Hyperbolic secants yield Gabor frames

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Abstract

We show that (g_2, a, b) is a Gabor frame when $a > 0, b > 0, ab < 1$ and $g_2(t) = (\frac{1}{2}\pi\gamma)^{\frac{1}{2}}(\cosh \pi\gamma t)^{-1}$ is a hyperbolic secant with scaling parameter $\gamma > 0$. This is accomplished by expressing the Zak transform of g_2 in terms of the Zak transform of the Gaussian $g_1(t) = (2\gamma)^{\frac{1}{4}} \exp(-\pi\gamma t^2)$, together with an appropriate use of the Ron-Shen criterion for being a Gabor frame. As a side result it follows that the windows, generating tight Gabor frames, that are canonically associated to g_2 and g_1 are the same at critical density $a = b = 1$. Also, we display the "singular" dual function corresponding to the hyperbolic secant at critical density.

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1 Introduction and results

Let $a > 0, b > 0$, and let $g \in \boldsymbol{L}^2(\mathbb{R})$. A Gabor system (g, a, b) consists of all time- and- frequency shifted functions $g_{na,mb}$ with integer n, m, where for $x, y \in \mathbb{R}$ we denote

$$g_{x,y}(t) = e^{2\pi i y t} g(t - x), \quad t \in \mathbb{R}. \tag{1}$$

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We say that (g, a, b) is a Gabor frame when there are $A > 0, B < \infty$ (lower, upper frame bound, respectively) such that for all $f \in \boldsymbol{L}^2(\mathbb{R})$ we have

$$A\|f\|^2 \leq \sum_{n,m} |\langle f, g_{na,mb}\rangle|^2 \leq B\|f\|^2. \tag{2}$$

Here $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the standard norm and inner product of $\boldsymbol{L}^2(\mathbb{R})$. We refer to [2, 3, 4] for generalities about frames for a Hilbert space and for both basic and in-depth information about Gabor frames, frame operators, dual frames, tight frames, etc. For a very recent and comprehensive treatment of Gabor frames, we refer to [5], Chs. 5-9, 11-13; many of the more advanced results of modern Gabor theory are covered there in a unified manner.

It is well known that a triple (g, a, b) cannot be a Gabor frame when $ab > 1$, see for instance [5], Corollary 7.5.1. Also such a system cannot be a Gabor frame when $ab = 1$ and $g' \in \boldsymbol{L}^2(\mathbb{R}), tg(t) \in \boldsymbol{L}^2(\mathbb{R})$ (extended Balian-Low theorem; see for instance [4], Ch. 2).

In this paper we demonstrate that (g_2, a, b) is a Gabor frame when $ab < 1$ and

$$g_{2,\gamma}(t) = \left(\frac{\pi\gamma}{2}\right)^{\frac{1}{2}} \frac{1}{\cosh \pi\gamma t}, \qquad t \in \mathbb{R}, \tag{3}$$

with $\gamma > 0$ (normalization such that $\|g_{2,\gamma}\| = 1$). The result is obtained by (i) relating the Zak transform

$$(Zg)(t,\nu) = \sum_{l=-\infty}^{\infty} g(t-l)e^{2\pi i l\nu}, \qquad t, \nu \in \mathbb{R}, \tag{4}$$

of $g = g_{2,\gamma}$ to the Zak transform of $g = g_{1,\gamma}$, where $g_{1,\gamma}$ is the normalized Gaussian

$$g_{1,\gamma}(t) = (2\gamma)^{\frac{1}{4}} e^{-\pi\gamma t^2}, \qquad t \in \mathbb{R}, \tag{5}$$

(ii) using the observation that $(g_{1,\gamma}, a, b)$ is a Gabor frame and (iii) an appropriate use of the Ron-Shen time domain criterion [10] for a Gabor system (g, a, b) to be a Gabor frame. Explicitly, we show that there is a positive constant E such that

$$(Zg_{2,\gamma})(t,\nu) = \frac{E(Zg_{1,\gamma})(t,\nu)}{\vartheta_4(\pi\nu; e^{-\pi\gamma})\vartheta_4(\pi t; e^{-\pi/\gamma})}, \quad t, \nu \in \mathbb{R}. \tag{6}$$

Here $\vartheta_4(z;q)$ is the theta function, see [11], Ch. 21,

$$\vartheta_4(z;q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2inz}, \qquad z \in \mathbb{C}. \tag{7}$$

This ϑ_4 is positive and bounded for real arguments z and parameter $q \in (0,1)$.

The rest of the paper is organized as follows. In Section 2 we present the proof or our main result. In Section 3 we prove formula (6) by using elementary properties of theta functions and basic complex analysis. These same methods, combined with formula (6), allow us to compute explicitly the "singular" dual function $g_{2,\gamma}^d$ (at critical density $a = b = 1$) canonically associated to $g_{2,\gamma}$ according to

$$g_{2,\gamma}^d(t) = \int_0^1 \frac{d\nu}{(Zg_{2,\gamma})^*(t,\nu)}, \qquad t \in \mathbb{R}. \tag{8}$$

This is briefly presented in Section 4. We also show in Section 4 that the tight Gabor frame generating windows $g_{1,\gamma}^t, g_{2,\gamma}^t$, canonically associated to $g_{1,\gamma}, g_{2,\gamma}$ and given at critical density $a = b = 1$ by

$$g^t(t) = \int_0^1 \frac{(Zg)(t,\nu)}{|(Zg(t,\nu)|} d\nu, \qquad t \in \mathbb{R}, \tag{9}$$

for $g = g_{1,\gamma}, g_{2,\gamma}$, are the same. In [9] a comprehensive study is made of the process, embodied by formula (9), of passing from windows g with few zeros in the Zak transform domain to tight Gabor frame generating windows g^t at critical density. One of the observations in [9] is that the operation in (9) seems to diminish distances between positive, even, unimodal windows enormously. The fact that $g_{1,\gamma}^t = g_{2,\gamma}^t$ is an absurdly accurate illustration of this phenomenon.

2 Proof of the main result

In this section we present a proof for the result that $(g_{2,\gamma}, a, b)$ is a Gabor frame when $a > 0, b > 0, ab < 1$ and $g_{2,\gamma}$ is given by (3). According to the

Ron-Shen criterion in the time-domain [10] we have that (g,a,b) is a Gabor frame with frame bounds $A>0, B<\infty$ if and only if

$$AI \leq \frac{1}{b} M_g(t) M_g^*(t) \leq BI, \qquad \text{a.e. } t \in \mathbb{R}. \tag{10}$$

Here I is the identity operator of $\boldsymbol{\ell}^2(\mathbb{Z})$ and $M_g(t)$ is the linear operator of $\boldsymbol{\ell}^2(\mathbb{Z})$ (in the notation of [4], Subsec. 1.3.2), whose matrix with respect to the standard basis of $\boldsymbol{\ell}^2(\mathbb{Z})$ is given by

$$M_g(t) = \big(g(t-na-l/b)\big)_{l\in\mathbb{Z}, n\in\mathbb{Z}}, \qquad \text{a.e. } t \in \mathbb{R} \tag{11}$$

(row index l, column index n). Since $g_{2,\gamma}$ is rapidly decaying, the finite frame upper bound condition is easily seen to be satisfied, and we therefore concentrate on the positive lower frame bound condition. We may restrict here to the case where $a<1, b=1$, since (g,a,b) is a Gabor frame if and only if $(D_c g, a/c, bc)$ is a Gabor frame. Here D_c is the dilation operator $(D_c f)(t) = c^{\frac{1}{2}} f(ct), t\in\mathbb{R}$, defined for $f \in \boldsymbol{L}^2(\mathbb{R})$ when $c>0$. Since $D_c g_{2,\gamma} = g_{2,\gamma c}$, we only need to replace $\gamma>0$ by $\gamma b>0$ when $b\neq 1$. Hence we shall show that there is an $A>0$ such that

$$\sum_{n=-\infty}^{\infty} \Big| \sum_{l=-\infty}^{\infty} c_l g(t-na-l) \Big|^2 \geq A \|\underline{c}\|^2, \quad \underline{c} = (c_l)_{l\in\mathbb{Z}} \in \boldsymbol{\ell}^2(\mathbb{Z}), t\in\mathbb{R}, \tag{12}$$

with $g = g_{2,\gamma}$.

Taking $\underline{c} \in \boldsymbol{\ell}^2(\mathbb{Z})$ it follows from Parseval's theorem for Fourier series and the definition of the Zak transform in (4) that for any $n\in\mathbb{Z}$

$$\sum_{l=-\infty}^{\infty} c_l g(t-na-l) = \int_0^1 (Zg)(t-na,\nu) C^*(\nu) d\nu, \tag{13}$$

where $C(\nu)$ is defined by

$$C(\nu) = \sum_{l=-\infty}^{\infty} c_l^* e^{2\pi i l \nu}, \qquad \text{a.e. } \nu \in \mathbb{R}. \tag{14}$$

Now assuming the result (6) (with $E > 0$) we have that

$$\sum_{l=-\infty}^{\infty} c_l g_{2,\gamma}(t - na - l) = \int_0^1 (Zg_{2,\gamma})(t - na, \nu) C^*(\nu) d\nu$$

$$= \frac{E}{\vartheta_4(\pi(t - na); e^{-\pi/\gamma})} \int_0^1 (Zg_{1,\gamma})(t - na, \nu) \frac{C^*(\nu)}{\vartheta_4(\pi\nu; e^{-\pi\gamma})} d\nu. \tag{15}$$

Define the 1-periodic function $D(\nu)$ by

$$D(\nu) = \sum_{l=-\infty}^{\infty} d_l^* e^{2\pi i l\nu} := \frac{C(\nu)}{\vartheta_4(\pi\nu; e^{-\pi\gamma})}, \qquad \nu \in \mathbb{R}. \tag{16}$$

It is well known that $(g_{1,\gamma}, a, 1)$ is a Gabor frame, see for instance [5], Theorem 7.5.3. Accordingly, see (12), there is an $A_{1,\gamma} > 0$ such that

$$\sum_{n=-\infty}^{\infty} \Big| \sum_{l=-\infty}^{\infty} d_l g_{1,\gamma}(t - na - l) \Big|^2 \geq A_{1,\gamma} \|\underline{d}\|^2, \quad \underline{d} \in \boldsymbol{\ell}^2(\mathbb{Z}), t \in \mathbb{R}. \tag{17}$$

Letting

$$m_\delta := \min_{z \in \mathbb{R}} \frac{1}{\vartheta_4^2(z; e^{-\pi\delta})} > 0 \tag{18}$$

for $\delta > 0$, we then see from (13)–(18) that

$$\sum_{n=-\infty}^{\infty} \Big| \sum_{l=-\infty}^{\infty} c_l g_{2,\gamma}(t - na - l) \Big|^2 \geq$$

$$m_{1/\gamma} E^2 \sum_{n=-\infty}^{\infty} \Big| \int_0^1 (Zg_{1,\gamma})(t - na, \nu) D^*(\nu) d\nu \Big|^2 =$$

$$m_{1/\gamma} E^2 \sum_{n=-\infty}^{\infty} \Big| \sum_{l=-\infty}^{\infty} d_l g_{1,\gamma}(t - na - l) \Big|^2 \geq m_{1/\gamma} E^2 A_{1,\gamma} \|\underline{d}\|^2. \tag{19}$$

Finally, with D and C related as in (16) we have from Parseval's theorem for Fourier series that

$$\|\underline{d}\|^2 = \int_0^1 |D(\nu)|^2 d\nu = \int_0^1 \frac{1}{|\vartheta_4(\pi\nu; e^{-\pi\gamma})|^2} |C(\nu)|^2 d\nu$$

$$\geq m_\gamma \int_0^1 |C(\nu)|^2 d\nu = m_\gamma \|\underline{c}\|^2. \tag{20}$$

Hence

$$\sum_{n=-\infty}^{\infty} |\sum_{l=-\infty}^{\infty} c_l g_{2,\gamma}(t - na - l)|^2 \geq m_\gamma m_{1/\gamma} E^2 A_{1,\gamma} \|\underline{c}\|^2, \tag{21}$$

as required.

3 Expressing $Zg_{2,\gamma}$ in terms of $Zg_{1,\gamma}$

In this section we express $Zg_{2,\gamma}$, with $g_{2,\gamma}$ given in (3), in terms of $Zg_{1,\gamma}$, with $g_{1,\gamma}$ given in (5). For this we need some basic facts of the theory of theta functions as they can be found in [11], Ch. 21. The four theta functions are for $q \in (0, 1)$ given by

$$\vartheta_1(z; q) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)iz}, \tag{22}$$

$$\vartheta_2(z; q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{(2n+1)iz}, \tag{23}$$

$$\vartheta_3(z; q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}, \tag{24}$$

$$\vartheta_4(z; q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz}, \tag{25}$$

where $z \in \mathbb{C}$. One thus easily gets from (4) and (5) that

$$(Zg_{1,\gamma})(t, \nu) = (2\gamma)^{\frac{1}{4}} e^{-\pi\gamma t^2} \vartheta_3(\pi(\nu - i\gamma t); e^{-\pi\gamma}), \quad t, \nu \in \mathbb{R}. \tag{26}$$

Theorem 3.1 *We have for $t, \nu \in \mathbb{R}$*

$$
\begin{aligned}
(Zg_{2,\gamma})(t,\nu) &= 2^{-\frac{1}{2}}\pi^{\frac{1}{2}}\gamma\vartheta_1'(0;e^{-\pi\gamma})e^{-\pi\gamma t^2}\frac{\vartheta_3(\pi(\nu-i\gamma t);e^{-\pi\gamma})}{\vartheta_4(\pi\nu;e^{-\pi\gamma})\vartheta_4(\pi t;e^{-\pi/\gamma})} \\
&= \pi^{\frac{1}{2}}\left(\frac{\gamma}{2}\right)^{\frac{3}{4}}\vartheta_1'(0;e^{-\pi\gamma})\frac{(Zg_{1,\gamma})(t,\nu)}{\vartheta_4(\pi\nu;e^{-\pi\gamma})\vartheta_4(\pi t;e^{-\pi/\gamma})}. \qquad (27)
\end{aligned}
$$

Proof: For fixed $t \in \mathbb{R}$ we compute the Fourier coefficients $b_n(t)$ of the 1-periodic function

$$
\frac{\vartheta_3(\pi(\nu-i\gamma t);e^{-\pi\gamma})}{\vartheta_4(\pi\nu;e^{-\pi\gamma})} = \sum_{n=-\infty}^{\infty} b_n(t)e^{2\pi in\nu}, \quad \nu \in \mathbb{R}, \qquad (28)
$$

by the method that can be found in [11], Sec. 22.6. For brevity we shall suppress the expression "$;e^{-\pi\gamma}$" in $\vartheta_3(\pi(\nu-i\gamma t);e^{-\pi\gamma})$, etc. in the remainder of this proof.

We thus have

$$
b_n(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\vartheta_3(\pi(\nu-i\gamma t))}{\vartheta_4(\pi\nu)}e^{-2\pi in\nu}d\nu, \quad n \in \mathbb{Z}, \qquad (29)
$$

and we consider for $n \in \mathbb{Z}$

$$
c_n(t) = \int_C \frac{\vartheta_3(\pi(\nu-i\gamma t))}{\vartheta_4(\pi\nu)}e^{-2\pi in\nu}d\nu, \qquad (30)
$$

where C is the edge of the rectangle with corner points $-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}+i\gamma, -\frac{1}{2}+i\gamma$, taken with positive orientation. It follows from [11], Sec. 21.12 that the function $\vartheta_4(\pi\nu)$ has a first order zero at $\nu = \frac{1}{2}i\gamma$ and no zeros elsewhere on or within C. Therefore, by Cauchy's theorem,

$$
c_n(t) = 2\pi i \operatorname*{Res}_{\nu=\frac{1}{2}i\gamma}\left[\frac{\vartheta_3(\pi(\nu-i\gamma t))}{\vartheta_4(\pi\nu)}e^{-2\pi in\nu}\right] = \frac{2i\vartheta_3(\pi i(\frac{1}{2}-t)\gamma)e^{\pi\gamma n}}{\vartheta_4'(\frac{1}{2}\pi i\gamma)}. \qquad (31)
$$

By [11], Ex. 2, first identity, on p. 464 we have ($\tau = i\gamma, q = \exp(-\pi\gamma), z = 0$)

$$
\vartheta_4'\left(\frac{1}{2}\pi i\gamma\right) = ie^{\frac{1}{4}\pi\gamma}\vartheta_1'(0). \qquad (32)
$$

Next, by the first formula on p. 475 of [11] ($z = \pi i(\frac{1}{2} - t)\gamma, \tau' = -1/\tau = i/\gamma, q' = \exp(\pi i \tau') = \exp(-\pi/\gamma)$)

$$\vartheta_3\big(\pi i(\frac{1}{2} - t)\gamma\big) = \gamma^{-\frac{1}{2}} \exp\big(\pi\gamma(\frac{1}{2} - t)^2\big)\vartheta_3\big(\pi(\frac{1}{2} - t); e^{-\pi/\gamma}\big). \tag{33}$$

and by [11], Ex. 2, fourth identity, on p. 464 ($z = -\pi t$, q' instead of q)

$$\vartheta_3\big(\pi(\frac{1}{2} - t); e^{-\pi/\gamma}\big) = \vartheta_4(-\pi t; e^{-\pi/\gamma}) = \vartheta_4(\pi t; e^{-\pi/\gamma}), \tag{34}$$

where we also have used that ϑ_4 is an even function. It thus follows from (31)–(34) that

$$c_n(t) = \frac{2}{\vartheta_1'(0)}\gamma^{-\frac{1}{2}} e^{-\frac{1}{4}\pi\gamma} e^{\pi\gamma(\frac{1}{2}-t)^2}\vartheta_4(\pi t; e^{-\pi/\gamma})e^{\pi\gamma n}. \tag{35}$$

On the other hand, we have by 1-periodicity of the integrand in (30) (in ν) that the two integrals along the vertical edges of C cancel one another. Hence

$$\begin{aligned} c_n(t) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\vartheta_3(\pi(\nu - i\gamma t))}{\vartheta_4(\pi\nu)} e^{-2\pi i n\nu} d\nu - \int_{-\frac{1}{2}+i\gamma}^{\frac{1}{2}+i\gamma} \frac{\vartheta_3(\pi(\nu - i\gamma t))}{\vartheta_4(\pi\nu)} e^{-2\pi i n\nu} d\nu \\ &= b_n(t) - e^{2\pi\gamma n} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\vartheta_3(\pi(\nu - i\gamma t) + \pi i\gamma)}{\vartheta_4(\pi\nu + \pi i\gamma)} e^{-2\pi i n\nu} d\nu. \end{aligned} \tag{36}$$

Furthermore, by the table in [11], Ex. 3 on p. 465 we have

$$\vartheta_3\big(\pi(\nu - i\gamma t) + \pi i\gamma\big) = e^{\pi\gamma} e^{-2i\pi(\nu - i\gamma t)}\vartheta_3\big(\pi(\nu - i\gamma t)\big), \tag{37}$$

and

$$\vartheta_4(\pi\nu + \pi i\gamma) = -e^{\pi\gamma} e^{-2i\pi\nu}\vartheta_4(\pi\nu). \tag{38}$$

It thus follows that

$$\begin{aligned} c_n(t) &= b_n(t) + e^{-2\pi\gamma(t-n)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\vartheta_3(\pi(\nu - i\gamma t))}{\vartheta_4(\pi\nu)} e^{-2\pi i n\nu} d\nu \\ &= b_n(t)(1 + e^{-2\pi\gamma(t-n)}). \end{aligned} \tag{39}$$

By (35) we then find that

$$\begin{aligned}
b_n(t) =& \frac{2\gamma^{-\frac{1}{2}}}{\vartheta_1'(0)} e^{-\frac{1}{4}\pi\gamma} e^{\pi\gamma(\frac{1}{2}-t)^2} \vartheta_4(\pi t; e^{-\pi/\gamma}) \frac{e^{\pi\gamma n}}{1+e^{-2\pi\gamma(t-n)}} \\
=& \frac{\gamma^{-\frac{1}{2}}}{\vartheta_1'(0)} e^{-\frac{1}{4}\pi\gamma} e^{\pi\gamma(\frac{1}{2}-t)^2+\pi\gamma t} \frac{\vartheta_4(\pi t; e^{-\pi/\gamma})}{\cosh \pi\gamma(t-n)} \\
=& \frac{\gamma^{-\frac{1}{2}}}{\vartheta_1'(0)} e^{\pi\gamma t^2} \frac{\vartheta_4(\pi t; e^{-\pi/\gamma})}{\cosh \pi\gamma(t-n)}. \qquad (40)
\end{aligned}$$

We thus conclude, see (28), that

$$\frac{\vartheta_3(\pi(\nu - i\gamma t))}{\vartheta_4(\pi\nu)} = \frac{\gamma^{-\frac{1}{2}}}{\vartheta_1'(0)} e^{\pi\gamma t^2} \vartheta_4(\pi t; e^{-\pi/\gamma}) \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n\nu}}{\cosh \pi\gamma(t-n)}. \qquad (41)$$

That is,

$$\begin{aligned}
(Zg_{2,\gamma})(t,\nu) &= \left(\frac{\pi\gamma}{2}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n\nu}}{\cosh \pi\gamma(t-n)} \\
&= 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} \gamma \vartheta_1'(0) \frac{e^{-\pi\gamma t^2} \vartheta_3(\pi(\nu - i\gamma t))}{\vartheta_4(\pi\nu)\vartheta_4(\pi t; e^{-\pi/\gamma})}, \qquad (42)
\end{aligned}$$

and this is the first line identity in (27). The second line identity in (27) follows from (26). This completes the proof. ❑

<u>Note:</u> We have that

$$\vartheta_1'(0; e^{-\pi\gamma}), \vartheta_4(\pi\nu; e^{-\pi\gamma}), \vartheta_4(\pi t; e^{-\pi/\gamma}) > 0 \qquad (43)$$

for $\nu \in \mathbb{R}, t \in \mathbb{R}$. Indeed, we have by [11], Sec.21.41 that

$$\vartheta_1'(0; e^{-\pi\gamma}) = \vartheta_2(0; e^{-\pi\gamma})\vartheta_3(0; e^{-\pi\gamma})\vartheta_4(0; e^{-\pi\gamma}). \qquad (44)$$

One sees directly from (23) that $\vartheta_2(0;q) > 0$. Also, from the formula on p. 476 of [11] just before Ex. 1, one sees that

$$\vartheta_3(z;q) = \vartheta_4\left(z + \frac{1}{2}\pi; q\right) > 0, \qquad z \in \mathbb{R}. \qquad (45)$$

Remark: Theorem 3.1 implies that $(Zg_2)(t,\nu)/(Zg_1)(t,\nu)$ can be factored into a function of t and a function of ν. This factorization is crucial in the proof of the main result in Section 2. It seems possible to extend the approach in Section 2 to other pairs of windows g_1, g_2 where (g_1, a, b) is a frame and Zg_2/Zg_1 (nearly) factorizes. We do not pursue this extension in this paper.

4 Canonical dual window and tight window at critical density

Theorem 3.1 allows us to calculate the Zak transform of the canonical dual $g_{2,\gamma}^d$ of $g_{2,\gamma}$ for rational values of $ab < 1$ using Zibulski-Zeevi matrices representing the frame operator in the Zak transform domain, see for instance [8], Sec. 1.5, [5], Sec. 8.3, and, of course, [12]. The dual window can then be obtained as

$$g_{2,\gamma}^d(t) = \int_0^1 (Zg_{2,\gamma}^d)(t,\nu)d\nu, \qquad t \in \mathbb{R}. \tag{46}$$

For $a = b = 1$ the Zibulski-Zeevi matrices are just scalars, viz. $|(Zg_{2,\gamma})(t,\nu)|^2$, and we have formally

$$(Zg_{2,\gamma}^d)(t,\nu) = \frac{1}{(Zg_{2,\gamma})^*(t,\nu)}, \qquad t,\nu \in \mathbb{R}, \tag{47}$$

and

$$g_{2,\gamma}^d(t) = \int_0^1 \frac{d\nu}{(Zg_{2,\gamma})^*(t,\nu)}, \qquad t \in \mathbb{R}. \tag{48}$$

The identities here hold only formally since $1/Zg_{2,\gamma} \neq \boldsymbol{L}^2{}_{\mathrm{loc}}(\mathbb{R}^2)$, and, as said, $(g_{2,\gamma}, 1, 1)$ is not a frame. However, nothing prevents us from computing the right-hand side of (48) for $t \in \mathbb{R}$ not of the form $n + \frac{1}{2}, n \in \mathbb{Z}$. For these t we have that $(Zg_{2,\gamma})(t,\nu)$ is a well-behaved zero-free function of $\nu \in \mathbb{R}$.

When we do this calculation for the case that $\gamma = 1$, we find, using the same methods as in the proof of Theorem 3.1, that for $t \in (-\frac{1}{2}, \frac{1}{2}), n \in \mathbb{Z}$

$$g_{2,1}^d(t+n) = \frac{2^{\frac{1}{2}}(-1)^n \vartheta_4^2(\pi t)e^{2\pi t^2 + 2\pi n t}}{\pi^{\frac{1}{2}}(\vartheta_1'(0))^2 \cosh \pi(t+n)}. \tag{49}$$

Here all theta functions are with parameter $q = e^{-\pi}$. The same procedure for the Gaussian $g_{1,1}$ yields for $t \in \mathbb{R}$ not of the form $n + 1/2, n \in \mathbb{Z}$

$$g_{1,1}^d(t) = \frac{2^{-\frac{1}{4}}}{\vartheta_1'(0)} e^{\pi t^2} \sum_{n-\frac{1}{2} \geq |t|} (-1)^n e^{-\pi(n-\frac{1}{2})^2}, \tag{50}$$

see for instance [1], [6], Subsec. 2.14, and [7], Subsec. 4.4. Although the two functions in (49)–(50) seem quite different, one can show that both functions are bounded but not in $\boldsymbol{L}^p(\mathbb{R})$ when $1 \le p < \infty$. Moreover, there is for both functions exponential decay as $|t| \to \infty$ away from the set of half-integers.

In a similar fashion one can consider the canonical tight windows g^t associated with $g = g_{1,\gamma}, g_{2,\gamma}$ at critical density $a = b = 1$. We have that these windows are given in the Zak transform domain through the formula $Zg^t = Zg/|Zg|$, whence

$$g^t(t) = \int_0^1 \frac{(Zg)(t,\nu)}{|(Zg)(t,\nu)|} d\nu, \qquad t \in \mathbb{R}. \tag{51}$$

In this case the definitions are more than just formal since $Zg/|Zg| \in \boldsymbol{L}^\infty(\mathbb{R}^2)$, while one can show, see [9] for details, that in $\boldsymbol{L}^2$-sense

$$g^t = \lim_{a \uparrow 1} S_a^{-\frac{1}{2}} g, \tag{52}$$

where for $g = g_{1,\gamma}, g_{2,\gamma}$ we have denoted the frame operator corresponding to the Gabor frame (g, a, a) by S_a. Now as a surprising consequence of Theorem 3.1 we have that $g_{1,\gamma}^t = g_{2,\gamma}^t$ since $Zg_{1,\gamma}/Zg_{2,\gamma}$ is positive everywhere. In [9] it has been observed that the mapping $g \to g^t$ in (51) diminishes distances between even, positive, unimodal windows g enormously when scaling parameters are set correctly. The fact that $g_{1,\gamma}^t = g_{2,\gamma}^t$ is an extremely accurate demonstration of this phenomenon.

The systems $(g^t, 1, 1)$ form orthonormal bases for $\boldsymbol{L}^2(\mathbb{R})$, see, for instance, [5], Corollary 7.5.2. Interestingly, in the case that $g_\gamma = g_{1,\gamma}$ or $g_{2,\gamma}$, we have that, in $\boldsymbol{L}^2$-sense,

$$\lim_{\gamma \downarrow 0} g_\gamma^t = \operatorname{sinc} \pi\cdot, \qquad \lim_{\gamma \to \infty} g_\gamma^t = \chi_{(-\frac{1}{2},\frac{1}{2})}. \tag{53}$$

Hence we have a family $g_\gamma^t, \gamma > 0$ of reasonably behaved tight frame generating windows at critical density $a = b = 1$ that interpolates between the Haar window $\chi_{(-\frac{1}{2},\frac{1}{2})}$ and its Fourier transform $\operatorname{sinc} \pi\cdot$ as γ varies between ∞ and 0. See [9] for details.

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