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Characterization and computation of canonical tight windows for Gabor frames*

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Abstract

Let $(g_{na,mb})_{n,m\in\mathbb{Z}}$ be a Gabor frame for $\boldsymbol{L}^2(\mathbb{R})$ for given window g. We show that the window $h^0 = S^{-\frac{1}{2}}g$ that generates the canonically associated tight Gabor frame minimizes $\|g-h\|$ among all windows h generating a normalized tight Gabor frame. We present and prove versions of this result in the time domain, the frequency domain, the time-frequency domain, and the Zak transform domain, where in each domain the canonical h^0 is expressed using functional calculus for Gabor frame operators. Furthermore, we derive a Wiener-Levy type theorem for rationally oversampled Gabor frames. Finally, a Newton-type method for a fast numerical calculation of h^0 is presented. We analyze the convergence behavior of this method and demonstrate the efficiency of the proposed algorithm by some numerical examples.

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*The authors wish to dedicate this work to Hans G. Feichtinger on occasion of his 50^{th} birthday

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1 Introduction

A Gabor system consists of functions of the form

$$g_{na,mb}(t) = e^{2\pi imbt} g(t - na), \quad n, m \in \mathbb{Z}, \tag{1.1}$$

for some window $g \in \boldsymbol{L}^2(\mathbb{R})$ and $a, b \in \mathbb{R}$. The parameters a and b are the time- and frequency translation parameters.

We say that the triple (g, a, b) generates a *Gabor frame* for $\boldsymbol{L}^2(\mathbb{R})$ for given a, b if there exist constants (*frame bounds*) $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{n,m\in\mathbb{Z}} |\langle f, g_{na,mb}\rangle|^2 \le B\|f\|^2, \tag{1.2}$$

for any $f \in \boldsymbol{L}^2(\mathbb{R})$.

The Gabor frame operator S is given by

$$Sf = \sum_{n,m\in\mathbb{Z}} \langle f, g_{na,mb}\rangle g_{na,mb}, \qquad f \in \boldsymbol{L}^2(\mathbb{R}), \tag{1.3}$$

and satisfies

$$AI \le S \le BI, \tag{1.4}$$

where I is the identity operator of $\boldsymbol{L}^2(\mathbb{R})$.

If $(g_{na,mb})_{n,m\in\mathbb{Z}}$ is a Gabor frame for $\boldsymbol{L}^2(\mathbb{R})$ then any f in $\boldsymbol{L}^2(\mathbb{R})$ can be written in the form

$$f = \sum_{n,m\in\mathbb{Z}} \langle f, \gamma^0_{na,mb}\rangle g_{na,mb} = \sum_{n,m\in\mathbb{Z}} \langle f, g_{na,mb}\rangle \gamma^0_{na,mb}, \tag{1.5}$$

where the *canonical dual frame* $(\gamma^0_{na,mb})_{n,m\in\mathbb{Z}}$ is given by

$$\gamma^0_{na,mb} = e^{2\pi imbt}\gamma^0(t - na), \quad n, m \in \mathbb{Z} \tag{1.6}$$

with $\gamma^0 = S^{-1}g$.

If $A = B$ the frame is called *tight*. In this case $S = AI$, whence $\gamma^0 = \frac{1}{A}g$. We say that a frame is *normalized tight* if in addition $A = B = 1$. Tight Gabor frames play an important role in signal processing and communications. They appear for instance in the construction of paraunitary modulated filter banks [8, 4] as well as in the construction of orthogonal frequency division multiplex (OFDM) systems [3, 32] in wireless communications. Tight

Gabor frames are also useful tools for the analysis of pseudodifferential operators [27].

Given a frame $(g_{na,mb})_{n,m\in\mathbb{Z}}$ with frame operator S a standard technique to construct a tight frame is the following. Set

$$h^0 = S^{-\frac{1}{2}}g, \tag{1.7}$$

then it is easy to see that $(h^0_{na,mb})_{n,m\in\mathbb{Z}}$ is a tight frame for $\boldsymbol{L}^2(\mathbb{R})$. We will refer to h^0 as *canonical tight window* and to $(h^0_{na,mb})_{n,m\in\mathbb{Z}}$ as *canonical tight frame*.

It is known that the canonical dual γ^0 minimizes

$$\left\| \frac{g}{\|g\|} - \frac{\gamma}{\|\gamma\|} \right\| \tag{1.8}$$

among all dual windows γ, cf. [19]. This minimality property of γ^0 can be expressed more precisely in the time-domain as: for a.e. t and all $\alpha > 0, \beta > 0$ (possibly depending on t) we have that

$$\sum_{n=-\infty}^{\infty} |\alpha g(t-na) - \beta\gamma(t-na)|^2 \tag{1.9}$$

is minimal among all dual γ for $\gamma = \gamma^0$. Such minimality results also hold in the frequency domain and in the Zak transform domain.

We will show in Section 2 that a result similar to (1.8) holds for tight Gabor frames. More precisely, we will demonstrate that h^0 solves

$$\operatorname*{arg\,min}_{h\in\boldsymbol{L}^2(\mathbb{R})} \|g-h\| \tag{1.10}$$

among all h such that (h, a, b) is a normalized tight frame.

This result is also important from a practical viewpoint, where one often selects a "nice" g and constructs the tight frame by using (1.7) with the goal in mind that the resulting tight frame window should have similar properties as g. Thus it is gratifying to know that h^0 is closest to g (in the $\boldsymbol{L}^2$-sense) among all possible normalized tight frames (h, a, b). For conditions under which h^0 inherits the decay properties of g see [35, 31, 5, 32].

It is sufficient to restrict our attention to normalized tight frames due to following reason. Assuming that (1.10) holds the reader may easily convince herself/himself that the window $\tilde{h}^0$ that minimizes $\|g-h\|$ among all h such that (h,a,b) is a tight frame is given by

$$\tilde{h}^0 = \frac{\langle h^0, g\rangle}{\langle h^0, h^0\rangle} h^0. \tag{1.11}$$

For the special case that $(g, a=1, b=1)$ constitutes a frame it is not difficult to see that (1.10) holds. In this case we can consider the problem in the Zak transform domain with $\lambda = p = q = 1$ (see Subsection 1.1 (1.24)–(1.26) for the definition of the Zak transform and the Zibulski-Zeevi matrices that are implicitly used in the next steps). Thus we have for any f such that $(f,1,1)$ has a finite frame upper bound that

$$(Z_1 Sf)(t,\nu) = |(Z_1 g)(t,\nu)|^2 (Z_1 f)(t,\nu), \quad \text{a.e.}\, t,\nu. \tag{1.12}$$

By functional calculus (cf. Subsection 1.1) we then have that

$$(Z_1 h^0)(t,\nu) = (Z_1 S^{-\frac{1}{2}} g)(t,\nu) = \Big(|(Z_1 g)(t,\nu)|^2\Big)^{-\frac{1}{2}} (Z_1 g)(t,\nu) \tag{1.13}$$

$$= \frac{(Z_1 g)(t,\nu)}{|(Z_1 g)(t,\nu)|}, \quad \text{a.e.}\, t,\nu. \tag{1.14}$$

Moreover, normalized tightness of $(h,1,1)$ is equivalently expressed as

$$|(Z_1 h)(t,\nu)| = 1, \quad \text{a.e.}\, t,\nu. \tag{1.15}$$

Now when we have a complex number z then

$$\min_{|\omega|=1} |z-\omega| = \left|z - \frac{z}{|z|}\right| = \Big||z|-1\Big|. \tag{1.16}$$

Thus we have that for a.e. t,ν and all normalized tight $(h,1,1)$

$$\Big|(Z_1 g)(t,\nu) - (Z_1 h^0)(t,\nu)\Big|^2 \le \Big|(Z_1 g)(t,\nu) - (Z_1 h)(t,\nu)\Big|^2. \tag{1.17}$$

Integrating the latter inequality over $[0,1)^2$ we get using unitarity of $Z_1 : \boldsymbol{L}^2(\mathbb{R}) \to \boldsymbol{L}^2([0,1)^2)$ that $\|g-h^0\|^2 \le \|g-h\|^2$.

The rest of the paper is organized as follows. Subsection 1.1 contains a brief review of various representations of the Gabor frame operator in different domains. We will make use of these representations in Section 2 when we prove the minimization result (1.10). We also discuss some refinements and extensions of this result. In Section 3 we present a Wiener-Levy type theorem for Gabor frames. Finally, in Section 4 we describe a Newton-type method for computing h^0 and analyze the convergence properties of this method.[1]

It is obvious that some or many of our results admit generalization to more general lattices in $\mathbb{R}^2$ as sets of time-frequency points to which the windows are shifted and to higher dimensions, but we have not pursued this point here.

1.1 Representations of the Gabor frame operator

We briefly review the representations of the Gabor frame operator in the time domain, in the frequency domain, in the time-frequency domain, and, for rational oversampling where $ab \in \mathbb{Q}$, in the Zak transform domain. We also present a lemma concerning the approximation of the inverse square root of an infinite-dimensional matrix by finite-dimensional inverse square roots.

Assume that (g, a, b) has a finite frame upper bound B, and define the Ron-Shen matrices

$$M_g(t) := \Big(g(t - na - k/b)\Big)_{k\in\mathbb{Z},n\in\mathbb{Z}}, \quad \text{a.e. } t, \tag{1.18}$$

$$H_g(\nu) := \Big(\hat{g}(\nu - mb - l/a)\Big)_{l\in\mathbb{Z},m\in\mathbb{Z}}, \quad \text{a.e. } \nu. \tag{1.19}$$

Now there holds that $M_g(t)$ and $H_g(\nu)$ are for a.e. t and a.e. ν bounded linear operators of $\ell^2(\mathbb{Z})$ (with operator norms $\le (bB)^{\frac{1}{2}}$ and $\le (aB)^{\frac{1}{2}}$). Now when (f, a, b) also has a finite frame upper bound, then there holds

$$M_{Sf}(t) = \frac{1}{b} M_g(t) M_g^*(t) M_f(t), \quad \text{a.e. } t, \tag{1.20}$$

$$H_{Sf}(\nu) = \frac{1}{a} H_g(\nu) H_g^*(\nu) H_f(\nu), \quad \text{a.e. } \nu. \tag{1.21}$$

For the case (1.19), (1.21) this can be found in [21], proof of Theorem 1.2.6 and Proposition 1.2.3, in the more general context of shift-invariant systems, and, in somewhat different form, in [28] or in [29].

[1] We were kindly informed by Hans G. Feichtinger that he discovered the algorithm of Section 4 independently of us, as early as 1995 and used it in numerical experiments to compute h^0 with satisfactory results.

Next, still assuming that (g, a, b) has a finite frame upper bound B, we define the analysis operator with respect to the adjoint lattice $\{(k/b, l/a) | k, l \in \mathbb{Z}\}$ by

$$m \in \boldsymbol{L}^2(\mathbb{R}) \to U_g m = \Big(\langle m, g_{k/b, l/a} \rangle \Big)_{k,l \in \mathbb{Z}}. \qquad (1.22)$$

Now there holds that U_g is a bounded linear operator (operator norm $\leq (abB)^{\frac{1}{2}}$). When (f, a, b) also has a finite upper frame bound, then there holds

$$U_{Sf} = \frac{1}{ab} U_g U_g^* U_f. \qquad (1.23)$$

These results can be found in [19], Propositions 2.7, 3.1, and also in [29] and [9] (in somewhat different form). Expression (1.23) is sometimes called the Janssen representation of the frame operator.

Finally still assuming that (g, a, b) has a finite frame upper bound B, but now with $ab = p/q \in \mathbb{Q}$ and $1 \leq p \leq q$, $\gcd(p, q) = 1$, we define the Zibulski-Zeevi matrices as

$$\Phi^g(t, \nu) = p^{-\frac{1}{2}} \Big((Z_{\frac{1}{b}} g)(t - l\frac{p}{q}, \nu + \frac{k}{p}) \Big)_{k=0,\dots,p-1; l=0,\dots,q-1} \qquad (1.24)$$

$$\Psi^g(t, \nu) = p^{-\frac{1}{2}} \Big((Z_a g)(t - k\frac{q}{p}, \nu - \frac{l}{q}) \Big)_{k=0,\dots,p-1; l=0,\dots,q-1} \qquad (1.25)$$

for a.e. $t, \nu \in \mathbb{R}$. Here we define for $\lambda > 0$ the Zak transform $Z_\lambda f$ of an $f \in \boldsymbol{L}^2(\mathbb{R})$ as

$$(Z_\lambda f)(t, \nu) = \lambda^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} f(\lambda(t - k)) e^{2\pi i k \nu}, \qquad \text{a.e. } t, \nu \in \mathbb{R}. \qquad (1.26)$$

Now there holds that $\Phi^g(t, \nu)$ and $\Psi^g(t, \nu)$ are for a.e. t, ν bounded linear mappings of $\mathbb{C}^p$ into $\mathbb{C}^q$ (matrix norms $\leq B^{\frac{1}{2}}$). When (f, a, b) also has a finite frame upper bound, then there holds

$$\Phi^{Sf}(t, \nu) = \Phi^g(t, \nu) \Big(\Phi^g(t, \nu) \Big)^* \Phi^f(t, \nu), \qquad \text{a.e. } t, \nu, \qquad (1.27)$$

$$\Psi^{Sf}(t, \nu) = \Psi^g(t, \nu) \Big(\Psi^g(t, \nu) \Big)^* \Psi^f(t, \nu), \qquad \text{a.e. } t, \nu. \qquad (1.28)$$

These results can be found in [20] and in [36].

The frame operator S is represented through the operators

$$S(t) := \frac{1}{b} M_g(t) M_g^*(t), \quad \hat{S}(\nu) := \frac{1}{a} H_g(\nu) H_g^*(\nu), \tag{1.29}$$

$$\underline{S} := \frac{1}{ab} U_g U_g^*, \tag{1.30}$$

and

$$S_1(t,\nu) := \Phi^g(t,\nu)\Big(\Phi^g(t,\nu)\Big)^*, \;\; S_2(t,\nu) := \Psi^g(t,\nu)\Big(\Psi^g(t,\nu)\Big)^* \tag{1.31}$$

in the following sense. When $A > 0, B < \infty, g \in \boldsymbol{L}^2(\mathbb{R})$, then

$$\begin{aligned} & (g,a,b) \text{ is a Gabor frame with frame bounds } A, B \\ \Longleftrightarrow \;\; & AI \le S(t) \le BI, \text{a.e.}\, t \\ \Longleftrightarrow \;\; & AI \le \hat{S}(\nu) \le BI, \text{a.e.}\, \nu \\ \Longleftrightarrow \;\; & AI \le \underline{S} \le BI \\ \Longleftrightarrow \;\; & AI \le S_1(t,\nu) \le BI, \text{a.e.}\, t,\nu \\ \Longleftrightarrow \;\; & AI \le S_2(t,\nu) \le BI, \text{a.e.}\, t,\nu. \end{aligned} \tag{1.32}$$

Here the I's denote identity operators of $\boldsymbol{\ell}^2(\mathbb{Z}), \boldsymbol{\ell}^2(\mathbb{Z}), \boldsymbol{\ell}^2(\mathbb{Z}\times\mathbb{Z}), \mathbb{C}^p$, and $\mathbb{C}^p$, respectively. See the references just given.

An important point for the developments in this paper is that the relations (1.20), (1.21), (1.23), (1.27), (1.28) can be extended as follows. Assume that (g,a,b) is a frame with frame bounds $A > 0, B < \infty$, and let φ be a function analytic in an open neighbourhood of $[A,B]$. When (f,a,b) has a finite frame upper bound, there holds

$$M_{\varphi(S)f}(t) = \varphi(S(t)) M_f(t), \qquad \text{a.e.}\, t, \tag{1.33}$$

$$H_{\varphi(S)f}(\nu) = \varphi(\hat{S}(\nu)) H_f(\nu), \qquad \text{a.e.}\, \nu, \tag{1.34}$$

$$U_{\varphi(S)f} = \varphi(\underline{S}) U_f, \tag{1.35}$$

$$\Phi^{\varphi(S)f}(t,\nu) = \varphi(S_1(t,\nu))\Phi^f(t,\nu), \quad \text{a.e.}\, t,\nu \tag{1.36}$$

$$\Psi^{\varphi(S)f}(t,\nu) = \varphi(S_2(t,\nu))\Psi^f(t,\nu), \quad \text{a.e.}\, t,\nu. \tag{1.37}$$

Indeed, the aforementioned relations can be shown to hold by iteration when φ is a polynomial, and then a standard argument in functional calculus noting (1.32) then yields the result for general φ analytic in an open neighbourhood of $[A,B]$. In the important special case that $\varphi(s) = s^\alpha$, one can do this extension explicitly by using

$$s^\alpha = \Big(\frac{B+A}{2}(1-v)\Big)^\alpha = \Big(\frac{B+A}{2}\Big)^\alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} (-v)^n, \tag{1.38}$$

where $v = 1-2s(B+A)^{-1}$ has modulus $\le (B-A)(B+A)^{-1}$ when $s \in [A,B]$.

A further result that we shall use is that for any frame (g,a,b) and any normalized tight frame (h,a,b) we have

$$\langle g, S^{-1}g\rangle = \|h\|^2 = ab. \tag{1.39}$$

See [9, 19, 29].

In the sequel we will also make use of the following result.

Lemma 1.1 *Let $T = (T_{kl})_{k,l\in\mathbb{Z}}$ be a hermitian positive definite biinfinite matrix with $c_1 I \le T \le c_2 I$. Define $T_N = (T_{kl})_{|k|,|l|\le N}$. Then $T_N^{-\frac{1}{2}} \to T^{-\frac{1}{2}}$ strongly for $N \to \infty$.*

Proof: Without loss of generality assume $c_2 < 1$, otherwise we can always set $T' = \frac{1}{c_2}T$. Note that

$$T^{-\frac{1}{2}} = \sum_{k=0}^{\infty} c_k (I-T)^k, \tag{1.40}$$

where $c_k = \binom{-\frac{1}{2}}{k}$ and $T_N^{-\frac{1}{2}} = \sum_{k=0}^{\infty} c_k (I_N - T_N)^k$. Define $T^{(K)} = \sum_{k=0}^{K} c_k (I-T)^k$ and $T_N^{(K)} = \sum_{k=0}^{K} c_k (I_N - T_N)^k$. Observe that

$$c_1 I \le \|T_N\| \le c_2 I \quad \text{uniformly in } N \tag{1.41}$$

and

$$T_N \to T \quad \text{strongly for } N \to \infty. \tag{1.42}$$

For fixed N there holds $T_N^{(K)} \to T_N^{-\frac{1}{2}}$ as $K \to \infty$. Furthermore, for each K we have that $T_N^{(K)} \to T^{(K)}$ for $N \to \infty$. This together with properties (1.41) and (1.42) implies that $T_N^{-\frac{1}{2}} \to T^{-\frac{1}{2}}$ strongly for $N \to \infty$. ❏

2 Canonical tight Gabor frames and the Fan-Hoffman inequality

Theorem 2.1 *Let (g, a, b) be a frame and let $h^0 = S^{-\frac{1}{2}}g$ be the canonically associated tight frame generating window. Then for any $h \in \boldsymbol{L}^2(\mathbb{R})$ for which (h, a, b) is a normalized tight Gabor frame there holds*

$$\|g - h^0\| \leq \|g - h\| \leq \|g + h^0\|. \tag{2.1}$$

In Subsections 2.1, 2.2, and 2.3 we will present different proofs of this inequality, making use of different representations of the Gabor frame operator.

Inequality (2.1) can be considered as a Fan-Hoffman type inequality [11]. The Fan-Hoffman inequality can be formulated as follows. Let the matrix $M \in \mathbb{C}^{m\times n}, m \geq n$ have the polar decomposition $M = UH$, where $U^*U = I_n$ and H is hermitian positive semidefinite. Then for any $m \times n$ matrix W with $W^*W = I_n$

$$\|M - U\| \leq \|M - W\| \leq \|M + U\| \tag{2.2}$$

for any unitarily invariant norm. See [11] for a proof for $M \in \mathbb{C}^{n\times n}$ and [17] for a discussion of the case $M \in \mathbb{C}^{m\times n}$.

An immediate consequence of the Fan-Hoffman inequality is the following result. Consider a frame for $\mathbb{C}^n$ consisting of m vectors $\varphi_k, k = 1, \ldots, m$ with frame operator S. Then the tight frame $(\psi_k^0)_{k=1}^m$ that minimizes

$$\sum_{k=1}^{m} \|\varphi_k - \psi_k\|^2 \tag{2.3}$$

among all tight frames $(\psi_k)_{k=1}^m$ in $\mathbb{C}^n$ is given by

$$\psi_k^0 = \frac{\operatorname{trace}(S^{\frac{1}{2}})}{n} S^{-\frac{1}{2}}\varphi_k. \tag{2.4}$$

This can be easily seen by considering the vectors φ_k as rows of an $m \times n$ matrix M and using (2.2) with the Frobenius norm, see also Section 1.5 in [33].

The case where $n = m$ and $(\varphi_k)_{k=1}^n$ is an orthonormal basis has been investigated by Löwdin in the context of quantum chemistry [16]. The solution – known under the name *Löwdin orthogonalization* – is of course a special case of (2.4).

Both, the Fan-Hoffman inequality and the Löwdin orthogonalization, have been extended to infinite dimensions, see e.g. Chapter VI in [15], or [1, 13]. These extensions usually require that the involved operators satisfy some Hilbert-Schmidt-type condition in order that

$$\sum_{k\in\mathbb{Z}} \|\varphi_k - \psi_k\|^2 \tag{2.5}$$

and the operator version of (2.2) are finite. A different notion of closeness of frames, based on work of Paley and Wiener [26], has been used in [2].

For Gabor frames the expression $\sum_{m,n\in\mathbb{Z}} \|g_{na,mb} - h_{na,mb}\|^2$ is of course not finite for $g \neq h$. But since $\|g_{na,mb} - h_{na,mb}\| = \|g - h\|$ for all n, m we can restrict ourselves to the minimization of $\|g - h\|$. The minimization result for tight Gabor frames presented in this paper can also be considered directly in the context of Löwdin orthogonalization, due to the following reason. The results in Subsection 1.1 imply that if (h, a, b) is a tight frame for $\boldsymbol{L}^2(\mathbb{R})$ then $(h, 1/b, 1/a)$ is an orthogonal basis for its closed linear span. Thus the statement that h^0 minimizes $\|g - h\|$ among all normalized tight frames (h, a, b) is equivalent to saying that h^0 minimizes $\|g - h\|$ among all h such that $(h, 1/b, 1/a)$ is an orthonormal basis for the closed linear span of $(g_{na,mb})_{n,m}$.

2.1 The problem in the time domain and in the frequency domain

2.1.1 The problem in the time domain

For the first proof of Theorem 2.1 we consider the Gabor frame operator in the time domain using Ron-Shen matrices.

We consider matrices

$$M_g(t) = \Big(g(t - na - \frac{k}{b})\Big)_{k\in\mathbb{Z}, n\in\mathbb{Z}} \qquad \text{a.e. } t, \tag{2.6}$$

with row index k and column index n. We have by functional calculus, see (1.33)

$$M_{h^0}(t) = \Big(\frac{1}{b} M_g(t) M_g^*(t)\Big)^{-\frac{1}{2}} M_g(t) \qquad \text{a.e. } t. \tag{2.7}$$

The condition of normalized tightness can be expressed equivalently as, see (1.32)

$$M_h(t) M_h^*(t) = bI \qquad \text{a.e. } t. \tag{2.8}$$

Note also that by normalized tightness $\|h\|^2 = ab$ (see (1.39)). Thus

$$\|h-g\|^2 = ab + \|g\|^2 - 2\,\mathrm{Re}\langle h, g\rangle, \tag{2.9}$$

and

$$\|h+g\|^2 = ab + \|g\|^2 + 2\,\mathrm{Re}\langle h, g\rangle, \tag{2.10}$$

and so we are to maximize $\mathrm{Re}\langle h, g\rangle$ over all h satisfying (2.8).

Let $K = 1, 2, \dots$ and consider the sections

$$M_{g,K}(t) = \Big(g(t - na - \frac{k}{b})\Big)_{|k|\le K, n\in\mathbb{Z}}. \tag{2.11}$$

$M_{g,K}$ consists of $(2K+1)$ rows and a biinfinite number of columns. Note that

$$M_{g,K}(t)M^*_{g,K}(t) = \Big(\big(M_g(t)M^*_g(t)\big)_{k,l}\Big)_{|k|\le K, |l|\le K}. \tag{2.12}$$

Thus we have that for any allowed h there holds

$$M_{h,K}(t)M^*_{h,K}(t) = b\, I_{2K+1}. \tag{2.13}$$

We are going to maximize, for fixed $t\in\mathbb{R}$, the quantity

$$\mathrm{trace}\,\big(M_K M^*_{g,K}(t)\big), \tag{2.14}$$

where $M_K = (M_{kn})_{|k|\le K, n\in\mathbb{Z}}$ is a matrix satisfying

$$M_K M_K^* = b\, I_{2K+1}. \tag{2.15}$$

Noting that the matrix in (2.12) is positive definite (as a $(2K+1)\times(2K+1)$ section of the positive definite operator $M_g(t)M^*_g(t)$ since (g, a, b) is a frame for $\boldsymbol{L}^2(\mathbb{R})$) we do a singular value decomposition of $M_{g,K}(t)$ according to

$$M_{g,K}(t) = \sum_{j=0}^{2K} \sigma_j u_j v_j^H, \tag{2.16}$$

where

$$\begin{aligned} v_j &= (v_{jn})^T_{n\in\mathbb{Z}} \quad \in \boldsymbol{\ell}^2(\mathbb{Z}) \\ u_j &= [u_{j,-K}, \dots, u_{j,K}]^T \quad \in \mathbb{C}^{2K+1} \end{aligned}$$

are orthonormal, σ_j^2 is the j-th eigenvalue of $M_{g,K}(t)M^*_{g,K}(t)$ and u_j is the corresponding eigenvector. Thus

$$\big(M_{g,K}(t)M^*_{g,K}(t)\big)u_j = \sigma_j^2 u_j \tag{2.17}$$

and

$$v_j = \frac{1}{\sigma_j}M^*_{g,K}(t)u_j. \tag{2.18}$$

Now we compute

$$\begin{aligned}
\big|\operatorname{trace}\big(M_K M^*_{g,K}(t)\big)\big| &= \Big|\operatorname{trace}\big(M_K \sum_{j=0}^{2K}\sigma_j v_j u_j^H\big)\Big| \\
= \Big|\operatorname{trace}\big(\sum_{j=0}^{2K}\sigma_j M_k v_j u_j^H\big)\Big| &= \Big|\operatorname{trace}\big(\sum_{j=0}^{2K}\sigma_j u_j^H M_k v_j\big)\Big| \\
= \Big|\sum_{j=0}^{2K}\sigma_j\langle M_K v_j, u_j\rangle\Big| &\le \sum_{j=0}^{2K}\sigma_j\|M_K v_j\|\|u_j\| \\
\le b^{\frac12}\sum_{j=0}^{2K}\sigma_j &= b^{\frac12}\operatorname{trace}\Big(M_{g,K}(t)M^*_{g,K}(t)\Big)^{\frac12}.
\end{aligned} \tag{2.19}$$

Taking

$$M_K(t) = \Big(\frac{1}{b}M_{g,K}(t)M^*_{g,K}(t)\Big)^{-\frac12}M_{g,K}(t) \tag{2.20}$$

we also see that the maximum value

$$b^{\frac12}\operatorname{trace}\Big(M_{g,K}(t)M^*_{g,K}(t)\Big)^{\frac12} \tag{2.21}$$

is assumed by $M_K(t)$ in (2.20).

Next we observe that for an allowed h we have

$$\operatorname{trace}[M_{h,K}(t)M^*_{g,K}(t)] = \sum_{k=-K}^{K}\sum_{n=-\infty}^{\infty} h(t-na-\frac{k}{b})g^*(t-na-\frac{k}{b}). \tag{2.22}$$

Taking the average over k and integrating over $t\in[0,a)$, we thus see that

$$\begin{aligned}
\langle h, g\rangle &= \tfrac{1}{2K+1}\int_0^a \textstyle\sum_{k=-K}^{K}\sum_{n=-\infty}^{\infty} h(t-na-\frac{k}{b})g^*(t-na-\frac{k}{b})\,dt \\
&= \tfrac{1}{2K+1}\int_0^a \operatorname{trace}[M_{h,K}(t)M^*_{g,K}(t)]\,dt.
\end{aligned} \tag{2.23}$$

The right-hand side of (2.23) has modulus $\leq$ what one would obtain by replacing $M_{h,K}(t)$ by $M_K(t)$ in (2.23). Due to Lemma 1.1 we have that

$$\lim_{K\to\infty} M_K(t) = \lim_{K\to\infty} \Big(\frac{1}{b} M_{g,K}(t) M^*_{g,K}(t)\Big)^{-\frac{1}{2}} M_{g,K}(t) = M_{h^0}(t) \tag{2.24}$$

in the strong operator topology. This implies that

$$\lim_{K\to\infty} \frac{1}{2K+1} \int_0^a \mathrm{trace}[M_K(t) M^*_{g,K}(t)]dt = \langle h^0, g\rangle, \tag{2.25}$$

and the proof of inequality (2.1) is complete.

Actually we have shown more than (2.1). Consider the a-periodic $\boldsymbol{L}^\infty$-functions

$$f(t) = \sum_{n=-\infty}^{\infty} h(t-na)g^*(t-na) = \langle h, g\rangle_a(t), \tag{2.26}$$

where we have used the bracket product notation $\langle\cdot,\cdot\rangle_a$ as in [7]. That $f \in \boldsymbol{L}^\infty$ follows from the frame bound conditions, together with the considerations in Subsection 1.1. Consider the case that $ab = p/q$ with integer $p, q, 1 \leq p \leq q, \gcd(p,q) = 1$. Then for a.e. t there holds (see the right-hand side of (2.22))

$$\begin{aligned} &\frac{1}{2K+1} \sum_{k=-K}^{K} f(t-k/b) \qquad (2.27)\\ &= \frac{1}{2K+1} \sum_{k=-K}^{K} f(t-\frac{q}{p}a) \\ &\underset{[k=lp+r]}{=} \frac{1}{2K+1} \sum_{r=0}^{p-1} \sum_{l;-K\leq lp+r\leq K} f(t-lqa-r\frac{q}{p}a) \\ &\underset{[\text{using the } a\text{-periodicity of } f]}{=} \frac{1}{2K+1} \sum_{r=0}^{p-1} \#_{l;-K\leq lp+r\leq K} \cdot f(t-r\frac{q}{p}a) \\ &\approx \frac{1}{2K+1} \cdot \frac{2K+1}{p} \sum_{r=0}^{p-1} f(t-r/b) \end{aligned}$$

$$= \frac{1}{p} \sum_{r=0}^{p-1} \langle h, g \rangle_a (t - r/b).$$

The $\approx$ in (2.27) becomes $=$ on letting $K \to \infty$. Thus there holds, more precisely, that for a.e. t

$$\frac{1}{p} \sum_{r=0}^{p-1} \langle h, g \rangle_a (t - r/b) \tag{2.28}$$

is maximized among all normalized tight (h, a, b) by h^0. And then, of course, integrating over $t \in [0, 1/b)$ gives the result.

In case that $ab \notin \mathbb{Q}$, it can be shown that

$$\frac{1}{2K+1} \sum_{k=-K}^{K} f(\cdot - k/b) \to \langle h, g \rangle \tag{2.29}$$

in the mean square sense when $K \to \infty$ (here f is as in (2.26)). Hence in this case we do not get a more precise result of the type given above for rational ab.

2.1.2 The problem in the frequency domain

We now present a proof of Theorem 2.1 where we consider the Gabor frame operator in the frequency domain, using frequency domain Ron-Shen matrices. We thus consider

$$H_g(\nu) = \Big(\hat{g}(\nu - mb - l/a) \Big)_{l \in \mathbb{Z}, m \in \mathbb{Z}}, \quad \text{a.e. } \nu, \tag{2.30}$$

and we have by functional calculus

$$H_{h^0}(\nu) = \Big(\frac{1}{a} H_g(\nu) H_g^*(\nu) \Big)^{-\frac{1}{2}} H_g(\nu), \quad \text{a.e. } \nu, \tag{2.31}$$

while the condition of normalized tightness can be expressed equivalently as

$$H_h(\nu) H_h^*(\nu) = aI, \quad \text{a.e. } \nu. \tag{2.32}$$

We therefore find ourselves in a situation that is similar to what we had in Subsection 2.1.1 in all respects, except that the Ron-Shen matrices involve now the Fourier transform of the windows, rather than the windows themselves. Accordingly, we know that $\|\hat{h}-\hat{g}\|^2$ is minimal among all h such that (h,a,b) is normalized tight for $h=h^0$. By Parseval's theorem we have that $\|\hat{h}-\hat{g}\|^2=\|h-g\|^2$, and the proof is complete. We also get a frequency domain sharpening of the result for $ab\in\mathbb{Q}$ in a similar manner as the one obtained at the end of Subsection 2.1.1. This latter sharpening reads that for a.e. ν

$$\frac{1}{p}\sum_{s=0}^{p-1}\langle\hat{h},\hat{g}\rangle_b(\nu-s/a) \tag{2.33}$$

is maximal among all h such that (h,a,b) is normalized tight for $h=h^0$.

The frequency domain result can be generalized to some extent to shift-invariant systems

$$(g_{n,m})_{n,m\in\mathbb{Z}}=\Big(g_m(\cdot-na)\Big)_{n,m\in\mathbb{Z}}, \tag{2.34}$$

where $g_m\in\boldsymbol{L}^2(\mathbb{R})$ and $a>0$. Following the presentation of the Ron-Shen theory [28] given in [21], Subsection 1.2, we now consider the frame operator

$$Sf=\sum_{n,m}\langle f,g_{n,m}\rangle g_{n,m},\qquad f\in\boldsymbol{L}^2(\mathbb{R}), \tag{2.35}$$

and the Ron-Shen matrices

$$H_g(\nu)=\Big(g_m(\nu-l/a)\Big)_{l\in\mathbb{Z},m\in\mathbb{Z}},\quad \text{a.e.}\,\nu. \tag{2.36}$$

There holds for $A>0, B<\infty$ that

$$AI\le S\le BI\Leftrightarrow AI\le\frac{1}{a}H_g(\nu)H_g^*(\nu)\le BI,\ \text{ a.e.}\,\nu. \tag{2.37}$$

The canonical dual system $(\gamma^0_{nm})_{n,m\in\mathbb{Z}}$ and the canonically associated tight system $(h^0{}_{n,m})_{n,m\in\mathbb{Z}}$ are given by

$$\Big(\gamma^0_m(\cdot-na)\Big)_{n,m\in\mathbb{Z}},\quad\Big(h^0{}_m(\cdot-na)\Big)_{n,m\in\mathbb{Z}} \tag{2.38}$$

with $\gamma^0_m=S^{-1}g_m, h^0_m=S^{-\frac{1}{2}}g_m$. Here we note that S in (2.35) commutes with all shift operators $f\to f(\cdot-na), n\in\mathbb{Z}$. The conditions of duality of $(\gamma_{nm})_{n,m\in\mathbb{Z}}$ and normalized tightness of $(h_{nm})_{n,m\in\mathbb{Z}}$ can be equivalently expressed as

$$H_g(\nu)H_\gamma^*(\nu) = aI, \quad H_h(\nu)H_h^*(\nu) = aI. \tag{2.39}$$

We have a functional calculus in terms of the matrices $a^{-1}H_g(\nu)H_g^*(\nu)$ by the formula

$$H_{\varphi(S)g}(\nu) = \varphi\Big(\frac{1}{a}H_g(\nu)H_g^*(\nu)\Big)H_g(\nu), \quad \text{a.e.}\, \nu, \tag{2.40}$$

where φ is a function analytic in an open neighbourhood of $[A, B]$. We have set here

$$\varphi(S)g = \Big((\varphi(S)g_m)(\cdot - na)\Big)_{n,m\in\mathbb{Z}}, \quad \text{a.e.}\, \nu. \tag{2.41}$$

In particular there holds

$$H_{h^0}(\nu) = \Big(\frac{1}{a}H_g(\nu)H_g^*(\nu)\Big)^{-\frac{1}{2}}H_g(\nu), \ \text{a.e.}\, \nu. \tag{2.42}$$

Now the developments as given in Subsection 2.1.1 can be mimicked to a large extent. Considering truncated matrices $H_{g,K}$ as in (2.11), and denoting the Frobenius norm by $\|.\|_F$, the minimization of (for a.e. ν)

$$\|H_K - H_{g,K}(\nu)\|_F^2 = \sum_{|l|\le K}\sum_{m=-\infty}^{\infty} |h_{lm} - \hat{g}(\nu - l/a)|^2 \tag{2.43}$$

over all $(2K+1)\times\infty$ matrices $H_K = (h_{lm})_{|l|\le K, m\in\mathbb{Z}}$ satisfying

$$H_K H_K^* = aI_{2K+1} \tag{2.44}$$

yields that the minimizing H_K is given by

$$H_K = \Big(\frac{1}{a}H_{g,K}(\nu)H_{g,K}^*(\nu)\Big)^{-\frac{1}{2}}H_{g,K}(\nu). \tag{2.45}$$

And, again by Lemma 1.1, this $H_K \to H_{h^0}$ as $K\to\infty$ in the strong operator topology.

2.2 The problem in the time-frequency domain

We next present a proof of Theorem 2.1 where we consider the Gabor frame operator in the time-frequency domain, using analysis operators U_g as defined in (1.22) based on the adjoint lattice $\{(k/b, l/a) | k, l \in \mathbb{Z}\}$.

Note that $U_g U_g^*$ has the matrix representation

$$U_g U_g^* = \Big(\langle g_{k'/b,l'/a}, g_{k/b,l/a} \rangle \Big)_{k,l \in \mathbb{Z}; k',l' \in \mathbb{Z}}. \tag{2.46}$$

Normalized tightness is equivalently expressed as

$$U_h U_h^* = ab\, I. \tag{2.47}$$

The canonical tight frame generating h^0 is given by

$$U_{h^0} = \Big(\frac{1}{ab} U_g U_g^* \Big)^{-\frac{1}{2}} U_g \tag{2.48}$$

by functional calculus. We also note that

$$U_h U_g^* = \Big(\langle g_{k'/b,l'/a}, h_{k/b,l/a} \rangle \Big)_{k,l \in \mathbb{Z}; k',l' \in \mathbb{Z}}. \tag{2.49}$$

Now consider the sections

$$U_{g,K,L} : m \in \boldsymbol{L}^2(\mathbb{R}) \to \Big(\langle m, g_{k/b,l/a} \rangle \Big)_{|k| \le K, |l| \le L} \in \mathbb{C}^{2K+1} \times \mathbb{C}^{2L+1} \tag{2.50}$$

with adjoints

$$U_{g,K,L}^* : \underline{c} \in \mathbb{C}^{2K+1} \times \mathbb{C}^{2L+1} \to \sum_{\substack{|k| \le K \\ |l| \le L}} c_{kl} g_{k/b,l/a} \tag{2.51}$$

and

$$U_{g,K,L} U_{g,K,L}^* = \Big((U_{g,K,L} U_{g,K,L}^*)_{k,l;k'l'} \Big)_{|k| \le K, |l| \le L; |k'| \le K, |l'| \le L}. \tag{2.52}$$

This we also do with allowed h's, see (2.47), and we thus see that

$$U_{h,K,L} U_{h,K,L}^* = ab\, I_{(2K+1)(2L+1)}. \tag{2.53}$$

Now consider the maximization of

$$\left| \operatorname{trace} \left[U_{K,L} U_{g,K,L}^* \right] \right| \tag{2.54}$$

over all

$$U_{K,L} = \Big(U_{kl;k'l'} \Big)_{|k| \le K, |l| \le L; |k'| \le K, |l'| \le L} \tag{2.55}$$

such that $U_{K,L} U_{K,L}^* = abI_{(2K+1)(2L+1)}$. This yields the maximum value

$$\operatorname{trace} \left[(ab)^{\frac{1}{2}} \Big(U_{g,K,L} U_{g,K,L}^* \Big)^{\frac{1}{2}} \right], \tag{2.56}$$

assumed by

$$U_{K,L} = \Big(\frac{1}{ab} U_{g,K,L} U_{g,K,L}^* \Big)^{-\frac{1}{2}} U_{g,K,L}. \tag{2.57}$$

Note that from (2.52)

$$\langle g, h \rangle = \frac{1}{(2K+1)(2L+1)} \operatorname{trace} \left[U_{h,K,L} U_{g,K,L}^* \right]. \tag{2.58}$$

By Lemma 1.1 $U_{K,L}$ in (2.57) converges strongly to $U_{h^0,K,L}$, hence

$$\langle g, h^0 \rangle = \lim_{K,L \to \infty} \frac{1}{(2K+1)(2L+1)} \operatorname{trace} \left[U_{K,L} U_{g,K,L}^* \right]. \tag{2.59}$$

Therefore, $|\langle g, h \rangle| \le \langle g, h^0 \rangle$.

2.3 The problem in the Zak transform domain

We proceed by presenting a proof of Theorem 2.1 by considering the Gabor frame operator in the Zak transform domain, using Zibulski-Zeevi matrices. The proof in this section is the simplest among all presented proofs, however it only applies for rational oversampling.

Assume $ab = p/q, 1 \le p \le q, \gcd(p,q) = 1$. By [21], Sec. 1.5, (1.5.11) we have

$$\langle h, g \rangle = p \int_0^{q^{-1}} \int_0^{p^{-1}} \sum_{k=0}^{p-1} \sum_{l=0}^{q-1} \Phi_{kl}^h(t,\nu) \big(\Phi_{kl}^g(t,\nu) \big)^* dt \, d\nu$$

$$= p \int_0^{q^{-1}} \int_0^{p^{-1}} \text{trace}\left[\Phi^h(t,\nu)\big(\Phi^g(t,\nu)\big)^*\right] dt\, d\nu.$$

Here $\Phi^f(t,\nu)$ is the Zibulski-Zeevi matrix as defined in (1.24). The condition of normalized tightness is expressed equivalently as

$$\Phi^h(t,\nu)\big(\Phi^h(t,\nu)\big)^* = I_p \qquad \text{a.e. } t,\nu. \tag{2.60}$$

The maximum of

$$\left|\text{trace}\left[\Phi\big(\Phi^g(t,\nu)\big)^*\right]\right| \tag{2.61}$$

over all $p \times q$ matrices Φ with $\Phi\Phi^* = I_p$ is equal to

$$\text{trace}\left[\Big(\Phi^g(t,\nu)\big(\Phi^g(t,\nu)\big)^*\Big)^{\frac{1}{2}}\right] \tag{2.62}$$

and assumed by

$$\Phi(t,\nu) := \Big(\Phi^g(t,\nu)\big(\Phi^g(t,\nu)\big)^*\Big)^{-\frac{1}{2}} \Phi^g(t,\nu). \tag{2.63}$$

The right-hand side of (2.63) equals $\Phi_{h^0}(t,\nu)$ by functional calculus for Zibulski-Zeevi matrices.

Hence we see that h^0 indeed maximizes $\langle h, g\rangle$, and that it does so in a pointwise manner in terms of Zibulski-Zeevi matrices.

The proof using the matrices Ψ^g as defined in (1.25) is almost identical to the one for the matrices Φ^g and is therefore left to the reader.

3 A Wiener-Levy theorem for Gabor frames

In [35, 31, 5, 32] it has been investigated under which conditions the canonical tight window inherits the decay properties of the window g. Loosely spoken h^0 inherits the decay properties of g whenever this is true for γ^0. We give a precise mathematical formulation of this observation. We start with the following result from the theory of Banach algebras [14].

Proposition 3.1 *Let $\mathcal{B}$ be a Banach algebra, and x an element of $\mathcal{B}$ with real-valued spectrum $\sigma(x) > 0$. Then there exists in $\mathcal{B}$ a unique square root of x with real-valued positive spectrum.*

This implies the following: Assume that the properties of g imply that S belongs to some Banach algebra $\mathcal{B}$. Whenever $S^{-1} \in \mathcal{B}$, it follows that $S^{-\frac{1}{2}} \in \mathcal{B}$.

Actually for rational values of ab a more general result holds.

Theorem 3.2 *Assume that $ab = p/q, 1 \leq p, q \in \mathbb{Z}, \gcd(p,q) = 1$. Assume that (g,a,b) is a frame, where g satisfies Tolimieri and Orr's condition A (cf. [34])*

$$\sum_{k,l} |\langle g, g_{k/b,l/a}\rangle| < \infty. \tag{3.1}$$

Let $\mathcal{W}$ be a time-frequency shift-invariant Banach space of windows w such that (w,a,b) has a finite frame upper bound when $w \in \mathcal{W}$. Finally, let φ be analytic in an open neighbourhood of $[A,B]$, where $A > 0, B < \infty$ are the frame bounds of (g,a,b). Then $\varphi(S)w \in \mathcal{W}$ for any $w \in \mathcal{W}$.

For the proof of Theorem 3.2 we first need a matrix-version of the classical Wiener-Levy theorem. We introduce the following

Definition 3.3 *A set U of functions $f(t) = \sum_{k\in\mathbb{Z}} a_k(f)e^{2\pi ikt}$ is said to have an absolutely uniformly convergent Fourier series if for every $\varepsilon > 0$ there is a $K > 0$ such that*

$$\sum_{|k|\geq K} |a_k(f)| \leq \varepsilon, \qquad \textit{for all } f \in U. \tag{3.2}$$

Proposition 3.4 *Let $H(t)$ be a periodic hermitian matrix-valued function with absolutely convergent Fourier series, i.e., $H(t) = \sum_{k\in\mathbb{Z}} H_k e^{2\pi ikt}, H_k = ((H_k)_{m,n})_{m,n=1}^N$ and*

$$\|H\|_{\mathcal{F}} := \sum_{m,n=1}^{N} \sum_{k\in\mathbb{Z}} |(H_k)_{m,n}| < \infty. \tag{3.3}$$

Let $A, B \in \mathbb{R}, A < B$ be such that all eigenvalues of all $H(t)$ are in $[A,B]$. If the function $\varphi(z)$ is analytic in an open neighbourhood of $[A,B]$ then $\varphi(H(t))$ also has an absolutely convergent Fourier series.

Proof: We have the Dunford representation [10]

$$\varphi(H(t)) = \frac{1}{2\pi i} \oint_{\Gamma} \varphi(z)\big(zI - H(t)\big)^{-1}\, dz, \tag{3.4}$$

where Γ is a closed contour containing $[A, B]$ in its interior. We write

$$\big(zI - H(t)\big)^{-1} = \det\big(zI - H(t)\big)^{-1} \operatorname{adj}\big(zI - H(t)\big). \tag{3.5}$$

The entries of $zI - H(t)$ have Fourier series absolutely uniformly convergent in $z \in \Gamma$. Hence it follows easily that $\operatorname{adj}\big(zI - H(t)\big)$ and $\det\big(zI - H(t)\big)$, as sums and products of the entries of $zI - H(t)$, have Fourier series that are absolutely uniformly convergent in $z \in \Gamma$. Furthermore, there are constants $C_1 > 0, C_2 < \infty$ such that

$$C_1 \leq \big|\det\big(zI - H(t)\big)\big| \leq C_2, \quad z \in \Gamma, t \in \mathbb{R}. \tag{3.6}$$

Thus by the uniform Wiener $1/f$-theorem [6] we have that $\det\big(zI - H(t)\big)^{-1}$ has a Fourier series that is absolutely uniformly convergent in $z \in \Gamma$. This implies that $\|(zI - H)^{-1}\|_{\mathcal{F}}$ is uniformly bounded in $z \in \Gamma$, by C, say. Hence

$$\|\varphi(H)\|_{\mathcal{F}} \leq \frac{1}{2\pi} C \oint_{\Gamma} |\varphi(z)||dz| < \infty, \tag{3.7}$$

where the boundedness of the integral in (3.7) follows from the continuity of φ. The proof is complete. ❏

The knowledgeable reader will have no difficulties to extend the argument concerning the uniform Wiener $1/f$-theorem given in [6] to the case that we are considering functions of two variables with absolutely uniformly convergent Fourier series. With the two-dimensional version of the uniform Wiener $1/f$-theorem at hand it is then easy to extend Proposition 3.4 to bivariate hermitian matrix-valued functions $H(t, \nu)$.

Proof of Theorem 3.2: Let $w \in \mathcal{W}$. By functional calculus we have

$$\varphi(S)w = \sum_{k,l} \Big(\varphi(\frac{1}{ab} U_g U_g^*)\Big)^*_{0,0;k,l} w_{k/b,l/a}, \tag{3.8}$$

with convergence of the right-hand series at least in weak $\boldsymbol{L}^2$-sense. It is therefore sufficient to show that

$$\sum_{k,l} \Big|\Big(\varphi(\frac{1}{ab} U_g U_g^*)\Big)^*_{0,0;k,l}\Big| < \infty \tag{3.9}$$

since $w_{k/b,l/a} \in \mathcal{W}$ for all $k, l \in \mathbb{Z}$. Formula (3.8) follows by considering for $m \in \boldsymbol{L}^2(\mathbb{R})$ the $(k = 0, l = 0)$-entry of $U_{\varphi(S)w} m$ per definition (1.22) while using (1.35) and the definition (1.30) of $\underline{S}$ for expressing it in terms of $\varphi(\underline{S})$.

To show (3.9) we take $w = g$ in (3.8). By biorthogonality we have

$$\Big(\varphi(\frac{1}{ab}U_g U_g^*)\Big)^*_{0,0;k,l} = \frac{1}{ab}\langle \varphi(S)g, \gamma^0_{k/b,l/a}\rangle. \tag{3.10}$$

From Prop. 1.1 in [20] we have for any $f, h \in \boldsymbol{L}^2(\mathbb{R})$

$$\sum_{k,l} |\langle f, h_{k/b,l/a}\rangle| < \infty \Longleftrightarrow \tag{3.11}$$

$$\Phi^f(t,\nu)\big(\Phi^h(t,\nu)\big)^* \text{ has an abs. conv. Fourier series.}$$

Now take

$$f = \varphi(S)g, \qquad h = \gamma^0 = S^{-1}g. \tag{3.12}$$

By functional calculus in the Zak transform domain, we have

$$\begin{aligned}\Phi^f(t,\nu)\big(\Phi^h(t,\nu)\big)^* = \qquad\qquad\qquad\qquad \qquad\qquad\qquad\qquad (3.13)\\ = \varphi\Big[\Phi^g(t,\nu)\big(\Phi^g(t,\nu)\big)^*\Big]\Phi^g(t,\nu)\Big(\big[\Phi^g(t,\nu)\big(\Phi^g(t,\nu)\big)^*\big]^{-1}\Phi^g(t,\nu)\Big)^*\\ = \varphi\Big[\Phi^g(t,\nu)\big(\Phi^g(t,\nu)\big)^*\Big].\end{aligned}$$

By the bivariate version of Proposition 3.4 we have that the right-hand side of (3.13) has an absolutely convergent Fourier series. Then by (3.10), (3.11), (3.12) we see that (3.9) holds, and the proof is complete. ❏

Example: Assume that $ab = p/q, 1 \le p, q \in \mathbb{Z}, \gcd(p,q) = 1$. If g is in the Feichtinger-algebra $\boldsymbol{S}_0$ [12], then by Theorem 3.2 $h^0 \in \boldsymbol{S}_0$.

4 A Newton method for the computation of tight Gabor frames

For applications and design purposes it is useful to have a fast algorithm for computing the canonical tight window h^0. It is clear that in the case of integer oversampling this can be done via the Zak transform. However for non-integer oversampling the frame operator can at best only be block-diagonalized. This means that at some point we would have to compute the inverse square root of a matrix, which is computationally much more expensive than the inversion of a matrix. In this section we present a Newton-type algorithm that allows a very efficient calculation of h^0.

Newton's method is well-known in the context of computing matrix-sign functions and square roots of finite-dimensional matrices [18, 24]. Let M be an $n \times n$ matrix with positive real-valued eigenvalues and polar decomposition $M = UH$. Set $M_0 = M, X_0 = M$, and define the iterations

$$M_{k+1} = \frac{1}{2}(M_k + [M_k^*]^{-1}M), \qquad k = 0, 1, \dots \tag{4.1}$$

$$X_{k+1} = \frac{1}{2}(X_k + [X_k^*]^{-1}), \qquad k = 0, 1, \dots. \tag{4.2}$$

Then M_k converges to $M^{\frac{1}{2}}$ and X_k converges to U, see [18, 24].

In Section 2 we have seen that h^0 minimizes $\|g - h\|$ among all windows h for which (h, a, b) is a normalized tight frame. Clearly in the same way h^0 minimizes $\|\gamma^0 - h\|$. Also recall that γ^0 minimizes

$$\left\| \frac{g}{\|g\|} - \frac{\gamma}{\|\gamma\|} \right\| \tag{4.3}$$

among all dual windows γ. This suggests to compute h^0 iteratively by setting $g_0 = g$ and defining the iteration

$$g_{k+1} = \frac{1}{2}\Big(\frac{g_k}{\|g_k\|} + \frac{\gamma_k^0}{\|\gamma_k^0\|}\Big), \qquad k = 0, 1, \dots, \tag{4.4}$$

where γ_k^0 is the dual window associated with the frame (g_k, a, b). Accordingly, we denote the frame bounds of the system (g_k, a, b) by A_k and B_k. Iteration (4.2) can be interpreted as a scaled Newton iteration. The proof of convergence of Newton's method as described in (4.1) and (4.2) relies on finite-dimensional methods (see [17]), which cannot be extended straightforwardly to infinite matrices or operators.

4.1 Convergence of the algorithm

In this section we analyze the convergence behavior of the iteration (4.4). We need the following

Lemma 4.1 *Assume that (g, a, b) is a frame, and let φ be a function analytic in an open neighbourhood of $[A, B]$, where A and B are the optimal frame bounds for (g, a, b), and assume that $\varphi(s) > 0, s \in [A, B]$. Then $(\varphi(S)g, a, b)$ is a frame with*

$$\textit{lower frame bound } \geq \min_{A \leq s \leq B} s\varphi^2(s), \tag{4.5}$$

$$\textit{upper frame bound} \leq \max_{A \leq s \leq B} s\varphi^2(s), \tag{4.6}$$

and the inequalities can be strict. Furthermore (g, a, b) *and* $(\varphi(S)g, a, b)$ *have the same canonically associated tight frame generating window, viz.* h^0*.*

Proof: We have for any $f \in \boldsymbol{L}^2(\mathbb{R})$ that

$$\sum_{n,m} \langle f, (\varphi(S)g)_{nm} \rangle (\varphi(S)g)_{nm} = \varphi(S)[\sum_{n,m} \langle (\varphi^*(S)f, g_{nm} \rangle] = \varphi(S)S\varphi^*(S)f. \tag{4.7}$$

Here we have used that S, and hence $\varphi(S)$, commutes with all relevant time-frequency shifts. Thus the frame operator corresponding to $(\varphi(S)g, a, b)$ is given by $\varphi(S)S\varphi^*(S)$. By the spectral mapping theorem we have that

$$\sigma[\varphi(S)S\varphi^*(S)] = \{\varphi(s)s\varphi^*(s) = s|\varphi(s)|^2 \; \big| s \in \sigma(S)\}, \tag{4.8}$$

and this gives the first part of the result.

Next we compute the tight frame generating window h^φ canonically associated to $(\varphi(S)g, a, b)$ according to

$$h^\varphi = \Big(\varphi(S)S\varphi(S)\Big)^{-\frac{1}{2}} \varphi(S)g = S^{-\frac{1}{2}}(\varphi^2(S))^{-\frac{1}{2}}\varphi(S)g = S^{-\frac{1}{2}}g = h^0. \tag{4.9}$$

Here we have used that $\varphi(s) > 0, s \in [A, B]$ so that $\varphi^*(S) = \varphi(S)$ and $\varphi(S) > 0$ so that $\Big((\varphi(S))^2\Big)^{-\frac{1}{2}} = \varphi(S)$. ❑

Theorem 4.2 *Let* (g, a, b) *be a frame for* $\boldsymbol{L}^2(\mathbb{R})$ *with optimal frame bounds* A, B *and canonical tight window* h^0*. Set* $g_0 = g$ *and define*

$$g_k = \frac{1}{2}\Big(\frac{g_{k-1}}{||g_{k-1}||} + \frac{\gamma^0_{k-1}}{||\gamma^0_{k-1}||}\Big), \qquad k = 1, 2, \ldots. \tag{4.10}$$

then g_k *converges quadratically to* $\frac{1}{\sqrt{ab}}h^0$*.*

Proof: Set

$$g_1 = \varphi(S)g \, : \, \varphi(x) = \alpha + \beta x^{-1}, \tag{4.11}$$

where for simplicity we have denoted $\alpha_k = \frac{1}{2||g_k||}, \beta_k = \frac{1}{2||\gamma^0_k||}$ and $\alpha = \alpha_0, \beta = \beta_0$. Note that

$$A \leq \frac{\beta}{\alpha} = \frac{||g||}{||S^{-1}g||} \leq B. \tag{4.12}$$

Using Lemma 4.1 we obtain that

$$A_1 \geq \min_{x\in[A,B]}(\alpha^2 x + 2\alpha\beta + \beta^2 x^{-1}), \tag{4.13}$$

$$B_1 \leq \max_{x\in[A,B]}(\alpha^2 x + 2\alpha\beta + \beta^2 x^{-1}). \tag{4.14}$$

As to A_1, we note that $\alpha^2 x + 2\alpha\beta + \beta^2 x^{-1}$ assumes its minimum value $4\alpha\beta$ at $x = \beta/\alpha \in [A, B]$, see (4.12). Hence $A_1 = 4\alpha\beta$, and we furthermore note that

$$4\alpha\beta = \frac{1}{\|g\|\|S^{-1}g\|} \leq \frac{1}{|\langle g, S^{-1}g\rangle|} = \frac{1}{ab}, \tag{4.15}$$

see (1.39). Hence $A_1 \leq \frac{4}{ab}$, no matter what g is. As to B_1 we note that $\alpha^2 x + 2\alpha\beta + \beta^2 x^{-1}$ is strictly convex on $[A, B]$ and assumes its minimum at $x = \beta/\alpha$. Hence

$$B_1 = \max_{x=A \text{ or } B}(\alpha^2 x + 2\alpha\beta + \beta^2 x^{-1}). \tag{4.16}$$

Therefore we find that

$$\begin{aligned} \frac{A_1}{B_1} &\geq \min\Big\{\frac{4\alpha\beta}{\alpha^2 A + 2\alpha\beta + \beta^2 A^{-1}}, \frac{4\alpha\beta}{\alpha^2 B + 2\alpha\beta + \beta^2 B^{-1}}\Big\} \\ &= \min\Big\{\frac{4A\frac{\beta}{\alpha}}{(A + \frac{\beta}{\alpha})^2}, \frac{4B\frac{\beta}{\alpha}}{(B + \frac{\beta}{\alpha})^2}\Big\}. \end{aligned} \tag{4.17}$$

The graph in Figure 1 shows the two functions $f_1(y) = \frac{4Ay}{(A+y)^2}$, $f_2(y) = \frac{4By}{(B+y)^2}$ for $y = \frac{\beta}{\alpha}$. Assuming that $\sqrt{AB}$ is in $\sigma(S)$ it is obvious that

$$\frac{4AB}{(A+B)^2} \leq \frac{A_1}{B_1} \leq \frac{4\sqrt{AB}}{(\sqrt{A} + \sqrt{B})^2}\,. \tag{4.18}$$

In general, the right-hand-side can be as large as 1; this occurs for instance when $\sigma(S) = \{A, B\}$ and $\alpha = (AB)^{-\frac{1}{4}}, \beta = (AB)^{\frac{1}{4}}$.

The above argument applies to any g_{k-1} as in (4.10). In particular, we have for $k = 1, 2, \ldots$

$$1 \geq \frac{A_k}{B_k} \geq \frac{4A_{k-1}B_{k-1}}{(A_{k-1} + B_{k-1})^2} = \frac{4\frac{A_{k-1}}{B_{k-1}}}{(1 + \frac{A_{k-1}}{B_{k-1}})^2} \tag{4.19}$$

Consider the recursion

$$C_k = \frac{4C_{k-1}}{(1 + C_{k-1})^2}, \qquad k = 1, 2, \ldots \tag{4.20}$$

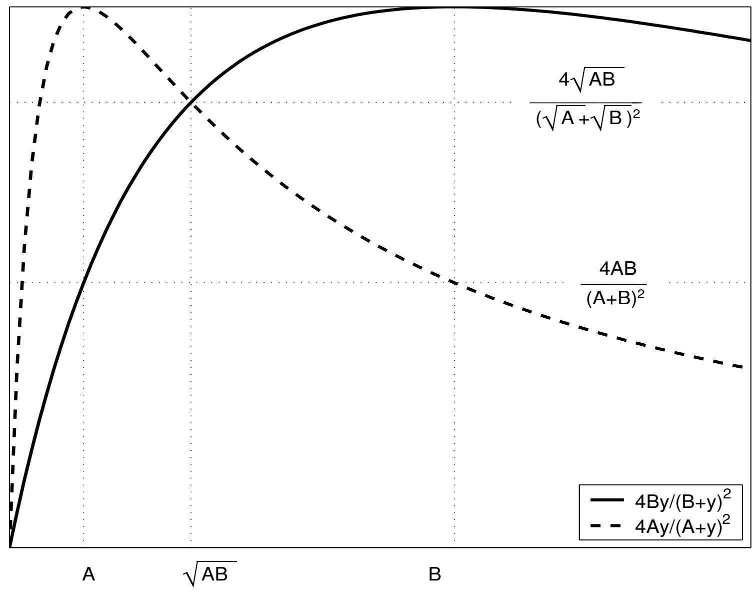

Figure 1:

with $C_0 \in (0,1)$. The recursion (4.20) converges *quadratically* to 1, since the function

$$f(y) = \frac{4y}{(1+y)^2}, \qquad y \in (0,1] \tag{4.21}$$

increases on $(0,1]$, satisfies $f(0) = 0, f(1) = 1,\ f(y) > y, y \in (0,1)$, and $f'(1) = 0, f''(1) = -\frac{1}{2}$. Evidently when we take $C_0 = A/B$, we obtain that

$$1 \geq \frac{A_k}{B_k} \geq C_k, \qquad k = 1, 2, \ldots \tag{4.22}$$

and $A_k/B_k \to 1$, quadratically.

It follows from (4.10) and (4.11) that the g_k are all of the form $\varphi_k(S)g$. Hence by Lemma 4.1 they all have h^0 as their canonically associated tight frame generator. At the same time, denoting by S_k the frame operator of the system (g_k, a, b), we have that this canonically associated tight frame generator is given by $S_k^{-\frac{1}{2}} g_k$. Hence

$$h^0 = S_k^{-\frac{1}{2}} g_k, \qquad k = 1, 2, \ldots. \tag{4.23}$$

Since $A_k/B_k \to 1$, we easily see from

$$g_k = \frac{1}{2}\Big(\frac{g_{k-1}}{\|g_{k-1}\|} + \frac{S_{k-1}^{-1} g_{k-1}}{\|S_{k-1}^{-1} g_{k-1}\|}\Big) \tag{4.24}$$

that $\|g_k\| \to 1$ as $k \to \infty$. Also $\|h^0\| = \sqrt{ab}$ by (1.39), and it then follows from (4.23) and the fact that $A_k/B_k \to 1$ that

$$A_k, B_k \to \frac{1}{ab}, \qquad k \to \infty. \tag{4.25}$$

All these convergences are quadratic. Consequently $S_k^{-\frac{1}{2}} \to \sqrt{ab}\, I$ quadratically, which completes the proof. ❑

Considering (4.17) and the left-hand-side of (4.18) it is not difficult to check that quadratic convergence of the algorithm holds for other choices of the scaling parameters. Writing the iteration as

$$g_{k+1} = \frac{1}{2}\Big(\alpha_k g_k + \beta_k \gamma_k^0\Big), \qquad k = 0, 1, \ldots \tag{4.26}$$

it is easy to see that the algorithm converges quadratically to a scaled version of h^0 if $\frac{\beta_k}{\alpha_k} \in [A, B]$. The right-hand side in (4.18) shows that the convergence cannot be expected to be faster than quadratic. Indeed, although the function

$$k(y) = \frac{4y^{\frac{1}{2}}}{(1 + y^{\frac{1}{2}})^2} = f(y^{\frac{1}{2}}), \tag{4.27}$$

(see (4.21)) is considerably flatter than $f(y)$ at $y = 1$, we have that $k(1) = 1, k'(1) = 0, k''(1) = -\frac{1}{8}$, hence we have at best that $1 - \frac{A_k}{B_k} = \frac{1}{16}(1 - \frac{A_{k-1}}{B_{k-1}})^2$ compared to $1 - \frac{A_k}{B_k} = \frac{1}{4}(1 - \frac{A_{k-1}}{B_{k-1}})^2$ for $f(y)$ in (4.21). This improves convergence behavior with respect to the constant, but not with respect to the order. However, since from a numerical viewpoint improving the constant is of considerable importance, we want to find out which choice of α_k, β_k leads to the optimal constant.

Optimizing (4.17) with respect to $\frac{\beta}{\alpha}$ yields that the optimum choice for the scaling parameters is

$$\alpha_k = c(A_k B_k)^{-\frac{1}{4}}, \beta_k = c(A_k B_k)^{\frac{1}{4}}, \tag{4.28}$$

for some $c > 0$, see also the graph in Figure 1. The same optimal scaling parameter arises in the finite-dimensional Newton method, cf. [17] as well in

Balan's construction of "nearest tight frames", see Section 3 in [2]. It is not difficult to see that for optimal scaling we get

$$A_1 = 4, \qquad B_1 = \frac{(\sqrt{A}+\sqrt{B})^2}{\sqrt{AB}}, \tag{4.29}$$

hence $\frac{A_1}{B_1}$ takes on the maximum value in the right-hand side of (4.18).

The choice (4.28) however is practically not feasible, since it requires the computation (or estimation) of the frame bounds A_k, B_k in each iteration step, which is in general computationally expensive.

An alternative is to find the scaling parameter α that minimizes

$$\|\alpha g - \alpha^{-1}\gamma^0\|. \tag{4.30}$$

A simple calculation shows that in this case

$$\alpha = \Big(\frac{\|\gamma^0\|}{\|g\|}\Big)^{\frac{1}{2}}, \tag{4.31}$$

which is similar to Frobenius-norm scaling in [24] (hence we will refer to it as Frobenius-norm scaling) and closely related to the scaling used in Theorem 4.2.

Next we estimate how the frame bounds get "tighter" from iteration k to $k+1$ when using the scaling proposed in Theorem 4.2.

Corollary 4.3 *Under the assumptions of Theorem 4.2 the frame bounds A_k, B_k of (g_k, a, b) can be estimated recursively by*

$$A_k \ge \frac{1}{ab}\frac{2\sqrt{A_{k-1}B_{k-1}}}{A_{k-1}+B_{k-1}}, \qquad B_k \le \frac{1}{ab}\frac{(A_{k-1}+B_{k-1})^2}{4A_{k-1}B_{k-1}}. \tag{4.32}$$

Proof: It is sufficient to consider the case $k = 1$. We begin with the observation that for S satisfying $AI \le S \le BI$ where $A > 0, B < \infty$ and $f \in \boldsymbol{L}^2(\mathbb{R})$ there holds

$$\frac{\langle S^{-1}f, f\rangle}{\|f\|\|S^{-1}f\|} \ge \frac{2\sqrt{AB}}{A+B}. \tag{4.33}$$

Inequality (4.33) is equivalent to the operator version of the Kantorovich inequality [22]. For the reader's convenience we include a short proof. Inequality (4.33) follows from

$$(A+B)^2\langle S^{-1}x, x\rangle^2 - 4AB\|x\|^2\|S^{-1}x\|^2 =$$

$$= \Big[2AB\|S^{-1}x\|^2 - (A+B)\|S^{-\frac{1}{2}}x\|^2\Big]^2 +$$
$$+4AB\langle (BI-S)(I-AS^{-1})S^{-\frac{1}{2}}x, S^{-\frac{1}{2}}x\rangle \|S^{-1}x\|^2 \geq 0, \tag{4.34}$$

since $(BI-S)(I-AS^{-1})$ is a non-negative operator.

As a consequence we have now immediately that

$$A_1 \geq \frac{1}{\|g\|\|S^{-1}g\|} = \frac{1}{ab}\frac{\langle S^{-1}g, g\rangle}{\|g\|\|S^{-1}g\|} \geq \frac{1}{ab}\frac{2\sqrt{AB}}{A+B}, \tag{4.35}$$

which is the first inequality in (4.32).

Furthermore, we have

$$B_1 = \max_{x=A,B}(\alpha^2 x + 2\alpha\beta + \beta^2 x^{-1}); \qquad \alpha = \frac{1}{2}\|g\|^{-1}, \beta = \frac{1}{2}\|S^{-1}g\|^{-1} \tag{4.36}$$

by (4.16). Now

$$\frac{x}{4\|g\|^2} + \frac{2}{4\|g\|\|S^{-1}g\|} + \frac{x^{-1}}{4\|S^{-1}g\|^2} = \tag{4.37}$$
$$= \frac{1}{4\|g\|\|S^{-1}g\|}\Big(x^{\frac{1}{2}}\frac{\|S^{-1}g\|^{\frac{1}{2}}}{\|g\|^{\frac{1}{2}}} + x^{-\frac{1}{2}}\frac{\|g\|^{\frac{1}{2}}}{\|S^{-1}g\|^{\frac{1}{2}}}\Big)^2 \tag{4.38}$$
$$\leq \frac{1}{4ab}(u+u^{-1})^2\,; \qquad u = (x\frac{\|S^{-1}g\|}{\|g\|})^{\frac{1}{2}}, \tag{4.39}$$

where we have used that $\|g\|\|S^{-1}g\| \geq \langle S^{-1}g, g\rangle = ab$, see (1.39). Next

$$\frac{A}{B} \leq A\frac{\|S^{-1}g\|}{\|g\|} \leq 1 \leq B\frac{\|S^{-1}g\|}{\|g\|} \leq \frac{B}{A}, \tag{4.40}$$

since $B^{-1}I \leq S^{-1} \leq A^{-1}I$. It thus follows that for either $x = A$ or B we have

$$\frac{x}{4\|g\|^2} + \frac{2}{4\|g\|\|S^{-1}g\|} + \frac{x^{-1}}{4\|S^{-1}g\|^2} \leq \frac{1}{4ab}\Big((\frac{A}{B})^{\frac{1}{2}} + (\frac{B}{A})^{\frac{1}{2}}\Big)^2 \tag{4.41}$$
$$= \frac{1}{ab}\frac{(A+B)^2}{4AB}, \tag{4.42}$$

and this is the second inequality in (4.32). ❑

Both estimates in (4.32) are sharp. This can be seen by considering the following example. Assume $a = b = 1$. Let $\varepsilon > 0$ be arbitrarily small, and take g such that $|Zg| = B^{\frac{1}{2}}, 1/|Zg| = |ZS^{-1}g| = B^{-\frac{1}{2}}$ on a set of measure $1 - \varepsilon$ in the unit square while $|Zg| = A^{\frac{1}{2}}, 1/|Zg| = |ZS^{-1}g| = A^{-\frac{1}{2}}$ on the complement of this set. Then we have $\|g\| \approx B^{\frac{1}{2}}$, $\|S^{-1}g\| \approx B^{-\frac{1}{2}}$, whence

$$B_1 = \max_{x=A,B}\Big(\frac{x}{4\|g\|^2} + \frac{1}{2\|g\|\|S^{-1}g\|} + \frac{x^{-1}}{4\|S^{-1}g\|^2}\Big) \tag{4.43}$$

$$\approx \max_{x=A,B}\Big(\frac{x}{4B} + \frac{1}{2} + \frac{x^{-1}}{4B^{-1}}\Big) = \frac{(A+B)^2}{4AB}. \tag{4.44}$$

Hence the second inequality in (4.32) is sharp.

As to sharpness in the first inequality in (4.32), we consider a g that has a Zak transform with $Zg = B^{\frac{1}{2}}$ on a set $\mathcal{T} \subset [0,1)^2$ of measure $t \in [0,1]$ and $Zg = A^{\frac{1}{2}}$ on the complementary set. For sufficiently nice $\mathcal{T}$ we can construct a function $\tilde{g}$ by an arbitrarily small perturbation of g such that $Z\tilde{g}$ is continuous and assumes all values between $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ on $[0,1)^2$. For such a $\tilde{g}$ there holds

$$A_1 = \frac{1}{\|\tilde{g}\|\|S^{-1}\tilde{g}\|} = \frac{1}{\|Z\tilde{g}\|\|ZS^{-1}\tilde{g}\|} \tag{4.45}$$

$$\approx \Big[(A + t(B-A))(\frac{1}{A} + t(\frac{1}{B} - \frac{1}{A}))\Big]^{-\frac{1}{2}} \tag{4.46}$$

$$= \Big(1 + t\frac{(B-A)^2}{AB} - t^2\frac{(B-A)^2}{AB}\Big)^{-\frac{1}{2}} \tag{4.47}$$

Taking $t = \frac{1}{2}$ here, we have

$$A_1 \approx \Big(1 + \frac{(B-A)^2}{4AB}\Big)^{-\frac{1}{2}} = \frac{2\sqrt{AB}}{A+B}, \tag{4.48}$$

which is the right-hand side of the first inequality in (4.32).

We compute Zg_1 for the "unperturbed" g in this last example. We find

$$Zg_1 = \begin{cases} \frac{1}{2}\Big(\frac{B}{A+t(B-A)}\Big)^{\frac{1}{2}} + \frac{1}{2}\Big(\frac{A}{B-t(B-A)}\Big)^{\frac{1}{2}} & \text{when } Zg = B^{\frac{1}{2}}, \\ \frac{1}{2}\Big(\frac{A}{A+t(B-A)}\Big)^{\frac{1}{2}} + \frac{1}{2}\Big(\frac{B}{B-t(B-A)}\Big)^{\frac{1}{2}} & \text{when } Zg = A^{\frac{1}{2}}. \end{cases} \tag{4.49}$$

The difference of upper and lower number in (4.49), ignoring terms of order $(2t-1)^n(B-A)^m$ with $n+m>4$ (there are no terms with $n+m=4$), is

$$(1-2t)\frac{(\sqrt{B}-\sqrt{A})^2}{2\sqrt{AB}}. \tag{4.50}$$

E.g., when $t=\frac{1}{2}$ the difference is zero. This example also illustrates that the algorithm may converge even much better than predicted by the theoretical estimates presented above.

4.2 Numerical simulations

The numerical simulations presented in this section are representative for a large number of situations. We consider a well-localized window g and oversampling rate $\frac{1}{ab}\in[1,2]$. Such a setup is typical for oversampled modulated filter banks as well as for OFDM systems (where k-times oversampling corresponds to k-times undersampling). We choose $g(t)=2^{\frac{1}{4}}e^{-\pi t^2}$ with $a=b=\sqrt{\frac{15}{16}}$ resulting in an oversampling rate of 16/15. For the numerical implementation we follow the finite-dimensional model in [33], Sec.1.6.

In the first experiment we consider different types of scaling for iteration (4.26). We compare the Newton method with *norm-scaling* as used in Theorem 4.2 to optimal scaling (cf. (4.28)), Frobenius-norm scaling (cf. (4.31)) and to the unscaled Newton iteration (i.e., $\alpha_k=\beta_k=1$). We precompute h^0, calculate in each iteration the normalized error $\|g_k/\|g_k\|-h^0/\|h^0\|\|$ and terminate the iterations when the accuracy is within 10^{-14}. The results are shown in Figure 2.

As expected, optimal scaling results in fastest convergence. Norm-scaling and Frobenius-norm scaling yield almost identical convergence, slightly slower than optimal scaling. Recall however, that the costs for computing the optimal scaling parameters are much larger than one iteration step for the other two scaling schemes. Thus the faster convergence comes at a prohibitive high price. Newton's method without scaling requires significantly more iterations in order to achieve the same approximation error. In terms of overall computational costs Newton's method with norm-scaling or Frobenius-norm scaling are clearly the most efficient methods.

An alternative to the algorithm presented in Theorem 4.2 may be to compute first iteratively $S^{-\frac{1}{2}}$ and at the end calculate $h^0=S^{-\frac{1}{2}}g$. Efficient iterative methods to compute the (inverse) square root of a positive definite

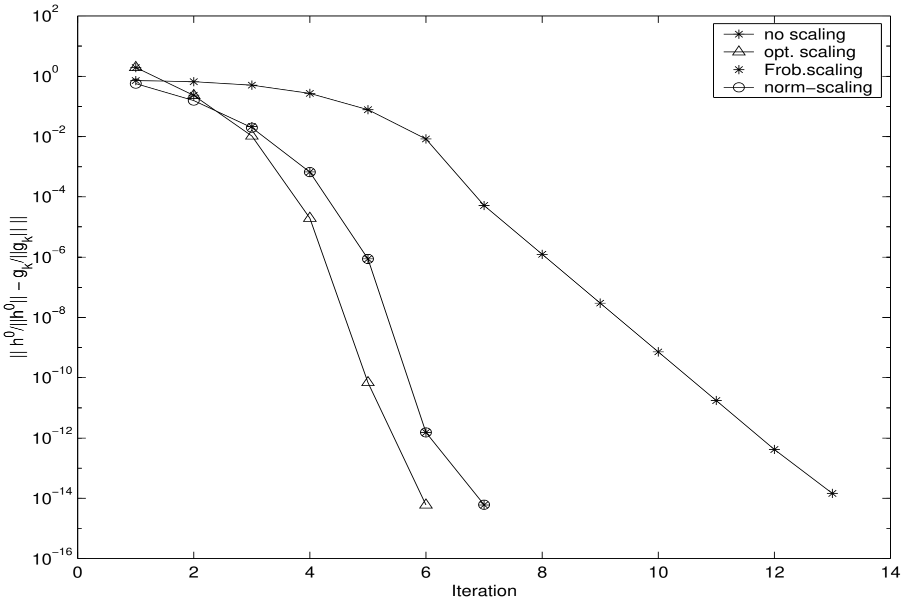

Figure 2: Comparison of different scalings for Newton's method.

matrix have been proposed in [30, 25]. These methods are different from Newton's method as described in (4.1).

Let M be a symmetric positive definite matrix. Then Sherif's method to compute $M^{-\frac{1}{2}}$ is given by (cf. [30]):

$$\begin{aligned} M_0 &= I \\ M_{k+1} &= 2M_k(I + MM_k^2)^{-1}, \qquad k = 0, 1, \ldots \end{aligned} \tag{4.51}$$

This method yields quadratic convergence. The method due to Lakic achieves even a cubic rate of convergence. Its iteration rule is (cf. [25]):

$$\begin{aligned} M_0 &= I \\ M_{k+1} &= \frac{1}{3}M_k(I + 8(I + 3MM_k^2)^{-1}), k = 0, 1, \ldots \end{aligned} \tag{4.52}$$

Lakic's method yields usually slightly faster convergence than our method, whereas Sherif's method is somewhat slower than ours, see Figure 3. For an overall comparison of the efficiency of iterative methods we have to take into account the computational effort needed for one iteration step. The

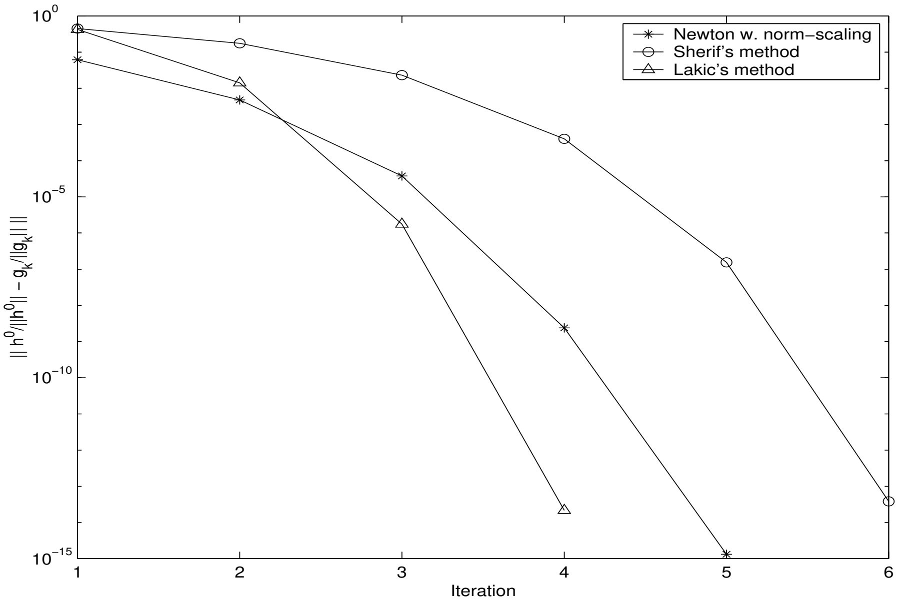

Figure 3: Comparison of different methods to compute the tight window h^0.

number of flops for the matrix-multiplications and inversions in (4.24), (4.51), and (4.52) depends on the chosen representation of the frame operator. Note that for the proposed method this representation has to be computed for each iteration (since we are dealing with a new frame in each iteration), which is not the case for the other two methods. However Newton's method needs only one matrix inversion per iteration, whereas the other two methods require three matrix-matrix multiplications in addition to one matrix inversion.

Fortunately the computation of the various representations of the frame operator can in general be done very efficiently. For the representation in the time-domain this amounts to simple data addressing, in the other three domains FFT-based algorithm can be used (possibly combined with data addressing). Thus, as long as the computation of this representation is cheaper than the inversion of the resulting matrix, the method proposed in Theorem 4.2 outperforms the other two methods in terms of computational efficiency.

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