b) is not a Gabor frame. Increasing the oversampling factor N does improve the odds for a g to have a positive $A_{\max}$ in (4) and/or (6), but there do not seem to exist simple conditions on a g that guarantee this. For instance, in [7] there is constructed an even, continuous, rapidly decaying, positive g which is convex on $[0,\infty)$ and that has a positive Fourier transform such that Zg vanishes at $(\frac{1}{4}, \frac{1}{2})$ and $(\frac{3}{4}, \frac{1}{2})$. For this g we have that $(g, \frac{1}{2}, 1)$ is not a Gabor frame since $A_{\max}$ in (4) vanishes.

In Sec. 2 we consider classes of windows g for which Zg has few zeros in $[0,1)^2$. In Subsec. 2.1 we show that for any integrable, non-negative window g that is supported by $[0,\infty)$ and strictly decreasing there, we have that Zg is bounded away from 0 on $[0,1)^2$. Hence $(g,1/N,1)$ and $(g,1,1/N)$ are Gabor frames for these g when $N=1,2,...$. In Subsec. 2.2 we show that for any even g having on $[0,\infty)$ the form

$$g(t)=b(t)+b(t+1)\ , \qquad t\geq 0\ , \tag{8}$$

with b an integrable, non-negative, strictly convex function on $[0,\infty)$, we have that Zg has exactly one zero, viz. at $(t,\nu)=(\frac{1}{2},\frac{1}{2})$. Hence $(g,1/N,1)$ and $(g,1,1/N)$ are Gabor frames for these g and $N=2,3,...$. We shall refer to the condition that g has the form (8) on $[0,\infty)$ with b integrable, non-negative and strictly convex on $[0,\infty)$ as super convexity. Safe from the non-negativity, integrability and continuity requirements on g, this condition of super convexity is somewhat stronger than strict convexity; it holds, for instance, when $g''(t)>0$, $g'''(t)<0$ for $t\in[0,\infty)$. The class of super convex functions is not dilation invariant. In Subsec. 2.3 we consider even, non-negative, integrable, continuous windows g that are strictly (but not necessarily super) convex on $[0,\infty)$. We show that the latter windows g have Zak transforms Zg whose zeros in $[0,1)^2$ are contained in the set $\{(\frac{1}{2},\nu)\,|\,0\leq\nu<1\}$. Hence $(g,1/N,1)$ is a Gabor frame for these g, and therefore, by dilation invariance of the class of considered windows, (g,a,b) is a Gabor frame for these g when $(ab)^{-1}=N$. In particular $(g,1,1/N)$ is then a Gabor frame, and, although Zg can have many zeros in the set $\{(\frac{1}{2},\nu)\,|\,0\leq\nu<1\}$, there is no ν such that $(Zg)(\frac{1}{2},\nu+l/N)=0$ for $l=0,...,N-1$, see (7).

Some of the results just mentioned were announced, but not proved, in [6], Sec. 5. We have, however, significant sharpenings of these results. In addition we present examples in Subsec. 2.4 showing that our result in Subsec. 2.1 ceases to hold when the g's are decreasing but not strictly decreasing, and that the result in Subsec. 2.2 ceases to hold when the b used in the representation (8) of g is convex but not strictly convex. Furthermore, we redisplay the example in [7] of a continuous, non-negative, even, integrable g that is (non-strictly) convex on $[0,\infty)$ such that $(g,\frac{1}{2},1)$ is not a Gabor frame, and we give an example of a continuous, non-negative, even, integrable g that is strictly convex on $[0,\infty)$ such that Zg has many zeros in $\{(\frac{1}{2},\nu)\,|\,0\leq\nu<1\}$.

Although the conditions of the results in Subsecs. 2.1– 2.2 are satisfied by some interesting windows (such as the one- and two-sided exponentials, respectively) they are rather far from being necessary conditions for having a Gabor frame. For instance, neither one of these conditions is satisfied by Gaussian windows g (for which (g,a,b) is a frame for any $a>0$, $b>0$ with $ab<1$).

In Sec. 3 we investigate in detail for which $a>0$, $b>0$ the triple (g,a,b) is a Gabor frame, where g is the characteristic function $\chi_{[0,c_0)}$ of an interval $[0,c_0)$ and $c_0>0$ is fixed. The result of this investigation is summarized in Fig. 1 for $c_0=2.15$ (this choice of c_0 is immaterial since any other c_0 gives qualitatively a similar picture). In Fig. 1 diagonal hatching means "no frame", vertical hatching means "frame without any proviso", horizontal hatching means "frame when ab is irrational", and in the unhatched regions all four combinations (ab rational or not, frame or no frame) occur. It is conjectured that in the unhatched regions we have "frame when ab is irrational and non-exceptional", where "exceptional" means that (a,b) lies on one of a set of curves. One such curve is given as

$$b=\frac{5}{c_0+3a}\ , \qquad \tfrac{3}{11}\,c_0\le a\le\tfrac{1}{2}\,c_0\ , \tag{9}$$

and is shown in Fig. 2 that enlarges the section $\{(a,b)\,|\,0\le a\le b^{-1}, c_0^{-1}\le b<2c_0^{-1}\}$ in Fig. 1. These investigations heavily use the basic result in what is called Ron-Shen theory, see [8] and [3], Sec. 1.2, that the triple (g,a,b) is a Gabor frame with frame bounds $A>0$, $B<\infty$ if and only if

$$AI\le\frac{1}{b}\,M_g(t)\,M_g^*(t)\le BI\ , \qquad \text{a.e. } t\in\mathbb{R}\ , \tag{10}$$

where $M_g(t)$ is the linear operator of $l^2(\mathbb{Z})$ whose matrix with respect to the standard basis of $l^2(\mathbb{Z})$ is given by

$$M_g(t)=(g(t-na-k/b))_{k\in\mathbb{Z},n\in\mathbb{Z}}\ , \qquad \text{a.e. } t\in\mathbb{R}\ . \tag{11}$$

A further ingredient, mainly used for settling what happens at the boundaries of the various regions, is the Zak transform. In all these investigations we have not spent particular effort in keeping track of the best frame bounds $A_{\max}$, $B_{\min}$.

2 Zak transforms with few zeros

In this section we consider two classes of non-negative, integrable windows g such that Zg has no and only one zero in $[0,1)^2$, respectively. The first class consists of windows g that are supported by and strictly decreasing on $[0,\infty)$, while the second class consists of windows g that are even and continuous on $\mathbb{R}$ and super convex on $[0,\infty)$; these two classes are considered in Subsecs. 2.1 and 2.2, respectively. In Subsec. 2.3 we study the zero sets of the Zak transforms Zg of g's as in Subsec. 2.2, with the condition of super convexity replaced by the condition of strict convexity on $[0,\infty)$. In Subsec. 2.4 we give various comments on the condition of super convexity, and we present counterexamples showing that the results in Subsecs. 2.1–2.2 are sharp.

2.1 One-sided, strictly decreasing windows g

In this subsection g is an integrable, non-negative function, supported by and strictly decreasing on $[0,\infty)$. For such a g the series at the right-hand side of (3), defining the Zak transform of g, converges absolutely and uniformly by boundedness of g $(0 \le g(t) \le g(0))$ and integrability and monotonicity of g on $[0,\infty)$.

We shall use the following result (Kakeya's theorem, see [9]).

Lemma.
Assume that $a_0 \ge a_1 \ge \ldots \ge 0$, $\sum_0^\infty a_k < \infty$. Then $\sum_0^\infty a_k\, z^k \ne 0$ for $|z| \le 1$ unless there is a $K = 2, 3, \ldots$ such that $a_k = a_{k-1}$ when k is not a multiple of K. In the latter exceptional case we have that $\sum_0^\infty a_k\, z^k = 0$ only for $z = \exp(2\pi i l/K)$ with $l = 1, \ldots, K-1$.

Proof. The proof of this lemma is based on the identity

$$(1-z)\sum_{k=1}^{\infty} a_k\, z^k = \sum_{k=1}^{\infty} (a_{k-1}-a_k)(1-z^k)\ , \qquad |z| \le 1\ , \tag{12}$$

where we note that the right-hand side series has only terms with non-negative real parts.

Theorem 1.
There is an $A > 0$ such that $|(Zg)(t,\nu)| \ge A$ for $(t,\nu) \in [0,1)^2$.

Proof. We have

$$(Zg)(t,\nu) = \sum_{k=0}^{\infty} g(t+k)\, e^{-2\pi i k\nu}\ , \qquad (t,\nu) \in [0,1)^2\ , \tag{13}$$

where the series at the right-hand side converges uniformly. Also,

$$(Zg)(t,0) \ge \sum_{k=0}^{\infty} g(k+1) > 0\ , \qquad t \in [0,1)\ . \tag{14}$$

Hence there is an $\varepsilon > 0$, $\delta > 0$ such that

$$|(Zg)(t,\nu)| \ge \delta\ , \qquad t \in [0,1) \ \text{ and } \ (0 \le \nu < \varepsilon \text{ or } 1-\varepsilon \le \nu < 1)\ . \tag{15}$$

For $\varepsilon \le \nu \le 1-\varepsilon$ and $t \in [0,1)$ we let $z = e^{-2\pi i\nu}$ and $a_k = g(t+k)$, $k = 0, 1, \ldots$. Then we have as in the proof of the Lemma that

$$\begin{aligned}\operatorname{Re}\Big[(1-z)\sum_{k=0}^{\infty} a_k\, z^k\Big] \ &= \ \operatorname{Re}\Big[\sum_{k=1}^{\infty} (a_{k-1}-a_k)(1-z^k)\Big) \ \ge \\ &\ge \ (a_0-a_1)(1-\cos 2\pi\nu)\ .\end{aligned}$$

Now it follows that for $\varepsilon \le \nu \le 1-\varepsilon$ and $t \in [0,1)$

$$|(Zg)(t,\nu)| \ge (a_0 - a_1)\,\frac{1-\cos 2\pi\nu}{|1-e^{-2\pi i\nu}|} \;=\; (a_0-a_1)\sin\pi\nu \;\ge\; \ge\; (a_0-a_1)\sin\pi\varepsilon \; . \qquad (16)$$

Finally, we have for $t \in [0,1)$ that

$$a_0 - a_1 = g(t) - g(t+1) \ge \min(g(\tfrac{1}{2}) - g(0), g(1) - g(\tfrac{3}{2})) > 0 \; , \qquad (17)$$

and this completes the proof.

In Subsec. 2.4 we shall discuss what can happen when the condition of strict decreasingness of g is weakened into (non-strict) decreasingness.

2.2 Even windows g that are super convex on $[0,\infty)$

In this subsection g is an even, continuous function having on $[0,\infty)$ the form

$$g(t) = b(t) + b(t+1) \; , \qquad t \ge 0 \; , \qquad (18)$$

with b an integrable, non-negative, strictly convex function on $[0,\infty)$. For brevity we call such a g super convex on $[0,\infty)$. The conditions on b imply that g is strictly convex on $[0,\infty)$, and that both g and b are integrable, positive and strictly decreasing on $[0,\infty)$. In Subsec. 2.4 we present some further observations on the condition of super convexity. We note that Zg is a continuous function so that it has at least one zero in $[0,1)^2$.

Theorem 2.
We have $(Zg)(\frac{1}{2},\frac{1}{2}) = 0$ and $(Zg)(t,\nu) \ne 0$ for $(t,\nu) \in [0,1]^2$, $(t,\nu) \ne (\frac{1}{2},\frac{1}{2})$.

For the proof of this result we make an analysis of the real part and the imaginary part of Zg. These are given respectively by

$$R(t,\nu) := \mathrm{Re}\,[(Zg)(t,\nu)] = g(t) + \sum_{k=1}^{\infty} (g(k-t) + g(k+t))\cos 2\pi k\nu \; , \quad (19)$$

and

$$I(t,\nu) := \mathrm{Im}\,[(Zg)(t,\nu)] = \sum_{k=1}^{\infty} (g(k-t) - g(k+t))\sin 2\pi k\nu \; , \qquad (20)$$

for $(t,\nu) \in [0,1]^2$. Letting E be the set

$$E = \{(0,\nu) \mid 0 \le \nu \le 1\} \cup \{(t,\nu) \mid 0 \le t \le 1, \; \nu = 0, \tfrac{1}{2}, 1\} \; , \qquad (21)$$

we shall show that for $(t,\nu) \in [0,1]^2$

(i) $R(t,\nu) \neq 0$ for $(t,\nu) \in E \backslash \{(\frac{1}{2},\frac{1}{2})\}$,

(ii) $I(t,\nu) = 0$ only for $(t,\nu) \in E$.

From (i) and (ii) it follows easily that $(Zg)(t,\nu) \neq 0$ for $(\frac{1}{2},\frac{1}{2}) \neq (t,\nu) \in [0,1]^2$, and it is easily seen directly that $(Zg)(\frac{1}{2},\frac{1}{2}) = 0$.

Proof of (i). **(a)** We have by partial summation

$$R(0,\nu) = g(0) + 2\sum_{k=1}^{\infty} g(k)\cos 2\pi k\nu = \sum_{k=0}^{\infty} (g(k) - g(k+1))\, S_k(\nu)\ , \qquad (22)$$

where we have set

$$S_k(\nu) = 1 + 2\sum_{j=1}^{k} \cos 2\pi j\nu\ , \quad k = 1,2,...\ ; \qquad S_0(\nu) = 1\ . \qquad (23)$$

By [10], Part VI, § 3, Ex. 18 we have

$$\sum_{l=0}^{k} S_l(\nu) = k + 1 + 2\sum_{l=1}^{k} (k+1-l)\cos 2\pi l\nu = \Big(\frac{\sin \pi(k+1)\nu}{\sin \pi\nu}\Big)^2 \qquad (24)$$

for $k = 0,1,...$. Hence, again by partial summation,

$$R(0,\nu) = \sum_{k=0}^{\infty} (g(k) - 2g(k+1) + g(k+2))\,\Big(\frac{\sin \pi(k+1)\nu}{\sin \pi\nu}\Big)^2\ . \qquad (25)$$

Hence $R(0,\nu) > 0$ for $\nu \in [0,1]$ by strict convexity of g on $[0,\infty)$.

(b) We have $R(t,0) = R(t,1) > 0$ from (19) and positivity of g.

(c) Since g is even, we have for $0 \leq t \leq 1$ that

$$\begin{aligned} R(t,\tfrac{1}{2}) &= \sum_{k=-\infty}^{\infty} (-1)^k\, g(t+k) = \\ &= \sum_{k=0}^{\infty} \{(g(2k+t) - g(2k+1-t)) - \\ &\qquad + (g(2k+1+t) - g(2k+1+(1-t)))\}\ . \end{aligned} \qquad (26)$$

By strict convexity of g on $[0,\infty)$ each of the terms between { } on the last two lines of (26) is > 0 or < 0 according as $0 \leq t < \frac{1}{2}$ or $\frac{1}{2} < t \leq 1$. This completes the proof of (i) (note that we have only used strict, and not super, convexity of g on $[0,\infty)$).

Proof of (ii). Assume that we have numbers c_k, $k = 1, 2, ...$ of the form

$$c_k = d_k + d_{k+1} \ , \qquad k = 1, 2, ... \ , \tag{27}$$

where $d_k \geq 0$, $\sum_k d_k < \infty$. Then

$$\sum_{k=1}^{\infty} c_k \sin 2\pi k\nu = \sum_{k=1}^{\infty} (d_k - d_{k+1}) \, T_k(\nu) \ , \tag{28}$$

where

$$T_k(\nu) = 2 \sum_{j=1}^{k-1} \sin 2\pi j\nu + \sin 2\pi k\nu \ . \tag{29}$$

Indeed, when we let $U_k(\nu) = \sum_1^k \sin 2\pi j\nu$, we have by partial summation

$$\sum_{k=1}^{\infty} d_k \sin 2\pi k\nu = \sum_{k=1}^{\infty} (d_k - d_{k+1}) \, U_k(\nu) \ =$$

$$= \ \tfrac{1}{2} \sum_{k=1}^{\infty} (d_k - d_{k+1})(T_k(\nu) + \sin 2\pi k\nu) \ =$$

$$= \ \tfrac{1}{2} \sum_{k=1}^{\infty} (d_k - d_{k+1}) \, T_k(\nu) + \tfrac{1}{2} \sum_{k=1}^{\infty})(d_k - d_{k+1}) \ \sin 2\pi k\nu \ . \tag{30}$$

Hence (28) follows upon bringing the last term on the third line of (30) to the far left-hand side.

Now let $t \in [0, 1]$, and consider the choice, see (18) and (20)

$$c_k = g(k - t) - g(k + t) \ ; \qquad d_k = b(k - t) - b(k + t) \ . \tag{31}$$

We then have

$$I(t, \nu) = \sum_{k=1}^{\infty} c_k \sin 2\pi k\nu = \sum_{k=1}^{\infty} (d_k - d_{k+1}) \, T_k(\nu) \ . \tag{32}$$

By [10], Part VI, § 3, Ex. 21 and its solution on p. 255, we have

$$T_k(\nu) = 2 \cos \pi\nu \, \frac{\sin^2 \ \pi k\nu}{\sin \pi\nu} \ , \tag{33}$$

whence

$$I(t, \nu) = 2 \cos \pi\nu \sum_{k=1}^{\infty} (d_k - d_{k+1}) \, \frac{\sin^2 \ \pi k\nu}{\sin \pi\nu} \ . \tag{34}$$

By strict convexity of b we have that $d_k - d_{k+1} \geq 0$, $k = 1, 2, \ldots$, with equality if and only if $t = 0$. When $t \neq 0$ it follows that the series at the right-hand side of (34) vanishes if and only if $\nu = 0$ or 1. Finally, since $\cos \pi\nu$ vanishes only for $\frac{1}{2} = \nu \in [0, 1]$, the proof of (ii) is complete.

By continuity of Zg an immediate consequence, see (4)–(7), of Theorem 2 is that both $(g, 1/N, 1)$ and $(g, 1, 1/N)$ are Gabor frames when N is an integer ≥ 2. In Subsec. 2.3 we shall investigate what becomes of Theorem 2 when the condition of super convexity on g is replaced by strict convexity, and in Subsec. 2.4 we give some examples to see what happens when the b in (18) is convex but not necessarily strictly convex.

2.3 Even windows g that are strictly convex on $[0, \infty)$

In this subsection g is an even, non-negative, continuous and integrable function that is strictly convex on $[0, \infty)$. Since the condition of strict convexity is weaker than that of super convexity, we cannot expect Theorem 2 to be true for g. However, it was already noted in the proof of Theorem 2 that one can already obtain significant insight into the zero set of Zg when g is strictly convex on $[0, \infty)$.

Theorem 3.
We have $(Zg)(t, \nu) \neq 0$ for $(t, \nu) \in [0, 1]^2$, $t \neq \frac{1}{2}$.

Proof. With $z = e^{-2\pi i\nu}$, $z^* = e^{2\pi i\nu}$ we have, since g is even, that

$$(Zg)(t, \nu) = \sum_{k=-\infty}^{\infty} g(t+k)\, e^{-2\pi i k\nu} = \sum_{k=0}^{\infty} g(k+t)\, z^k + \sum_{k=1}^{\infty} g(k-t)(z^*)^k\ . \quad (35)$$

Next we compute

$$\begin{aligned} (1-z)(Zg)(t,\nu) \;&=\\ =\; &(1-z) \sum_{k=0}^{\infty} g(k+t)\, z^k + (1-z)\, z^* \sum_{k=1}^{\infty} g(k-t)(z^*)^{k-1} \;=\\ =\; &(1-z) \sum_{k=0}^{\infty} g(k+t)\, z^k - (1-z^*) \sum_{k=0}^{\infty} g(k+1-t)(z^*)^k\ , \end{aligned} \quad (36)$$

where we have used that $zz^* = 1$. From (12) it follows that

$$(1-z)(Zg)(t,\nu) \;=\; \sum_{k=1}^{\infty} (g(k-1+t) - g(k+t))(1-z^k) \;+$$

$$- \sum_{k=1}^{\infty} (g(k-t) - g(k+1-t))(1-(z^*)^k) \ . \quad (37)$$

Since $\mathrm{Re}(z^*)^k = \mathrm{Re}\, z^k = \cos 2\pi k\nu$ we then get that

$$\mathrm{Re}\,[(1-e^{-2\pi i\nu})(Zg)(t,\nu)] \ =$$

$$= \sum_{k=1}^{\infty} \{g(k-1+t) - g(k+t) - g(k-t) + g(k+1-t)\} \ \cdot$$

$$\cdot\, (1-\cos 2\pi k\nu) \ . \qquad (38)$$

The expression between { } on the second line of (38) can be written as

$$[g(k+1-t) - g(k+t)] - [g(k-t) - g(k-(1-t))] \ . \qquad (39)$$

When $0 \leq t < \frac{1}{2}$, we see that (39) is the difference of the increments of g over the intervals $(k+1-t, k+t)$ and $(k-t, k-(1-t))$ that have equal length and that lie entirely to the right of k and to the left of k, respectively. Hence $(39) > 0$ when $0 \leq t < \frac{1}{2}$ by strict convexity of g. Similarly, we have that $(39) < 0$ when $\frac{1}{2} < t \leq 1$. Hence (38) vanishes only when $\nu = 0$ or 1 and when $t = \frac{1}{2}$. Since $(Zg)(t,0) = (Zg)(t,1) > 0$ by positivity of g the result is proved.

As a consequence of Theorem 3 we have that $(g, 1/N, 1)$ is a Gabor frame for any $N = 2, 3, ...$ as follows from (4)–(5) and continuity of Zg. Noting that the class of g as we have them here is invariant under dilations, it follows that (g, a, b) is a Gabor frame when $ab = N$; in particular $(g, 1, 1/N)$ is a Gabor frame.

In Subsec. 2.4 we shall display a g of the type considered in this subsection with as many zeros in $\{(t,\nu) \mid t = \frac{1}{2}\}$ as one wants. By (6) and the fact that $(g, 1, 1/N)$ is a Gabor frame, it follows that there is no ν such that $(Zg)(\frac{1}{2}, \nu + l/N) = 0$ for $l = 0, 1, ..., N-1$. We finally note the following result.

Theorem 4.
We have $(Zg)\Big(\dfrac{1}{2}, \dfrac{1}{2N}\Big) \neq 0$ when $N = 2, 3, ...$.

Proof. Setting $b_k = g(k+\frac{1}{2}) - g(k-\frac{1}{2})$, we have from (20) by partial summation

$$I(\tfrac{1}{2},\nu)\sin \pi\nu \ = \ \tfrac{1}{2} \sum_{k=1}^{\infty} b_k (\cos 2\pi(k-\tfrac{1}{2})\nu - \cos 2\pi(k+\tfrac{1}{2})\nu) \ =$$

$$= \tfrac{1}{2}\, b_1 \cos \pi\nu - \tfrac{1}{2} \sum_{k=2}^{\infty} (b_{k-1} - b_k) \cos 2\pi (k - \tfrac{1}{2})\nu \ . \quad (40)$$

Consider the expression on the second line of (40) with $\nu = 1/2N$ where N is an integer $\geq\ 2$. There holds for $k = 0, 1, ...$ that

$$\cos 2\pi (k - \tfrac{1}{2})/2N \leq \cos \pi/2N \quad (41)$$

with equality if and only if k is of the form $2pN$ or $2pN+1$ with $p = 0, 1, ...$. Since $b_{k-1} - b_k > 0$ for all $k = 2, 3, ...$, it thus follows that

$$I\Big(\frac{1}{2}, \frac{1}{2N}\Big) \sin \frac{\pi}{2N} > \frac{1}{2}\, b_1 \cos \frac{\pi}{2N} - \frac{1}{2} \sum_{k=2}^{\infty} (b_{k-1} - b_k) \cos \frac{\pi}{2N} = 0 \ , \quad (42)$$

and this completes the proof.

2.4 Comments and counterexamples

In this subsection we present examples for Theorems 1–4 that show that the various strictness requirements on g cannot be weakened. We give furthermore some comments on the super convexity condition in Theorem 2.

2.4.1

When g is as in Subsec. 2.1, except that g is non-strictly decreasing instead of strictly decreasing, Theorem 1 may fail to hold. Let $t_0 \in [0, 1)$, $K = 2, 3, ...$, and take g such that g is constant on all intervals $[t_0 + mK, t_0 + (m+1)K - 1]$, $m = 0, 1, ...$. Then Zg vanishes at all points $(t_0, l/K)$ with $l = 1, ..., K-1$.

2.4.2

For the proof of (i) in the proof of Theorem 2 we only needed that g is strictly (and not super) convex on $[0, \infty)$. However, when g is not strictly convex, it may happen that $R(t, \frac{1}{2})$ vanishes. Then also $(Zg)(t, \frac{1}{2})$ vanishes. To construct such a g we take $t_0 \in [0, \frac{1}{2})$ and we let g be linear on each of the intervals $[2k + t_0, 2k + 2 - t_0]$, $k = 0, 1, ...$. Now (26) shows that $R(t, \frac{1}{2}) = 0$ when $\min(t, 1-t) \geq t_0$. Such a g, viz.

$$g(t) \;=\; (e^{-|t|} - e^{-1/4})\, \chi_{[-\frac{1}{4}, \frac{1}{4}]}(t) \;+ \;+ \sum_{k=0}^{\infty} e^{-k} \Big(1 - \frac{|t|}{2k + 7/4}\Big)\, \chi_{[-1,1]}\Big(\frac{t}{2k + 7/4}\Big) \ , \qquad t \in \mathbb{R} \ , \quad (43)$$

was considered in [7]. This g is strictly positive, just as its Fourier transform, while its Zak transform vanishes at $(t, \frac{1}{2})$ with $\frac{1}{4} \leq t \leq \frac{3}{4}$.

2.4.3

Assume that g is as in Subsec. 2.2, except that b in (18) is convex but not strictly convex on $[0,\infty)$. Then it still holds that $R(t,\frac{1}{2}) \neq 0$ for $t \neq \frac{1}{2}$. Indeed, when $R(t_0,\frac{1}{2}) = 0$ for some $t_0 \neq \frac{1}{2}$, we have from (26) and convexity of b that for $l = 0, 1, \ldots$

$$b(l+t_0) - b(l+(1-t_0)) = b(l+1+t_0) - b(l+1+(1-t_0)) \ . \qquad (44)$$

Therefore $b(l+t_0) - b(l+(1-t_0))$ does not depend on $l = 0, 1, \ldots$. But then convexity of b on $[0,\infty)$ implies that b is linear on $[1,\infty)$ which is absurd in view of the other assumptions on g.

2.4.4

Assume that g is non-negative, integrable and decreasing on $[0,\infty)$. Then we have for g the representation (18) with

$$b(t) = \sum_{k=0}^{\infty} \left(g(2k+t) - g(2k+1+t)\right) \ , \qquad t \geq 0 \ . \qquad (45)$$

Therefore, when $g'''(t) < 0$ for $t \geq 0$, it follows in the first place that $g''(t) > 0$, $t \geq 0$, and secondly that b is strictly convex. We note that (ii) in the proof of Theorem 2 was announced in [6], Sec. 5 for g's satisfying $g'''(t) < 0$, $t \geq 0$. Evidently, a g of the form (18) with b strictly convex on $[0,\infty)$ does not need to satisfy $g'''(t) < 0$ for $t \geq 0$.

2.4.5

One easily checks that the b obtained by (45) from the g in (43) is not convex on $[-,\infty)$.

2.4.6

Assume that g is as in Subsec. 2.2, except that b is convex but not strictly convex on $[0,\infty)$. Then $(Zg)(t,\nu)$ can vanish at certain points $(t,\nu) \neq (\frac{1}{2},\frac{1}{2})$. Indeed, with g even and of the form (18) on $[0,\infty)$, we calculate from (19) and (20)

$$R(\tfrac{1}{2},\tfrac{1}{4}) = I(\tfrac{1}{2},\tfrac{1}{4}) = \sum_{m=0}^{\infty} \left(b(4m+\tfrac{1}{2}) - 2b(4m+\tfrac{5}{2}) + b(4m+\tfrac{9}{2})\right) \ . \qquad (46)$$

Hence, when b is linear on each of the intervals $[4m+\frac{1}{2}, 4m+\frac{9}{2}]$, $m = 0, 1, \ldots$, we have that $(Zg)(\frac{1}{2},\frac{1}{4}) = 0$. Interestingly, we have for this g that

$(Zg)(\frac{1}{2},\frac{3}{4})=0$ as well, and this implies that $(g,1,\frac{1}{2})$ is not a Gabor frame, see (6). By Theorem 4 such a g cannot be strictly convex on $[0,\infty)$.

The example just given is more or less typical for all cases where $I(t,\nu)=0$ for some $t\neq 0,1$ and $\nu\neq 0,\frac{1}{2},1$ with g as above. Indeed, one easily sees that the right-hand side of (34) can only vanish for $\nu=0,\frac{1}{2},1$ and for ν of the form p/M with integer p, M and $\gcd(p,M)=1$. In the latter case we also need that $d_k=d_{k+1}$ for $k=1,2,...$ unless k is a multiple of M. By convexity of b this implies that b is linear on each of the intervals $[mM+1-t,(m+1)M+t]$, $m=0,1,...$ (in order that this latter condition can hold with an integrable, non-trivial b we also need that $t\leq\frac{1}{2}$).

2.4.7

We present an example of a g as in Subsec. 2.3 such that $(Zg)(\frac{1}{2},\nu)$ has as many zeros as one wants. It is an interesting fact that $\exp(-\pi i\nu)(Zg)(\frac{1}{2},\nu)\in\mathbb{R}$. Consequently, we have that $R(\frac{1}{2},\nu)\cos\pi\nu=I(\frac{1}{2},\nu)\sin\pi\nu$, whence it suffices to construct a g such that $I(\frac{1}{2},\nu)$ vanishes. From (20) and setting $b_k=g(k-\frac{1}{2})-g(k+\frac{1}{2})$ we get by partial summation

$$I(\tfrac{1}{2},\nu)=\sum_{k=1}^{\infty}(b_k-b_{k+1})\,W_k(\nu)=\sum_{k=1}^{\infty}(g(k-\tfrac{1}{2})-2g(k+\tfrac{1}{2})+g(k+\tfrac{3}{2}))\,W_k(\nu)\ , \tag{47}$$

where, see [10], Part VI, § 3, Ex. 16

$$W_k(\nu)=\sum_{l=1}^{k}\sin 2\pi l\nu=\frac{\sin\pi k\nu\sin\pi(k+1)\nu}{\sin\pi\nu}\ . \tag{48}$$

Now let $K=2,3,...$, and take g such that

$$g(k-\tfrac{1}{2})-2g(k+\tfrac{1}{2})+g(k+\tfrac{3}{2}) \tag{49}$$

is small but positive for all $k\neq K$ and unity for $k=K$. Then

$$I(\tfrac{1}{2},\nu)=W_K(\nu)+\varepsilon(\nu)=\frac{\sin\pi K\nu\sin\pi(K+1)\nu}{\sin\pi\nu}+\varepsilon(\nu) \tag{50}$$

with ε, ε' as small as one wishes. Now $W_K(\nu)$ has many zeros, and these zeros are only slightly disturbed when ε is added to W_k. Hence $I(\frac{1}{2},\nu)$ has many zeros as well.

We also observe that Theorem 4 fails to hold when g is allowed to be just non-strictly convex instead of strictly convex. Indeed, then the g can be taken such that the ε in (50) vanishes identically and counterexamples for Theorem 4 are obtained by taking K an even multiple of N.

3 When is $(\chi_{[0,c_0)}, a, b)$ a Gabor frame?

In this section we consider the following problem. Let $c_0 > 0$. Find all $a > 0$, $b > 0$ such that $(\chi_{[0,c_0)}, a, b)$ is a Gabor frame. In Subsec. 3.1 we give some basic observations, and we reduce the problem to finding all $a > 0$, $c > 0$ such that $(\chi_{[0,c)}, a, 1)$ is a Gabor frame. For the latter problem we use the criterion of having a Gabor frame in terms of (the adjoints of) the Ron-Shen matrices

$$M_g^*(t) = (g(t - na - k))_{n\in\mathbb{Z},k\in\mathbb{Z}} \;, \qquad t \in \mathbb{R} \;, \tag{51}$$

with $g = \chi_{[0,c)}$, to obtain some of the more basic results. It so turns out that we can restrict attention to the case that $a \leq 1$ and $c \geq 1$. In Subsec. 3.2 we use the Zak transform to find out for which $c > 0$ the triple $(\chi_{[0,c)}, q^{-1}, 1)$ is a frame when $q = 1, 2, \ldots$. In particular, we find that $(\chi_{[0,c)}, 1, 1)$ is a frame if and only if $c = 1$, and that $(\chi_{[0,c)}, \frac{1}{2}, 1)$ is a frame for all c of the form $c = M + \frac{1}{2}$ with $M = 1, 2, \ldots$. In Subsec. 3.3 we give some more advanced observations, based on the frame characterization in terms of Ron-Shen matrices, leading to the result that $(\chi_{[0,c)}, a, 1)$ is a frame

- when $1 \leq c < 2$ and $a \leq 2 - c$,
- when $c \geq 2$ and $a \leq \min(c - \lfloor c \rfloor, 1 - (c - \lfloor c \rfloor))$.

It is also shown that $(\chi_{[0,c)}, a, 1)$ is not a frame when $a \leq 1$ and $c = 2, 3, \ldots$. At the end of Subsec. 3.3 we present some results indicating that rationality of a often prevents $(\chi_{[0,c)}, a, 1)$ from being a frame. In Subsec. 3.4 we consider irrational $a < 1$, and we show that $(\chi_{[0,c)}, a, 1)$ is a frame

- when $1 \leq c < 2$,
- when $c \geq 2$ and $a \leq \max(c - \lfloor c \rfloor, 1 - (c - \lfloor c \rfloor))$.

For the proof of this result we need some basic facts of the theory of uniform distribution in $[0, 1)$ of the numbers $na - \lfloor na \rfloor$. In Subsec. 3.5 we consider the case that a and c are rationally related, and we produce non-trivial examples $a \notin \mathbb{Q}$, $c > 2$ such that $(\chi_{[0,c)}, a, 1)$ is or is not a frame. In fact, it is conjectured that when $a \notin \mathbb{Q}$, $a < 1$ and a and c are not rationally related, then $(\chi_{[0,c)}, a, 1)$ is a frame. In Subsec. 3.6 we present an algorithm that decides whether $(\chi_{[0,c)}, a, 1)$ is a frame for $a \in \mathbb{Q}$ and $c \geq 1$. For a given $a = p/q$ with integer p, q, $\gcd(p, q) = 1$, it is first shown that it suffices to consider t in (51) and c of the form l/q with integer l. A further observation is that the Ron-Shen matrices are periodic in the sense that

$$(M_g^*(t))_{n+q,k+p} = (M_g^*(t))_{n,k} \;, \qquad n, k \in \mathbb{Z} \;. \tag{52}$$

Since all rows of $M_g^*(t)$ consist of $\lfloor c \rfloor$ or $\lfloor c \rfloor + 1$ consecutive 1's (and 0's otherwise), it is fairly easy to determine the range and nullspace of a $q \times p$ section of $M_g^*(t)$. This then gives rise to a directed graph on $\lfloor c \rfloor$ points, each of them having indegree and outdegree ≤ 1. Now $(\chi_{[0,c)}, a, 1)$ is a frame if and only if this graph has no cycles. This simple criterion is used to produce a number of examples, not covered by the results in the earlier sections, with $a \in \mathbb{Q}$ and $c \geq 1$ such that $(\chi_{[0,c)}, a, 1)$ is or is not a frame. In Subsec. 3.7 we return to the problem of determining for a given $c_0 > 0$ the set of all $a > 0$, $b > 0$ such that $(\chi_{[0,c_0)}, a, b)$ is a frame. This then gives rise to the Figs. 1–2, which were already presented in Sec. 1, that nicely display and illustrate the results of Subsecs. 3.1–3.6.

As to the notation in this section the following. We denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the largest integer $\leq x$ and the smallest integer $> x$. Consequently

$$\lfloor -5 \rfloor = \lfloor -4\tfrac{1}{2} \rfloor = -5 \ , \qquad \lceil -5 \rceil = \lceil -4\tfrac{1}{2} \rceil = -4 \ . \tag{53}$$

3.1 Basic observations

We present in this subsection some basic observations.

3.1.1

We have that $(\chi_{[0,c_0)}, a, b)$ is a Gabor frame with frame bounds $A > 0$, $B < \infty$ if and only if $(\chi_{[0,bc_0)}, ba, 1)$ is a Gabor frame with frame bounds bA, bB. In Subsecs. 3.1–3.6 we shall therefore be concerned with the problem of determining for which $c > 0$, $a > 0$ the triple $(\chi_{0,c)}, a, 1)$ is a Gabor frame.

3.1.2

We have that $(\chi_{[0,c)}, a, 1)$ is not a Gabor frame when $a > 1$. We shall restrict henceforth to the case $a \leq 1$.

3.1.3

According to Ron-Shen theory, $(g, a, 1)$ is a Gabor frame with frame bounds $A > 0$, $B < \infty$ if and only if

$$AI \leq M_g(t)\, M_g^*(t) \leq BI \ , \qquad \text{a.e. } t \ , \tag{54}$$

where for $t \in \mathbb{R}$ the operator $M_g(t)$ of $l^2(\mathbb{Z})$ is defined through its matrix with respect to the standard basis of $l^2(\mathbb{Z})$ as

$$M_g(t) = (g(t + na - k))_{k \in \mathbb{Z}, n \in \mathbb{Z}} \ . \tag{55}$$

In (55) k is the row index and n is the column index. In the special case that $g = \chi_{[0,c)}$, we have that each column of $M_g(t)$ has $\lfloor c \rfloor$ or $\lfloor c \rfloor + 1$ consecutive 1's and 0's otherwise. It thus easily follows that the upper frame bound condition in (54) is always satisfied. Hence, we shall only be concerned with the lower frame bound condition in (54).

We compute for $\mathbf{x} = (x_k)_{k\in\mathbb{Z}}$

$$\langle M_g(t)\, M_g^*(t)\, \mathbf{x}, \mathbf{x} \rangle = \| M_g^*(t)\, \mathbf{x} \|^2 \; =$$

$$= \sum_{n\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} \chi_{[0,c)}(t + na - k)\, x_k \Big|^2 = \sum_{n\in\mathbb{Z}} \Big| \sum_{k=\lceil t+na-c \rceil}^{\lfloor t+na \rfloor} x_k \Big|^2 , \tag{56}$$

where we recall our conventions for the definition of the floor and the ceiling function, see just before the beginning of this subsection.

3.1.4

We have that $(\chi_{[0,c)}, a, 1)$ is not a frame when $a > c$.

Proof. We have

$$M_g(t)\, M_g^*(t) = \Big(\sum_{n\in\mathbb{Z}} g(t + na - k)\, g^*(t + na - l) \Big)_{k,l\in\mathbb{Z}} . \tag{57}$$

Now in the case of $g = \chi_{[0,c)}$ with $a > c$ the diagonal element

$$(M_g(t)\, M_g^*(t))_{00} = \sum_n \chi^2_{[0,c)}(t + na) = 0 \tag{58}$$

when $c \le t < a$. Hence A in (54) vanishes and this completes the proof.

3.1.5

We have that $(\chi_{[0,c)}, a, 1)$ is a Gabor frame when $a \le c \le 1$.

Proof. The matrix $M_g(t)\, M_g^*(t)$ is a diagonal matrix now, with diagonal elements

$$(M_g(t)\, M_g^*(t))_{kk} = \sum_n \chi^2_{[0,c)}(t + na - k) \ge 1 \tag{59}$$

for all $t \in \mathbb{R}$ and all $k \in \mathbb{Z}$. This completes the proof.

3.2 Observations for integer value of a^{-1} using the Zak transform

We present in this subsection some observations for the case that a^{-1} is an integer using the Zak transform, see (3)– (7).

3.2.1

We have that $(\chi_{[0,c)},1,1)$ is a frame if and only if $c=1$.

Proof. We compute the Zak transform of $g=\chi_{[0,c)}$ as

$$(Z\chi_{[0,c)})(t,\nu)=\sum_{k=-\infty}^{\infty} g(t+k)\,e^{-2\pi ik\nu} \;=\; e^{-2\pi i\lfloor c-t\rfloor\nu}\,\frac{\sin\pi(\lfloor c-t\rfloor+1)\nu}{\sin\pi\nu}\ ,$$

$$(t,\nu)\in[0,1)^2\ . \qquad (60)$$

The case that $c<1$ has already been dealt with in 3.1.4. The case that $c=1$ yields $Z\chi_{[0,c)}\equiv 1$ on $[0,1)^2$. When $c>1$, we have that $Z\chi_{[0,c)}$ is a continuous function on $[0,1)^2$, $t\neq c-\lfloor c\rfloor$, that vanishes whenever $(\lfloor c-t\rfloor+1)\nu$ is an integer >1. Hence by (4) with $N=1$ we have that $A_{\max}=0$. This completes the proof.

3.2.2

Let $N=2,3,...$. When $1\leq c<2$ we have that

$$(\chi_{[0,c)},N^{-1},1)\text{ is a frame}\Leftrightarrow c\in[1,2-N^{-1}]\ . \qquad (61)$$

When $c\geq 2$ we have that

$$(\chi_{[0,c)},N^{-1},1)\text{ is a frame}\Leftrightarrow c-\lfloor c\rfloor\in[N^{-1},1-N^{-1}]\ . \qquad (62)$$

Proof. We have for $t\in[0,1)$ that $\lfloor c-t\rfloor+1=\lfloor c\rfloor+1$ or $\lfloor c\rfloor$ according as $t\leq c-\lfloor c\rfloor$ or $t>c-\lfloor c\rfloor$.

Case $\lfloor c\rfloor=1$. When $1\leq c\leq 2-N^{-1}$ and $t\in[0,1)$, at least one of the numbers $c-t-l/N$, $l=0,1,...,N-1$, falls in the interval $(c,2](\mathrm{mod}\,1)$. Hence

$$\sum_{l=0}^{N-1}\Big|(Z\chi_{[0,c)})\Big(t+\frac{l}{N},\nu\Big)\Big|^2\geq 1\ ,\qquad \nu\in[0,1)\ , \qquad (63)$$

so that the ess inf in (4) $\geq$ 1. When $2-N^{-1}<c<2$ and $t\in(1-N^{-1},c-1)$, all points $c-t-l/N$, $l=0,...,N-1$, fall in the interval $[1,c)$. And then

$$\sum_{l=0}^{N-1}\Big|(Z\chi_{[0,c)})\Big(t+\frac{l}{N},\nu\Big)\Big|^2=4N\cos^2\pi\nu\ ,\qquad \nu\in[0,1)\ , \qquad (64)$$

vanishes at $\nu=\frac{1}{2}$. Hence the ess inf in (4) $=$ 0.

Case $\lfloor c \rfloor > 1$. The proof for this case is similar to the proof for the case $\lfloor c \rfloor = 1$, except that for a positive $A_{\max}$ in (4) we must check that for any $t \in [0, 1)$ the series at the left-hand side of (63) contains both numbers

$$\Big(\frac{\sin \pi(\lfloor c \rfloor + 1)\nu}{\sin \pi\nu}\Big)^2 , \ \Big(\frac{\sin \pi \lfloor c \rfloor \nu}{\pi\nu}\Big)^2 , \tag{65}$$

ensuring that the series stays away from 0 when $\nu \in [0, 1)$. This leads then to the necessary and sufficient condition $c - \lfloor c \rfloor \in [N^{-1}, 1 - N^{-1}]$ for $(\chi_{[0,c)}, N^{-1}, 1)$ to be a frame.

Corollary.
$(\chi_{[0,M+\frac{1}{2})}, \frac{1}{2}, 1)$ is a frame for $M = 1, 2, \ldots$.

3.3 More advanced observations

In this subsection we present some more advanced observations based on the equivalent condition for $(\chi_{[0,c)}, a, 1)$ to be a frame in terms of the Ron-Shen matrices in (55). We first develop some notations that will be used in this and the subsequent subsections. We let $c > 1$ and we set for $t \in \mathbb{R}$ and $n \in \mathbb{Z}$

$$m_n = \lfloor t + na \rfloor , \qquad r_n = \lceil t + na - c \rceil , \qquad \tau_n = t + na - \lfloor t + na \rfloor , \tag{66}$$

and

$$M_n = m_n - r_n + 1 = \begin{cases} \lfloor c \rfloor + 1 \ , & 0 \le \tau_n < c - \lfloor c \rfloor \ , \\ \lfloor c \rfloor \ , & c - \lfloor c \rfloor \le \tau_n < 1 \ . \end{cases} \tag{67}$$

Hence for any $\mathbf{x} = (x_k)_{k \in \mathbb{Z}}$ we have

$$\langle M_g(t)\, M_g^*(t)\, \mathbf{x}, \mathbf{x} \rangle = \sum_{n \in \mathbb{Z}} \Big| \sum_{k=r_n}^{m_n} x_k \Big|^2 , \tag{68}$$

and M_n is the number of terms in the series at the right-hand side of (68) between $|\ |^2$.

3.3.1

We have that $(\chi_{[0,c)}, a, 1)$ is not a frame when $c = 2, 3, \ldots$.

Proof. When $c = 2, 3, \ldots$, we have that $M_n = c$ for all $n \in \mathbb{Z}$. For any $l \in \mathbb{Z}$ there are at most $\lfloor a^{-1} \rfloor + 1$ integers n such that $m_n = l$. Hence

$$\langle M_g(t)\, M_g^*(t)\, \mathbf{x}, \mathbf{x} \rangle \le (\lfloor a^{-1} \rfloor + 1) \sum_{l=-\infty}^{\infty} |x_l + \ldots + x_{l-c+1}|^2 . \tag{69}$$

For $\mathbf{x} \in l^2(\mathbb{Z})$ we write

$$X(\nu) = \sum_{l=-\infty}^{\infty} x_l \, e^{2\pi i l \nu} \; , \qquad \nu \in \mathbb{R} \; . \tag{70}$$

Then we have for $m = 0, 1, ...$ that

$$\begin{aligned} \sum_{l=-\infty}^{\infty} |x_l + ... + x_{l-m}|^2 &= \int_0^1 |X(\nu)(1 + ... + e^{2\pi i m \nu})|^2 \, d\nu = \\ &= \int_0^1 |X(\nu)|^2 \Big(\frac{\sin \pi (m+1)\nu}{\sin \pi \nu}\Big)^2 \, d\nu \; . \end{aligned} \tag{71}$$

With $m + 1 = c = 2, 3, ...$ we have that $\sin \pi (m+1)\nu / \sin \pi \nu$ has $\operatorname{ess\,inf} = 0$. Hence the infimum of the expression on the second line of (71) over all $\mathbf{x} \in l^2(\mathbb{Z})$ with $\|\mathbf{x}\| = 1$ equals 0. This completes the proof.

3.3.2

We have that $(\chi_{[0,c)}, a, 1)$ is a frame when $a \leq \min(c - \lfloor c \rfloor, 1 - (c - \lfloor c \rfloor))$.

Proof. Assume that $a \leq \min(c - \lfloor c \rfloor, 1 - (c - \lfloor c \rfloor))$, and let $t \in \mathbb{R}$. For every $l \in \mathbb{Z}$ there is at least one $n \in \mathbb{Z}$ such that $(m_n, r_n) = (l, l - \lfloor c \rfloor)$ and at least one $n \in \mathbb{Z}$ such that $(m_n, r_n) = (l, l - \lfloor c \rfloor + 1)$. Therefore

$$\langle M_g(t) \, M_g^*(t) \, \mathbf{x}, \mathbf{x} \rangle \geq \sum_{l=-\infty}^{\infty} \{ |x_l + ... + x_{l-\lfloor c \rfloor}|^2 + |x_l + ... + x_{l-\lfloor c \rfloor + 1}|^2 \} \; . \tag{72}$$

Arguing as in the proof of 3.3.1 we see that the right-hand side of (72) equals

$$\int_0^1 |X(\nu)|^2 \Big\{ \Big(\frac{\sin \pi (\lfloor c \rfloor + 1)\nu}{\sin \pi \nu}\Big)^2 + \Big(\frac{\sin \pi \lfloor c \rfloor \nu}{\sin \pi \nu}\Big)^2 \Big\} \, d\nu \; . \tag{73}$$

The expression in the integrand in (73) within $\{ \; \}$ has a positive $\operatorname{ess\,inf}$, whence there is an $A > 0$ such that (73) is at least

$$A \int_0^1 |X(\nu)|^2 \, d\nu = A \sum_{l=-\infty}^{\infty} |x_l|^2 \; . \tag{74}$$

This completes the proof.

3.3.3

We have that $(\chi_{[0,c)}, a, 1)$ is a frame when $1 \leq c < 2$ and $a \leq 1 - (c - \lfloor c \rfloor) = 2 - c$.

Proof. Following the argument in the proof of 3.3.2 we have, when $1 \leq c < 2$ and $a \leq 2 - c$, that for every $l \in \mathbb{Z}$ there is at least one $n \in \mathbb{Z}$ such that $(m_n, r_n) = (l, l - \lfloor c \rfloor + 1) = (l, l)$. Hence

$$\langle M_g(t)\, M_g^*(t)\, \mathbf{x}, \mathbf{x} \rangle \geq \sum_{l=-\infty}^{\infty} |x_l|^2 \tag{75}$$

when $\mathbf{x} \in l^2(\mathbb{Z})$. This completes the proof.

3.3.4

Assume that $a = p/q$ with integer p, q, $\gcd(p,q) = 1$, and that $c \geq 2$ satisfies $c - \lfloor c \rfloor \in [0, q^{-1}) \cup (1 - q^{-1}, 1)$. Then $(\chi_{[0,c)}, a, 1)$ is not a frame.

Proof. Consider the case that $c - \lfloor c \rfloor \in [0, 1/q)$, and take any $t \in (c - \lfloor c \rfloor, 1/q)$. With τ_n as in (66) we note that

$$\{\tau_n \mid n \in \mathbb{Z}\} = \left\{ t + \frac{k}{q} \mid k = 0, ..., q-1 \right\} \subset (c - \lfloor c \rfloor, 1) \; . \tag{76}$$

Hence $M_n = \lfloor c \rfloor$ for all $n \in \mathbb{Z}$, and we conclude as in the proof of 3.3.1 that $(\chi_{[0,c)}, a, 1)$ is not a frame.

The proof for the case that $c - \lfloor c \rfloor \in (1 - 1/q, 1)$ is similar.

3.3.5

Assume that $a = p/q$ with integer p, q, $\gcd(p,q) = 1$ and that c, $1 \leq c < 2$, satisfies $c \in (2 - 1/q, 2)$. Then $(\chi_{[0,c)}, a, 1)$ is not a frame.

Proof. Similar to the proof of 3.3.4.

3.3.6

Assume that $a = p/q$ with integer p, q, $\gcd(p,q) = 1$ and that c, $1 \leq c < 2$, satisfies $c \leq 2 - 1/q$. Then $(\chi_{[0,c)}, a, 1)$ is a frame.

Proof. We may assume that $a < 1$ by 3.1.2 and 3.2.1. As in the proofs of the earlier results in this subsection, we have for $t \in [0,1)$ and $\mathbf{x} \in l^2(\mathbb{Z})$ that

$$\langle M_g(t)\, M_g^*(t)\, \mathbf{x}, \mathbf{x} \rangle \geq \sum_{l=-\infty}^{\infty} (\varepsilon_l\, |x_l + x_{l-1}|^2 + \delta_l\, |x_l|^2) =: S \; , \tag{77}$$

where ε_l, δ_l are all 0 or 1 and at least one of ε_l, δ_l equals 1. We have $\delta_l = 1$ when there is an $n \in \mathbb{Z}$ such that $m_n = l$ and $\tau_n \in [c-1, 1)$. These numbers τ_n are periodic in $n \in \mathbb{Z}$ with period q and visit each of the intervals $[k/q, (k+1)/q)$, $k = 0, 1, ..., q-1$, exactly once when n runs through q consecutive integers. Thus when we order the l for which $\delta_l = 1$ as an increasing sequence $(l_j)_{j\in\mathbb{Z}}$, we have from $a = p/q$ that $l_{j+1} - l_j = p$.

We write the S defined in (77) as

$$S = \sum_{j=-\infty}^{\infty} \{\tfrac{1}{2}\,|x_{l_j}|^2 + \sum_{l_j < l \le l_{j+1}} |x_l + x_{l-1}|^2 + \tfrac{1}{2}\,|x_{l_{j+1}}|^2\} \ , \tag{78}$$

and for $L = 0, 1, ...$ we consider the quadratic form

$$\tfrac{1}{2}\,|y_0|^2 + \sum_{l=1}^{L} |y_l + y_{l-1}|^2 + \tfrac{1}{2}\,|y_L|^2 = \mathbf{y}^H A\mathbf{y} \ , \qquad \mathbf{y} \in \mathbb{C}^{L+1} \ , \tag{79}$$

where A is the tridiagonal matrix

$$A = \begin{bmatrix} \alpha & 1 & & \\ 1 & 2 & & \\ & & 2 & 1 \\ & & 1 & \alpha \end{bmatrix}_{(L+1)\times(L+1)} \quad ; \qquad \alpha = 3/2 \ . \tag{80}$$

When $\alpha \in (1, 2]$, the smallest eigenvalues $\lambda_{\min}(L)$ of A is given as

$$\lambda_{\min}(L) = 2(1 - \cos\vartheta) \ , \tag{81}$$

where ϑ is the smallest positive solution of the equation

$$2(L+2)\vartheta + 4\arctan\Big[\frac{(2-\alpha)\sin\vartheta}{1-(2-\alpha)\cos\vartheta}\Big] = 2\pi \ . \tag{82}$$

Hence we have $\lambda_{\min}(L) > 0$ and $\mathbf{y}^H A\mathbf{y} \ge \lambda_{\min}(L)\,\|\mathbf{y}\|^2$. Now since $l_{j+1} - l_j = p$ for all j, it follows that

$$S \ge \sum_{j=-\infty}^{\infty} \lambda_{\min}(p) \sum_{l=l_j}^{l_{j+1}} |x_l|^2 \ge \lambda_{\min}(p)\,\|\mathbf{x}\|^2 \ , \tag{83}$$

and this completes the proof.

3.4 Irrational a

In this subsection we consider the case that a is irrational. We first give two lemmas on uniform distribution of the $\tau_n = t + na - \lfloor t + na \rfloor$ when $a \notin \mathbb{Q}$.

3.4.1 Lemma 1.

Let $a \notin \mathbb{Q}$, $L > 0$. There is an integer $K > 0$ such that for all $t \in \mathbb{R}$, all $N \in \mathbb{Z}$ and all intervals $I \subset (0, 1)$ of length $> L$ there is an $n \in \{N, ..., N + K\}$ with $\tau_n \in I$.

Proof. It is sufficient to show that there is an integer $K > 0$ such that for all $t \in \mathbb{R}$ and all intervals $I \subset (0, 1)$ of length $> L$ there is an $n = 0, ..., K$ with $\tau_n \in I$; indeed, this follows upon replacing t by $t + Na$.

Let q be an integer $< 2/L$, and consider the intervals $I_k = (k/q, (k+1)/q)$, $k = 0, ..., q-1$. Since the numbers $na - \lfloor na \rfloor$, $n = 0, 1, \ldots$, are dense in $(0, 1)$, for any $k = 0, ..., q-1$ there is an integer $K_k > 0$ such that $K_k - \lfloor K_k a \rfloor \in I_k$. Now let $t \in \mathbb{R}$ and let $I \subset (0, 1)$ be an interval of length $> L$. At least one of the sets $(t + I_k) \bmod [0, 1)$ is completely contained in I. For this k we have

$$\begin{aligned} \tau_{K_k} = t + K_k a - \lfloor t + K_k a \rfloor \;&\equiv\; (t + K_k a - \lfloor K_k a \rfloor) \bmod [0,1) \;\in \\ &\in\; (t + I_k) \bmod [0, 1) \subset I \;. \end{aligned} \tag{84}$$

Hence it suffices to take $K = \max(K_0, ..., K_{q-1})$ to prove the lemma.

3.4.2 Lemma 2.

Let $J \subset (0, 1)$ be an open interval, let $a \notin \mathbb{Q}$, and let M be a positive integer. There is a $K > 0$ such that for all $N \in \mathbb{Z}$ and all $t \in \mathbb{R}$

$$\{(n - N) \bmod M \mid N \le n \le N + K - 1,\ \tau_n \in J\} = \{0, 1, ..., M - 1\} \;. \tag{85}$$

Proof. As in the proof of Lemma 1 we assume that $N = 0$. With $L =$ length of J we take an integer $q > 2/L$, and we define the intervals I_k as in the proof of Lemma 1. Next take an $l = \{0, 1, ..., M - 1\}$. The numbers $na - \lfloor na \rfloor$ with n of the form $pM + l$, $p = 0, 1, ...$ are dense in $(0, 1)$, whence there is a $p_{kl} = 0, 1, ...$ such that $(p_{kl}M + l)a - \lfloor (p_{kl}M + l)a \rfloor \in I_k$. Now take $t \in \mathbb{R}$. At least one of the sets $(t + I_k) \bmod [0, 1)$ is completely contained in J. Thus, as in (84), we have for any $l = 0, 1, ..., M - 1$ that

$$t + (p_{kl}M + l)a - \lfloor t + (p_{kl}M + l)a \rfloor \in J \;. \tag{86}$$

It follows that we can take $K = M - 1 + \max p_{kl}$ to prove the lemma.

3.4.3

Assume that $a \notin \mathbb{Q}$, $a < 1$ and that $\lfloor c \rfloor = 1$. Then $(\chi_{[0,c)}, a, 1)$ is a frame.

Proof. We may assume that $c > 1$. As in the proof of 3.3.6 we have for any $\mathbf{x} \in l^2(\mathbb{Z})$ that

$$\langle M_g(t)\, M_g^*(t)\, \mathbf{x}, \mathbf{x} \rangle \geq \sum_{l=-\infty}^{\infty} (\varepsilon_l\, |x_l + x_{l-1}|^2 + \delta_l\, |x_l|^2) = S\ . \tag{87}$$

Note that $\delta_l = 1$ when l occurs as $\lfloor t + na \rfloor$ with $\tau_n \in [c-1, 1)$. By Lemma 1 there is a K, independent of t, such that two consecutive n's with $\tau_n \in [c-1, 1)$ have distance bounded by K. Since $a < 1$ the corresponding $l = \lfloor t + na \rfloor$ also have distance bounded by K. Then the proof is completed in the same way as the proof of 3.3.6, where we also note that

$$\lambda_{\min}(K) = \min \{\lambda_{\min}(1), ..., \lambda_{\min}(K)\}\ . \tag{88}$$

3.4.4

Assume that $a \notin \mathbb{Q}$, $0 < a < 1$, and that $c \geq 2$, $a \leq \max(c - \lfloor c \rfloor, 1 - (c - \lfloor c \rfloor))$. Then $(\chi_{[0,c)}, a, 1)$ is a frame.

Proof. Assume that $a \leq c - \lfloor c \rfloor$; the case that $a \leq 1 - (c - \lfloor c \rfloor)$ can be handled in practically the same way. By 3.3.2 we may assume that $a > 1 - (c - \lfloor c \rfloor)$. Since $a \leq c - \lfloor c \rfloor$ there holds for any $t \in \mathbb{R}$ that

$$\{\lfloor t + na \rfloor \mid t + na - \lfloor t + na \rfloor \in [0, c - \lfloor c \rfloor)\} = \mathbb{Z}\ . \tag{89}$$

Therefore, for any $t \in \mathbb{R}$ and any $\mathbf{x} \in l^2(\mathbb{Z})$ there holds

$$\langle M_g(t)\, M_g^*(t)\, \mathbf{x}, \mathbf{x} \rangle \geq \sum_{l=-\infty}^{\infty} \Big| \sum_{k=0}^{\lfloor c \rfloor} x_{l+k} \Big|^2 + \sum\nolimits_{II} \Big| \sum_{k=0}^{\lfloor c \rfloor - 1} x_{l+k} \Big|^2\ , \tag{90}$$

where $\sum_{II}$ is the series over all l such that l occurs as $\lceil t + na - c \rceil$ with $t + na - \lfloor t + na \rfloor \geq c - \lfloor c \rfloor$.

We shall show below that there is a $K > 0$ such that for all $t \in \mathbb{R}$, $N \in \mathbb{Z}$ there holds

$$\sum_{l=N}^{N+K} \Big| \sum_{k=0}^{\lfloor c \rfloor} x_{l+k} \Big|^2 + \sum_{l=N}^{N+K}{}_{II} \Big| \sum_{k=0}^{\lfloor c \rfloor - 1} x_{l+k} \Big|^2 = 0$$

$$\Rightarrow\ x_N = x_{N+1} = ... = x_{N+K+\lfloor c \rfloor} = 0\ . \tag{91}$$

Assuming this result for the moment, it thus follows that there is a $C > 0$ such that the left-hand side expression on the first line of (91) $\geq C(|x_N|^2 + ... + |x_{N+K+\lfloor c \rfloor}|^2)$. Indeed, although the quadratic form in $(x_N, ..., x_{N+K+\lfloor c \rfloor})$

on the first line of (91) depends on t and N, there are only finitely many instances of it ($\leq 2^{K+1}$) that may occur in this way when t and N vary, and of these instances we only have to consider the positive definite ones. We therefore conclude that for any t and any $\mathbf{x} \in l^2(\mathbb{Z})$

$$
\begin{aligned}
&(M_g(t)\, M_g^*(t)\,\mathbf{x},\mathbf{x}) \;\geq \\
&\geq \;\sum_{p=-\infty}^{\infty} \Big\{ \sum_{l=p(K+1)}^{(p+1)(K+1)-1} \Big| \sum_{k=0}^{\lfloor c \rfloor} x_{l+k} \Big|^2 + {\sum_{l=p(K+1)}^{(p+1)(K+1)-1}}_{II} \Big| \sum_{k=0}^{\lfloor c \rfloor -1} x_{l+k} \Big|^2 \Big\} \;\geq \\
&\geq \;C \sum_{p=-\infty}^{\infty} \sum_{l=p(K+1)}^{(p+1)(K+1)+\lfloor c \rfloor -1} |x_l|^2 \geq C\,\|\mathbf{x}\|^2 \;, \qquad (92)
\end{aligned}
$$

and this proves the result.

We now prove the claim made in connection with the implication in (91). As to $\sum_{II}$ we are concerned with the set of all $l' = l + \lfloor c \rfloor - 1$ that occur as $\lfloor t+na \rfloor$ with $n \in \mathbb{Z}$ and $t + na - \lfloor t+na \rfloor \geq c - \lfloor c \rfloor$. Thus all l' occur for which there is an integer n such that

$$
n < \frac{l'}{a} + \frac{1-t}{a} \leq n + \frac{1-(c-\lfloor c \rfloor)}{a} \;. \qquad (93)
$$

Note that $a^{-1}(1-(c-\lfloor c \rfloor)) < 1$ by assumption. We can apply Lemma 2 with l' instead of n, $J = (0, a^{-1}(1-(c-\lfloor c \rfloor)))$, $M = \lfloor c \rfloor + 1$, and a replaced by $1/a \notin \mathbb{Q}$ and t replaced by $a^{-1}(1-t)$. Accordingly, we take $K > 0$ so large that for all $N \in \mathbb{Z}$ and all $t \in \mathbb{R}$ there holds

$$
\begin{aligned}
&\{ (l'-N) \bmod (\lfloor c \rfloor + 1) \mid l' = N, ..., N+K-1 \text{ of the form } l' = \lfloor t+na \rfloor \\
&\text{with } n \in \mathbb{Z} \text{ and } t+na-\lfloor t+na \rfloor \geq c - \lfloor c \rfloor \} = \{0, 1, ..., \lfloor c \rfloor\} \;. \qquad (94)
\end{aligned}
$$

Take now $t \in \mathbb{R}$, $N \in \mathbb{N}$ and an $\mathbf{x} \in l^2(\mathbb{Z})$ such that the left-hand side member of the implication in (91) is valid. Then

$$
\sum_{k=0}^{\lfloor c \rfloor} x_{l+k} = 0 \;, \qquad l = N, ..., N+K \;, \qquad (95)
$$

and therefore

$$
x_{l+\lfloor c \rfloor +1} = x_l \;, \qquad l = N, ..., N+K-1 \;. \qquad (96)
$$

Thus $\mathbf{x}$ is periodic with period $\lfloor c \rfloor + 1$ on the indicated range. Now assume that we have an l such that $l' = \lfloor t+na \rfloor$ with $n \in \mathbb{Z}$ and $t+na-\lfloor t+na \rfloor \geq c - \lfloor c \rfloor$. Then we have from

$$
x_l + ... + x_{l+\lfloor c \rfloor} = 0 = x_{l+1} + ... + x_{l+\lfloor c \rfloor} \qquad (97)
$$

that $x_l = 0$. Note that $l' = l + \lfloor c \rfloor - 1$. Taking now $\lfloor c \rfloor + 1$ different values of l' in accordance with (94), we conclude from $(\lfloor c \rfloor + 1)$-periodicity that $x_N = x_{N+1} = \ldots = x_{N+K+\lfloor c \rfloor} = 0$, and this completes the proof.

3.5 Rationally related a and c

In this subsection we present some results for the case that a and c are rationally related. In particular, we shall find a and c such that $a > \max(c - \lfloor c \rfloor, 1 - (c - \lfloor c \rfloor))$ and $(\chi_{[0,c)}, a, 1)$ is or is not a frame. We furthermore conjecture that $(\chi_{[0,c)}, a, 1)$ is a frame when $0 < a < 1$, $a \notin \mathbb{Q}$, and a and c not rationally related.

3.5.1

Let $K = 2, 3, \ldots$, and assume that $1 - (K+2)^{-1} \leq a \leq 1$, $c = K + (K+1)(1-a)$. Then $(\chi_{[0,c)}, a, 1)$ is not a frame.

Note. It is easy to see that $a \geq \max(c - \lfloor c \rfloor, 1 - (c - \lfloor c \rfloor))$, and when we have equality here then $a \in \mathbb{Q}$. Hence there is no contradiction with 3.4.4.

Proof. Take any $t \in [c - \lfloor c \rfloor, 1)$. We shall find a sequence of $\mathbf{x}_m \in l^2(\mathbb{Z})$ with $\|\mathbf{x}_m\| \to \infty$ while $\langle M_g(t)\, M_g^*(t)\, \mathbf{x}_m, \mathbf{x}_m \rangle$ stays bounded as $m \to \infty$. Let $\mathbf{x} \in l^2(\mathbb{Z})$ and consider

$$\langle M_g(t)\, M_g^*(t)\, \mathbf{x}, \mathbf{x} \rangle = \sum_{n \in \mathbb{Z}} \Big| \sum_{k=r_n}^{m_n} x_k \Big|^2 , \tag{98}$$

where we adopt the notation of the beginning of Subsec. 3.3. Note that M_0, the number of terms in the series over k at the right-hand side of (98) between $|\ |^2$ for $n = 0$, equals $\lfloor c \rfloor = K$ since $t \in [c - \lfloor c \rfloor, 1)$. We let $y_k = x_{k-K-1}$, and we consider $\mathbf{x}$ such that $y_k = 0$ for $k < 0$ and

$$\sum_{k=0}^{K-1} y_k = 0 . \tag{99}$$

Now we have

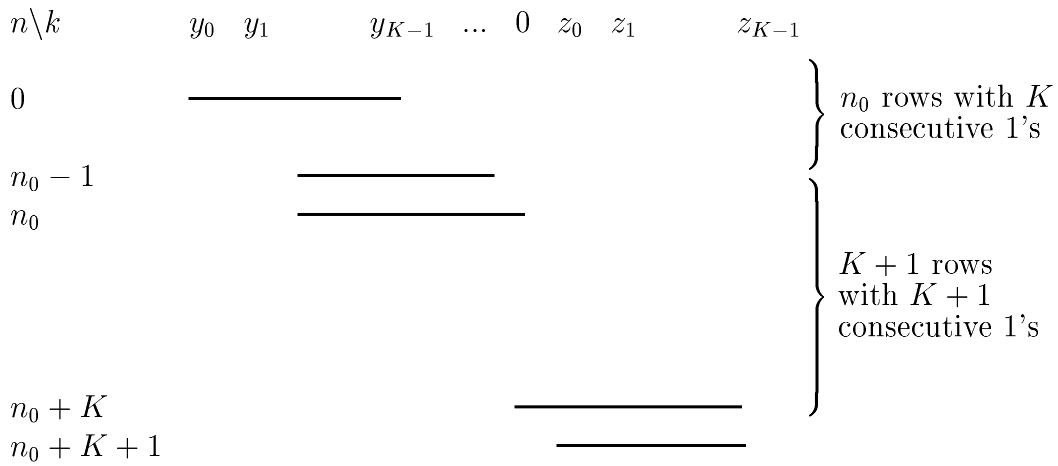
$$\langle M_g(t)\, M_g^*(t)\, \mathbf{x}, \mathbf{x} \rangle = \sum_{n=-\infty}^{\infty} \Big| \sum_{k=r_n+K-1}^{m_n+K-1} y_k \Big|^2 =$$

$$= \sum_{n=0}^{\infty} \Big| \sum_{k=r_n+K-1}^{m_n+K-1} y_k \Big|^2 + \sum_{n=-K-2}^{-1} \Big| \sum_{k=r_n+K-1}^{m_n+K-1} y_k \Big|^2 . \tag{100}$$

We shall show that we can find y_k with $y_k \not\to 0$ such that the first term on the second line of (100) vanishes. Hence we have to find $y_K, y_{K+1}, \ldots$ such that

$$\sum_{k=r_n+K-1}^{m_n+K-1} y_k = 0 \ , \qquad n = 0, 1, \ldots \ , \tag{101}$$

where we note that (101) with $n = 0$ agrees with (99) since $r_0 = -K+1$, $m_n = 0$.



Schematic representation of the first $n_0 + K + 2$ rows of the linear system (101) where $y_{n_0+K}, \ldots, y_{n_0+2K-1}$ are denoted as $z_0, \ldots, z_{K-1}$.

In the above scheme we have indicated the first rows involved in the linear system (101). Here n_0 is the smallest integer $n > 0$ such that $M_n = K+1$. The rows with index $n = 0, \ldots, n_0 - 1$ all have $M_n = K$ consecutive 1's, and they are obtained from one another by unit shifts to the right. These n_0 rows are followed by exactly $K+1$ rows with $K+1$ consecutive 1's, and the latter rows are also obtained from one another by unit shifts to the right. Next, we let n_1 be the smallest index $n > 0$ such that $M_{n_0+K+1+n} = K+1$, etc. Due to $a \geq \max(c - \lfloor c \rfloor, 1 - (c - \lfloor c \rfloor))$ we have that sections of a positive number of rows having K consecutive 1's are alternated by sections of exactly $K+1$ rows each having $K+1$ consecutive 1's. Due to the requirement (101) we can evaluate the y_k subsequently for $k = K, K+1, \ldots$. On comparison of the rows with index $n_0 - 1$ and n_0 we obtain $y_{n_0+K-1} = 0$. Furthermore, $y_{n_0+K}, \ldots, y_{n_0+2K-1}$ are obtained from $y_0, \ldots, y_{K-1}$ by a cyclic permutation. It thus appears that from row index $n = n_0 + K + 1$ onwards we have a situation which is in all respects the same as the situation from row index $n = 0$ onwards. Continuing this process of computing y_k's from previous y_k's, we see that at any row index n, where a row with K consecutive 1's is preceded by a row with $K+1$ consecutive 1's, the involved y_k are a cyclic permutation of the $y_0, ..., y_{K-1}$. It follows that $y_k \not\to 0$.

When we let for $m = 0, 1, ...$

$$(\mathbf{x}_m)_k = \begin{cases} y_{k+K+1} \ , & k \le m \\ 0 \ , & k > m \ , \end{cases} \tag{102}$$

it is easily seen that $\|\mathbf{x}_m\| \to \infty$ while $\langle M_g(t)\, M_g^*(t)\, \mathbf{x}_m, \mathbf{x}_m \rangle$ stays bounded, and this completes the proof.

3.5.2

Let $K = 2, 3, ...\,$, $I = 3, 4, ...$ with $I \equiv 0 \bmod (K+1)$. When $1 - (I+1)^{-1} \le a \le 1$ and $c = K + I(1-a)$, then $(\chi_{[0,c)}, a, 1)$ is not a frame.

Proof. The proof of this generalization of 3.5.1 consists of a simple adaptation of the proof of 3.5.1, where we now have sections of exactly I rows with $K+1$ consecutive 1's alternated by sections of a positive number of rows with K consecutive 1's.

3.5.3

Let $K = 2, 3, ...\,$, $I = 1, 2, ...$ with $I \not\equiv 0 \bmod (K+1)$. When $a \notin \mathbb{Q}$, $1 - (I+1)^{-1} < a < 1$ and $c = K + I(1-a)$, then $(\chi_{[0,c)}, a, 1)$ is a frame.

Proof. In the process described in the proof of 3.5.1, one of the $y_{n_0+K}, ...,$ y_{n_0+2K-1} has to vanish since the number I of rows with $K+1$ consecutive 1's is not a multiple of $K+1$. Hence one of the $y_0, ..., y_{K-1}$ has to vanish (as $y_{n_0+K}, ..., y_{n_0+2K-1}$ is a cyclic permutation of $y_0, ..., y_{K-1}$). Furthermore, since $a \notin \mathbb{Q}$, all cyclic permutations of $y_0, ..., y_{K-1}$ have appeared (in conjunction with a row having K consecutive 1's following a section of I rows having $K+1$ consecutive 1's) after a number $K(t)$ of evaluations of (101) where $K(t)$ is bounded in t. For this, see the proof of 3.4.4 and the argument developed there using Lemma 2 in 3.4.2. The proof is then completed in the same way as the proof of 3.4.4.

3.5.4

We conjecture that the only cases with $a \notin \mathbb{Q}$, $a < 1$ where $(\chi_{[0,c)}, a, 1)$ is not a frame are described by 3.5.2. In particular, we think that $(\chi_{[0,c)}, a, 1)$ is a frame when $a \notin \mathbb{Q}$ and a, c are not rationally related. This should follow from the fact that in the process described in the proof of 3.5.1, the section lengths of rows with $\lfloor c \rfloor$, $\lfloor c \rfloor + 1$ consecutive 1's are sufficiently random (when $a \notin \mathbb{Q}$ and a, c are not rationally related).

3.6 Algorithm for rational a and examples

Let $a = p/q$ with $p, q \in \mathbb{Z}$, $1 \leq p < q$, $\gcd(p,q) = 1$. In this subsection we present an algorithm to decide whether for a given $c \geq 1$ the triple $(\chi_{[0,c)}, a, 1)$ is a frame. Note that we may assume here that $c \geq 2$ (when $1 \leq c \leq 2$, we have by 3.3.5 and 3.3.6 that $(\chi_{[0,c)}, p/q, 1)$ is a frame if and only if $1 \leq c \leq 2 - 1/q$). We write, with $c \geq 2$,

$$c - \lfloor c \rfloor = \frac{l}{q} + r \tag{103}$$

where $l \in \{0, ..., q-1\}$ and $0 \leq r < 1/q$. By 3.3.4 we can restrict to $l \in \{1, ..., q-2\}$ when $r > 0$ and to $l \in \{1, ..., q-1\}$ when $r = 0$.

We have to decide whether there is a $C > 0$ such that for a.e. $t \in \mathbb{R}$ we have for all $\mathbf{x} \in l^2(\mathbb{Z})$ that

$$\langle M_g(t)\, M_g^*(t)\, \mathbf{x}, \mathbf{x} \rangle = \sum_{n=-\infty}^{\infty} \Big| \sum_{k=\lceil t+na-c \rceil}^{\lfloor t+na \rfloor} x_k \Big|^2 \geq C\, \|\mathbf{x}\|^2 \,. \tag{104}$$

3.6.1

It is sufficient to consider (104)

- for $\hat{c} = \lfloor c \rfloor + l/q$ and $\hat{t} = l/q$ when $r = 0$,
- for $\hat{c} = \lfloor c \rfloor + l/q$, $\lfloor c \rfloor + (l+1)/q$ and $t = l/q$, $(l+1)/q$ when $r > 0$.

Proof. Let $t \in \mathbb{R}$ and write

$$t = \frac{m}{q} + s\ ; \qquad m \in \mathbb{Z}\,, \quad 0 \leq s < 1/q\ . \tag{105}$$

Since $\gcd(p,q) = 1$ there is j_0, $n_0 \in \mathbb{Z}$ such that $m + n_0 p = j_0 q$. Therefore, by appropriately shifting summation ranges in (104), we have

$$\begin{aligned}\langle M_g(t)\, M_g^*(t)\, \mathbf{x}, \mathbf{x} \rangle &= \sum_{n=-\infty}^{\infty} \Big| \sum_{k=\lceil \frac{m+np}{q}+s-c \rceil}^{\lfloor \frac{m+np}{q}+s \rfloor} x_k \Big|^2 = \\ &= \langle M_g(s)\, M_g^*(s)\, T_{j_0}\mathbf{x}, T_{j_0}\mathbf{x} \rangle\ ,\end{aligned} \tag{106}$$

where $T_{j_0}\mathbf{x} = (x_{k+j_0})_{k \in \mathbb{Z}}$. Hence for checking (104) we may choose any $m \in \mathbb{Z}$ in (105) that we find convenient.

We next observe that

$$s - c = s - r - (\lfloor c \rfloor + l/q)\ , \tag{107}$$

where $|s-r|<1/q$. Hence for all $n,m\in\mathbb{Z}$ there holds

$$\Big\lfloor\frac{m+np}{q}+s\Big\rfloor=\Big\lfloor\frac{m+np}{q}\Big\rfloor\ , \tag{108}$$

and

$$\Big\lceil\frac{m+np}{q}+s-c\Big\rceil=\begin{cases}\Big\lceil\dfrac{m+np}{q}-\Big(\lfloor c\rfloor+\dfrac{l+1}{q}\Big)\Big\rceil\ , & s-r<0\\ \Big\lceil\dfrac{m+np}{q}-\Big(\lfloor c\rfloor+\dfrac{l}{q}\Big)\Big\rceil\ , & s-r\geq 0\ .\end{cases} \tag{109}$$

When $r=0$ we only need to bother about the lower case in (109), and when $r>0$ we need to consider both options in (109). For the lower case in (109) it is convenient to choose $m=l$ while for the upper case it is convenient to choose $m=l+1$. This then proves the result.

3.6.2

We let

$$(\hat{c},\hat{t})=(\lfloor c\rfloor+l/q,l/q),(\lfloor c\rfloor+(l+1)/q,(l+1)/q) \tag{110}$$

in accordance with 3.6.1. We furthermore adopt the notations and conventions as we had them in the proof of 3.5.1. Thus, we let $K=\lfloor c\rfloor$, and we put $y_k=x_{k-K-1}$. We take $y_0,...,y_{K-1}$ with $\sum_0^{K-1}y_k=0$, and we consider the linear equations

$$\sum_{k=k_{\min}}^{k_{\max}}y_k=0\ ,\qquad n=0,1,...,q\ , \tag{111}$$

where $k_{\min}=\lceil\hat{t}+na-\hat{c}\rceil+\lfloor\hat{c}\rfloor-1$ and $k_{\max}=\lfloor\hat{t}+na\rfloor+\lfloor\hat{c}\rfloor-1$. The choice $n=0$ in (111) corresponds to $\sum_0^{K-1}y_k=0$. Furthermore, the integer q is the smallest $n>0$ such that $\hat{t}+na-\lfloor\hat{t}+na\rfloor=l/q$.

The equation (111) with $n=q$ involves the samples $y_p,...,y_{p+K-1}$, and these can be computed recursively using (111) repeatedly. This computation is done efficiently by displaying the rows of the matrix $M_g^*(t)$ in the same manner as was done in the proof of 3.5.1. In Subsec. 3.6.3 we give a number of examples where this computation scheme is applied. Accordingly, we see that sections of rows having K consecutive 1's (short rows) are alternated by sections of rows having $K+1$ consecutive 1's (long rows). The transitions from short rows to long rows are characterized by the fact that the right most involved y_k gets assigned the value 0. The transitions from long rows to short rows are characterized by the fact that the left most involved y_k turns out to be a 0 and disappears completely from the scene hereafter. Hence the sequence

$$z_0, ..., z_{K-1} \equiv y_p, ..., y_{p+K-1} \tag{112}$$

can be described in terms of the $y_0, ..., y_{K-1}$, two subsets I and J of $\{0, ..., K-1\}$ of the same cardinality, and an injection φ from I into J such that

$$z_{\varphi(i)} = y_i\ ,\quad i \in I\ ;\qquad z_j = 0\ ,\quad j \notin J\ . \tag{113}$$

Moreover, as a consequence of (111) we have that $y_i = 0$ when $i \notin I$. Finally, there is an $i_0 \in I$ such that $\varphi(i(\operatorname{mod} K))$ increases in $i \in \{i_0, ..., i_0 + K - 1\} \cap (I \cup (I + K - 1))$; that is, φ preserves cyclic ordering.

We can also describe the mapping of $\mathbf{y} = (y_0, ..., y_{K-1})^T$ onto $\mathbf{z} = (z_0, ..., z_{K-1})^T$ with the aid of a matrix A according to

$$z_j = (A\mathbf{y})_j = \sum_{k=0}^{K-1} A_{jk} y_k\ ,\qquad j = 0, ..., K-1\ , \tag{114}$$

where

$$A_{\varphi(i)k} = \delta_{ik}\ ,\qquad i \in I\ ,\quad k = 0, ..., K-1\ , \tag{115}$$

$$A_{jk} = 0\ ,\qquad j \notin \varphi(I) = J\ ,\quad k = 0, ..., K-1\ . \tag{116}$$

Hence A is a $(0,1)$-matrix with at most one 1 in each of its rows and columns. We note that

$$\sum_{j=0}^{K-1} z_j = \sum_{k=0}^{K-1} y_k = 0\ . \tag{117}$$

The matrix $M_g^*(t)$ has the following periodicity property:

$$(M_g^*(t))_{n+q,k+p} = (M_g^*(t))_{n,k}\ ,\qquad n, k \in \mathbb{Z}\ . \tag{118}$$

Accordingly, when $n = q$ in (111) we arrive at a situation that is similar in all respects to the situation we had at $n = 0$, with $z_0, ..., z_{K-1}$ replacing $y_0, ..., y_{K-1}$ (also see (117)). In particular, we can compute $\mathbf{v} = (y_{2p}, ..., y_{2p+K-1})^T$ according to

$$\mathbf{v} = A\mathbf{z} = A^2\mathbf{y}\ , \tag{119}$$

etc. From basic matrix theory we know that $\operatorname{rank}(A^m)$, $m = 0, 1, \ldots$, decreases until it gets constant, and this constant value of the rank is reached for a value of $m \le K$. As in the proof of 3.5.1 we see then that

$F\ :\ (\chi_{[0,c)}, a, 1)$ is a frame $\ \Leftrightarrow$

$A^K = 0$ for the one or two relevant instances $(\hat{c}, \hat{t})$ in (110) .

The equivalent condition F for having a frame can also be expressed in terms of the mapping φ as

$$F' \ : \ \text{there is no non-empty set } I_0 \subset I \text{ such that } I_0 = \varphi(I_0) \ .$$

Furthermore, the directed graph Γ canonically associated with φ and A according to

$$\text{edge from } j \text{ to } k \Leftrightarrow A_{jk} = 1 \Leftrightarrow (j \in J, k \in I, j = \varphi(k)) \ , \qquad (120)$$

has indegree and outdegree ≤ 1 for all points $j, k \in \{0, ..., K-1\}$, and gives a third way for expressing the fact that $(\chi_{[0,c)}, a, 1)$ is a frame, viz. acoording to

$$F'' \ : \ \Gamma \text{ has no cycles} \ .$$

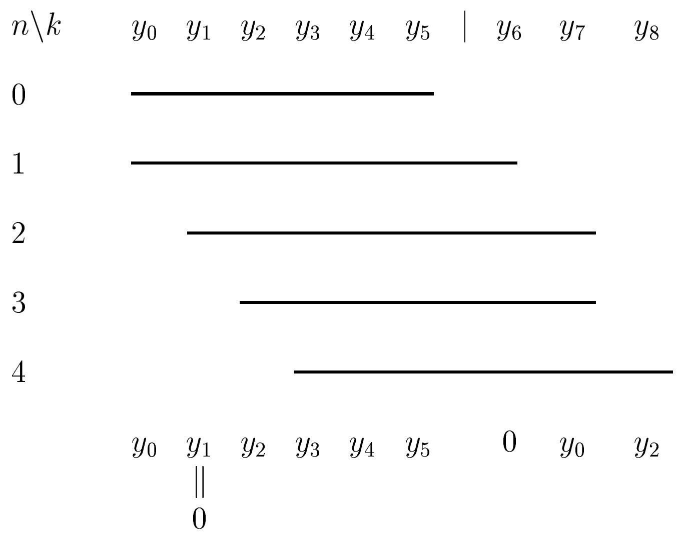
3.6.3

We shall now illustrate the above algorithm by means of a number of examples.

(a) Consider the case that $a = 3/4 = p/q$ and that $c = 6\frac{1}{2}$. According to 3.6.1 we only have to check for $(\hat{c}, \hat{t}) = (6\frac{1}{2}, \frac{1}{2})$, and we have $K = 6$ here. There is the table

n	0	1	2	3	4
$\lfloor \hat{t} + na \rfloor + \lfloor \hat{c} \rfloor - 1$	5	6	7	7	8
$\lceil \hat{t} + na - \hat{c} \rceil + \lfloor \hat{c} \rfloor - 1$	0	0	1	2	3

for $k_{\max}$ and $k_{\min}$ in (111). The equations (111) for $n = 0, ..., q = 4$, with the consequences of solving y_k's from previous y_k's can be represented schematically as

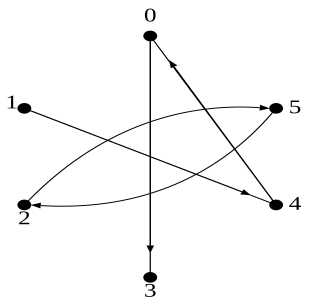


Thus we have that

$$A\,[y_0 \;\; y_1 \;\; y_2 \;\; y_3 \;\; y_4 \;\; y_5]^T = [y_3 \;\; y_4 \;\; y_5 \;\; 0 \;\; y_0 \;\; y_2]^T \,, \tag{121}$$

with the additional information that $y_1 = 0$. The matrix A and the associated directed graph Γ are given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \Gamma \;:$$



Evidently, Γ has the cycle $(2 \;\; 5)$, and also

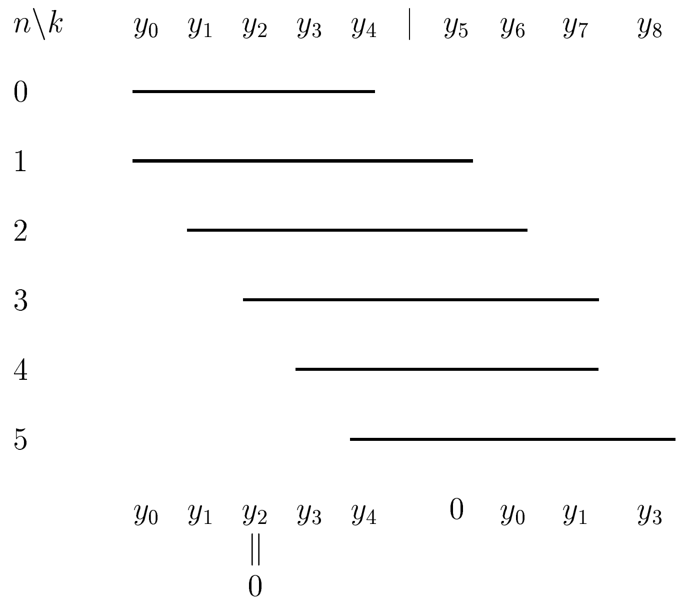
$$A^3\,[y_0 \;\; y_1 \;\; y_2 \;\; y_3 \;\; y_4 \;\; y_5]^T = [0 \;\; 0 \;\; y_2 \;\; 0 \;\; 0 \;\; y_5]^T \,. \tag{122}$$

Hence $(\chi_{[0,c)}, a, 1)$ is not a frame.

(b) Consider the case that $a = 4/5 = p/q$ and that $c = 5\frac{3}{5}$. According to 3.6.1 we only have to check for $(\hat{c}, \hat{t}) = (5\frac{3}{5}, \frac{3}{5})$, and we have $K = 5$ here. There is the table

n	0	1	2	3	4	5
$\lfloor \hat{t} + na \rfloor + \lfloor \hat{c} \rfloor - 1$	4	5	6	7	7	8
$\lceil \hat{t} + na - \hat{c} \rceil + \lfloor \hat{c} \rfloor - 1$	0	0	1	2	3	4

for $k_{\max}$ and $k_{\min}$ in (111). The equations (111) for $n = 0, \ldots, q = 5$, with the consequences for the y_k's, are represented now as

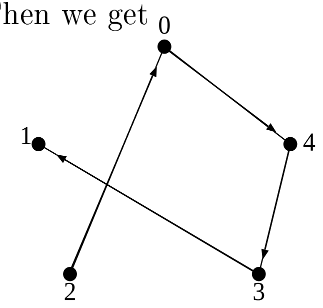


We thus have that

$$A\,[y_0 \;\; y_1 \;\; y_2 \;\; y_3 \;\; y_4]^T = [y_4 \;\; 0 \;\; y_0 \;\; y_1 \;\; y_3]^T \;, \tag{123}$$

with the additional information that $y_2 = 0$. Then we get

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} , \qquad \Gamma \;:$$



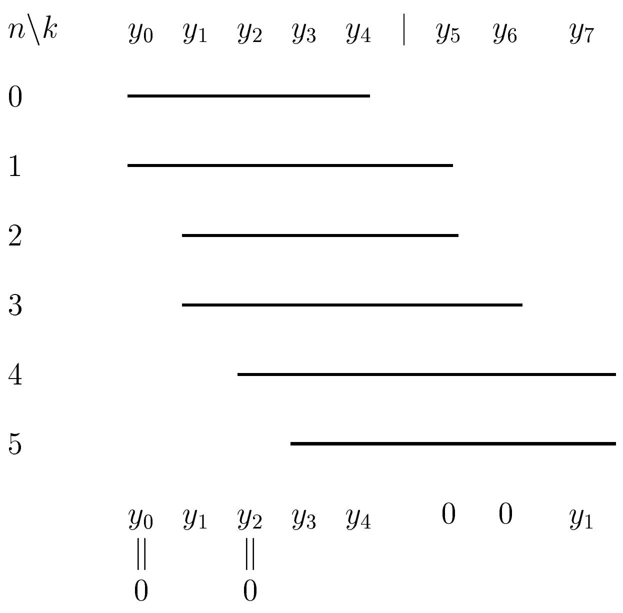
Evidently, Γ has no cycles this time, and we have $A^5 = 0$. Hence $(\chi_{[0,c)}, a, 1)$ is a frame.

(c) Consider the case that $a = 3/5 = p/q$ and that $c = 5\frac{7}{10}$. According to 3.6.1 we must check now for $(\hat{c}, \hat{t}) = (5\frac{3}{5}, \frac{3}{5})$ and $(5\frac{4}{5}, \frac{4}{5})$, and we have $K = 5$ here.

For the case $(\hat{c}, \hat{t}) = (5\frac{3}{5}, \frac{3}{5})$ we have the table

n	0	1	2	3	4	5
$\lfloor \hat{t} + na \rfloor + \lfloor \hat{c} \rfloor - 1$	4	5	5	6	7	7
$\lceil \hat{t} + na - \hat{c} \rceil + \lfloor \hat{c} \rfloor - 1$	0	0	1	1	2	3

for $k_{\max}$ and $k_{\min}$ in (111). The equations (111) for $n = 0, 1, \dots, q = 5$, with the consequences for the y_k's, are represented as follows

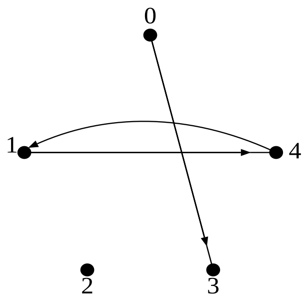


We thus have that

$$A\,[y_0 \;\; y_1 \;\; y_2 \;\; y_3 \;\; y_4]^T = [y_3 \;\; y_4 \;\; 0 \;\; 0 \;\; y_1]^T \,, \tag{124}$$

with the additional information that y_0 and y_2 are 0. The matrix A and the associated directed graph are given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} , \qquad \Gamma \;:$$



Evidently, Γ has the cycle $(1 \;\; 4)$, and also

$$A^2\,[y_0 \;\; y_1 \;\; y_2 \;\; y_3 \;\; y_4]^T = [0 \;\; y_1 \;\; 0 \;\; 0 \;\; y_4]^T \,, \tag{125}$$

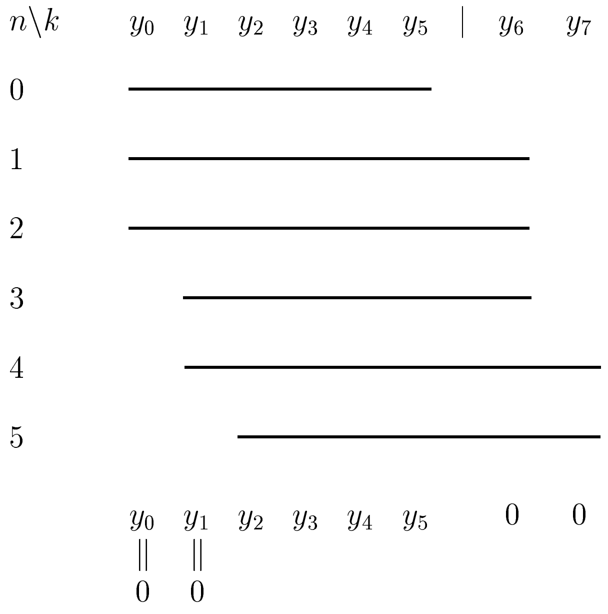
whence $(\chi_{[0,c)}, a, 1)$ is not a frame. Also, there is no need to check $(\hat{c}, \hat{t}) = (5\frac{4}{5}, \frac{4}{5})$ here.

(d) Consider the case that $a = 2/5 = p/q$ and that $c = 6\frac{7}{10}$. According to 3.6.1 we must check now for $(\hat{c}, \hat{t}) = (6\frac{3}{5}, \frac{3}{5})$ and $(\hat{c}, \hat{t}) = (6\frac{4}{5}, \frac{4}{5})$, and we have $K = 6$ here.

For the case $(\hat{c}, \hat{t}) = (6\frac{3}{5}, \frac{3}{5})$ we have the table

n	0	1	2	3	4	5
$\lfloor \hat{t} + na \rfloor + \lfloor \hat{c} \rfloor - 1$	5	6	6	6	7	7
$\lceil \hat{t} + na - \hat{c} \rceil + \lfloor \hat{c} \rfloor - 1$	0	0	0	1	1	2

for $k_{\max}$ and $k_{\min}$ in (111). The equations (111) for $n = 0, 1, \ldots, q = 5$, with the consequences for the y_k's, are schematically represented as

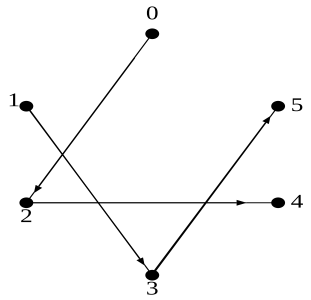


Thus we have that

$$A\,[y_0 \;\; y_1 \;\; y_2 \;\; y_3 \;\; y_4 \;\; y_5]^T = [y_2 \;\; y_3 \;\; y_4 \;\; y_5 \;\; 0 \;\; 0]^T\,, \tag{126}$$

with the additional information that $y_0 = y_1 = 0$. The matrix A and the associated directed graph Γ are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} , \qquad \Gamma \ :$$

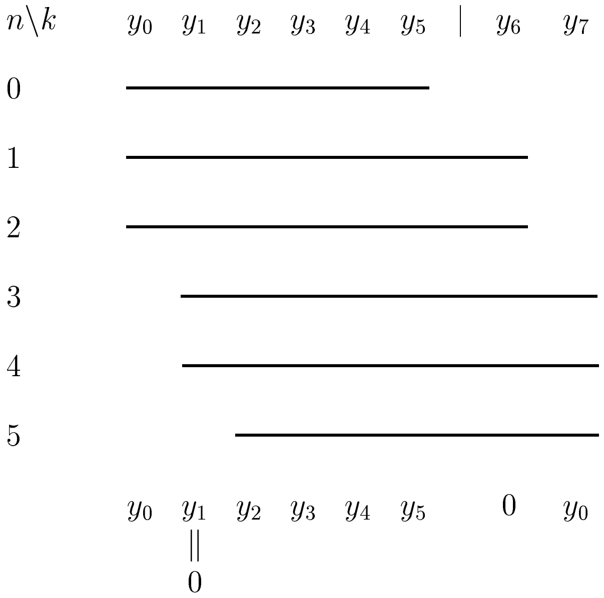


Evidently, Γ has no cycles, and $A^3 = 0$.

We next check the case that $(\hat{c}, \hat{t}) = (6\frac{4}{5}, \frac{4}{5})$. We now have the table

$$\begin{array}{lcccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ & & & & & & \\ \lfloor \hat{t} + na \rfloor + \lfloor \hat{c} \rfloor - 1 & 5 & 6 & 6 & 7 & 7 & 7 \\ \lceil \hat{t} + na - \hat{c} \rceil + \lfloor \hat{c} \rfloor - 1 & 0 & 0 & 0 & 1 & 1 & 2 \end{array}$$

for $k_{\max}$ and $k_{\min}$ in (111). The equations (111) for $n = 0, 1, \ldots, q = 5$, with the consequences for the y_k's, are schematically represented as

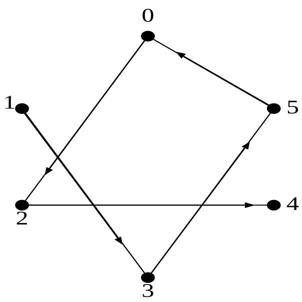


We thus have that

$$A\,[y_0 \;\; y_1 \;\; y_2 \;\; y_3 \;\; y_4 \;\; y_5]^T = [y_2 \;\; y_3 \;\; y_4 \;\; y_5 \;\; 0 \;\; y_1]^T \ , \tag{127}$$

with the additional information that $y_1 = 0$. The matrix A and the associated directed graph are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} , \qquad \Gamma \ :$$



Evidently, Γ has no cycles, and we have $A^6 = 0$.

Since in either case $(\hat{c}, \hat{t}) = (6\frac{3}{5}, \frac{3}{5})$ and $(6\frac{4}{5}, \frac{4}{5})$ we have that the graph Γ has no cycles, we conclude that $(\chi_{[0,c)}, a, 1)$ is a frame.

3.6.4

Summarizing the examples in 3.6.3 we find the following. In 3.6.3 (a) we have

$$a > \max(c - \lfloor c \rfloor, 1 - (c - \lfloor c \rfloor)) \ , \qquad \text{no frame} \ . \tag{128}$$

In 3.6.3 (b) we have

$$a > \max(c - \lfloor c \rfloor, 1 - (c - \lfloor c \rfloor)) \ , \qquad \text{frame} \ . \tag{129}$$

In 3.6.3 (c) we have

$$1 - (c - \lfloor c \rfloor) < a < c - \lfloor c \rfloor \ , \qquad \text{no frame} \ . \tag{130}$$

In 3.6.3 (d) we have

$$1 - (c - \lfloor c \rfloor) < a < c - \lfloor c \rfloor \ , \qquad \text{frame} \ . \tag{131}$$

We furthermore mention that for (128)–(131) there are examples with the inequality signs replaced by equality signs at the various places, while for (128)–(131) there are examples with the positions $1 - (c - \lfloor c \rfloor)$ and $c - \lfloor c \rfloor$ in the inequalities interchanged. Finally, it is interesting to note, see 3.5.3, that we have $c = K + I(1-a)$ with $a = 3/4$, $K = 6$, $I = 2 \not\equiv 0 \bmod 7$ in 3.6.3(a), and with $a = 4/5$, $K = 5$, $I = 3 \not\equiv 0 \bmod 6$ in 3.6.3(b). Hence the assumption that $a \notin \mathbb{Q}$ in 3.5.3 is essential.

3.7 The tie

In this subsection we discuss Figs. 1–2, given in Sec. 1, that display in the positive (a,b)-quadrant the set of all a, b such that $(\chi_{[0,c_0)}, a, b)$ is a Gabor frame, where c_0 is fixed at the value $c_0 = 2.15$. As already noted in 3.1.1, we have that $(\chi_{[0,c_0)}, a, b)$ is a frame if and only if $(\chi_{[0,bc_0)}, ba, 1)$ is a frame, whence the results of Subsecs. 3.1–3.6 carry over straightforwardly. Also, pictures with other values $c_0 > 0$ are qualitatively the same for this reason.

As to Fig. 1 we have the following observations.

(a) From 3.1.3 we get that there is no frame when $ab > 1$; this region has hatching, diagonally from NW to SE, in Fig. 1. Furthermore, from 3.2.1 on the curve $ab = 1$ we have a frame only when $a = 1/b = c_0$.

(b) From 3.1.4 we get that there is no frame when $a > c_0$; this region has hatching, diagonally from SW to NE, in Fig. 1.

(c) From 3.1.5 we get that there is a frame when $a \le c_0$ and $b \le 1/c_0$; this region has vertical hatching in Fig. 1.

(d) Let $c_0^{-1} \le b < 2c_0^{-1}$. From 3.3.3 we get that there is a frame when $ab \le 1$ and

$$b \le \frac{1}{c_0 + a}\ ; \tag{132}$$

this region has vertical hatching in Fig. 1. Furthermore, from 3.4.3, we get that there is a frame when $ab \notin \mathbb{Q}$, $ab < 1$; this region has horizontal hatching in Fig. 1. And from 3.2.2 we get that there is a frame when $ab = 1/2$ and $b \le \frac{3}{2}\, c_0^{-1}$, and there is no frame when $ab = 1/2$ and $b > \frac{3}{2}\, c_0^{-1}$.

(e) From 3.3.1 we have that there is no frame when $ab \le 1$ and $b = Kc_0^{-1}$ with $K = 2, 3, \ldots$. These sets are indicated as horizontal lines in Fig. 1. We recall from (c) above that there is a frame when $ab \le 1$ and $b = c_0^{-1}$.

(f) Let $K = 2, 3, \ldots$ and assume that $Kc_0^{-1} \le b < (K+1)c_0^{-1}$. From 3.3.2 we get that there is a frame when $ab < 1$ and

$$\frac{K}{c_0 - a} \le b \le \frac{K+1}{c_0 + a}\ ; \tag{133}$$

this region has vertical hatching in Fig. 1. The two curves $b = K(c_0 - a)^{-1}$ and $b = (K+1)(c_0 + a)^{-1}$ are indicated in Fig. 1, and they intersect at

$$(a, b) = \Big(\frac{1}{2}\, \frac{c_0}{K + \frac{1}{2}}\, ,\ \frac{K + \frac{1}{2}}{c_0}\Big)\ , \tag{134}$$

where $ab = 1/2$. This agrees with the Corollary at the end of 3.2.2. Furthermore, from 3.4.3 we get that there is a frame when $ab \notin \mathbb{Q}$, $ab < 1$ and

$$b \ge \frac{K}{c_0 - a} \quad \text{or} \quad b \le \frac{K+1}{c_0 + a}\ ; \tag{135}$$

this region has horizontal hatching in Fig. 1.

As to Fig. 2 we have the following observations. Fig. 2 is a detail picture from Fig. 1 of the region $ab < 1$, $2c_0^{-1} \le b < 3c_0^{-1}$. Accordingly, there is a frame when $2(c_0 - a)^{-1} \le b \le 3(c_0 + a)^{-1}$, see (133), and this region has vertical hatching in Fig. 2. Also, there is a frame when $ab \notin \mathbb{Q}$ and $b \ge 2(c_0 - a)^{-1}$ or $b \le 3(c_0 + a)^{-1}$, see (135), and this region has horizontal hatching in Fig. 2. The curves $b = 2(c_0 - a)^{-1}$ and $b = 3(c_0 + a)^{-1}$ intersect at $(a, b) = (\frac{1}{5}\,c_0, \frac{5}{2}\,c_0^{-1})$ where $ab = 1/2$. From 3.5.2 we get that there is no frame when $K = 2, 3, \ldots$, $I = 0 \bmod (K + 1)$ and

$$b = \frac{K+I}{c_0 + Ia} \in [Kc_0^{-1}, (K+1)c_0^{-1}) \ , \qquad a \in \Big[\frac{Ic_0}{KI + K + I}\,,\ \frac{c_0}{K}\Big] \ . \tag{136}$$

For $K = 2$, $I = 3$ we get $b = 5(c_0 + 3a)^{-1}$, and this curve is shown in Fig. 2. We note that all curves described by (136) intersect the curve $ab = 1$ at $(a, b) = (K^{-1}c_0, Kc_0^{-1})$. Finally, the curve $b = 5(c_0 + 3a)^{-1}$ intersects the curve $b = 2(c_0 - a)^{-1}$ at $(a, b) = (\frac{3}{11}\,c_0, \frac{11}{4}\,c_0^{-1})$. Since at the latter point we have that $ab \in \mathbb{Q}$, this does not yield a contradiction with (134).

Acknowledgement

The author wishes to thank M. Keane for encouraging discussions regarding the material presented in Sec. 3.

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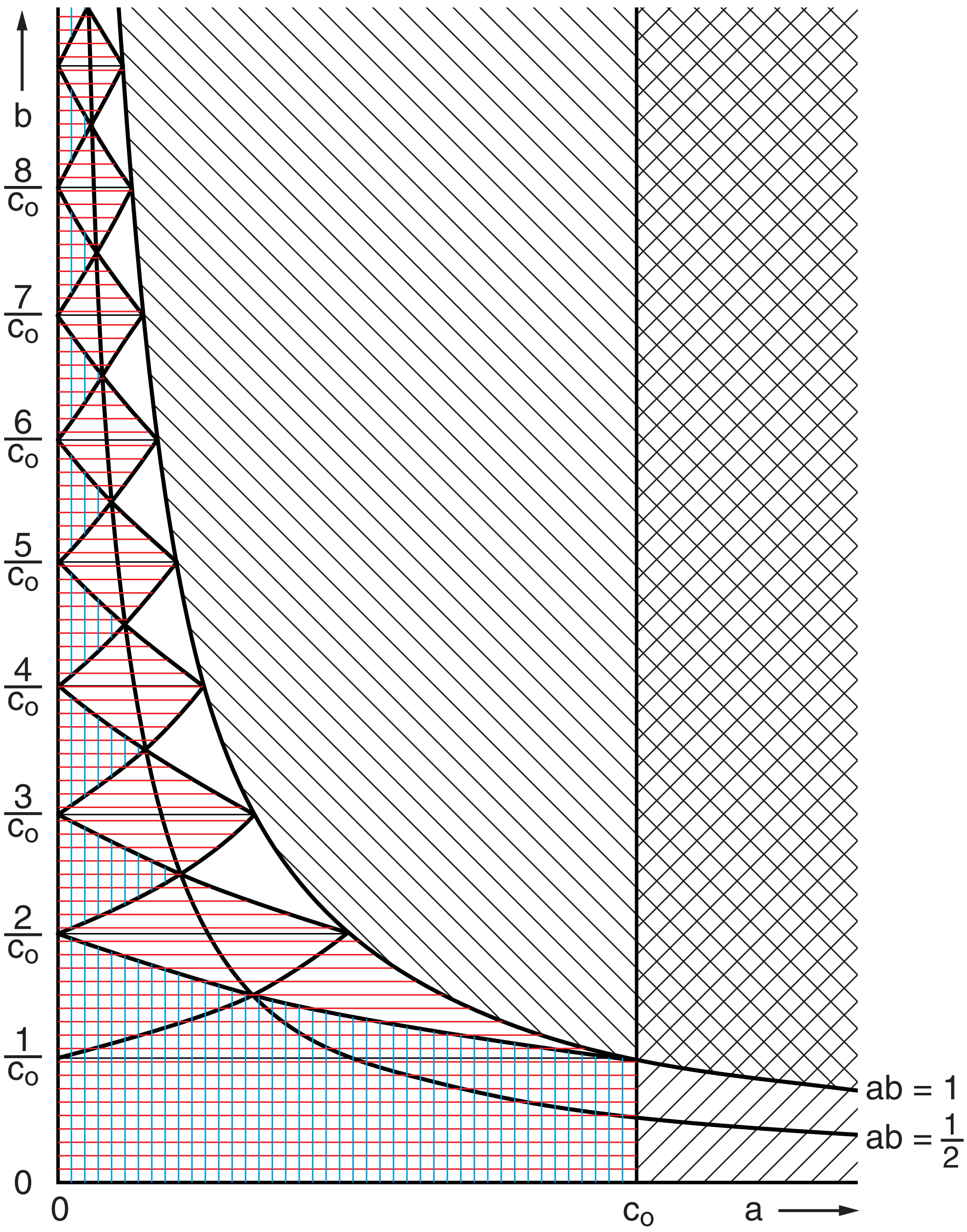
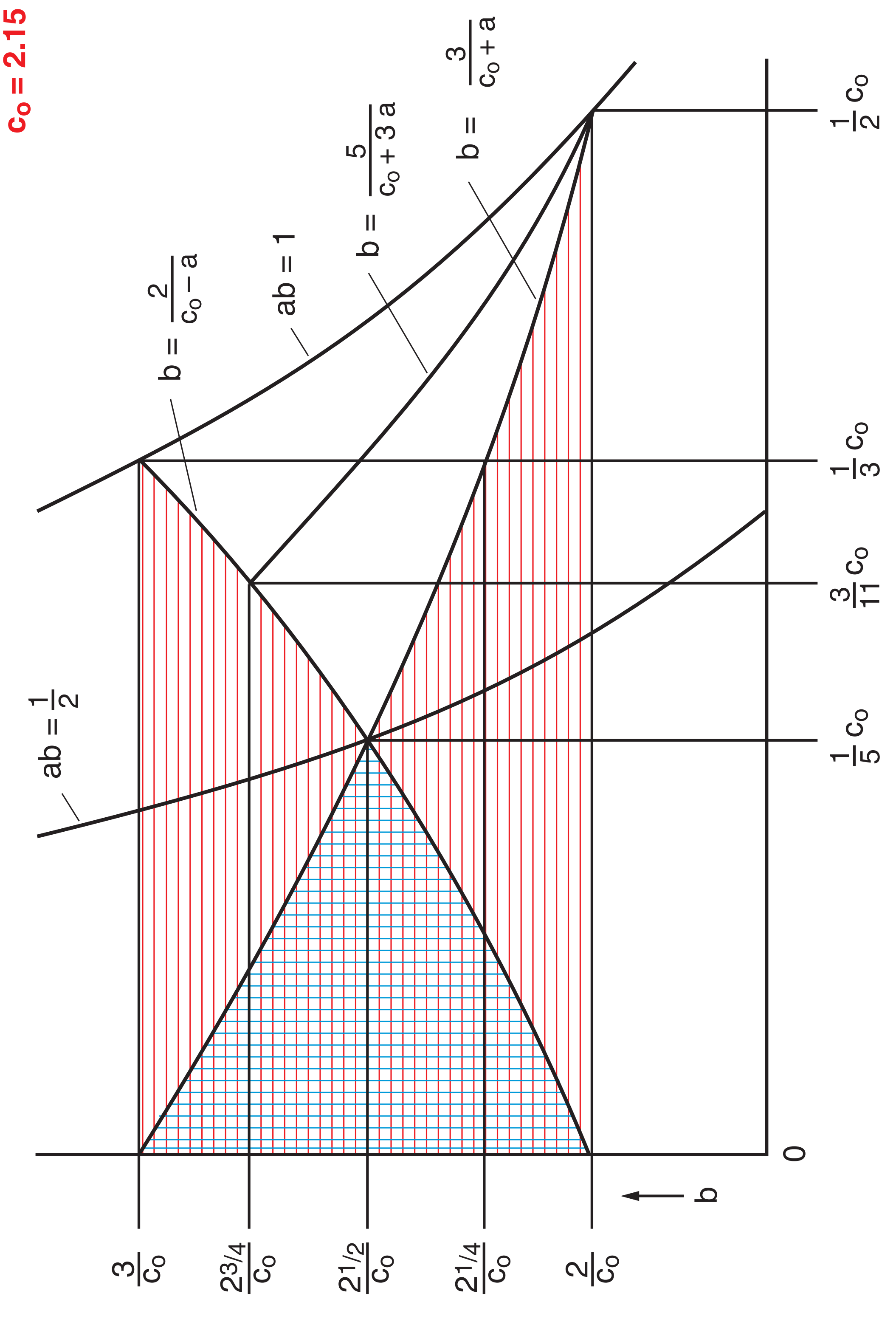
Figure captions

Fig. 1. Regions in the positive (a,b)-quadrant where $(\chi_{[0,c_0)},a,b)$ is or is not a frame for $c_0 = 2.15$. There is no frame when (a,b) is in the diagonally hatched regions. There is a frame when (a,b) is in the vertically hatched regions or when (a,b) is in the horizontally hatched regions and ab is irrational. More details in Subsec. 3.7.

Fig. 2. Detail picture from Fig. 1 of the region $ab < 1$, $2c_0^{-1} \leq b < 3c_0^{-1}$ with emphasis on the unhatched region. More details in Subsec. 3.7.


b
8/c_o
7/c_o
6/c_o
5/c_o
4/c_o
3/c_o
2/c_o
1/c_o
0
0
c_o
a
ab = 1
ab = 1/2


$c_o = 2.15$
$ab = \frac{1}{2}$
$b = \frac{2}{c_o - a}$
$ab = 1$
$b = \frac{5}{c_o + 3a}$
$b = \frac{3}{c_o + a}$
$\frac{1}{2}c_o$
$\frac{1}{3}c_o$
$\frac{3}{11}c_o$
$\frac{1}{5}c_o$
0
b
$\frac{3}{c_o}$
$\frac{2^{3/4}}{c_o}$
$\frac{2^{1/2}}{c_o}$
$\frac{2^{1/4}}{c_o}$
$\frac{2}{c_o}$