Representations of Gabor frame operators

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Abstract.
Gabor theory is concerned with expanding signals f as linear combinations of elementary signals that are obtained from a single function g (the window) by shifting it in time and frequency over integer multiples of a time shift parameter a and a frequency shift parameter b. In these expansion problems a key role is played by the Gabor frame operator associated with the set of elementary signals used in the expansions. The Gabor frame operator determines whether stable expansions exist for any finite-energy signal f (that is, whether we have indeed a frame), and, if so, gives a recipe for computing the expansion coefficients by using the canonically associated dual frame. In this contribution we consider the Gabor frame operator and associated dual frames in the time domain, the frequency domain, the time-frequency domain, and, for rational values of the sampling factor $(ab)^{-1}$, the Zak transform domain. We thus have the opportunity to address the basic problems – whether we have a Gabor frame and how we can compute a dual frame – in any of these domains we find, depending on g and a, b, convenient. The representations in the time domain and the frequency domain are conveniently discussed in the more general context of shift-invariant systems, and for this we present certain parts of what is known as Ron-Shen theory, adapted to our needs with emphasis on computational aspects.

This contribution contains many examples, counter-intuitive and confusing results, statements that one would like to be true but that are not and vice versa, etc., to show that Gabor theory, despite its rapid development in the last ten years, is still far from being completed.

1. Introduction

To introduce the subject matter of this contribution conveniently, we start with a survey of the various origins and early developments of Gabor theory. We do not aim here at a complete historical account, certainly not for the later developments, but rather refer for this to some excellent recent and less recent papers and books containing such surveys. Gabor systems are systems of functions of a real variable t that are built from a single function g (called the window) by shifting it in time and frequency over integer multiples of a time shift parameter a and a frequency shift parameter b. That is, denoting for real x, y by $g_{x,y}$ the time-frequency shifted version

$$g_{x,y}(t) = e^{2\pi iyt}\, g(t - x)\ , \qquad t \in \mathbb{R}\ , \tag{1.1}$$

of g, a Gabor system with shift parameters a, b and window g consists of the functions $g_{na,mb}$ with integer n, m. We denote this system by (g, a, b). These systems were considered by Gabor [1] in 1946, with the window g a Gaussian and $(ab)^{-1} = 1$, with the aim of constructing efficient, time-frequency localized, non-redundant expansions of finite-energy signals as linear combinations of the system's elements in which the coefficients "represent" the expanded signal. Gabor's choice of Gaussian elementary building blocks and densities a^{-1}, b^{-1}, with product equal to unity was motivated by his desire for non-redundant, unique expansions that should exist for any finite-energy signal. Indeed, Gaussians uniquely achieve equality in the uncertainty inequality $\Delta t \cdot \Delta f \geq 1/2$ (with the deltas referring to the standard deviations in the time and the frequency domain, respectively), whence they occupy in a sense the least amount of area in the time-frequency plane. Furthermore, the setting of Nyquist's theorem, saying that band-limited signals are uniquely determined by their sample values at regularly spaced sample points with the spacing determined by the bandwidth, can be recast into a limiting case of a Gabor expansion problem, and this suggests to take $(ab)^{-1} = 1$ as critical density when non-redundant expansions have to exist for all signals.

Already much earlier, in 1932, systems (g, a, b) with Gaussian g and $(ab)^{-1} = 1$ were considered by von Neumann [2] in a quantum mechanical context, and, apparently, he established the completenss of these systems in L^2. For that reason one also finds in the literature the name von Neumann lattice systems, and also Weyl-Heisenberg systems to emphasize the underlying continuous Weyl-Heisenberg group of translations in the phase plane, for what we have called Gabor systems.

Gabor's 1946 paper certainly did not go unnoticed by the engineering community, but it lasted until 1980 that the attention for Gabor expansions was revived through the work of Portnoff [3], Bastiaans [4] and Janssen [5]. This revival coincided, not entirely by accident, with the increasing interest in the electrical engineering community for time-frequency tools, such as the Wigner-Ville distribution and the short-time Fourier transform. (It should, however, be noted that as early as 1961 Lerner [6] presented a theory of signal representations in which Gabor expansions play a dominant role; in [6] one can already find orthogonalization procedures reminiscent of what we presently would call the construction of canonically associated dual frames.) From the von Neumann lattice side, completeness results for the Gaussian window and $(ab)^{-1} = 1$ were already obtained by Perelomov [7] and Bargmann et al. [8] using the Bargmann transform (and, in [8], the Zak transform in guise form) and by Bacry, Grossmann and Zak [9] using the Zak transform.

Although Perelomov in [7] already presented some considerations on dual functions, it was Bastiaans in [4] who analytically computed a dual function for the case of Gaussian g and $(ab)^{-1} = 1$. These dual functions are important since they allow one to exhibit the expansion coefficients for a particular signal f as inner products of f with the dual function shifted in a similar way as the window g itself. The mathematical analysis given by Janssen in [5] and [10] of the convergence properties of Gabor expansions and of Bastiaans' dual function showed that Gabor systems with Gaussian g and $(ab)^{-1} = 1$ yield unstable expansions that do not properly reflect time-frequency localization of the signals to be expanded. This point was also observed by Davis and Heller in 1979 [11]; they suggested to consider Gabor systems with Gaussian window g and $(ab)^{-1} > 1$ (oversampling) and thus obtained expansions with much better convergence properties.

The interest of mathematicians for Gabor systems dates from around 1980 with Janssen's work [5], [10] on the connection between Bargmann transform, Zak transform and Gabor expansions, and that of Feichtinger (joined later on by Gröchenig) focusing on the more functional analytic (modulation spaces) and group theoretic aspects of Gabor expansions [12], [13]. A major development in the mathematical theory of Gabor expansions was made in 1986 by Daubechies, Grossmann and Meyer [14] who placed the Gabor expansion problem in the context of frames for a Hilbert space. The latter concept was introduced by Duffin and Schaeffer [15] in 1951 for addressing completeness and expansion problems involving sets of exponentials in spaces of band-limited functions. For a Gabor system (g, a, b) one thus considers the frame operator S, defined for $f \in L^2$ by

$$Sf = \sum_{n,m} (f, g_{na,mb})\, g_{na,mb}\ . \tag{1.2}$$

By definition, the Gabor system (g, a, b) is a frame when the frame operator S is bounded and positive definite. In this case, the Gabor system $(^\circ\gamma, a, b)$ with

$$^\circ\gamma = S^{-1} g \tag{1.3}$$

is also a frame, called the canonical dual frame, and for any $f \in L^2$ we have the L^2-convergent expansions

$$f = \sum_{n,m} (f, {}^\circ\gamma_{na,mb})\, g_{na,mb} = \sum_{n,m} (f, g_{na,mb})\, {}^\circ\gamma_{na,mb}\ . \tag{1.4}$$

Many of the research efforts in Gabor theory after 1986 were directed at studying frame operators, finding criteria for when a Gabor system is a frame, identification of the tight Gabor frames (for which g and $^\circ\gamma$ coincide, except for a factor) and how to compute efficiently the canonical dual and the expansion coefficients in (1.4). These problems and their solutions have attracted many scientists from quite diverse disciplines and fields, such as theoretical eletrical engineering, mathematical physics, Fourier analysis, numerical analysis, complex function theory, functional analysis, where it should be noted that especially the last field has increased its share of practitioners considerably over the last few years.

We have now arrived at a point where we are able to describe the technical content of this contribution, and so we finish the historical survey by pointing at two basic references for the developments in Gabor theory (before and) after 1986. These are Ch. 4 of the book [16] by Daubechies (having an enormous influence on the more recent developments in time-frequency analysis, and, in particular, Gabor theory) and the book [17], edited by Feichtinger and Strohmer which is entirely devoted to Gabor analysis and applications with an extensive and up-to-date bibliography. For a survey of Gabor theory until 1989 and an excellent tutorial for both Gabor theory and wavelet theory, one should also consult the survey paper [18] by Heil and Walnut.

In this contribution we focus on the various representations of the frame operator S in (1.2). To that end, we first present the basics of frame theory, specialized to Gabor systems and to the more general shift-invariant systems. For the latter type of systems we provide our version of certain parts of a theory developed by Ron and Shen [19], [20], where we pay special attention to the issue of how to compute canonical dual systems. This is applied in two ways to Gabor systems, yielding a description of the frame bound conditions, the Gabor frame operator, the duality relation (1.4) and a characterization of and a computaton method for the canonical dual function, both in the frequency domain and the time domain. The representation of the frame operator in the time domain is well known as Walnut's representation [21]. We shall also consider Gabor systems in the time-frequency domain using spectrograms, and this yields the Tolimieri-Orr-Janssen representation [22], [23] of the Gabor frame operator with a corresponding description of the frame bound conditions, the Wexler-Raz biorthogonality condition [24] for the duality relation (1.4), and a characterization of the canonical dual function as the minimum-energy Wexler-Raz dual. For rational values of the sampling factor $(ab)^{-1}$, we can also consider Gabor systems in the Zak transform domain which yields the Zibulski-Zeevi description [25] of the frame bound conditions, frame operator, duality relation and characterization/computation of the canonical dual function in terms of Zak matrices. Each of the four representations just given is potentially useful as a tool for finding out whether a Gabor system (g, a, b) is indeed a frame, and, if so, offers a means for computation of (canonical) dual functions in the considered domain.

We conclude this contribution with various counter-intuitive and confusing results, statements that one would obviously like to be true but that are not and vice versa, comments on the basic and hard problem of when a particular triple (g, a, b) is a Gabor frame, etc. As examples of this we have the Balian-Low theorem (conflicting with the relaxed attitude of von Neumann, Gabor himself and Lerner towards completeness, existence and convergence issues for the Gaussian window Gabor system at critical density) and the beating of this same Balian-Low phenomenon by considering Wilson systems at critical density; the existence of a well-behaved, positive g with positive Fourier transform such that $(g, \frac{1}{2}, 1)$ is not a Gabor frame; the difficulties of deciding whether (g, a, b) is a frame with $ab < 1$ for windows g as elementary as a Gaussian or the characteristic function of an interval. All this shows that Gabor theory, despite the great progress that has been made in recent years, is still far from being completed, with various basic questions still waiting to be answered.

Almost all results presented here are proved somewhere in the literature; we shall therefore omit all proofs and we shall give appropriate references instead. In Secs. 2–6 of this contribution we follow roughly the developments of Secs. 1.1–5 of [17], Ch. 1; however, the presentation of the results has been considerably enhanced by adopting a uniform organization per section, while some of the results have been worked out in more detail.

2. Basics from frame theory

In this section we present some basic facts from frame theory, with particular attention for Gabor systems and shift-invariant systems. A shift-invariant system consists of a collection of functions g_{nm}, $n, m \in \mathbb{Z}$, of the form

$$g_{nm}(t) = g_m(t - na) \ , \qquad t \in \mathbb{R} \ , \tag{2.1}$$

where $g_m \in L^2$, $m \in \mathbb{Z}$, and $a > 0$. We are interested in finding dual systems γ_{nm}, $n, m \in \mathbb{Z}$, with $\gamma_{nm}(t) = \gamma_m(t - na)$, $t \in \mathbb{R}$, by which we mean that any $f \in L^2$ has the L^2-convergent expansions

$$f = \sum_{n,m} (f, \gamma_{nm})\, g_{nm} = \sum_{n,m} (f, g_{nm})\, \gamma_{nm} \ . \tag{2.2}$$

For this to be meaningful, we require the two systems to have a finite frame upper bound. A system g_{nm}, $n, m \in \mathbb{Z}$, as in (2.1), has a finite frame upper bound when there is a $B_g < \infty$ such that

$$\sum_{n,m} |(f, g_{nm})|^2 \leq B_g \, \|f\|^2 \ , \qquad f \in L^2 \ , \tag{2.3}$$

and any $B_g < \infty$ such that (2.3) holds, is called a frame upper bound.

When g_{nm}, $n, m \in \mathbb{Z}$, has a finite frame upper bound B_g, one can define the operators T_g (analysis operator) and T_g^* (synthesis operator) by

$$T_g : f \in L^2 \to T_g f = ((f, g_{nm}))_{n,m \in \mathbb{Z}} \in l^2(\mathbb{Z}^2) \tag{2.4}$$

and

$$T_g^* : \alpha \in l^2(\mathbb{Z}^2) \to T_g^* \alpha = \sum_{n,m} \alpha_{nm} g_{nm} \in L^2 \ , \tag{2.5}$$

respectively. These T_g and T_g^* are bounded linear operators with operator norm $\leq \ B_g^{\frac{1}{2}}$, and they are indeed adjoint operators when the standard inner products for L^2 and $l^2(\mathbb{Z}^2)$ are taken. When the system γ_{nm}, $n, m \in \mathbb{Z}$, has a finite frame upper bound as well, the duality condition (2.2) can be written as

$$T_g^* \, T_\gamma = T_\gamma^* \, T_g = I \ , \tag{2.6}$$

where I denotes the identity operator of L^2.

When the system g_{nm}, $n, m \in \mathbb{Z}$, has a finite frame upper bound B_g, the frame operator S_g is defined by $S_g = T_g^* T_g$. Explicitly,

$$S_g : f \in L^2 \to S_g f = \sum_{n,m} (f, g_{nm})\, g_{nm} \in L^2 \ , \tag{2.7}$$

and there holds $S_g \leq B_g I$. When there is, in addition, an $A_g > 0$ such that

$$\sum_{n,m} |(f, g_{nm})|^2 \geq A_g \|f\|^2 \;, \qquad f \in L^2 \;, \tag{2.8}$$

so that S_g is invertible with $S_g \geq A_g I$, we say that the system g_{nm}, $n, m \in \mathbb{Z}$, has a positive frame lower bound, and any $A_g > 0$ such that (2.8) holds is called a frame lower bound. A system g_{nm}, $n, m \in \mathbb{Z}$, having both a finite frame upper bound and a positive frame lower bound is called a frame. When we have $A_g = B_g$ in (2.3) and (2.8), we say that the frame is tight, and then we have $S_g = A_g I = B_g I$.

When the system g_{nm}, $n, m \in \mathbb{Z}$, is a frame, a dual system is given by

$$^{\circ}\gamma_{nm} = S_g^{-1}\, g_{nm} \;, \qquad n, m \in \mathbb{Z} \;, \tag{2.9}$$

and this system is also a frame with frame bounds $A_{^{\circ}\gamma} = B_g^{-1}$, $B_{^{\circ}\gamma} = A_g^{-1}$. Since S_g, and hence S_g^{-1}, commutes with all relevant time-shift operators $f \in L^2 \to f(\cdot - na) \in L^2$, $n \in \mathbb{Z}$, we have that

$$^{\circ}\gamma_{nm} = S_g^{-1}\, g_{nm} = (S_g^{-1}\, g_m)(\cdot - na) \;, \qquad n, m \in \mathbb{Z} \;. \tag{2.10}$$

We have, furthermore, that $S_g S_{^{\circ}\gamma} = I$, whence $S_{^{\circ}\gamma}$ is the inverse of the frame operator S_g and vice versa. The system in (2.9) is called the canonical dual system.

When the system g_{nm}, $n, m \in \mathbb{Z}$, is a frame, there are, in general, other dual systems γ_{nm}, $n, m \in \mathbb{Z}$, than the canonical dual system in (2.9). When we have two systems g_{nm}, $n, m \in \mathbb{Z}$, and γ_{nm}, $n, m \in \mathbb{Z}$, both with a finite frame upper bound, such that the duality condition (2.2) with L^2-convergence for all $f \in L^2$ holds, then both systems are a frame. This can be a useful method for checking whether a particular system g_{nm}, $n, m \in \mathbb{Z}$, is indeed a frame, viz. in those cases that one can easily produce a dual system γ_{nm}, $n, m \in \mathbb{Z}$, that does not need to be the canonical dual frame in (2.9).

The dual system in (2.9) is special for several reasons. For any $f \in L^2$ and any $\alpha \in l^2(\mathbb{Z}^2)$ with

$$f = \sum_{n,m} \alpha_{nm} g_{nm} \;, \tag{2.11}$$

there holds

$$\sum_{n,m} |(f, {}^{\circ}\gamma_{nm})|^2 \leq \sum_{n,m} |\alpha_{nm}|^2 \;, \tag{2.12}$$

with equality if and only if $\alpha_{nm} = (f, {}^{\circ}\gamma_{nm})$ for all $n, m \in \mathbb{Z}$. By applying this to the trivial representation for $n, m \in \mathbb{Z}$

$$g_{nm} = \sum_{n',m'} (g_{nm}, {}^{\circ}\gamma_{n'm'})\, g_{n'm'} = g_{nm} + \sum_{(n',m') \neq (n,m)} 0 \cdot g_{n'm'} \;, \tag{2.13}$$

we find that

$$|(g_{nm}, {}^\circ\gamma_{nm})|^2 \le \sum_{n',m'} |(g_{nm}, {}^\circ\gamma_{n'm'})|^2 \le 1 \ . \tag{2.14}$$

A different way to characterize the dual system in (2.9) is as follows. Assume that $\alpha \in l^2$ is given and that we consider the α_{nm} as noisy/distorted versions of the numbers (f, g_{nm}), $n, m \in \mathbb{Z}$, of some $f \in L^2$. Then an estimate of f can be obtained by minimizing

$$J(f) = \sum_{n,m} |(f, g_{nm}) - \alpha_{nm}|^2 \ . \tag{2.15}$$

When the system g_{nm}, $n, m \in \mathbb{Z}$, is a frame, this yields for f the unique solution

$$f = S_g^{-1}\Big(\sum_{n,m} \alpha_{nm} g_{nm}\Big) = \sum_{n,m} \alpha_{nm} {}^\circ\gamma_{nm} \ . \tag{2.16}$$

In particular, when

$$\alpha_{nm} = \delta_{nn_0}\delta_{mm_0} \ , \qquad n, m \in \mathbb{Z} \tag{2.17}$$

with some $n_0, m_0 \in \mathbb{Z}$ (the deltas denote Kronecker's delta), we obtain $f = {}^\circ\gamma_{n_0 m_0}$. For more generalities about frames and shift-invariant systems we refer to [16], Sec. 3.2, [26], Sec. I.C and [19], Sec. 1.3.

A particular example of a shift-invariant system arises when we take for $m \in \mathbb{Z}$

$$g_m(t) = e^{2\pi imbt}\, g(t) \ , \qquad t \in \mathbb{R} \ , \tag{2.18}$$

with $b > 0$ and $g \in L^2$. It is customary here to ignore the phase factors in g_{nm}, γ_{nm}, given by $\exp(-2\pi inmab)$ for $n, m \in \mathbb{Z}$, when studying duality questions, since these vanish anyway at the right-hand sides of (2.2). Thus one considers

$$g_{na,mb}(t) = e^{2\pi imbt}\, g(t - na)$$

rather than (2.19)

$$g_{nm}(t) = e^{2\pi imb(t-na)}\, g(t - na)$$

for $n, m \in \mathbb{Z}$, and one arrives at a Gabor system (g, a, b) as in Sec. 1 with window g and shift parameters $a > 0$, $b > 0$.

In the case of a Gabor frame (g, a, b), the frame operator S_g commutes with all relevant time-frequency shift operators $f \in L^2 \to \exp(2\pi imb\cdot)\, f(\cdot - na) \in L^2$, $n, m \in \mathbb{Z}$. As a consequence we have then that

$$^\circ\gamma_{nm} = S_g^{-1}\, g_{nm} = ({}^\circ\gamma)_{na,mb} \tag{2.20}$$

with ${}^\circ\gamma = S_g^{-1}\, g$ the canonical dual window.

3. Shift-invariant systems

In this section we consider shift-invariant systems g_{nm}, $n, m \in \mathbb{Z}$, and γ_{nm}, $n, m \in \mathbb{Z}$, and we present, in the frequency domain an equivalent condition for a shift-invariant system to have a finite frame upper bound and to be a frame, a representation result for the frame operators, an equivalent condition for the two systems to be dual, and a characterization of and a computation method for the canonical dual system. Many of the results in this section can be found in [19] by Ron and Shen. However, the presentation of the results we give here is rather different from the one in [19], and, for instance, the results on frame operator representation as well as those on the characterization and computation of canonical dual systems cannot be found in [19], at least not in the form we present them here. For full details and proofs we refer to [27], Sec. 1.2.

We consider $L^2 = L^2(\mathbb{R})$ with the standard inner product and norm

$$(f, h) = \int_{-\infty}^{\infty} f(t)\, h^*(t)\, dt \; ; \qquad \|f\|^2 = (f, f) \, , \;\; f, h \in L^2 \, . \tag{3.1}$$

Furthermore, we denote for $f \in L^2$ by $\hat{f} = \mathcal{F} f$ the Fourier transform of f, given as

$$\hat{f}(\nu) = (\mathcal{F} f)(\nu) = \int_{-\infty}^{\infty} e^{-2\pi i \nu t}\, f(t)\, dt \, , \qquad \text{a.e. } \nu \in \mathbb{R} \, . \tag{3.2}$$

With $f_m \in L^2$, $m \in \mathbb{Z}$, we define the "matrices"

$$H_g(\nu) := (\hat{g}_m(\nu - k/a))_{k \in \mathbb{Z}, m \in \mathbb{Z}} \, , \qquad \text{a.e. } \nu \in \mathbb{R} \, , \tag{3.3}$$

whose k^{th} "row" consists of the sample values $\hat{g}_m(\nu - k/a)$, $m \in \mathbb{Z}$.

Theorem 3.1. The system g_{nm}, $n, m \in \mathbb{Z}$, has a finite frame upper bound B_g if and only if $H_g(\nu)$ and $H_g^*(\nu)$ define for a.e. $\nu \in \mathbb{R}$ a bounded linear operator of $l^2(\mathbb{Z})$ with operator norm $\leq \; (a\, B_g)^{\frac{1}{2}}$. In particular, there then holds for $k, m \in \mathbb{Z}$

$$\sum_m |\hat{g}_m(\nu - k/a)|^2 \leq a\, B_g \, , \qquad \sum_k |\hat{g}_m(\nu - k/a)|^2 \leq a\, B_g \tag{3.4}$$

for a.e. $\nu \in \mathbb{R}$.

Theorem 3.2. Let $A \geq 0$, $B < \infty$. Then we have

$$A\, \|f\|^2 \leq \sum_{n,m} |(f, g_{nm})|^2 \leq B\, \|f\|^2 \, , \qquad f \in L^2 \, , \tag{3.5}$$

if and only if

$$AI \leq \frac{1}{a}\, H_g(\nu)\, H_g^*(\nu) \leq BI\ , \qquad \text{a.e. } \nu \in \mathbb{R}\ , \tag{3.6}$$

where the I in (3.6) denotes the identity operator of $l^2(\mathbb{Z})$.

We observe that $H_g(\nu)\, H_g^*(\nu)$ is given as the "matrix"

$$H_g(\nu)\, H_g^*(\nu) = \Big(\sum_m \hat{g}_m(\nu - k/a)\, \hat{g}_m^*(\nu - l/a)\Big)_{k,l\in\mathbb{Z}}\ , \qquad \text{a.e. } \nu \in \mathbb{R}\ , \tag{3.7}$$

and that for all $j,k,l \in \mathbb{Z}$ there holds

$$(H_g(\nu - l/a)\, H_g^*(\nu - l/a))_{jk} = (H_g(\nu)\, H_g^*(\nu))_{l+j,l+k}\ , \qquad \text{a.e. } \nu \in \mathbb{R}\ . \tag{3.8}$$

Hence for checking (3.6) is is sufficient to consider ν in an interval of length $1/a$.

Also note that the system g_{nm}, $n,m \in \mathbb{Z}$, is a tight frame if and only if there is a constant c such that for a.e. $\nu \in \mathbb{R}$

$$\sum_m \hat{g}_m(\nu - k/a)\, \hat{g}_m^*(\nu - l/a) = c\,\delta_{kl}\ , \qquad k,l \in \mathbb{Z}\ . \tag{3.9}$$

Theorem 3.3. Assume that the system g_{nm}, $n,m \in \mathbb{Z}$, has a finite frame upper bound B_g, and let $f \in L^2$. Then we have, with S_g the frame operator,

$$\widehat{S_g f}(\nu) = \frac{1}{a} \sum_k d_k(\nu)\, \hat{f}(\nu - k/a)\ , \qquad \text{a.e. } \nu \in \mathbb{R}\ , \tag{3.10}$$

with absolute convergence of the right-hand series for a.e. $\nu \in \mathbb{R}$. Here

$$d_k(\nu) = (H_g(\nu)\, H_g^*(\nu))_{0k} = \sum_m \hat{g}_m(\nu)\, \hat{g}_m^*(\nu - k/a)\ ,$$

$$\text{a.e. } \nu \in \mathbb{R}\,,\quad k \in \mathbb{Z}\ . \tag{3.11}$$

Because of (3.8) there holds, more generally, for $f \in L^2$ and a.e. $\nu \in \mathbb{R}$

$$\widehat{S_g f}(\nu - l/a) = \frac{1}{a} \sum_k (H_g(\nu)\, H_g^*(\nu))_{lk}\, \hat{f}(\nu - k/a)\ , \qquad l \in \mathbb{Z}\ . \tag{3.12}$$

This gives a frame operator representation in the Fourier domain in terms of the matrices $H_g(\nu)\, H_g^*(\nu)$ where $L^2(\hat{\mathbb{R}})$ is identified with $L^2([0,1/a) \times \mathbb{Z})$. The relation (3.12) can be extended as follows. Let A_g be a frame lower bound for the system g_{nm}, $n,m \in \mathbb{Z}$, and let φ be a function analytic in an open set containing the closed segment $[A_g, B_g]$. Then we have for $f \in L^2$ and a.e. $\nu \in \mathbb{R}$

$$\widehat{\varphi(S_g)f}(\nu - l/a) = \sum_k \Big(\varphi\Big(\frac{1}{a}\, H_g(\nu)\, H_g^*(\nu)\Big)\Big)_{lk} \hat{f}(\nu - k/a)\ , \qquad l \in \mathbb{Z}\ . \tag{3.13}$$

In particular, when $A_g > 0$, so that g_{nm}, $n, m \in \mathbb{Z}$, is a frame, the choice $\varphi(x) = x^{-1}$ yields a representation of the inverse of the frame operator S_g according to

$$\widehat{S_g^{-1}f}(\nu - l/a) = \widehat{S_{{}^\circ\gamma}f}(\nu - l/a) = \sum_k \Big(\frac{1}{a}\, H_g(\nu)\, H_g^*(\nu)\Big)^{-1}_{lk} \hat{f}(\nu - k/a)\ , \qquad l \in \mathbb{Z}\ , \tag{3.14}$$

for $f \in L^2$ and a.e. $\nu \in \mathbb{R}$. Specialization of (3.14) to the case $f = g_m$, where $m \in \mathbb{Z}$, and $l = 0$ yields for a.e. $\nu \in \mathbb{R}$

$$\hat{{}^\circ\gamma}_m(\nu) = \widehat{S_g^{-1}g_m}(\nu) = \sum_k \Big(\frac{1}{a}\, H_g(\nu)\, H_g^*(\nu)\Big)^{-1}_{0k} \hat{g}_m(\nu - k/a)\ . \tag{3.15}$$

<u>Theorem 3.4</u>. Assume that the systems g_{nm}, $n, m \in \mathbb{Z}$, and γ_{nm}, $n, m \in \mathbb{Z}$, have finite frame upper bounds. Then the two systems are dual in the sense of (2.2) if and only if

$$H_g(\nu)\, H_\gamma^*(\nu) = H_\gamma(\nu)\, H_g^*(\nu) = aI\ , \qquad \text{a.e. } \nu \in \mathbb{R}\ , \tag{3.16}$$

if and only if

$$\sum_m \hat{g}_m(\nu - k/a)\, \hat{\gamma}_m^*(\nu) = \sum_m \hat{\gamma}_m(\nu - k/a)\, \hat{g}_m^*(\nu) = a\, \delta_{k0}\ , \qquad k \in \mathbb{Z}\ , \ \ \text{a.e. } \nu \in \mathbb{R}\ . \tag{3.17}$$

Note that (3.16) says that $a^{-1}H_\gamma^*(\nu)$ is a right-inverse of $H_g(\nu)$ and that $a^{-1}H_\gamma(\nu)$ is a left-inverse of $H_g^*(\nu)$ for a.e. $\nu \in \mathbb{R}$. The next theorem shows that the canonical dual system ${}^\circ\gamma_{nm}$, $n, m \in \mathbb{Z}$, is special in the sense that $a^{-1}H_{{}^\circ\gamma}^*(\nu)$ is "the" generalized inverse of $H_g(\nu)$ for a.e. $\nu \in \mathbb{R}$.

<u>Theorem 3.5</u>. Assume that g_{nm}, $n, m \in \mathbb{Z}$, is a frame. Then

$$H_{{}^\circ\gamma}^*(\nu) = a\, H_g^*(\nu)(H_g(\nu)\, H_g^*(\nu))^{-1}\ , \qquad \text{a.e. } \nu \in \mathbb{R}\ , \tag{3.18}$$

and

$$\Big(\frac{1}{a}\, H_{{}^\circ\gamma}(\nu)\, H_{{}^\circ\gamma}^*(\nu)\Big)^{-1} = \Big(\frac{1}{a}\, H_g(\nu)\, H_g^*(\nu)\Big)^{-1}\ , \qquad \text{a.e. } \nu \in \mathbb{R}\ . \tag{3.19}$$

Theorem 3.5 can be made more explicit for the purpose of calculating the canonical dual functions ${}^\circ\gamma_m$ as follows.

Theorem 3.6. Assume that the system g_{nm}, $n, m \in \mathbb{Z}$, is a frame, and denote by $\mathbf{c}(\nu) \in l^2(\mathbb{Z})$ for a.e. $\nu \in \mathbb{R}$ the least-norm solution $\mathbf{c} = (c_m)_{m\in\mathbb{Z}}$ of the linear system

$$\sum_m \hat{g}_m(\nu - k/a)\, c_m = a\, \delta_{k0}\ , \qquad k \in \mathbb{Z}\ . \tag{3.20}$$

Then there holds

$$^\circ\hat{\gamma}_m(\nu) = c_m^*(\nu)\ , \qquad m \in \mathbb{Z}\ ,\ \ \text{a.e. } \nu \in \mathbb{R}\ . \tag{3.21}$$

As a consequence of Theorem 3.6 we have the following. Assume that g_{nm}, $n, m \in \mathbb{Z}$, and γ_{nm}, $n, m \in \mathbb{Z}$, are dual frames. Then we have

$$\sum_m |^\circ\hat{\gamma}_m(\nu)|^2 \le \sum_m |\hat{\gamma}_m(\nu)|^2\ , \qquad \text{a.e. } \nu \in \mathbb{R}\ , \tag{3.22}$$

with equality if and only if $^\circ\hat{\gamma}_m = \hat{\gamma}_m$ a.e.

4. Gabor systems as shift-invariant systems

In this section we specialize the results of Sec. 3 to the case of a Gabor system (g, a, b), so that we have now a $g \in L^2$ and a $b > 0$ such that

$$g_m(t) = e^{2\pi i m b t}\, g(t)\ , \qquad m \in \mathbb{Z}\ ,\ \ t \in \mathbb{R}\ ; \tag{4.1}$$

as already said it is customary in Gabor theory to consider $g_{na,mb}$ rather than $g_{nm} = \exp(2\pi i m b\,(\cdot - na))\, g(\cdot - na)$ which amounts to dropping the phase factors $\exp(-2\pi i n m a b)$. Since (g, a, b) is a Gabor system if and only if $(\hat{g}, b, a)$ is a Gabor system, we have two ways of specialization of the results of Sec. 3, yielding a description of the various notions and conditions in the frequency domain and the time domain, respectively.

4.1. Frequency domain results

We first show what form the results of Sec. 3 take when we choose the g_m as in (4.1). This then yields a description of the frame bound conditions, the frame operators, the duality condition and a characterization of and a computation method for the canonical dual function $^\circ\gamma$. Since now

$$\hat{g}_m(\nu) = (\mathcal{F} g_m)(\nu) = \hat{g}(\nu - mb)\ , \qquad m \in \mathbb{Z}\ ,\ \ \text{a.e. } \nu \in \mathbb{R}\ , \tag{4.2}$$

the "matrix" $H_g(\nu)$ in (3.3) is given by

$$H_g(\nu) = (\hat{g}(\nu - mb - k/a))_{k\in\mathbb{Z},m\in\mathbb{Z}}\,, \qquad \text{a.e. } \nu \in \mathbb{R}\,, \tag{4.3}$$

and the "matrix" $H_g(\nu)\,H_g^*(\nu)$ in Theorem 3.2 is given by

$$H_g(\nu)\,H_g^*(\nu) = \Big(\sum_{m=-\infty}^{\infty} \hat{g}(\nu - mb - k/a)\,\hat{g}^*(\nu - mb - l/a)\Big)_{k,l\in\mathbb{Z}}\,, \qquad \text{a.e. } \nu \in \mathbb{R}\,. \tag{4.4}$$

Hence, Theorem 3.1 gives that (g, a, b) is a Gabor system with a finite frame upper bound B_g if and only if $H_g(\nu)$ in (4.3) and $H_g^*(\nu)$ define for a.e. $\nu \in \mathbb{R}$ a bounded linear operator of $l^2(\mathbb{Z})$ with operator norm $\leq\ (a\,B_g)^{\frac{1}{2}}$; in particular, we then have that

$$\sum_m |\hat{g}(\nu - mb)|^2 \leq a\,B_g\,, \quad \sum_k |\hat{g}(\nu - k/a)|^2 \leq a\,B_g\,, \qquad \text{a.e. } \nu \in \mathbb{R}\,. \tag{4.5}$$

Furthermore, from Theorem 3.2 we see that (g, a, b) is a Gabor frame with frame bounds $A > 0$, $B < \infty$ if and only if the matrices $H_g(\nu)\,H_g^*(\nu)$ in (4.4) satisfy

$$AI \leq \frac{1}{a}\,H_g(\nu)\,H_g^*(\nu) \leq BI\,, \qquad \text{a.e. } \nu \in \mathbb{R}\,. \tag{4.6}$$

Moreover, by Theorem 3.3 the frame operator of the Gabor frame (g, a, b) has the representation

$$\widehat{S_g f}(\nu) = \frac{1}{a}\sum_k d_k(\nu)\,\hat{f}(\nu - k/a)\,, \qquad \text{a.e. } \nu \in \mathbb{R}\,, \tag{4.7}$$

for $f \in L^2$ with absolute convergence of the right-hand side series for a.e. $\nu \in \mathbb{R}$, in which the $d_k(\nu)$ are given by

$$d_k(\nu) = \sum_m \hat{g}(\nu - mb)\,\hat{g}^*(\nu - mb - k/a)\,, \qquad k \in \mathbb{Z}\,,\ \text{a.e. } \nu \in \mathbb{R}\,. \tag{4.8}$$

Also, tightness of the frame (g, a, b) is equivalent with

$$\sum_m \hat{g}(\nu - mb - k/a)\,\hat{g}^*(\nu - mb - l/a) = c\,\delta_{kl}\,, \qquad k, l \in \mathbb{Z} \tag{4.9}$$

for a.e. $\nu \in \mathbb{R}$ with c some constant. And duality of two Gabor systems (g, a, b) and (γ, a, b) having a finite frame upper bound is equivalent with

$$\sum_m \hat{g}(\nu - mb - k/a)\,\gamma^*(\nu - mb) = \sum_m \hat{\gamma}(\nu - mb - k/a)\,g^*(\nu - mb) = $$
$$= a\,\delta_{k0}\,, \qquad k \in \mathbb{Z}\,,\ \text{a.e. } \nu \in \mathbb{R}\,. \tag{4.10}$$

Finally, the canonical dual functions ${}^\circ\gamma_m = \exp(2\pi imb\cdot)\,{}^\circ\gamma$ can be found by using that

$$
{}^\circ\gamma(\nu) = \sum_k \Big(\frac{1}{a}\,H_g(\nu)\,H_g^*(\nu)\Big)^{-1}_{0k}\,\hat{g}(\nu - k/a)\;, \qquad \text{a.e. } \nu \in \mathbb{R}\;. \tag{4.11}
$$

This canonical dual function ${}^\circ\gamma$ is minimal in the sense that for any dual frame (γ, a, b) we have for a.e. $\nu \in \mathbb{R}$ that

$$
\sum_m |{}^\circ\hat{\gamma}(\nu - mb)|^2 \le \sum_m |\hat{\gamma}(\nu - mb)|^2\;, \tag{4.12}
$$

with equality if and only if ${}^\circ\hat{\gamma}(\nu - mb) = \hat{\gamma}(\nu - mb)$, $m \in \mathbb{Z}$. When we integrate (4.12) over an interval of length b and use Parseval's theorem, we obtain

$$
\|{}^\circ\gamma\|^2 \le \|\gamma\|^2 \tag{4.13}
$$

with equality if and only if ${}^\circ\gamma = \gamma$ a.e. Hence the canonical dual ${}^\circ\gamma$ has the least L^2 -norm among all dual functions.

4.2. Time-domain results

We next show how the results of Sec. 3 can be applied to yield a description of the frame bound conditions, the frame operator, the condition of duality, and a characterization of and a computation method for the canonical dual ${}^\circ\gamma$ in the time domain. To do so, we note that

$$
\mathcal{F}^{-1}[g_{na,mb}]\,(\nu) = e^{2\pi inmab}(\check{g})_{-mb,na}\;, \qquad n, m \in \mathbb{Z}\,, \;\; \text{a.e. } \nu \in \mathbb{R}\;. \tag{4.14}
$$

Here $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and $\check{g}$ is the inverse Fourier transform of g, so that

$$
\check{g}(\nu) = (\mathcal{F}^{-1}g)(\nu) = \int_{-\infty}^{\infty} e^{2\pi i\nu t}\,g(t)\,dt\;, \qquad \text{a.e. } \nu \in \mathbb{R}\;. \tag{4.15}
$$

Furthermore, we have for $f \in L^2$

$$
(f, g_{na,mb}) = e^{-2\pi inmab}(\check{f}, (\check{g})_{-mb,na})\;, \qquad n, m \in \mathbb{Z}\;. \tag{4.16}
$$

Consequently, when the system (g, a, b) has a finite frame upper bound or a positive frame lower bound, then, by Parseval's theorem, so has the system $(\check{g}, b, a)$, and the respective frame bounds can be taken equal. And, in the case of finite frame upper bounds, the two frame operators are related according to

$$
S_{(g,a,b)}f = \widehat{S_{(\check{g},b,a)}\check{f}}\;, \qquad f \in L^2\;. \tag{4.17}
$$

Furthermore, the two Gabor systems (g,a,b) and (γ,a,b) are dual if and only if the systems $(\check{g},b,a)$ and $(\check{\gamma},b,a)$ are dual. Also, when (g,a,b) is a Gabor frame with canonical dual Gabor frame $({}^\circ\gamma,a,b)$, then $(\check{g},b,a)$ is a Gabor frame as well with canonical dual $(({}^\circ\gamma)\check{},b,a)$ (for the latter fact we have used (4.17) together with $(\hat{h})\check{} = h$ for $h \in L^2$).

Accordingly, we consider now the "matrix"

$$M_g(t) := (g(t-na-i/b))_{i\in\mathbb{Z},n\in\mathbb{Z}}\ , \qquad \text{a.e. } t \in \mathbb{R}\ , \tag{4.18}$$

instead of the "matrix" $H_g(\nu)$ in (3.3) and (4.3), and in Theorem 3.2 and (4.4) we have now the "matrix"

$$M_g(t)\,M_g^*(t) = \Big(\sum_{n=-\infty}^{\infty} g(t-na-i/b)\,g^*(t-na-j/b)\Big)_{i,j\in\mathbb{Z}}\ , \qquad \text{a.e. } t \in \mathbb{R}\ , \tag{4.19}$$

instead of the matrix $H_g(\nu)\,H_g^*(\nu)$. Theorem 3.1 gives that (g,a,b) has a finite frame upper bound B_g if and only if $M_g(t)$ in (4.18) and $M_g^*(t)$ define for a.e. $t \in \mathbb{R}$ a bounded linear operator of $l^2(\mathbb{Z})$ with operator norm $\leq\ (b\,B_g)^{\frac{1}{2}}$; in particular, we then have that

$$\sum_n |g(t-na)|^2 \leq b\,B_g\ , \quad \sum_i |g(t-i/b)|^2 \leq b\,B_g\ , \qquad \text{a.e. } t \in \mathbb{R}\ . \tag{4.20}$$

Furthermore, by Theorem 3.2 we see that (g,a,b) is a frame with frame bounds $A>0$, $B<\infty$ if and only if

$$AI \leq \frac{1}{b}\,M_g(t)\,M_g^*(t) \leq BI\ , \qquad \text{a.e. } t \in \mathbb{R}\ . \tag{4.21}$$

Next, by Theorem 3.3 the frame operator of the Gabor frame (g,a,b) has the representation

$$(S_{(g,a,b)}f)(t) = \frac{1}{b}\sum_i e_i(t)\,f(t-i/b)\ , \qquad \text{a.e. } t \in \mathbb{R}\ , \tag{4.22}$$

for $f \in L^2$ with absolute convergence of the right-hand side series for a.e. $t \in \mathbb{R}$, in which the $e_i(t)$ are given by

$$e_i(t) = \sum_n g(t-na)\,g^*(t-na-i/b)\ , \qquad i \in \mathbb{Z}\,,\ \text{a.e. } t \in \mathbb{R}\ . \tag{4.23}$$

Also, tightness of the frame (g,a,b) is equivalent with

$$\sum_n g(t-na-i/b)\,g^*(t-na-j/b) = c\,\delta_{ij}\ , \qquad i,j \in \mathbb{Z}\ , \tag{4.24}$$

for a.e. $t \in \mathbb{R}$ with c some constant. Moreover, the duality condition between two Gabor frames

(g,a,b) and (γ,a,b) can be expressed as

$$\sum_n g(t-na-i/b)\,\gamma^*(t-na) = \sum_n \gamma(t-na-i/b)\,g^*(t-na) =$$

$$= b\,\delta_{i0}\,, \qquad i\in\mathbb{Z}\,,\ \text{a.e. } t\in\mathbb{R}\,. \tag{4.25}$$

Finally, the canonical dual function ${}^\circ\gamma$ can be computed as

$${}^\circ\gamma(t) = \sum_i \Big(\frac{1}{b}\,M_g(t)\,M_g^*(t)\Big)^{-1}_{0i}\, g(t-i/b)\,, \qquad \text{a.e. } t\in\mathbb{R}\,, \tag{4.26}$$

and this ${}^\circ\gamma$ is minimal in the sense that for any other dual function γ we have that $\|{}^\circ\gamma\|^2 \le \|\gamma\|^2$ with equality if and only if ${}^\circ\gamma = \gamma$ a.e.

We note that the representation (4.22–23) of the frame operator S_g (with shift parameters a, b) is called the Walnut representation [21] of the frame operator. Note that this representation holds for any $f\in L^2$ with absolute convergence for a.e. t. A detailed study of the convergence of the right-hand side of (4.22) as an operator of L^2 has been carried out in [28]. A sufficient condition that the Gabor system has a finite frame upper bound while the representation (4.22) converges unconditionally is that g satisfies the CC-condition: there is an $M<\infty$ such that

$$\sum_i |e_i(t)| \le M\,, \qquad \text{a.e. } t\in\mathbb{R}\,, \tag{4.27}$$

with the e_i given in (4.23), see [28], Theorem 4.1 and 6.9.

5. Gabor systems in the time-frequency domain

In this section we consider Gabor systems (g,a,b) with $g\in L^2$ and $a>0$, $b>0$ in the time-frequency domain. We shall thus obtain a description in the time-frequency domain of the frame bound conditions and the frame operator, of the duality condition and of the canonical dual function ${}^\circ\gamma$. We define time-frequency shift operators U_{kl} for $k,l\in\mathbb{Z}$ by

$$U_{kl}\,h = h_{k/b,l/a}\,, \qquad h\in L^2\,. \tag{5.1}$$

The proofs of the main results in this section can be found in [23], [27], Sec. 1.4, while many of these main results can also be found in [20], [29]. It should be noted that the approaches used in [23], [27], Sec. 1.4 and in [20] and in [29] are quite different; indeed, [20], [23] and [29] were written independently of one another and more or less simultaneously. We follow here the the approach in [23], [27], Sec. 1.4 which is based upon what we call the Fundamental Identity. This identity can be traced back to the work of Tolimieri and Orr [22], and the sharp form that we present below is due to Janssen, [30], Proof of Prop. A, [23], Props. 2.3 and 2.4, [27], Subsec. 1.4.1.

Fundamental Identity. Let $f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)} \in L^2$, and assume that at least one of the systems $(f^{(1)}, a, b)$, $(f^{(2)}, a, b)$ and at least one of the systems $(f^{(3)}, a, b)$, $(f^{(4)}, a, b)$ has a finite frame upper bound. Also assume that

$$\sum_{k,l} |(f^{(3)}, f^{(2)}_{k/b,l/a})|\,|(f^{(1)}_{k/b,l/a}, f^{(4)})| < \infty \,. \tag{5.2}$$

Then

$$\sum_{n,m} (f^{(1)}, f^{(2)}_{na,mb})(f^{(3)}_{na,mb}, f^{(4)}) = \frac{1}{ab} \sum_{k,l} (f^{(3)}, f^{(2)}_{k/b,l/a})(f^{(1)}_{k/b,l/a}, f^{(4)}) \,. \tag{5.3}$$

The proof of this result consists of a careful inspection of the proof of the Poisson summation formula for functions of two variables and their $2D$-Fourier transforms where the $2D$-Fourier transform pair

$$(x,y) \to (f^{(1)}, f^{(2)}_{x,y})(f^{(3)}_{x,y}, f^{(4)}) \,; \qquad (v,w) \to (f^{(3)}, f^{(2)}_{w,-v})(f^{(1)}_{w,-v}, f^{(4)}) \tag{5.4}$$

is taken.

Now let $g \in L^2$ and define the linear mapping U_g of L^2 by

$$U_g f = ((f, g_{k/b,l/a}))_{k,l\in\mathbb{Z}} \,, \qquad f \in L^2 \,. \tag{5.5}$$

Theorem 5.1. The Gabor system (g, a, b) has a finite frame upper bound B_g if and only if U_g and U_g^* are bounded linear mappings of L^2 into $l^2(\mathbb{Z}^2)$ and $l^2(\mathbb{Z}^2)$ into L^2, respectively, with operator norms $\leq \ (abB_g)^{\frac{1}{2}}$. In particular, the Gabor system $(g, 1/b, 1/a)$ then has the finite frame upper bound abB_g.

Note that the mapping U_g^* is given by

$$U_g^*\,\mathbf{c} = \sum_{k,l} c_{kl}\, g_{k/b,l/a} \,, \qquad \mathbf{c} \in l^2(\mathbb{Z}^2) \,. \tag{5.6}$$

Theorem 5.2. Let $A \geq 0$, $B < \infty$. Then we have

$$A\,\|f\|^2 \leq \sum_{n,m} |(f, g_{na,mb})|^2 \leq B\,\|f\|^2 \,, \qquad f \in L^2 \,, \tag{5.7}$$

if and only if

$$AI \leq \frac{1}{ab}\, U_g\, U_g^* \leq BI \,, \tag{5.8}$$

where I is now the identity operator of $l^2(\mathbb{Z}^2)$.

We observe that $U_g\, U_g^*$ maps $l^2(\mathbb{Z}^2)$ into $l^2(\mathbb{Z}^2)$ (when (5.8) holds), with matrix elements given by

$$(U_g\, U_g^*)_{k,l;k',l'} = (g_{k'/b,l'/a}, g_{k/b,l/a}) \ , \qquad k,l \in \mathbb{Z}\,; \;\; k',l' \in \mathbb{Z}\ . \tag{5.9}$$

Hence, the frame upper bound conditions and tightness of the Gabor frame (g,a,b) can be read off from the operator $U_g\, U_g^*$ whose matrix elements are given in (5.9). In particular, (g,a,b) is a tight frame if and only if

$$(g_{k'/b,l'/a}, g_{k/b,l/a}) = c\, \delta_{kk'}\, \delta_{ll'} \ , \qquad k,l \in \mathbb{Z}\,; \;\; k',l' \in \mathbb{Z}\ , \tag{5.10}$$

for some constant c.

We next give a result, Theorem 5.3 below, on frame operator representation. We first introduce a norm-preserving mapping of L^2 into $L^2(\mathbb{Z}^2 \times [0,b^{-1}) \times [0,a^{-1}))$. Let h be any member of Schwartz space $\mathcal{S}$ with $\|h\| = 1$. Then the mapping $STFT_h$, defined for $f \in L^2$ by

$$(STFT_h f)(x,y) = (f, h_{x,y}) \ , \qquad y \in \mathbb{R}\ , \tag{5.11}$$

is a norm-preserving mapping of L^2 into $L^2(\mathbb{R}^2)$. Now there holds, see (5.1), for $f \in L^2$, $k,l \in \mathbb{Z}$ and $x,y \in \mathbb{R}$ that

$$(U_{kl} f, h_{x,y}) = (f, h_{x-k/b,y-l/a})\, e^{-2\pi iky/b + 2\pi ikl/ab}\ . \tag{5.12}$$

Hence the mapping V_h, defined for $f \in L^2$ by

$$(V_h f)(k,l\,;\,x,y) = (U_{kl} f, h_{x,y})\ , \;\; k,l \in \mathbb{Z}\,; \qquad x \in [0,b^{-1})\ , \;\; y \in [0,a^{-1})\ , \tag{5.13}$$

is a norm-preserving mapping from L^2 into $L^2(\mathbb{Z}^2 \times [0,b^{-1}) \times [0,a^{-1}))$.

<u>Theorem 5.3</u>. When the system (g,a,b) has a finite frame upper bound B_g, we have for $f \in L^2$

$$(V_h S_g f)(\cdot,\cdot\,;\,x,y) = \frac{1}{ab}\,(U_g U_g^*)^T\,(V_h f)(\cdot,\cdot\,;\,x,y)\ , \qquad x \in [0,b^{-1})\ , \;\; y \in [0,a^{-1})\ , \tag{5.14}$$

where $(U_g U_g^*)^T$ is the transpose of the "matrix" $U_g U_g^*$ in (5.9).

Note that in this representation of the frame operator S_g the matrix $(ab)^{-1}\,(U_g U_g^*)^T$ is independent of x,y. The relation (5.14) extends as follows. Assume that A_g is a frame lower bound of the Gabor frame (g,a,b) and that φ is analytic on an open set containing the closed segment $[A_g,B_g]$. Then (5.14) holds with S_g at the left-hand side replaced by $\varphi(S_g)$ and $(ab)^{-1}\,(U_g U_g^*)^T$ at the right-hand side replaced by $\varphi((ab)^{-1}\,(U_g U_g^*)^T)$.

The representation in Theorem 5.3 can be rephrased more loosely as follows. Assume that the Gabor system (g,a,b) has a finite frame upper bound B_g. Then the frame operator S_g has the representation

$$S_g = \frac{1}{ab} \sum_{k,l} (g, g_{k/b,l/a})\, U_{kl} \tag{5.15}$$

in the sense that for any $f, h \in L^2$ such that $((U_{kl}f, h))_{k,l\in\mathbb{Z}} \in l^2(\mathbb{Z}^2)$ there holds

$$(S_g f, h) = \frac{1}{ab} \sum_{k,l} (g, g_{k/b,l/a})(U_{kl} f, h) \ . \tag{5.16}$$

In the case that g satisfies the condition

$$A \ : \ \ E := \sum_{k,l} |(g, g_{k/b,l/a})| < \infty \tag{5.17}$$

of Tolimieri and Orr [22], Sec. 3, the system (g,a,b) has the finite frame upper bound E/ab, and the convergence in (5.15) is without any proviso.

<u>Theorem 5.4</u>. Assume that the systems (g,a,b) and (γ,a,b) have finite frame upper bounds. Then the two systems are dual if and only if

$$U_g U_\gamma^* = U_\gamma U_g^* = abI \ . \tag{5.18}$$

Moreover, we have for the canonical dual ${}^\circ\gamma$ that

$$U_{{}^\circ\gamma}^* = ab\, U_g^* (U_g U_g^*)^{-1} \ , \tag{5.19}$$

and

$$\frac{1}{ab}\, U_{{}^\circ\gamma} U_{{}^\circ\gamma}^* = \Big(\frac{1}{ab}\, U_g U_g^*\Big)^{-1} \ , \tag{5.20}$$

so that the inverse frame operator has the representation

$$S_g^{-1} = S_{{}^\circ\gamma} = \sum_{k,l} \Big(\frac{1}{ab}\, U_g U_g^*\Big)^{-1}_{kl;00} U_{kl} \tag{5.21}$$

with the same proviso as in (5.15).

The duality condition in (5.18) can be made more explicit as follows. We have that two Gabor systems (g,a,b) and (γ,a,b), both with a finite frame upper bound, are dual if and only if

$$(\gamma, g_{k/b,l/a}) = ab\, \delta_{k0}\delta_{l0} \ , \qquad k,l \in \mathbb{Z} \ . \tag{5.22}$$

This is a rigourous form of the celebrated Wexler-Raz biorthogonality condition [24]. Also, (5.19) can be made more explicitly as follows. We have that ${}^{\circ}\gamma$ is the unique element $\gamma \in L^2$ of minimum norm such that (5.22) holds. The latter result has become known as the "Wexler-Raz dual equals the frame dual"-result.

6. Gabor systems in the Zak transform domain

We consider in this section Gabor systems (g, a, b) for the special case that $(ab)^{-1} = q/p$ with integer q and p satisfying $\gcd(q, p) = 1$.

Let $\lambda > 0$. We define for $h \in L^2$ the Zak transform $Z_\lambda h$ of h by

$$(Z_\lambda h)(t, \nu) = \lambda^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} h(\lambda(t-k))\, e^{2\pi i k \nu} \ , \qquad \text{a.e. } t, \nu \in \mathbb{R} \ . \tag{6.1}$$

The following properties hold for the Zak transform. Let $f, h \in L^2$. Then $Zf, Zh \in L^2_{\text{loc}}(\mathbb{R}^2)$, they are quasi-periodic according to

$$F(t+1, \nu) = e^{2\pi i \nu}\, F(t, \nu) \ ; \qquad F(t, \nu+1) = F(t, \nu) \ , \ \text{ a.e. } t, \nu \in \mathbb{R} \ , \tag{6.2}$$

and there holds

$$(f, h) = (Zf, Zh) \ , \tag{6.3}$$

where the inner product at the right-hand side involves any unit square in $\mathbb{R}^2$. Furthermore, any $F \in L^2_{\text{loc}}$ satisfying (6.2) is of the form $F = Zf$ with some unique $f \in L^2$. Some other properties are

$$\lambda^{\frac{1}{2}}\, f(\lambda t) = \int_0^1 (Z_\lambda f)(t, \nu) \ , \qquad \text{a.e. } t \in \mathbb{R} \ , \tag{6.4}$$

and

$$(Z_\lambda \hat{f})(t, \nu) = e^{2\pi i \nu t} (Z_{1/\lambda} f)(-\nu, t) \ , \qquad \text{a.e. } t, \nu \in \mathbb{R} \ . \tag{6.5}$$

Finally, any continuous F satisfying (6.2) has a zero in any unit square in $\mathbb{R}^2$. See [31] for these and many more properties of the Zak transform.

The usefulness of the Zak transform for description of frame bound conditions, frame operator, the duality condition and characterization and computation of the canonical dual function was recognized and elaborated by Zibulski and Zeevi, see, for instance, [25]. Also see [26], p. 978 and 981 and [32]. We make the choice $\lambda = b^{-1}$ and suppress the subscript λ in Z_λ so that

$$(Zh)(t, \nu) = b^{-\frac{1}{2}} \sum_{k=-\infty}^{\infty} h\Big(\frac{t-k}{b}\Big)\, e^{2\pi i k \nu} \ , \qquad \text{a.e. } t, \nu \in \mathbb{R} \ , \tag{6.6}$$

for $h \in L^2$. See [27], Subsec. 1.5.7, where it is shown that the choice $\lambda = a$ yields equally useful results.

We set for $f, h \in L^2$ and a.e. $t, \nu \in \mathbb{R}$

$$\Phi^f(t,\nu) = p^{-\frac{1}{2}} \Big((Zf)\Big(t - l\,\frac{p}{q}\,, \nu + \frac{k}{p}\Big)\Big)_{\substack{k=0,\ldots,p-1 \\ l=0,\ldots,q-1}} \,, \tag{6.7}$$

$$A^{fh}(t,\nu) = \Phi^f(t,\nu)(\Phi^h(t,\nu))^* \,. \tag{6.8}$$

Theorem 6.1. The Gabor system (g, a, b) has a finite frame upper bound B_g if and only if $\Phi^g(t,\nu)$ and $(\Phi^g(t,\nu))^*$ are for a.e. $t, \nu \in [0,1)$ bounded linear mappings of $\mathbb{C}^q$ into $\mathbb{C}^p$ and $\mathbb{C}^p$ into $\mathbb{C}^q$, respectively, with norm $\leq B_g^{\frac{1}{2}}$. In particular, there holds

$$|(Zg)(t,\nu)|^2 \leq p\,B_g \,, \qquad \text{a.e. } t,\nu \in \mathbb{R} \,. \tag{6.9}$$

We note here that it thus follows that (g, a, b) has a finite frame upper bound if and only if Zg is essentially bounded.

Theorem 6.2. Let $A \geq 0$, $B < \infty$. Then we have

$$A\,\|f\|^2 \leq \sum_{n,m} |(f, g_{na,mb})|^2 \leq B\,\|f\|^2 \,, \qquad f \in L^2 \,, \tag{6.10}$$

if and only if

$$A\,I_{p\times p} \leq A^{gg}(t,\nu) \leq B\,I_{p\times p} \,, \qquad \text{a.e. } t,\nu \in \mathbb{R} \,, \tag{6.11}$$

where $I_{p\times p}$ denotes the identity operator of $\mathbb{C}^p$.

We observe that one can restrict oneselves in checking the condition (6.11) to the set $(t,\nu) \in [0, q^{-1}) \times [0, p^{-1})$. We also note that the Gabor frame (g, a, b) is tight if and only if

$$A^{gg}(t,\nu) = c\,I_{p\times p} \,, \qquad \text{a.e. } t,\nu \in \mathbb{R} \,, \tag{6.12}$$

for some constant c.

Theorem 6.3. Assume that the Gabor system (g, a, b) has a finite frame upper bound. With S_g the frame operator, there holds for $f \in L^2$

$$\Phi^{S_g f}(t,\nu) = A^{gg}(t,\nu)\,\Phi^f(t,\nu) \,, \qquad \text{a.e. } t,\nu \in \mathbb{R} \,. \tag{6.13}$$

Theorem 6.3 gives the representation of the frame operator S_g in the Zak transform domain via the matrices Φ^f in (6.7). More generally, when A_g is a frame lower bound for (g, a, b) and φ is analytic in an open set containing the closed segment $[A_g, B_g]$, we can replace S_g at the left-hand side of (6.13) by $\varphi(S_g)$ and $A^{gg}(t, \nu)$ at the right-hand side of (6.13) by $\varphi(A^{gg}(t, \nu))$.

Theorem 6.4. Assume that the systems (g, a, b) and (γ, a, b) have finite frame upper bounds. Then the two systems are dual if and only if

$$\Phi^g(t,\nu)(\Phi^\gamma(t,\nu))^* = \Phi^\gamma(t,\nu)(\Phi^g(t,\nu))^* = I_{p\times p}\ , \qquad \text{a.e. } t,\nu \in \mathbb{R}\ . \tag{6.14}$$

Moreover, we have for the canonical dual that

$$(\Phi^{{}^\circ\gamma}(t,\nu))^* = (\Phi^g(t,\nu))^*\,(\Phi^g(t,\nu)\,(\Phi^g(t,\nu))^*)^{-1}\ , \qquad \text{a.e. } t,\nu \in \mathbb{R}\ , \tag{6.15}$$

and

$$A^{{}^\circ\gamma{}^\circ\gamma}(t,\nu) = (A^{gg}(t,\nu))^{-1}\ , \qquad \text{a.e. } t,\nu \in \mathbb{R}\ . \tag{6.16}$$

The condition of duality can be written more explicitly as follows. The systems (g, a, b) and (γ, a, b), both having a finite frame upper bound, are dual if and only if for a.e. $t, \nu \in \mathbb{R}$ we have

$$\frac{1}{p}\sum_{l=0}^{q-1}(Zg)\Big(t - l\,\frac{p}{q}\,,\nu+\frac{k}{p}\Big)(Z\gamma)^*\Big(t-l\,\frac{p}{q}\,,\nu\Big) = \delta_{k0}\ , \qquad k=0,...,p-1\ . \tag{6.17}$$

Furthermore, for any dual system (γ, a, b) and a.e. $t, \nu \in \mathbb{R}$ we have

$$\sum_{k=0}^{q-1}\Big|(Z^\circ\gamma)\Big(t+\frac{k}{q}\,,\nu\Big)\Big|^2 \le \sum_{k=0}^{q-1}\Big|(Z\gamma)\Big(t+\frac{k}{q}\,,\nu\Big)\Big|^2\ , \tag{6.18}$$

with equality if and only if $(Z^\circ\gamma)(t + k/q, \nu) = (Z\gamma)(t + k/q, \nu)$ for $k = 0, ..., q - 1$.

7. When is (g, a, b) a Gabor frame?

In this section we present a collection of results, comments, observations, (counter)examples, open problems, etc., on the basic problem of deciding whether a triple (g, a, b) with $g \in L^2$ and $a > 0$, $b > 0$ is a frame. While the finite frame upper bound condition is reasonably easy to deal with by imposing rather mild smoothness and decay conditions, the positivity of the frame lower bound presents a much harder problem. The basic problem can be considered in each of the four domains of Secs. 4–6, and any one of these domains can come with a particular advantage. By nature, this section is not too well organized, with things that could be added and other things that could be deleted.

7.1. We start with the well-known result that when $g \in L^2$ and (g, a, b) is a frame then we must have $(ab)^{-1} \geq 1$. This result has a long and complicated history, see for this [16], Sec. 4.1, [26], p. 978, [30], Sec. 1. There is a stronger result, due to Howe and Steger using results of Rieffel in [33], that says that completeness of the systen (g, a, b) in L^2 implies that $(ab)^{-1} \geq 1$. In [34], Sec. 2, Benedetto, Heil and Walnut present a somewhat unsettling example of a $g \in L^2$ and an irregular Gabor system (i.e. the time-frequency points involved in the shifts do not form a lattice) of arbitrarily low density such that the system is complete in L^2, but not a frame. Restricting again to frames, several proofs of the fact that $(ab)^{-1} \geq 1$ when (g, a, b) is a frame are known now. When $(ab)^{-1} = q/p$ is rational with integer q, p such that $\gcd(q, p) = 1$, a simple rank consideration of the matrices in (6.7) and (6.11) suffices to show that $p \leq q$, i.e. $(ab)^{-1} \geq 1$. For general values of $(ab)^{-1}$, a simple proof can be based upon the Wexler-Raz biorthogonality condition (5.22) together with the inequality $|(g, {}^\circ\gamma)| \leq 1$ that follows from (2.14). Yet another proof follows upon integrating the duality condition (4.10) for the canonical dual ${}^\circ\gamma$ over an interval of length b and using (as in the previous proof) that $|(g, {}^\circ\gamma)| \leq 1$. It is the latter approach that can be generalized to shift-invariant systems, the windows of which have certain frequency localization properties, see [35].

7.2. We now consider the case that $(ab)^{-1} = 1$. This case is particularly conveniently discussed in the Zak transform domain since now we have $p = q = 1$ in $(ab)^{-1} = q/p$ so that the matrices in (6.7–8) reduce to scalars. As already said in Sec. 6, when Zg is continuous it must have a zero in any unit square. Hence, when $ab = 1$ and (g, a, b) is a frame, it cannot be true that g is continuous and rapidly decaying (for then Zg is continuous and has a zero, whence the lower frame bound in (6.11) is zero). Another result for the case $(ab)^{-1} = 1$ is the Balian-Low theorem (for the complicated history of this result, see [34], Subsec. 1.1) according to which at least one of $g'(t)$ and $tg(t)$ is not square integrable as a function of $t \in \mathbb{R}$ when (g, a, b) is a frame. Somewhat surprisingly, there is a construction involving cosines and sines rather than exponentials where one does get a frame at critical density $(ab)^{-1} = 1$ with a well-behaved window g (Wilson bases, see [16], Subsec. 4.2.2 for history and details of the construction).

In the remainder of this section we shall consider, with few exceptions, the case that $(ab)^{-1} > 1$.

7.3. We shall first indicate a class of windows g such that (g, a, b) is a Gabor frame. To that end we choose a continuous g, positive on and vanishing outside an interval $(-\frac{1}{2}c, \frac{1}{2}c)$, where c is any number in the non-empty interval (a, b^{-1}). Now the "matrix" $M_g(t)\, M_g^*(g)$ in (4.19) is a diagonal matrix with strictly positive diagonal elements

$$D(t) = \sum_n |g(t - na)|^2 \ , \qquad t \in \mathbb{R} \ , \tag{7.1}$$

that are bounded away from 0 and ∞. Hency by (4.21) there are the frame bounds $b^{-1} \min D$, $b^{-1} \max D$. One sees furthermore easily that a tight frame (h, a, b) is obtained by choosing $h = g/D^{\frac{1}{2}}$.

7.4. There are a few cases where one can show that a system (g, a, b) with a finite frame upper bound is a Gabor frame by displaying a dual function γ such that (γ, a, b) has a finite frame upper bound. This is so, for instance, for the case of the Gaussian window

$$g_\alpha(t) = (2\alpha)^{\frac{1}{4}} \exp(-\pi\alpha t^2)\ , \qquad t \in \mathbb{R}\ , \tag{7.2}$$

and for the one-sided exponential considered in 7.5 below. In [30], Sec. 3, the Wexler-Raz biorthogonality condition (5.22) for $g = g_\alpha$ is written out and elaborated to yield for any $\varepsilon > 0$ with $\varepsilon < \alpha^{-1}(1-ab)$ the biorthogonal function (apart from a constant factor)

$$\gamma_{\varepsilon,\alpha}(t) = \int_0^t e^{-\pi\alpha c s^2} \sum_{k=-\infty}^{\infty} (-1)^k \exp\Big(\frac{-\pi a}{\varepsilon b c}\,(k+\tfrac{1}{2}-bcs)^2\Big)\, ds\ , \qquad t \in \mathbb{R}\ , \tag{7.3}$$

where $c = (\alpha\varepsilon + ab)^{-1} > 1$. It can be shown that this $\gamma_{\varepsilon,\alpha}$ can be extended to an entire function of $t = x + iy\ \in$ satisfying

$$\gamma_{\varepsilon,\alpha}(x+iy) = O(\exp(-\pi\alpha x^2 + \pi\varepsilon^{-1}y^2))\ , \qquad x, y \in \mathbb{R}\ . \tag{7.4}$$

This implies that both $\gamma_{\varepsilon,\alpha}$ and $\hat{\gamma}_{\varepsilon,\alpha}$ have Gaussian decay, just like g_α and $\hat{g}_\alpha$. Such a $\gamma_{\varepsilon,\alpha}$ can also be constructed by using the Bargmann transform, see, for instance [7], [8]. Interestingly, when $\alpha = 1$ and we take $\varepsilon \downarrow 0$ we obtain a function $\gamma_{0,1}$ that would coincide with Bastiaans singular function in [4] when $ab = 1$. The latter singular function can be shown to be in any $L^\infty \backslash L^p$ with $1 \le p < \infty$, see [10], Subsec. 4.4.

The result that (g, a, b) is a frame for Gaussian g and $(ab)^{-1} > 1$, and several generalization of it, has a rich history for which we refer to [16], Subsec. 3.4.4.B, [26], pp. 980–982, [30], Sec. 1. In [36], Lyubarskii and Seip give a very careful and detailed analysis of what happens for Gaussian g and $(ab)^{-1} = 1$ when the lattice of the relevant time-frequency points is slightly disturbed.

It is unlikely that any of the $\gamma_{\varepsilon,\alpha}$ in (7.3) coincides with the canonical dual ${}^\circ\gamma$. For even values of $(ab)^{-1}$ and $\alpha = 1$ the canonical dual ${}^\circ\gamma_\alpha$ was computed in [37], Sec. 6 as

$$^\circ\gamma_\alpha(t) = \frac{ab}{\vartheta_3(\pi t/a\,;\, \exp(-\pi/2a^2))} \sum_{k=-\infty}^{\infty} c_k \exp(-\pi(t-k/b)^2)\ , \qquad t \in \mathbb{R}\ , \tag{7.5}$$

with ϑ_3 a theta function, see [37], (6.5), and c_k certain numbers decaying like $\exp(-\pi|k|/2b^2)$ as $|k| \to \infty$. This ${}^\circ\gamma_\alpha$ cannot be extended to an entire function and it decays only like $\exp(-\pi|t|/2b)$ as $t \in \mathbb{R}$, $|t| \to \infty$. For non-integral values of $(ab)^{-1}$ it does not seem easy to determine the canonical dual (even the case that $(ab)^{-1}$ is an odd integer presents serious problems).

7.5. We consider next the one-sided exponential

$$_\alpha g(t) = (2\alpha)^{\frac{1}{2}}\, e^{-\alpha t}\, \chi_{[0,\infty)}(t)\ , \qquad t \in \mathbb{R}\ , \tag{7.6}$$

with $\alpha > 0$. One can again guess a dual ${}_\alpha\gamma$ by looking at the Wexler-Raz biorthogonality condition (5.22), and one obtains as a dual function

$$ {}_{\alpha}\gamma(t) = \frac{b}{\sqrt{2\alpha}}\, e^{\alpha t}(\chi_{[0,a)}(t) - \chi_{[-a,0)}(t)) \;, \qquad t \in \mathbb{R} \;. \tag{7.7} $$

We note that this ${}_{\alpha}\gamma$ works also for the case that $(ab)^{-1} = 1$, whence $({}_{\alpha}g, a, b)$ is a Gabor frame for any $a > 0$, $b > 0$ with $(ab)^{-1} \geq 1$. When $(ab)^{-1}$ is an integer, one can compute the canonical dual ${}^{\circ}_{\alpha}\gamma$, see [37], (4.11), as

$$ \begin{aligned} {}^{\circ}_{\alpha}\gamma(t) = {} & \frac{b}{\sqrt{2\alpha}}\, \frac{1 - e^{-2\alpha a}}{1 - e^{-2\alpha/b}}\, e^{\alpha t - 2\alpha a \lfloor t/a \rfloor} \times \\ & \times (\chi_{[0,1/b)}(t) - e^{-2\alpha/b}\, \chi_{[-1/b,0)}(t)) \;, \qquad t \in \mathbb{R} \;, \end{aligned} \tag{7.8} $$

and this ${}^{\circ}_{\alpha}\gamma$ differs from the ${}_{\alpha}\gamma$ in (7.7) (unless $(ab)^{-1} = 1$).

We note that in [37] there are some more specific examples, such as two sided exponentials, hyperbolic secants, for which it is shown that they yield a Gabor frame with integer $(ab)^{-1} > 1$.

7.6. While 1 is the lower bound for $(ab)^{-1}$ so that (g, a, b) can be a frame, it appears that the chances of having a frame increase with increasing value of $(ab)^{-1}$. For integer values N of $(ab)^{-1}$ this is apparent from Sec. 6 since now the matrix $A^{gg}(t, \nu)$ is a scalar ($p = 1$, $q = N$), given by

$$ A^{gg}(t, \nu) = \sum_{l=0}^{N-1} \Big|(Zg)\Big(t - \frac{l}{q}, \nu\Big)\Big|^2 \;, \qquad \text{a.e. } t, \nu \in \mathbb{R} \;, \tag{7.9} $$

and this quantity is positive and bounded for many windows g, including certain smooth and rapidly decaying g's. More precisely, according to [26], Theorems 2.5–6, for any sufficiently well-behaved g there are $a_c > 0$, $b_c > 0$ such that (g, a, b) is a frame when $0 < a < a_c$, $0 < b < b_c$. Also see [13], Part 1, Theorem 6.1. Such a result can also be obtained from the frame operator representation (5.15) (holding under condition A in (5.17)) when $(g, g_{x,y})$ decays sufficiently rapidly when $x^2 + y^2 \to \infty$. Then in the right-hand side of (5.15) the terms with $(k, l) \neq (0, 0)$ are small compared to the term $(ab)^{-1}I$, corresponding to $(k, l) = (0, 0)$, as $a^{-2} + b^{-2} \to \infty$. When one allows g's that are not well-behaved, one gets problems with results of this type. In [38] there is constructed for any irrational α a smooth, bounded $g \in L^2$ such that for any rational $a > 0$, $b > 0$ the system (g, a, b) has a finite frame upper bound while for any $\beta > 0$ and any rational $c > 0$ the system $(g, c\alpha, \beta)$ has no such bound. Also in [38], there is an example of a g, bounded and supported by $[0, 1]$, such that the above a_c, b_c do not exist, and an example of a g such that 0 is accumulation point of points a such that (g, a, b) has frame lower bound 0 and, at the same time, accumulation point of points a such that (g, a, b) has frame lower bound $\geq$ 1 (arbitrary $b \in (0, 1)$).

7.7. It seems hard to find a general condition on windows $g \in L^2$ (with reasonable smoothness and decay properties) ensuring (g, a, b) to be a frame for all or even some specific values of $a > 0$, $b > 0$ with $(ab)^{-1} > 1$. In [39] two such classes of windows, for integer values of $(ab)^{-1}$, are found. The first class consists of all g supported, positive, strictly decreasing and continuous and integrable on $[0, \infty)$, for which $(ab)^{-1}$ is allowed to take any positive integer $\geq$ 1 as its value. This class contains the one-sided exponentials in 7.5. The second class consists of all even, positive, continuous and integrable g such that g has on $[0, \infty)$ the form

$$g(t) = b(t) + b(t + 1)\ , \qquad t \geq 0\ , \tag{7.10}$$

with b strictly convex and positive on $[0, \infty)$. In these classes, the respective conditions of strictness cannot be weakened yielding, for instance, a well-behaved rapidly decaying g such that both g and $\hat{g}$ are strictly positive while $(g, \frac{1}{2}, 1)$ is not a frame. We also observe that the Gaussians do not belong to either one of these two classes.

7.8. An example of a g that just fails to belong to the first class in 7.7 is the characteristic function $\chi_{[0,c)}$ of the interval $[0, c)$. Assuming without restriction that $b = 1$ the author has obtained the following results for this g. The frame upper bound condition is always satisfied in this case, and we can restrict to $a < 1$, $c > 1$ for it is not difficult to show that

(a) $c < a$ or $a > 1 \Rightarrow$ no frame,

(b) $c \geq a$, $a \leq 1 \Rightarrow$ frame,

(c) $a = 1$, $c > 1 \Rightarrow$ no frame.

Assuming $a < 1$, $c > 1$ one can show furthermore that

(d) $a \notin \mathbb{Q}$, $c \in (1, 2) \Rightarrow$ frame,

(e) $a = p/q \in \mathbb{Q}$, $\gcd(p, q) = 1$, $2 - \dfrac{1}{q} < c < 2 \Rightarrow$ no frame,

(f) $a > \frac{3}{4}$, $c = L - 1 + L(1 - a)$ with integer $L \geq 3 \Rightarrow$ no frame,

(g) $|c - \lfloor c \rfloor - \frac{1}{2}| < \frac{1}{2} - a \Rightarrow$ frame.

While (f) shows that one may fail to have a frame for irrational a, it seems to hold that one does have a frame when a, c, a/c are irrational. And for rational $a = p/q$, $\gcd(p, q) = 1$, one can give an algorithm that determines whether one has a frame for any $c > 1$ (complexity determined by q, p). In particular one thus can find rational $a > \frac{1}{2}$ and $c > 1$ such that one has a frame.

This shows that the answer to the basic question is already rather bewildering for windows g as elementary as a characteristic function.

7.9. We continue by giving some comments on inheritence of certain desirable properties of a $g \in L^2$ for which (g, a, b) is a frame by the canonical dual. We recall from Subsec. 4.1 the CC-condition of essential boundedness of $\sum_i |e_i(t)|$, with e_i given by (4.23), guaranteeing the Walnut representation (4.22) of the frame operator to converge unconditionally as an operator of L^2. It is an open problem whether ${}^\circ\gamma$ inherits the CC-condition from g. In [28], Theorem 4.14, it is shown that for rational values of $(ab)^{-1}$ a slightly stronger condition, viz. the uniform CC-condition, is inherited by ${}^\circ\gamma$ from g.

A similar situation occurs for condition A of Tolimieri and Orr, see (5.17). While it is an open problem whether this condition A is inherited for all values of $(ab)^{-1} > 1$, it has been shown in [32] that it does so for rational values of $(ab)^{-1}$.

7.10. The next topic concerns the construction of a tight frame canonically associated to a Gabor frame. Given a Gabor frame (g, a, b) with frame operator S_g, the Gabor frame (h, a, b) with

$$h = S_g^{-\frac{1}{2}}\, g \tag{7.11}$$

is tight. It should be noted that this h can be computed using the functional calculus (with $\varphi(x) = x^{-\frac{1}{2}}$) for the frame operator in the various domains (see the comments after Theorems 3.3, 5.3 and 6.3). For instance, when $(ab)^{-1} = q/p$ is rational, the h in (7.11) is given in the Zak transform domain according to

$$\Phi^h(t,\nu) = (A^{gg}(t,\nu))^{-\frac{1}{2}}\, \Phi^g(t,\nu)\ , \qquad \text{a.e. } t,\nu \in \mathbb{R}\ . \tag{7.12}$$

In the particular case that $(ab)^{-1}$ is an integer N (so that $q = N, p = 1$) we get

$$\Phi^h(t,\nu) = \frac{\Phi^g(t,\nu)}{\Big(\sum\limits_{l=0}^{N-1} \Big|(Zg)\Big(t + \dfrac{l}{N}, \nu\Big)\Big|^2\Big)^{\frac{1}{2}}}\ , \qquad \text{a.e. } t,\nu \in \mathbb{R}\ , \tag{7.13}$$

i.e.

$$(Zh)(t,\nu) = \frac{(Zg)(t,\nu)}{\Big(\sum\limits_{l=0}^{N-1} \Big|(Zg)\Big(t + \dfrac{l}{N}, \nu\Big)\Big|^2\Big)^{\frac{1}{2}}}\ , \qquad \text{a.e. } t,\nu \in \mathbb{R}\ . \tag{7.14}$$

7.14. A further interesting point is the problem of finding out whether certain decay and smoothness properties of a g generating a Gabor frame are inherited by the canonical dual $^\circ\gamma$ or by the tight frame generating h of (7.11). In [23], see Properties 5.5–6, it is shown, by using Banach algebra methods, that when $g \in \mathcal{S}$ generates a Gabor frame, then both $^\circ\gamma$ and h of (7.11) are in $\mathcal{S}$ as well. A different proof of this fact, as pointed out by K.-H. Gröchenig, can be based on the frame operator representations of Sec. 4 together with a result of S. Jaffard [40] on inheritance of decay of the off-diagonal elements of an invertible operator of $l^2(\mathbb{Z})$ by the inverse and the inverse square root of the operator. For inheritance of exponential decay in the time and/or frequency domain the latter approach has been worked out in [41], Secs. 2 and 4.

There are some more results in [41] on inheritance of smoothness and decay by $^\circ\gamma$ and h from g. For instance, it is shown that in the case of integer values of $(ab)^{-1}$, there is no inheritance of faster-than-exponential decay unless we are dealing with tight frames. Whether there do exist tight frames (g, a, b) with a g having faster-than-exponential decay in both the time domain and the frequency domain is an open problem.

7.12. We finally make some comments on discrete-time Gabor systems. We now take $N, M \in \mathbb{N}$ and $g \in l^2(\mathbb{Z})$, and we consider the system of sequences $g_{nN,m/M}$ with $n \in \mathbb{Z}$, $m = 0, 1, ..., M-1$, defined by

$$g_{nN,m/M} = (e^{2\pi imr/M}\, g(r-nN))_{r\in\mathbb{Z}}\ . \tag{7.15}$$

It turns out, see [27], Sec. 1.6, that the theory of discrete-time Gabor systems with respect to frame bound conditions, frame operator representation, the duality condition and characterization and computation of canonical dual systems can be developed in largely the same way as for continuous-time Gabor systems.

A basic question that arises here is whether one can obtain a discrete-time Gabor frame by sampling the windows of a continuous-time Gabor frame $(g, N, 1/M)$ at the integers, and how the respective frame operators, representations, canonical duals etc. are related. In [42] it is shown that (and how) this can be done under a decay condition and a regularity condition on the window g. These conditions are the condition A of Tolimieri and Orr, see (5.17), for the system $(g, N, 1/M)$ and the regularity condition

$$R : \lim_{\varepsilon\downarrow 0} \sum_{r=-\infty}^{\infty} \frac{1}{\varepsilon} \int\limits_{-\frac{1}{2}\varepsilon}^{\frac{1}{2}\varepsilon} |g(r+u)-g(r)|^2\, du = 0\ . \tag{7.16}$$

It is shown in [42] that when $(g, N, 1/M)$ is a Gabor frame with g satisfying A and R, then the canonical dual function ${}^{\circ}\gamma$ satisfies A and R as well (so that it can be sampled at the integers, just like g). Furthermore, denoting the sampling operation by a superscript D, we have that $(g^D, N, 1/M)$ is a discrete-time Gabor frame with the same frame bounds as the continuous-time frame. Denoting the frame operator of $(g^D, N, 1/M)$ by S_g^D, we have for a large class of $h \in L^2$ (including g and ${}^{\circ}\gamma$) that

$$(S_g\, h)^D = S_g^D\, h^D\ . \tag{7.17}$$

In particular we get that ${}^{\circ}\gamma^D$ is the canonical dual sequence for the discrete-time Gabor system $(g^D, N, 1/M)$.

Acknowledgement

The author wishes to thank H.G. Feichtinger and D. Walnut for a careful reading of an early version of Sec. 1 of this contribution.

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