On generating tight Gabor frames at critical density

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Abstract.
We consider the construction of tight Gabor frames (h, $a = 1$, $b = 1$) from Gabor systems (g, $a = 1$, $b = 1$) with g a window having few zeros in the Zak transform domain via the operation $h = Z^{-1}(Zg/|Zg|)$, where Z is the standard Zak transform. We consider this operation with g the Gaussian, the hyperbolic secant, and for g belonging to a class of positive, even, unimodal, rapidly decaying windows of which the two-sided exponential is a typical example. All these windows g have the property that Zg has a single zero, viz. at $(\frac{1}{2}, \frac{1}{2})$, in the unit square $[0, 1)^2$. The Gaussian and hyperbolic secant yield a frame for any $a, b > 0$ with $ab < 1$, and we show that so does the two-sided exponential. For these three windows it holds that $S_a^{-1/2} g \to h$ as $a \uparrow 1$, where S_a is the frame operator corresponding to the Gabor frame (g, a, a). It turns out that the h's corresponding to g's of the above type look and behave quite similarly when scaling parameters are set appropriately. We give a particular detailed analysis of the h corresponding to the two-sided exponential. We give several representations of this h, and we show that $h \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and is continuous and differentiable everywhere except at the half-integers, etc., and we pay particular attention to the cases that the time constant of the two-sided exponential g tends to ∞. We also consider the cases that the time constants of the Gaussian and of the hyperbolic secant tend to 0 or to ∞. It so turns out that h thus obtained changes from the box function $\chi_{(-1/2,1/2)}$ into its Fourier transform $\mathrm{sinc}\, \pi \cdot$ when the time constant changes from 0 to ∞.

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1 Introduction

We consider in this paper Gabor systems and frames, in particular tight Gabor frames, at critical density. We assume the reader to be somewhat familiar with Gabor theory and the basic notions in this theory, such as frame, frame operator, frame bounds, tight frame, dual frame, Zak transform, etc. To fix notations, we denote for $x, y \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$

$$g(x,y)(t) = e^{2\pi i y t}\, g(t-x)\ , \qquad \text{a.e. } t \in \mathbb{R}\ . \tag{1}$$

When $a > 0$, $b > 0$ and $g \in L^2(\mathbb{R})$, we call the collection of functions $\{g(na, mb)\,|\, n, m \in \mathbb{Z}\}$ a Gabor system which we denote by (g, a, b). Such a system is called a Gabor frame when there are $A > 0$, $B < \infty$ such that for all $f \in L^2(\mathbb{R})$

$$A\,\|f\|^2 \le \sum_{n,m} |(f, g(na, mb))|^2 \le B\,\|f\|^2\ . \tag{2}$$

Here $\|\ \|$ and $(\ ,\ )$ are the standard norm and inner product of $L^2(\mathbb{R})$. We call A a lower frame bound and B an upper frame bound of the Gabor system (g, a, b). We refer to [1]–[3] for generalities about frames for a Hilbert space and for both basic and in- depth information about Gabor frames and related matters. A very recent and comprehensive treatment of modern Gabor theory can be found in [4], in particular Chs. 5–9, 11–13.

In this paper we are interested in the tight frames at critical density $a = b = 1$, where tightness means that $A = B = 1$ in (2). It is well known, see for instance [4], Corollary 7.5.2, that $(h,\ a = 1,\ b = 1)$ is a tight Gabor frame if and only if $(h(n,m))_{n,m\in\mathbb{Z}}$ is an orthonormal base for $L^2(\mathbb{R})$. Another characterization of tightness is obtained in terms of the Zak transform. When $f \in L^2(\mathbb{R})$ we let

$$(Zf)(t,\nu) = \sum_{k=-\infty}^{\infty} f(t-k)\, e^{2\pi i k\nu}\ , \qquad \text{a.e. } t, \nu \in \mathbb{R}\ , \tag{3}$$

and we call Zf the Zak transform of f, see [4], Ch. 8. Then we have that $(h,\ a = 1,\ b = 1)$ is a tight Gabor frame if and only if

$$|(Zh)(t,\nu)| = 1\ , \qquad \text{a.e. } t, \nu \in \mathbb{R}\ . \tag{4}$$

It is well known that a window h such that $(h,\ a = 1,\ b = 1)$ is a tight frame, cannot be too well-behaved in terms of smoothness and decay, see for instance [4], Theorem 8.4.5 (Balian-Low theorem). Two examples of an h for which $(h,\ a = 1,\ b = 1)$ is a tight frame are

$$h = \chi_{(-1/2,1/2)}\ , \qquad h = \mathcal{F}\,\chi_{(-1/2,1/2)} = \operatorname{sinc} \pi\,\cdot\ \ , \tag{5}$$

where $\mathcal{F}$ denotes the Fourier transform, $(\mathcal{F}f)(\nu) = \int \exp(-2\pi i\nu t)\, f(t)\, dt$ for $f \in L^2(\mathbb{R})$. The Zak transform of the two windows in (5) are given by

$$(Z\,\chi_{(-1/2,1/2)})(t,\nu) = 1\,, \quad (Z\,\mathcal{F}\,\chi_{(-1/2,1/2)})(t,\nu) = e^{2\pi i t\nu}\,, \quad \text{a.e. } (t,\nu) \in [0,1)^2\,, \tag{6}$$

respectively.

One can generate tight frames (h, $a=1$, $b=1$) as follows. Given a well-behaved window $g \in L^2(\mathbb{R})$ such that Zg is a continuous function with few zeros per unit square, set

$$h = Z^{-1}(Zg/|Zg|)\,, \tag{7}$$

with Z^{-1} the inverse Zak transform. That is

$$h(t) = \int_0^1 \frac{(Zg)(t,\nu)}{|(Zg)(t,\nu)|}\, d\nu\,, \qquad \text{a.e. } t \in \mathbb{R}\,. \tag{8}$$

Obviously this h satisfies (4). In this paper we consider in particular the choices

$$g_{1,\gamma}(t) = (2\gamma)^{1/4}\, e^{-\pi\gamma t^2}\,, \qquad g_{2,\gamma}(t) = \Big(\frac{\pi\gamma}{2}\Big)^{1/2} \frac{1}{\cosh \pi\gamma t}\,, \qquad t \in \mathbb{R}\,, \tag{9}$$

(Gaussian and hyperbolic secant) with $\gamma > 0$, and

$$g_{3,\alpha}(t) = \alpha^{1/2}\, e^{-\alpha|t|}\,, \qquad t \in \mathbb{R}\,, \tag{10}$$

(two-sided exponential) with $\alpha > 0$. The normalizations in (9) and (10) are such that the g's have unit L^2-norm. Note that the two windows in (9) are smooth and rapidly decaying; they even have a Fourier invariance property in the sense that $\mathcal{F}\, g_{1,\gamma} = g_{1,1/\gamma}$, $\mathcal{F}\, g_{2,\gamma} = g_{2,1/\gamma}$. The two-sided exponential in (10) is an example of a window of, what we call, type II, see [5], Subsec. 2.2. The latter windows are even, positive and integrable on $\mathbb{R}$, and satisfy a convexity condition on $[0,\infty)$; in particular, g' has a jump at $t=0$ for such a window g. The Zak transforms of the windows in (9) and those of windows of type II all have exactly one zero, viz. at $(\frac{1}{2},\frac{1}{2})$, in the unit square $[0,1)^2$, whence the definition of h according to (7) makes sense. In this paper we shall investigate the windows h produced by the operation (7), where we pay particular attention to the cases that g is one of the windows in (9) and (10).

Although we present all our results directly in terms of $h(t)$ itself, it is well worth noting that a number of these results hold when h is replaced by its Fourier transform $\mathcal{F}\,h$. Indeed, it follows from the relation

$$(Z\,\mathcal{F}\,f)(t,\nu) = e^{2\pi i\nu t}(Zf)(-\nu,t)\,, \qquad \text{a.e. } t,\nu \in \mathbb{R}\,, \tag{11}$$

holding for $f \in L^2(\mathbb{R})$, that the operation $g \to h = Z^{-1}(Zg/|Zg|)$ commutes with the Fourier transform. Moreover, g is even and real if and only if $\mathcal{F}g$ is. Finally, by the quasi-periodicity relations of the Zak transform and (11), we have that Zg has a single zero at $(\frac{1}{2}, \frac{1}{2})$ in the unit square $[0,1)^2$ if and only if $Z\mathcal{F}g$ has, and the nature of these zeros is the same.

2 Overview of results

In Fig. 1 we display the h defined by (7) for the choice $g = g_{1,\gamma}$ and $g = g_{3,\alpha}$ with $\gamma = 1$ and $\alpha = \sqrt{2\pi}$, respectively. The choice of γ, α is such that $g_{1,\gamma}$ and $g_{3,\alpha}$ yield the same variances $\int t^2\,|g(t)|^2\,dt = 1/4\pi\gamma = 1/2\alpha^2$. We observe a striking similarity between the two h's: only inspection of Fig. 1 using a looking glass reveals that there are indeed *two* graphs. We have not displayed the h corresponding to $g_{2,\gamma}$: in [6] it has been shown that the h's corresponding to $g_{1,\gamma}$ and $g_{2,\gamma}$ are identical for all $\gamma > 0$. Hence it so seems that the operation embodied by formulas (7–8) diminishes distances between the windows considerably when scaling parameters are chosen appropriately. There are more qualitative statements of this type. For instance, we have for the windows in (9) and any window of type II that the corresponding h satisfies

$$h(n) = \delta_{no}\ , \qquad h(n+\tfrac{1}{2}) = \frac{(-1)^n}{\pi(n+\frac{1}{2})}\ , \qquad n \in \mathbb{Z}\ , \tag{12}$$

(Kronecker's delta), and

$$h(t) + h(1-t) \le \frac{4}{\pi}\,,\quad t \in [0,1]\ ; \qquad |h(t)| \le 1\,,\quad t \in \mathbb{R}\ . \tag{13}$$

Moreover, all these h's are continuous, even, real and, under a mild condition, in $L^1(\mathbb{R})$. Although a window h is by no means uniquely determined by the above properties, equalities and inequalities, it seems that a detailed investigation of one carefully selected h already will reveal a great deal of the more salient properties of the h's.

The window g that lends itself best for proving analytic properties of the corresponding h is the two-sided exponential $g_{3,\alpha}$ in (10). The Zak transform of $g_{3,\alpha}$ assumes a particular simple form, viz.

$$(Zg_{3,\alpha})(t,\nu) = 2\alpha^{1/2}e^{-\alpha}\,\frac{\sinh\alpha(1-t) + e^{2\pi i\nu}\sinh\alpha t}{1 - 2e^{-\alpha}\cos 2\pi\nu + e^{-2\alpha}}\ , \qquad (t,\nu) \in [0,1)^2\ , \tag{14}$$

see [7] and Prop. 5.1. Accordingly, $h_{3,\alpha}$ (the h corresponding to $g = g_{3,\alpha}$ under the mapping (7–8)) is given by

$$h_{3,\alpha}(t+n) = \int_0^1 \frac{1 + r(t)\,e^{2\pi i\nu}}{|1 + r(t)\,e^{2\pi i\nu}|}\,e^{2\pi i n\nu}\,d\nu\ , \qquad t \in [0,1)\ , \quad n \in \mathbb{Z}\ , \tag{15}$$

where

$$r(t) = \sinh \alpha t / \sinh \alpha(1-t)\ , \qquad 0 \le t < 1\ . \tag{16}$$

The forms (15–16) for $h_{3,\alpha}$ give rise to a number of representations of $h_{3,\alpha}$ that are useful for showing analytic results.

Let us now describe the further contents of this paper. In Sec. 3 we show the following. Assume that g is a smooth and rapidly decaying window such that Zg has a finite number of zeros in the unit square $[0,1)^2$. Also assume that (g,a,a) is a Gabor frame for all $a \in (0,1)$, and denote by S_a the corresponding frame operator, i.e.

$$S_a\,f = \sum_{n,m} (f, g(na,ma))\,g(na,ma)\ , \qquad f \in L^2(\mathbb{R})\ . \tag{17}$$

Then

$$\lim_{a \uparrow 1} S_a^{-1/2}\,g = h = Z^{-1}(Zg/|Zg|)\ , \tag{18}$$

where the limit is in $L^2(\mathbb{R})$-sense. We observe that $S_a^{-1/2}\,g$ is the tight frame generating window canonically associated with the Gabor frame (g,a,a), see [8] for characterization and computation of these windows $S_a^{-1/2}\,g$.

In Sec. 4 we consider even, positive, continuous, rapidly decaying windows g, and we list how symmetry properties of g are reflected by corresponding properties of the Zak transforms Zg. These symmetry properties are then also shown to hold for $h = Z^{-1}(Zg/|Zg|)$. We furthermore show in Sec. 4 that $h \in L^1(\mathbb{R})$ when Zg has a single zero, at $(\frac{1}{2},\frac{1}{2})$, in the unit square $[0,1)^2$ and

$$\frac{\partial\,Zg}{\partial t}\,(\tfrac{1}{2},\tfrac{1}{2}) \ne 0 \ne \frac{\partial\,Zg}{\partial \nu}\,(\tfrac{1}{2},\tfrac{1}{2})\ . \tag{19}$$

We also show the properties, equalities and inequalities mentioned in connection with (12) and (13).

It is well known that $(g_{1,\gamma},a,b)$ is a frame when $a,b > 0$ and $ab < 1$; recently a corresponding result has been shown for the Gabor systems $(g_{2,\gamma},a,b)$, see [6]. In Sec. 5 we shall show that $(g_{3,\alpha},a,b)$ is a frame when $a,b > 0$ and $ab < 1$. The approach to the proof of the latter result is basically the same as the one to the main result in [6], but the details are quite different.

In Sec. 6 we shall present a detailed analysis of $h_{3,\alpha} = Z^{-1}(Zg_{3,\alpha}/|Zg_{3,\alpha}|)$. This analysis is based on integral representations of $h_{3,\alpha}$, like the direct one in (15) or the representation (we set $r = r(t)$ for convenience)

$$h_{3,\alpha}(t+n) = \frac{(-1)^n}{\pi} \int_0^{\min(r,1/r)} v^{n-1/2} \sqrt{\frac{1-rv}{r-v}}\, dv \,, \qquad t \in [0,1)\,, \quad n = 0,1,\ldots\,, \tag{20}$$

that follows from the direct one, as well as on certain series representations of $h_{3,\alpha}(t+n)$. These representations are used to show that

(a) $h_{3,\alpha}$ is positive and decreasing on $[0,1)$,

(b) $(-1)^n\, h_{3,\alpha}(t+n) > 0$, $t \in [0,1)$, $n = 0,1,\ldots$,

(c) $(-1)^n\, h'_{3,\alpha}(t+n) < 0$, $t \in [\frac{1}{2},1)$, $n = 0,1,\ldots$,

(d) $(-1)^n\, h'_{3,\alpha}(n+\frac{1}{2}) = -\infty$, $n = 0,1,\ldots$.

Furthermore, bounds on $h_{3,\alpha}$ and asymptotics of $h_{3,\alpha}(t+n)$ as $n \to \infty$ and $t \in [0,1)$ is fixed can be derived. As a consequence we can get a more precise version of the statement shown in Sec. 4 that $h \in L^1(\mathbb{R})$. We also pay attention in Sec. 6 to the way the misbehaviour of $g_{3,\alpha}$ at $t = 0$ is reflected by corresponding misbehaviour or $h_{3,\alpha}$ at $t = n$, n integer.

In Sec. 7 we consider the case that $\alpha \downarrow 0$ in more detail. Then we get $r(t) = t/(1-t)$ in (16). Note that $h_{3,\alpha}$ can be expressed in terms of $h_{3,0}$ by the warping operation

$$h_{3,\alpha}(t+n) = h_{3,0}(\psi_\alpha(t)+n)\,; \qquad \psi_\alpha(t) = \frac{r(t)}{1+r(t)}\,, \quad 0 \le t < 1\,, \quad n = 0,1,\ldots\,. \tag{21}$$

We shall compute explicitly in Sec. 7 that for $n = 0,1,\ldots$

$$\int_0^{1/2} |h_{3,0}(t+n)|\, dt = \Big(\frac{1}{2}+\frac{1}{\pi}\Big)\, H_n\,, \qquad \int_{1/2}^{1} |h_{3,0}(t+n)|\, dt = \Big(\frac{1}{2}-\frac{1}{\pi}\Big)\, H_n\,, \tag{22}$$

where

$$H_n = \int_0^1 |h_{3,0}(t+n)|\, dt = 2(-1)^n \Big(\frac{\pi}{4} - \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} - \frac{(-1)^n}{2(2n+1)}\Big)\,. \tag{23}$$

From this it follows that $\|h_{3,0}\|_1 = \frac{1}{2}\,\pi$.

In Sec. 8 we consider the case that $g_\gamma = g_{1,\gamma}$ or $g_{2,\gamma}$ and $\gamma \to \infty$ or $\gamma \downarrow 0$; recall that in [6] it has been shown that $h_\gamma := h_{1,\gamma} = h_{2,\gamma}$ for all $\gamma > 0$. We show that

$$\lim_{\gamma\to\infty} h_\gamma = \chi_{(-1/2,1/2)} \ , \qquad \lim_{\gamma\downarrow 0} h_\gamma = \operatorname{sinc}\pi\,\cdot \ , \tag{24}$$

where the limits are in $L^2(\mathbb{R})$-sense. We shall furthermore present a simple condition on a window g such that the limit formulas (24) hold, more generally, for the tight frame generating window h_γ corresponding to the window $\gamma^{1/2}\,g(\gamma\cdot)$. Thus one obtains a whole class of families $((h_\gamma)_{n,m})_{n,m\in\mathbb{Z}}$ of reasonably behaved orthonormal Gabor bases (tight Gabor frames at critical density $a = b = 1$) that interpolate smoothly between the Gabor bases generated by the box function at $\gamma = \infty$ and its Fourier transform $\operatorname{sinc}\pi\cdot$ at $\gamma = 0$, compare (5).

3 Zak tight windows as limits of frame tight windows

In this section we prove the following result.

Theorem 3.1. Assume that we have a $g \in L^2(\mathbb{R})$ such that $(g, g(x, y))$ is rapidly decaying in $x, y \in \mathbb{R}$, see (1), and that Zg has a finite number of zeros in the unit square $[0, 1)^2$. Also assume that (g, a, a) is a frame for all $a \in (0, 1)$, with frame operator S_a. Then there holds

$$\lim_{a\uparrow 1} S_a^{-1/2}\, g = h = Z^{-1}(Zg/|Zg|) \tag{25}$$

in $L^2(\mathbb{R})$-sense.

Proof. Denote ${}^a h = S_a^{-1/2}\, g$ for $a \in (0, 1)$. The proof uses the following steps.

(a) $\|{}^a h\| = a$, $\|h\| = 1$,

(b) $S_a \to S_1$ strongly as $a \uparrow 1$ (S_1 is the operator in (17) with $a = 1$),

(c) $S_a^{1/2} \to S_1^{1/2}$ strongly as $a \uparrow 1$,

(d) $({}^a h, k) \to (h, k)$ as $a \uparrow 1$ for all k of the form $k = S_1^{1/2}\, f$ with $f \in L^2(\mathbb{R})$,

(e) ${}^a h \to h$ weakly as $a \uparrow 1$,

(f) $\|{}^a h - h\| \to 0$ as $a \uparrow 1$.

Ad (a). This follows from the well-known Wexler-Raz duality condition and the definition of h, ${}^a h$.

Ad (b). In Janssen's representation, see [4], Subsec. 7.2,

$$S_a f = \frac{1}{a^2} \sum_{k,l} (g, g(k/a, l/a))\, f(k/a, l/a) \ , \qquad f \in L^2(\mathbb{R}) \ , \tag{26}$$

there is absolute convergence of the right-hand side series. By continuity and rapid decay of $(g, g(x,y))$ (also see Note 3 below and (33)–(35)) we have that $S_a f \to S_1 f$ as $a \uparrow 1$ and $f \in L^2(\mathbb{R})$.

Ad (c). Approximate $x^{1/2}$ uniformly on $[0, B]$ by polynomials, where B is an upper frame bound for all systems (g, a, a) with $\frac{1}{2} \le a \le 1$. Use the spectral mapping theorem for the positive semi-definite operators S_a, S_1 to get polynomial approximations to $S_a^{1/2}$, $S_1^{1/2}$ in the strong operator topology, and finally we use that $S_a \to S_1$ strongly implies that $S_a^n \to S_1^n$ strongly for all $n = 0, 1, \ldots$.

Ad (d)). Let $k = S_1^{1/2} f$ with $f \in L^2(\mathbb{R})$. Below we show that for $0 < a < 1$

$$g = S_1^{1/2}\, h = S_a^{1/2}\, {}^a h \ . \tag{27}$$

From this it follows that

$$({}^a h - h, k) = ({}^a h - h, S_1^{1/2} f) = ({}^a h, S_1^{1/2} f) - (g, f) \ . \tag{28}$$

We also have

$$\begin{aligned} ({}^a h, S_1^{1/2} f) &= ({}^a h, S_a^{1/2} f) + ({}^a h, (S_1^{1/2} - S_a^{1/2}) f) = \\ &= (g, f) + o(1) \ , \qquad a \uparrow 1 \ , \end{aligned} \tag{29}$$

by (27) and (a), (c). Hence $({}^a h - h, k) \to 0$ as $a \uparrow 1$.

We now show (27). Clearly we have $g = S_1^{1/2}\, {}^a h$ since ${}^a h = S_a^{-1/2} g$. To show that $g = S_1^{1/2} h$ we let $\varepsilon > 0$. By functional calculus in the Zak transform domain for S_1, see for instance [8], Subsec. 1.1 (in particular formulas (37) or (38)), we have

$$Z\,[(S_1 + \varepsilon I)^{1/2} f] = (|Zg|^2 + \varepsilon)^{1/2}\, Zf \ , \qquad f \in L^2(\mathbb{R}) \ . \tag{30}$$

Letting $\varepsilon \downarrow 0$ and using strong convergence of $(S_1 + \varepsilon I)^{1/2}$ to $S_1^{1/2}$, we see that

$$Z\,[S_1^{1/2} f] = |Zg|\, Zf \ , \qquad f \in L^2(\mathbb{R}) \ . \tag{31}$$

Now taking $f = h$ in (31) and using the definition of h we easily obtain $Zg = Z(S_1^{1/2}\, h)$, i.e. $g = S_1^{1/2}\, h$.

Ad (e). We shall show that the set of all $k = S_1^{1/2}\, f$ with $f \in L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. From (a) and (d) it then follows that ${}^a h \to h$ weakly as $a \uparrow 1$.

By assumption Zg is continuous and has finitely many zeros in $[0,1)^2$. Hence the set $|Zg| \cdot F$ with $F \in L^2([0,1)^2)$ is dense in $L^2([0,1)^2)$. The mapping Z is a unitary operator from $L^2(\mathbb{R})$ onto $L^2([0,1)^2)$, whence the set $|Zg| \cdot Zf$ with $f \in L^2(\mathbb{R})$ is dense in $L^2([0,1)^2)$. Then from (31) and unitarity of Z it follows that the set $S_1^{1/2}\, f$ with $f \in L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

(f). This follows easily from (a) and (e).

Notes.

1. We could have considered equally well Gabor frames (g, a, b) or $(g, a, 1)$ or $(g, 1, b)$ as $a \uparrow 1$ and/or $b \uparrow 1$ with corresponding frame operators.

2. Assume that $(g, 1, 1)$ has a finite upper frame bound while Zg is continuous and has finitely many zeros in the unit square. Then

$$\lim_{\varepsilon \downarrow 0} (S_1 + \varepsilon I)^{-1/2}\, g = h \tag{32}$$

   in $L^2(\mathbb{R})$-sense. This follows on inspecting the arguments used to prove (d) and (e).

3. The author was kindly informed by H.G. Feichtinger that he and G. Zimmermann have shown that (b) in the proof holds on the assumption that $g \in S_0(\mathbb{R})$, see [3], ch. 3 for information on $S_0(\mathbb{R})$ (private communication).

We check the condition of continuity and rapid decay of $(g, g(x,y))$ as a function of $x, y \in \mathbb{R}$ for the cases that $g = g_{1,\gamma}, g_{2,\gamma}, g_{3,\alpha}$, see (9), (10). There holds, explicitly,

$$(g_{1,\gamma}, g_{1,\gamma}(x,y)) = e^{-\pi i x y}\, e^{-\frac{1}{2}\pi\gamma x^2 - \frac{1}{2}\pi y^2/\gamma}\ , \tag{33}$$

$$(g_{2,\gamma}, g_{2,\gamma}(x,y)) = \frac{\pi\, e^{-\pi i x y} \sin \pi x y}{\sinh \pi\gamma x \sinh \pi y/\gamma}\ , \tag{34}$$

$$(g_{3,\alpha}, g_{3,\alpha}(x,y)) = \alpha\, e^{-\pi i x y}\, e^{-\alpha|x|} \left\{ \frac{\alpha \cos \pi\, |x|\, y}{\alpha^2 + \pi^2 y^2} + \frac{\alpha^2 \sin \pi\, |x|\, y}{y(\alpha^2 + \pi^2 y^2)} \right\} \tag{35}$$

for $x, y \in \mathbb{R}$. This amount of decay is sufficient for the proof of Theorem 3.1 to work (also see Note 3 above).

As already said the Gabor systems $(g_{1,\gamma}, a, b)$, $(g_{2,\gamma}, a, b)$ are Gabor frames for any $\gamma > 0$, $a > 0$, $b > 0$ with $ab < 1$. In Sec. 5 we shall show that $(g_{3,\alpha}, a, b)$ is a Gabor frame as well when $\alpha > 0$, $a > 0$, $b > 0$ and $ab < 1$.

In the three cases $g = g_{1,\gamma}, g_{2,\gamma}, g_{3,\alpha}$ we have that Zg has a single zero, viz. at $(\frac{1}{2}, \frac{1}{2})$ in the unit square $[0,1)^2$. There holds, more precisely,

$$(Zg_{1,\gamma})(t,\nu) = \pi(2\gamma)^{1/4}\, \vartheta_1'(0)(\gamma(\tfrac{1}{2} - t) + i(\tfrac{1}{2} - \nu)) + O_2 \ , \tag{36}$$

$$(Zg_{2,\gamma})(t,\nu) = \pi^{3/2}\, 2^{-1/2}\, \gamma\Big(\frac{\vartheta_1'(0)}{\vartheta_3(0)}\Big)^2 (\gamma(\tfrac{1}{2} - t) + i(\tfrac{1}{2} - \nu)) + O_2 \ , \tag{37}$$

$$(Zg_{3,\alpha})(t,\nu) = \frac{\alpha^{3/2}}{\cosh \frac{1}{2}\alpha} \Big(\tfrac{1}{2} - t + \pi i \, \frac{\tanh \frac{1}{2}\alpha}{\alpha}\, (\tfrac{1}{2} - \nu)\Big) + O_2 \ , \tag{38}$$

where O_2 abbreviates a term of order $(t - \frac{1}{2})^2 + (\nu - \frac{1}{2})^2$. In (36–37) we have the same conventions about the theta functions as in [6].

In Fig. 2 we show $h_{3,\alpha}$ and ${}^a h_{3,\alpha}$ for $\alpha = \sqrt{2\pi}$ and $a = b = \sqrt{0.9}$, and in Fig. 3 we show $h_{3,\alpha}$ and ${}^a h_{3,\alpha}$ for $\alpha = \sqrt{2\pi}$ and $a = b = \sqrt{0.99}$. The $h_{3,\alpha}$ was obtained from (8) while the ${}^a h_{3,\alpha}$'s were obtained using the algorithm to compute $S_a^{-1/2}\, h$ as can be found in [8], Sec. 4.

4 Even, positive windows with one Zak transform zero

In this section we consider general even, positive continuous windows g with sufficient decay so that Zg is a continuous function on $\mathbb{R}^2$. We are particularly interested here in windows g such that Zg has a single zero in $[0,1)^2$, which then must occur at $(\frac{1}{2}, \frac{1}{2})$. A class of windows g having all these properties was studied in [5], Sec. 2. The windows g considered there are even and continuous, and have on $[0, \infty)$ the form

$$g(t) = b(t) + b(t+1) \ , \qquad t \geq 0 \ , \tag{39}$$

with b integrable, non-negative and strictly convex on $[0, \infty)$. The latter condition is referred to as super convexity in [5]. A disadvantage of this class is that it is not dilation invariant because of the occurrence of the shift-by-one operation in (39). A class of windows that is dilation invariant and whose members are super convex with an arbitrary shift operator $b \to (b \cdot\ + \Delta)$ in (39) is considered in [5], Subsec. 2.4.4: g even, integrable and continuous with $(-1)^n\, g^{(n)}(t) > 0$ for $t \geq 0$, $n = 0, 1, 2, 3$.

Symmetry properties and the Zak transform. Let $g : \mathbb{R} \to \mathbb{C}$ be continuous and (rapidly) decaying so that we can consider $Z = Zg$ in a pointwise manner. There hold the quasi-periodicity relations

$$Z(t+1,\nu) = e^{2\pi i\nu}\, Z(t,\nu) \ , \qquad Z(t,\nu+1) = Z(t,\nu) \ , \qquad t,\nu \in \mathbb{R} \ . \quad (40)$$

There holds furthermore

$$g \text{ real} \Leftrightarrow Z^*(t,\nu) = Z(t,-\nu) \ , \qquad t,\nu \in \mathbb{R} \ , \quad (41)$$

$$g \text{ even} \Leftrightarrow Z(t,\nu) = Z(-t,-\nu) \ , \qquad t,\nu \in \mathbb{R} \ . \quad (42)$$

From the quasi-periodicity relations in (40) and (41), (42) one gets the following further properties of Z when g is real and even:

$$Z(1-t,\nu) = e^{2\pi i\nu}\, Z(-t,\nu) = e^{2\pi i\nu}\, Z^*(t,\nu) \ , \quad (43)$$

$$Z(t,1-\nu) = Z(t,-\nu) = Z^*(t,\nu) = Z(-t,\nu) \ , \quad (44)$$

$$e^{-\pi i\nu}\, Z(\tfrac{1}{2},\nu) = e^{\pi i\nu}\, Z^*(\tfrac{1}{2},\nu) \in \mathbb{R} \ , \quad (45)$$

$$Z(t,0) \in \mathbb{R} \ , \qquad Z(t,\tfrac{1}{2}) = -Z(1-t,\tfrac{1}{2}) \in \mathbb{R} \ , \quad (46)$$

$$e^{-\pi i\nu}\, Z(\tfrac{1}{2},\nu) = -e^{-\pi i(1-\nu)}\, Z(\tfrac{1}{2},1-\nu) \ , \quad (47)$$

$$Z(0,\nu) = Z(1,\nu)\, e^{-2\pi i\nu} \in \mathbb{R} \ , \quad (48)$$

$$Z(\tfrac{1}{2},\tfrac{1}{2}) = 0 \ , \quad (49)$$

where $t,\nu \in \mathbb{R}$.

We next consider for real and even windows g with single zero of Zg in $[0,1)^2$ at $(\frac{1}{2},\frac{1}{2})$ the tight frame generating $h = Z^{-1}(Zg/|Zg|)$ that is given explicitly by (I any interval of length 1)

$$h(t) = \int\limits_I \frac{(Zg)(t,\nu)}{|(Zg)(t,\nu)|}\, d\nu \ , \qquad t \in \mathbb{R} \ . \quad (50)$$

We collect the following obvious properties of h.

Proposition 4.1. We have

(a) h is real and even,

(b) h is continuous,

(c) h is bounded: $|h(t)| \leq 1$, $t \in \mathbb{R}$.

Proof. Note that $Zh = Zg/|Zg|$. Then

(a) follows from the equivalences in (41), (42),

(b) follows from continuity of Zg and the fact that Zg has just one zero per unit square,

(c) follows trivially from (50).

The following result is slightly less obvious.

Proposition 4.2. Assume that $g(t) > 0$, $t \in \mathbb{R}$. Then we have

$$h(n) = \delta_n \ , \qquad h(n + \tfrac{1}{2}) = \frac{(-1)^n}{\pi(n + \frac{1}{2})} \ , \tag{51}$$

where δ_n denotes Kronecker's delta.

Proof. We have for $t \in [0,1)$, $n \in \mathbb{Z}$ by the first relation in (40)

$$h(n+t) = \int_I (Zh)(n+t,\nu)\,d\nu = \int_I \frac{(Zg)(t,\nu)}{|(Zg)(t,\nu)|}\,e^{2\pi i n\nu}\,d\nu \ . \tag{52}$$

Since $(Zg)(0,\nu)$ is real, $\neq\ 0$ (as Zg vanishes only at $(\frac{1}{2},\frac{1}{2})$ in $[0,1)^2$) and $(Zg)(0,0) > 0$, we have that $(Zg)(0,\nu) > 0$ for $\nu \in \mathbb{R}$. Consequently, $(Zg)(0,\nu)/|(Zg)(0,\nu)| = 1$ for $\nu \in \mathbb{R}$ and it follows at once that $h(n) = \delta_n$, $n \in \mathbb{Z}$.

Next we have from (45) that $\exp(-\pi i\nu)(Zg)(\frac{1}{2},\nu)$ is real, $\neq\ 0$, when $\nu \in [0,1)$, $\nu \neq \frac{1}{2}$. Moreover, by (47) we see that $\exp(-\pi i\nu)(Zg)(\frac{1}{2},\nu)$ is positive for $0 \leq \nu < \frac{1}{2}$ and negative for $\frac{1}{2} < \nu \leq 1$. Hence

$$\begin{aligned}
h(n+\tfrac{1}{2}) \ &= \ \int_0^1 \frac{(Zg)(\frac{1}{2},\nu)}{|(Zg)(\frac{1}{2},\nu)|}\,e^{2\pi i n\nu}\,d\nu \ = \\
&= \ \int_0^1 \frac{(Zg)(\frac{1}{2},\nu)\,e^{-\pi i\nu}}{|(Zg)(\frac{1}{2},\nu)\,e^{-\pi i\nu}|}\,e^{2\pi i(n+\frac{1}{2})\nu}\,d\nu \ = \\
&= \ \int_0^1 \mathrm{sgn}(\tfrac{1}{2}-\nu)\,e^{2\pi i(n+\frac{1}{2})\nu}\,d\nu = \frac{(-1)^n}{\pi(n+\frac{1}{2})} \ ,
\end{aligned} \tag{53}$$

as claimed.

We next give an inequality for $h(t)$, $t \in [0,1)$.

Proposition 4.3. We have

$$h(t) + h(1-t) \le \frac{4}{\pi}\ , \qquad t \in [0,1)\ . \tag{54}$$

Proof. There holds by (43) for any $t \in [0,1)$

$$\begin{aligned} h(t) + h(1-t) &= \int\limits_{-1/2}^{1/2} \frac{(Zg)(t,\nu) + e^{2\pi i\nu}(Zg)^*(t,\nu)}{|(Zg)(t,\nu)|}\, d\nu = \\ &= 2 \int\limits_{-1/2}^{1/2} \frac{\mathrm{Re}(e^{-\pi i\nu}(Zg)(t,\nu))}{|(Zg)(t,\nu)|}\, e^{\pi i\nu}\, d\nu\ . \end{aligned} \tag{55}$$

Since $h(t) + h(1-t)$ is real we can take the real part of (55). Then the inequality follows from the fact that

$$\cos \pi\nu \ge 0\ , \qquad \frac{\mathrm{Re}(e^{-\pi i\nu}(Zg)(t,\nu))}{|(Zg)(t,\nu)|} \le 1\ , \qquad -\tfrac{1}{2} \le \nu \le \tfrac{1}{2}\ , \tag{56}$$

and $\int\limits_{-1/2}^{1/2} \cos \pi\nu\, d\nu = 2/\pi$.

Note. For g super convex there is also the inequality $h(t) + h(1-t) \ge 0$, $t \in [0,1)$; this requires a rather delicate analysis of the Zak transforms of super convex functions, compare [5], Subsec. 2.2. In Sec. 6 this latter inequality will be strengthened to $h(t) + h(1-t) \ge 1$, $t \in [0,1)$, for the case that $g = g_{3,\alpha}$.

We conclude this section by showing that, under some further conditions on g, the h of (50) is in $L^1(\mathbb{R})$. These conditions are that Zg is smooth on $[0,1)^2$ and that there are real $A \ne 0 \ne B$ such that (as $(t,\nu) \to (\frac{1}{2},\frac{1}{2})$)

$$(Zg)(t,\nu) = A(t-\tfrac{1}{2}) + iB(\nu-\tfrac{1}{2}) + O((t-\tfrac{1}{2})^2 + (\nu-\tfrac{1}{2})^2)\ . \tag{57}$$

Observe that

$$\begin{aligned} A &= \frac{\partial\, Zg}{\partial t}\,(\tfrac{1}{2},\tfrac{1}{2}) = 2 \sum_{k=0}^{\infty} (-1)^k\, g'(k+\tfrac{1}{2})\ , \\ B &= \frac{1}{i}\,\frac{\partial\, Zg}{\partial \nu}\,(\tfrac{1}{2},\tfrac{1}{2}) = -2\pi \sum_{k=0}^{\infty} (-1)^k (2k+1)\, g(k+\tfrac{1}{2})\ . \end{aligned} \tag{58}$$

Theorem 4.1. Under the above conditions on g we have

$$\sum_{n=-\infty}^{\infty} |h(t+n)| = O(|t-\tfrac{1}{2}|^{-1/2}) \;, \qquad t \in [0,1) \;. \tag{59}$$

In particular we have that $h \in L^1(\mathbb{R})$.

Proof. We have for $t \in [0,1)$, $t \neq \frac{1}{2}$ by partial integration

$$\begin{aligned} h(t+n) &= \int_0^1 e^{2\pi i n\nu} (Zh)(t,\nu)\, d\nu \;= \\ &= \frac{-1}{2\pi i n} \int_0^1 \frac{\partial\, Zh}{\partial \nu}\,(t,\nu)\, e^{2\pi i n \nu}\, d\nu \;, \qquad n \in \mathbb{Z} \;. \end{aligned} \tag{60}$$

Hence by the Cauchy-Schwarz inequality and Parseval's theorem for Fourier series

$$\begin{aligned} \sum_{n\neq 0} |h(t+n)| &\leq \Big(\sum_{n\neq 0} \frac{1}{4\pi^2 n^2}\Big)^{1/2} \Big(\sum_{n \neq 0} \Big|\int_0^1 \frac{\partial Zh}{\partial \nu}\,(t,\nu)\, e^{2\pi i n\nu}\, d\nu\Big|^2\Big)^{1/2} \leq \\ &\leq \Big(\frac{1}{12} \int_0^1 \Big|\frac{\partial\, Zh}{\partial \nu}\,(t,\nu)\Big|^2 d\nu\Big)^{1/2} \;. \end{aligned} \tag{61}$$

We shall estimate

$$I(t) := \int_0^1 \Big|\frac{\partial\, Zh}{\partial \nu}\,(t,\nu)\Big|^2 d\nu \;. \tag{62}$$

We have for $(t,\nu) \neq (\frac{1}{2}, \frac{1}{2})$ from $Zh = Zg/|Zg|$ that

$$\frac{\partial\, Zh}{\partial \nu} = \frac{\dfrac{\partial\, Zg}{\partial \nu}}{|Zg|} - \frac{\dfrac{\partial}{\partial \nu}\,|Zg|}{(Zg)^*} \;. \tag{63}$$

Furthermore there holds for $(t,\nu) \neq (\frac{1}{2}, \frac{1}{2})$

$$\frac{\partial}{\partial \nu}\,|(Zg)(t,\nu)| \leq \Big|\frac{\partial\, Zg}{\partial \nu}\,(t,\nu)\Big| \;. \tag{64}$$

Now by the assumptions on Zg, see (57), there are $\varepsilon > 0$, $\delta > 0$ such that

$$|(Zg)(t,\nu)| \geq (\tfrac{1}{2}\, A^2 (t-\tfrac{1}{2})^2 + \tfrac{1}{2}\, B^2(\nu - \tfrac{1}{2})^2)^{1/2} \tag{65}$$

when $|t-\frac{1}{2}| \le \varepsilon$, $|\nu-\frac{1}{2}| \le \delta$. Moreover, since Zg only vanishes at $(\frac{1}{2},\frac{1}{2})$ within $[0,1)^2$ there is a $c>0$ such that

$$|(Zg)(t,\nu)| \ge c \tag{66}$$

when $|t-\frac{1}{2}| \ge \varepsilon$ or $|\nu-\frac{1}{2}| \ge \delta$. Next letting C be an upper bound for $\Big|\dfrac{\partial Zg}{\partial \nu}\Big|$ we conclude that

$$\Big|\frac{\partial Zh}{\partial \nu}(t,\nu)\Big| \le 2C(\tfrac{1}{2}A^2(t-\tfrac{1}{2})^2+\tfrac{1}{2}B^2(\nu-\tfrac{1}{2})^2)^{-1/2} \tag{67}$$

when $|t-\frac{1}{2}| \le \varepsilon$, $|\nu-\frac{1}{2}| \le \delta$, and that

$$\Big|\frac{\partial Zh}{\partial \nu}(t,\nu)\Big| \le 2C/c \tag{68}$$

when $|t-\frac{1}{2}| \ge \varepsilon$ or $|\nu-\frac{1}{2}| \ge \delta$. We conclude from (67), (68) that

$$I(t) \le 4C^2/c^2 \ , \qquad |t-\tfrac{1}{2}| \ge \varepsilon \ , \tag{69}$$

and that

$$I(t) \le (1-2\delta)\,4C^2/c^2 + \int\limits_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} \frac{8C^2}{A^2(t-\frac{1}{2})^2+B^2(\nu-\frac{1}{2})^2}\,d\nu \ , \qquad |t-\tfrac{1}{2}| \le \varepsilon \ . \tag{70}$$

Finally

$$\int\limits_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} \frac{d\nu}{A^2(t-\frac{1}{2})^2+B^2(\nu-\frac{1}{2})^2} \le 2\int\limits_0^\infty \frac{d\nu}{A^2(t-\frac{1}{2})^2+B^2\nu^2} = \frac{\pi}{AB|t-\frac{1}{2}|} \ , \tag{71}$$

and the proof is complete.

Note. We observe that the condition $A \in \mathbb{R}$, $B \in \mathbb{R}$, $A \ne 0 \ne B$, see (57), is satisfied by $g = g_{1,\gamma}, g_{2,\gamma}, g_{3,\alpha}$, see (36)– (38). In Sec. 6 we shall improve Thm. 4.1 for the case that $g = g_{3,\alpha}$.

5 Two-sided exponentials yield Gabor frames

In this section we show that $(g_{3,\alpha}, a, b)$ is a Gabor frame when $a>0$, $b>0$, $ab<1$ and $\alpha>0$. For the proof we follow largely the approach in [6], Sec. 2, where it was shown that $(g_{2,\gamma}, a, b)$ is a Gabor frame when $a > 0$, $b > 0$, $ab < 1$ and $\gamma > 0$. Accordingly, by scale invariance of the class of two-sided exponentials and the fact that (g, a, b) is a frame if and only if $(c^{1/2}\, g(c\cdot), a/c, bc)$ is a frame (any $c > 0$), we can assume that $b = 1$, $a < 1$. We thus consider the Ron-Shen matrices, see [4], Sec. 7.4,

$$(g_{3,\alpha}(t - na - l))_{l\in\mathbb{Z},n\in\mathbb{Z}} \ , \tag{72}$$

and we should show that there are $A > 0$, $B < \infty$ such that

$$A\,\|\mathbf{c}\|^2 \leq \sum_{n=-\infty}^{\infty} \Big|\sum_{l=-\infty}^{\infty} c_l\, g(t - na - l)\Big|^2 \leq B\,\|\mathbf{c}\|^2 \tag{73}$$

for all $t \in \mathbb{R}$ and all $\mathbf{c} \in l^2(\mathbb{Z})$. When $A > 0$ and $B < \infty$ as in (73) exist, they are a lower and upper frame bound, respectively, for the Gabor system $(g_{3,\alpha}, a, 1)$. Evidently, by exponential decay of g, we do not need to worry about the existence of a finite B such that the right inequality in (73) holds for all $t \in \mathbb{R}$, $\mathbf{c} \in l^2(\mathbb{Z})$, and so we concentrate on the left inequality.

We write, see [6], Sec. 2, for $\mathbf{c} \in l^2(\mathbb{Z})$, $t \in \mathbb{R}$, $n \in \mathbb{Z}$

$$\begin{aligned}\sum_l c_l\, g_{3,\alpha}(t - na - l) &= \int_0^1 (Zg_{3,\alpha})(t - na, \nu)\, C^*(\nu)\, d\nu = \\ &= \int_0^1 e^{2\pi i \lfloor t-na \rfloor \nu} (Zg_{3,\alpha})(\langle t - na\rangle, \nu)\, C^*(\nu)\, d\nu,\end{aligned} \tag{74}$$

where for $x \in \mathbb{R}$

$$\lfloor x \rfloor = \text{largest integer} \leq x \ , \qquad \langle x \rangle = x - \lfloor x \rfloor \ , \tag{75}$$

and

$$C(\nu) = \sum_l c_l^*\, e^{2\pi i l \nu} \ , \qquad \text{a.e. } \nu \in \mathbb{R} \ . \tag{76}$$

In (74) the quasi-periodicity relations in (40) have been used.

To proceed we need to calculate $Zg_{3,\alpha}$.

Proposition 5.1. We have for $(t, \nu) \in [0, 1)^2$

$$(Zg_{3,\alpha})(t, \nu) = (r_2(t) + r_1(t)\, e^{2\pi i \nu})\, \Phi(\nu) \ , \tag{77}$$

where

$$\Phi(\nu) = \frac{2\alpha^{1/2}\, e^{-\alpha}}{1 - 2e^{-\alpha} \cos 2\pi\nu + e^{-2\alpha}} \ , \qquad r_1(t) = r_2(1 - t) = \sinh \alpha t \ . \tag{78}$$

Proof. Let $t \in [0,1)$, $\nu \in \mathbb{R}$. Then

$$(Zg_{3,\alpha})(t,\nu) = \sum_{k=-\infty}^{\infty} \alpha^{1/2}\, e^{-\alpha|t-k|}\, e^{2\pi ik\nu} =$$

$$= \alpha^{1/2}\, e^{-\alpha(1-t)} \cdot \frac{e^{2\pi i\nu}}{1-e^{-\alpha}\, e^{2\pi i\nu}} + \alpha^{1/2}\, e^{-\alpha t} \cdot \frac{1}{1-e^{-\alpha}\, e^{-2\pi i\nu}}\ , \qquad (79)$$

where the second identity in (79) follows from splitting up the summation range of the first line series into $k \geq 1$ (so that $t-k > 0$) and $k \leq 0$ (so that $t-k \geq 0$) and summing the two geometric series. Then (77)–(78) follows easily.

We conclude from (74) and Proposition 5.1 that for $\mathbf{c} \in l^2(\mathbb{Z})$, $t \in \mathbb{R}$, $n \in \mathbb{Z}$

$$\sum_l c_l\, g_{3,\alpha}(t-na-l) = r_2(\langle t-na\rangle)\, d_{\lfloor t-na\rfloor} + r_1(\langle t-na\rangle)\, d_{\lfloor t-na\rfloor+1}\ , \qquad (80)$$

when the d_k's are defined by

$$\Phi(\nu)\, C(\nu) = \sum_k d_k^*\, e^{2\pi ik\nu}\ , \qquad \text{a.e. } \nu \in \mathbb{R}\ . \qquad (81)$$

Note that by Parseval's formula for Fourier series we have

$$m\, \|\mathbf{c}\| \leq \|\mathbf{d}\| \leq M\, \|\mathbf{c}\|\ , \qquad (82)$$

where $m > 0$, $M < \infty$ are the minimum and maximum of Φ, and $\mathbf{c}$ and $\mathbf{d}$ are related to one another according to (76), (81).

We can express (80) in the form

$$\sum_l c_l\, g_{3,\alpha}(t-na-l) = (R(t)\, \mathbf{d})_n\ , \qquad (83)$$

where $R(t)$ is the linear operator of $l^2(\mathbb{Z})$ given by its matrix elements with respect to the standard basis of $l^2(\mathbb{Z})$ as

$$R_{nk}(t) = \begin{cases} r_2(\langle t-na\rangle)\ , & k = \lfloor t-na\rfloor \\ r_1(\langle t-na\rangle)\ , & k = \lfloor t-na\rfloor + 1 \\ 0\ , & \text{otherwise}\ , \end{cases} \qquad (84)$$

with n the row index and k the column index. On account of (82) we should thus show that there is a $K > 0$ such that

$$\|R(t)\, \mathbf{d}\|^2 = \sum_{n=-\infty}^{\infty} |(R(t)\, \mathbf{d})_n|^2 \geq K\, \|\mathbf{d}\|^2 \qquad (85)$$

for all $t \in \mathbb{R}$, $\mathbf{d} \in l^2(\mathbb{Z})$.

We distinguish now between the cases that $a \leq \frac{1}{2}$ and $a > \frac{1}{2}$.

Case $a \leq \frac{1}{2}$**.** Take $t \in \mathbb{R}$. For any $k \in \mathbb{Z}$ there are at least two consecutive $n \in \mathbb{Z}$ such that $k = \lfloor t - na \rfloor$; denote the largest, smallest such n by $n_{k,+}$, $n_{k,-}$. Evidently

$$\|R(t)\,\mathbf{d}\|^2 \geq \sum_{k=-\infty}^{\infty} \{|(R(t)\,\mathbf{d})_{n_{k,+}}|^2 + |(R(t)\,\mathbf{d})_{n_{k,-}}|^2\} \ . \tag{86}$$

We can write for $k \in \mathbb{Z}$

$$|(R(t)\,\mathbf{d})_{n_{k,+}}|^a + |(R(t)\,\mathbf{d})_{n_{k,-}}|^2 = \left\| \begin{bmatrix} r_2(s_1) & r_1(s_1) \\ r_2(s_2) & r_1(s_2) \end{bmatrix} \begin{bmatrix} d_k \\ d_{k+1} \end{bmatrix} \right\|^2 \ , \tag{87}$$

where s_1, s_2 are numbers satisfying $0 \leq s_1 \leq s_2 - \frac{1}{2} < \frac{1}{2}$. A plot or $r_1(s)$ and $r_2(s)$, $s \in [0,1]$, suffices to see that there is a constant $C > 0$ such that for all $\mathbf{v} = [v_1 \ v_2]^T \in \mathbb{C}^2$ and all s_1, s_2 with $0 \leq s_1 \leq s_2 - \frac{1}{2} < \frac{1}{2}$ we have

$$\left\| \begin{bmatrix} r_2(s_1) & r_1(s_1) \\ r_2(s_2) & r_1(s_2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|^2 \geq C\,\|\mathbf{v}\|^2 = C(|v_1|^2 + |v_2|^2) \ . \tag{88}$$

Hence we see that

$$\|R(t)\,\mathbf{d}\|^2 \geq C \sum_{k=-\infty}^{\infty} (|d_k|^2 + |d_{k+1}|^2) = 2C\,\|\mathbf{d}\|^2 \ , \tag{89}$$

and this completes the proof for the case $a \leq \frac{1}{2}$.

Case $a \in (m/(m+1), (m+1)/(m+2)]$**,** $m = 1, 2, \ldots$**.** Let $t \in \mathbb{R}$. We order the set of $n \in \mathbb{Z}$ such that $\lfloor t - na \rfloor = \lfloor t - (n+1)\,a \rfloor$ as an increasing sequence $(n_j)_{j \in \mathbb{Z}}$, where we observe that for all $j \in \mathbb{Z}$

$$m + 1 \leq n_{j+1} - n_j \leq m + 2 \ . \tag{90}$$

Next we write

$$\begin{aligned} \|R(t)\,\mathbf{d}\|^2 &= \sum_{n=-\infty}^{\infty} |(R(t)\,\mathbf{d})_n|^2 \ = \\ &= \sum_{j=-\infty}^{\infty} \Big\{ \tfrac{1}{2}\,|(R(t)\,\mathbf{d})_{n_j}|^2 + \tfrac{1}{2}\,|(R(t)\,\mathbf{d})_{n_j+1}|^2 + \sum_{n_j < n < n_{j+1}} |(R(t)\,\mathbf{d})_n|^2 \ + \\ &\quad + \tfrac{1}{2}\,|(R(t)\,\mathbf{d})_{n_{j+1}}|^2 + \tfrac{1}{2}\,|(R(t)\,\mathbf{d})_{n_{j+1}+1}|^2 \Big\} \ . \end{aligned} \tag{91}$$

That is

$$\|R(t)\,\mathbf{d}\|^2 = \sum_{j=-\infty}^{\infty} \|R^{(j)}(t)\,\mathbf{d}^{(j)}\|^2 \ , \tag{92}$$

where $\mathbf{d}^{(j)}$ is the subvector of $\mathbf{d}$ made up from the consecutive entries d_k of $\mathbf{d}$ with index $k = \lfloor t - n_j a \rfloor, ..., \lfloor t - n_{j+1} a \rfloor + 1$, and $R^{(j)}(t)$ is the submatrix of $R(t)$ made up from the entries $R_{nk}(t)$ of $R(t)$ with row index $n = n_j, ..., n_{j+1}+1$ and column index $k = \lfloor t - n_j a \rfloor, ..., \lfloor t - n_{j+1} a \rfloor + 1$, where the entries with row index $n = n_j, n_j + 1, n_{j+1}, n_{j+1} + 1$ are divided by $\sqrt{2}$.

The matrices $R^{(j)}$ are of size $(m+3) \times (m+2)$ or $(m+4) \times (m+3)$ and look like

$$\begin{bmatrix} r'_{20} & r'_{10} & & & & & & \\ r'_{21} & r'_{11} & & & & & & \\ & r_{22} & r_{12} & & & & & \\ & & r_{23} & r_{13} & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \\ & & & & & & r_{2p} & r_{1p} & \\ & & & & & & & r'_{2,p+1} & r'_{1,p+1} \\ & & & & & & & r'_{2,p+2} & r'_{1,p+1} \end{bmatrix} \tag{93}$$

where $p = m$ or $m+1$. The primes at the entries in the first two and the last two rows indicate that these entries are of the form $r_{2i}/\sqrt{2}$, $r_{1i}/\sqrt{2}$, $i = 0, 1, p+1, p+2$. The numbers r_{2i}, r_{1i} are given in the form

$$r_{2i} = r_2(s_i)\,, \quad r_{1i} = r_1(s_i) \ , \qquad i = 0, 1, ..., p+1, p+2 \ , \tag{94}$$

where the $s_0, ..., s_{p+2}$ are numbers in $[0, 1)$ satisfying

$$\begin{cases} 0 \le s_0 < 1 - a\,, \quad s_1 = a + s_0 \ , & \\ s_i = s_1 - (1-a)(i-1) & , \qquad i = 1, ..., p+1 \ , \\ s_{p+2} = s_1 - (1-a)(p+1) + 1 \ge a \ . & \end{cases} \tag{95}$$

In Fig. 4 we have drawn for $a = \frac{49}{60} \in (\frac{4}{5}, \frac{5}{6}]$ and $s_0 = \frac{1}{24}$ (which implies that $p = 4$ in (95)) the points $s_0, ..., s_6$ in $[0, 1)$ together with the graphs of $r_2(s)$, $r_1(s)$, $s \in [0, 1]$.

The 2×2-matrices

$$\begin{bmatrix} r'_{20} & r'_{10} \\ r'_{21} & r'_{11} \end{bmatrix} , \qquad \begin{bmatrix} r'_{2,p+1} & r'_{1,p+1} \\ r'_{2,p+2} & r'_{1,p+2} \end{bmatrix} \tag{96}$$

are, except for the factors $1/\sqrt{2}$, of a similar type as the 2×2-matrix occurring at the right-hand side of (87). Hence they satisfy an inequality of the type (88) with a constant $C > 0$ which is independent of the particular configuration $s_0, ..., s_{p+2}$ as in (95). We next consider the submatrix of (93) obtained by deleting the first and last column and the first 2 and last 2 rows, and write it in the form

$$\begin{bmatrix} a_1 & b_1 & & & & & & & & & \\ & a_2 & b_2 & & & & & & & & \\ & & \cdot & \cdot & & & & & & & \\ & & & \cdot & \cdot & & & & & & \\ & & & & \cdot & \cdot & & & & & \\ & & & & & a_k & b_k & & & & \\ & & & & & & a_{k+1} & b_{k+1} & & & \\ & & & & & & & \cdot & \cdot & & \\ & & & & & & & & \cdot & \cdot & \\ & & & & & & & & & \cdot & \cdot \\ & & & & & & & & & a_{K-1} & b_{K-1} & \\ & & & & & & & & & & a_K & b_K \end{bmatrix} , \tag{97}$$

where $K = p - 1$ and k is to be specified below. The matrix in (97) has two non-zero diagonals where the lower diagonal entries increase from a value $a_1 = r_{22} < r_{12} = b_2$ to a value $a_K = r_{2p} > r_{1p} = b_K$ while the upper diagonal entries decrease from a value $b_1 = r_{12} > r_{22} = a_1$ to a value $b_K = r_{1p} < r_{2p} = a_K$. We take k such that $a_k \leq b_k$, $a_{k+1} \geq b_{k+1}$. Then

$$a_1 < a_2 < ... < a_k \leq b_k < b_{k-1} < ... < b_2 < b_1 \ , \tag{98}$$

$$b_K < b_{K-1} < ... < b_{k+1} \leq a_{k+1} < a_{k+2} < ... < a_{K-1} < a_K \ . \tag{99}$$

Accordingly there holds for any $\mathbf{x} = [x_1, ..., x_{K+1}]^T \in \mathbb{C}^{K+1}$

$$\sum_{i=1}^{K} |a_i x_i + b_i x_{i+1}| \geq \sum_{i=1}^{K} (b_i |x_{i+1}| - a_i |x_i|) + \sum_{i=k+1}^{K} (a_i |x_i| - b_i |x_{i+1}|) \ =$$

$$= \ -a_1 |x_1| + \sum_{i=2}^{k} (b_{i-1} - a_i) |x_i| + b_k |x_{k+1}| \ +$$

$$- \, b_K \, |x_{K+1}| + \sum_{i=k+2}^{K} (a_i - b_{i-1}) \, |x_i| + a_{k+1} \, |x_{k+1}| \ . \tag{100}$$

Now suppose that we have a vector $\mathbf{e} = [e_0, ..., e_{p+1}] \in \mathbb{C}^{p+2}$ such that

$$\|R^{(j)}(t) \, \mathbf{e}\|^2 = \sum_{i=0}^{p+2} |(R^{(j)}(t) \, \mathbf{e})_i|^2 \leq 1 \ . \tag{101}$$

The considerations of the matrices in (96) yield that there is a constant C (only depending on α, a) such that

$$|e_0|, |e_1|, |e_p|, |e_{p+1}| \leq C \ . \tag{102}$$

Next, also see Fig. 4 and (93), (94), (97), there is a $\delta > 0$, only depending on α and a, such that

$$b_{i-1} - a_i \geq \delta \, , \quad i = 2, ..., k \ ; \qquad a_i - b_{i-1} \geq \delta \, , \quad i = k+2, ..., K \ , \tag{103}$$

while also $b_k \geq \delta$, $a_{k+1} \geq \delta$. Therefore

$$\sum_{i=2}^{p} |(R^{(j)}(t) \, \mathbf{e})_i| \geq \delta \sum_{i=2}^{p-1} |e_i| - r_{22}|e_1| - r_{1p}|e_p| \ . \tag{104}$$

It thus follows from (101), (102) that

$$\sum_{i=2}^{p-1} |e_i| \leq \frac{1}{\delta} \, (2MC + \sqrt{p-1}) \ , \tag{105}$$

where $M = \max(r_{22}, r_{1p}) \leq \sinh(\alpha/2)$. Hence

$$\|\mathbf{e}\|^2 = \sum_{i=0}^{p+1} |e_i|^2 \leq \Big(\sum_{i=0}^{p} |e_i| \Big)^2 \leq (4C + \delta^{-1}(2MC + \sqrt{p-1}))^2 \ . \tag{106}$$

We thus conclude that there is a constant $D > 0$, only depending on α and a, such that

$$\|R^{(j)}(t) \, \mathbf{e}\|^2 \geq D \, \|\mathbf{e}\|^2 \ , \qquad t \in \mathbb{R} \, , \quad \mathbf{e} \in \mathbb{C}^{p+1} \ , \quad j \in \mathbb{Z} \ . \tag{107}$$

This implies that there is a constant $D > 0$ such that for $t \in \mathbb{R}$, $\mathbf{d} \in l^2(\mathbb{Z})$

$$\|R(t) \, \mathbf{d}\|^2 = \sum_{j=-\infty}^{\infty} \|R^{(j)}(t) \, \mathbf{d}^{(j)}\|^2 \geq D \sum_{j=-\infty}^{\infty} \|\mathbf{d}^{(j)}\|^2 \geq D \, \|\mathbf{d}\|^2 \ , \tag{108}$$

and this completes the proof for the case $a \in (m/(m+1), (m+1)/(m+2)]$.

Note. In [7] Einziger computes the canonical dual $g^d_{3,\alpha}$ of $g_{3,\alpha}$ formally for $t \in [0,1)$, $n \in \mathbb{Z}$ as

$$g^d_{3,\alpha}(t+n) = \int_0^1 \frac{1}{(Zg_{3,\alpha})^*\,(t,\nu)}\, e^{2\pi i n\nu}\, d\nu \tag{109}$$

by using Proposition 5.1. Carrying through a minor correction in the second member of [7], (13) there holds that $g^d_{3,\alpha}$ is even and

$$g^d_{3,\alpha}(t) = \begin{cases} \dfrac{\alpha^{-1/2}\sinh[\alpha(2-t)]}{2\sinh^2[\alpha(1-t)]} \;, & 0 \le t < \frac{1}{2}\;, \\ \dfrac{-\alpha^{1/2}}{2\sinh \alpha t} \;, & \frac{1}{2} < t \le 1\;, \end{cases} \tag{110}$$

while for $n = 1, 2, ...$ there holds

$$g^d_{3,\alpha}(t+n) = \begin{cases} \dfrac{\alpha^{-1/2}(-1)^n(\sinh \alpha t)^{n-1}\sinh^2\alpha}{2(\sinh \alpha(1-t))^{n+2}} \;, & 0 \le t < \frac{1}{2}\;, \\ 0 \;, & \frac{1}{2} < t < 1\;. \end{cases} \tag{111}$$

This $g^d_{3,\alpha}$ is shown in Fig. 5 for the case $\alpha = \sqrt{2\pi}$.

6 A detailed analysis of $h_{3,\alpha}$

In this section we give a detailed analysis of $h_{3,\alpha}$, where

$$h_{3,\alpha}(s) = \int_0^1 \frac{(Zg_{3,\alpha})(s,\nu)}{|(Zg_{3,\alpha})(s,\nu)|}\, d\nu = \int_0^1 \frac{(Zg_{3,\alpha})(t,\nu)}{|(Zg_{3,\alpha})(t,\nu)|}\, e^{2\pi i n\nu}\, d\nu \tag{112}$$

for $s = t+n$ with $t \in [0,1)$ and $n \in \mathbb{Z}$; as already noted this $h_{3,\alpha}$ is a key example of the windows considered in this paper that generate orthonormal Gabor bases. These results are based on Proposition 5.1 which yields

$$h_{3,\alpha}(s) = \int_0^1 \frac{1 + r(t)\, e^{2\pi i\nu}}{|1 + r(t)\, e^{2\pi i\nu}|}\, e^{2\pi i n\nu}\, d\nu \tag{113}$$

for $s = t+n$ with $t \in [0,1)$ and $n \in \mathbb{Z}$, where

$$r(t) = \frac{\sinh \alpha t}{\sinh \alpha(1-t)} \;, \qquad 0 \le t < 1\;. \tag{114}$$

Since $h_{3,\alpha}$ is even it is therefore sufficient to consider for $n = 0, 1, \ldots$ the integrals

$$I_n(r) = \int_0^1 \frac{1 + r\, e^{2\pi i\nu}}{|1 + r\, e^{2\pi i\nu}|}\, e^{2\pi i n\nu}\, d\nu\ , \qquad r \geq 0\ . \tag{115}$$

A. Representations of $I_n(r)$

In this subsection we present a number of representations of $I_n(r)$. We let for $r \geq 0$

$$k^2 := \frac{4r}{(1+r)^2} = \frac{4r^{-1}}{(1+r^{-1})^2} \in [0, 1]\ . \tag{116}$$

A.1. Representation in terms of elliptic integrals

We have

$$I_n(r) = \frac{2}{\pi(1+r)}\, (K_n(k^2) + r\, K_{n+1}(k^2))\ , \tag{117}$$

where

$$K_n(k^2) = \int_0^{\pi/2} \frac{\cos\, 2nx}{\sqrt{1 - k^2\, \sin^2\, x}}\, dx\ , \qquad n = 0, 1, \ldots\ . \tag{118}$$

For $n = 0$ there holds specifically

$$I_0(r) = \frac{1}{\pi}\, (1 - r)\, K(k^2) + \frac{1}{\pi}\, (1 + r)\, E(k^2)\ , \tag{119}$$

with

$$K(k^2) = \int_0^{\pi/2} (1 - k^2\, \sin^2\, x)^{-1/2}\, dx\ , \qquad E(k^2) = \int_0^{\pi/2} (1 - k^2\, \sin^2\, x)^{1/2}\, dx \tag{120}$$

the complete elliptic integrals of the first and second kind, respectively, see [9], p. 590, for $m = k^2 < 1$. The proof of these results is elementary.

From [10], 806.01 on p. 292 we have for K_n the series representation

$$K_n(k^2) = (-1)^n\, \frac{\pi}{2} \sum_{j=n}^{\infty} \binom{2j}{j} \binom{2j}{j-n} (\tfrac{1}{4}\, k)^{2j}\ . \tag{121}$$

Accordingly we get for $I_n(r)$ the series representation

$$I_n(r) = \frac{(-1)^n}{1+r} \sum_{j=n}^{\infty} \binom{2j}{j} \binom{2j}{j-n} (\tfrac{1}{4}\, k)^{2j} \Big\{1 - rk^2\, \frac{(j+\frac{1}{2})^2}{(j+n+1)(j+n+2)}\Big\}\ . \tag{122}$$

While the series in (121) only converges for $0 \leq k^2 < 1$, the series in (122) converges for $k^2 = 1$ as well by Stirling's formula and the fact that the factor in { } is $O(j^{-1})$.

A.2. Integral representations of $I_n(r)$

Aside from the direct representation of $I_n(r)$ through the definition (115), we have the integral representations for $n = 0, 1, ..., r > 0$

$$
\begin{aligned}
I_n(r) &= \frac{(-1)^n}{\pi} \int_0^{\min(r,r^{-1})} v^{n-1/2} \sqrt{\frac{1-rv}{r-v}}\, dv = \\
&= (-1)^n \int_{-1/2}^{1/2} R^n(\Theta)\, \frac{1+R(\Theta)}{1+r}\, \frac{1-r\,R(\Theta)}{1-R(\Theta)}\, d\Theta\ . \qquad (123)
\end{aligned}
$$

Here $R(\Theta) = R(r\,;\,\Theta)$ is for $r \geq 0$ and $\Theta \in [-\frac{1}{2}, \frac{1}{2}]$ the unique solution $R \in [0, 1]$ of the equation

$$\frac{4R}{(1+R)^2} = \frac{4r}{(1+r)^2} \cos^2 2\pi\Theta\ . \qquad (124)$$

Note that $R(\Theta) \leq \min(r, 1/r)$, whence the second integrand in (123) is bounded since

$$0 \leq \frac{1-r\,R(\Theta)}{1-R(\Theta)} \leq 1+r\,, \quad 0 \leq r < 1\,; \qquad 0 \leq \frac{1-r\,R(\Theta)}{1-R(\Theta)} \leq 1\,, \quad 1 \leq r < \infty\ . \qquad (125)$$

The first integrand in (123) has for $0 < r < 1$ a $(r-v)^{-1/2}$-behaviour when $v \uparrow r$.

Proof of the first representation in (123). Consider the case that $0 < r \leq 1$. Letting $z = e^{2\pi i\nu}$ in the right-hand side of (115), we have

$$I_n(r) = \frac{1}{2\pi i} \int_{|z|=1} \frac{1+rz}{|1+rz|}\, z^{n-1}\, dz\ . \qquad (126)$$

On $|z| = 1$ we have

$$|1+rz| = \sqrt{(1+rz)(1+rz^{-1})} \qquad (127)$$

when we choose the principal square root. The right-hand side of (127) is analytic in the whole complex plane, except for the branch cuts $[-r, 0]$,

$(-\infty, -1/r]$. From a consideration of the right-hand side of (127) near $z = 0$, $\operatorname{Im} z \neq 0$ it follows that

$$\sqrt{(1 + r(-v \pm i0))(1 + r(-v \pm i0)^{-1})} = \mp i \sqrt{(1 - rv)(rv^{-1} - 1)}\ , \qquad 0 \le v \le r\ , \tag{128}$$

with a non-negative square root at the right-hand side of (128). Now deforming the integration contour $|z| = 1$ in (126) into a contour tightly fitting around the branch cut $[-r, 0]$ we get the first representation in (123) for $0 < r \le 1$ by Cauchy's theorem.

We observe that

$$\lim_{r \downarrow 0} \int_0^r v^{n-1/2} \sqrt{\frac{1 - rv}{r - v}}\, dv = \pi\, \delta_{no}\ , \qquad n = 0, 1, \ldots\ , \tag{129}$$

where δ_{no} is Kronecker's delta, which is consistent with (115) for $r = 0$.

The case that $r \ge 1$ is handled in a similar manner, except that the branch cuts for (127) are now $[-1/r, 0]$, $(-\infty, -r]$.

Proof of the second representation in (123). We have from (117), (121) when $r \neq 1$

$$(-1)^n (1 + r)\, I_n(r) = \sum_{j=n}^{\infty} \frac{1}{4^j} \binom{2j}{j} \binom{2j}{j - n} (\tfrac{1}{4} k^2)^j\ + $$

$$-\ \tfrac{1}{4} r k^2 \sum_{j=n}^{\infty} \frac{1}{4^{j+1}} \binom{2j + 2}{j + 1} \binom{2j + 2}{j - n} (\tfrac{1}{4} k^2)^j\ . \tag{130}$$

Using that

$$\frac{1}{4^m} \binom{2m}{m} = \int_{-1/2}^{1/2} (\cos^2 2\pi\Theta)^m\, d\Theta\ , \qquad m = 0, 1, \ldots\ , \tag{131}$$

we get

$$(-1)^n (1 + r)\, I_n(r) = \int_{-1/2}^{1/2} \Big\{ \sum_{j=n}^{\infty} \binom{2j}{j - n} (\tfrac{1}{4} k^2 \cos^2 2\pi\Theta)^j\ + $$

$$-\ \tfrac{1}{4} r k^2 \cos^2 2\pi\Theta \sum_{j=n}^{\infty} \binom{2j + 2}{j - n} (\tfrac{1}{4} k^2 \cos^2 2\pi\Theta)^j \Big\}\, d\Theta\,. \tag{132}$$

Next we use [11], 5.2.13.29 on p. 713 with $\mu = 2n+1$ and $2n+3$, $v=2$ and $x = \frac{1}{4}\,k^2\,\cos^2\,2\pi\Theta \in [0,\frac{1}{4}]$, so that

$$\sum_{j=n}^{\infty} \Big(\begin{matrix} 2j \\ j-n \end{matrix} \Big)\, x^j = x^n\, \frac{y^{2n+1}}{-y+2} \;, \qquad \sum_{j=n}^{\infty} \Big(\begin{matrix} 2j+2 \\ j-n \end{matrix} \Big)\, x^j = x^n\, \frac{y^{2n+3}}{-y+2} \;, \tag{133}$$

where $x \in [0,\frac{1}{4}]$ and $y \in [1,2]$ are related according to

$$x = \frac{y-1}{y^2}\;. \tag{134}$$

Recalling the definitions (124) of $R(\Theta)$, (116) of k^2 and $x = \frac{1}{4}\,k^2\,\cos^2\,2\pi\Theta$, we easily see that

$$y(\Theta) = y = 1 + R(\Theta)\;, \qquad R(\Theta) = \tfrac{1}{4}\,k^2 y^2\,\cos^2\,2\pi\Theta\;. \tag{135}$$

Using this in (132) we obtain

$$\begin{aligned}(-1)^n (1+r)\, I_n(r) \;&=\; \int_{-1/2}^{1/2} \frac{R^n(\Theta)\, y(\Theta)}{-y(\Theta)+2}\,(1 - r\,R(\Theta))\,d\Theta \;= \\ &=\; \int_{-1/2}^{1/2} R^n(\Theta)(1+R(\Theta))\,\frac{1-r\,R(\Theta)}{1-R(\Theta)}\,d\Theta\;,\end{aligned} \tag{136}$$

as required.

The second representation in (123) can also be written as

$$I_n(r) = 4(-1)^n \int_0^{1/4} R^n(\Theta)\,\frac{1+R(\Theta)}{1-R(\Theta)}\;\frac{1-r\,R(\Theta)}{1+r}\,d\Theta\;, \tag{137}$$

and this yields the first representation in (123) by the substitution $v = R(\Theta) \in [0, \min\, r, 1/r]$, $\Theta \in [0,\frac{1}{4}]$. To see this we set

$$V = \sqrt{\sin^2\,2\pi\Theta + \Big(\frac{1-r}{1+r}\Big)^2\,\cos^2\,2\pi\Theta}\;. \tag{138}$$

Then we find from (124) and $0 \le R(\Theta) \le 1$ that

$$\frac{1+R(\Theta)}{1-R(\Theta)} = \frac{1}{V}\;, \qquad R(\Theta) = \frac{1-V}{1+V} = \frac{4r\,\cos^2\,2\pi\Theta}{(1+r)^2}\,\Big(\frac{1}{1+V}\Big)^2\;, \tag{139}$$

$$R'(\Theta) = \frac{-16\pi r \sin 2\pi\Theta \cos 2\pi\Theta}{(1+r)^2\, V(1+V)^2} \;, \tag{140}$$

so that

$$\frac{1+R(\Theta)}{1-R(\Theta)} = \frac{-1}{4\pi}\;\frac{\cos 2\pi\Theta}{\sin 2\pi\Theta}\;\frac{R'(\Theta)}{R(\Theta)} \;. \tag{141}$$

Furthermore, one has

$$(r - R(\Theta))(1 - r\,R(\Theta)) = (1+r)^2\,R(\Theta)\,\frac{\sin^2\, 2\pi\Theta}{\cos^2\, 2\pi\Theta} \;, \tag{142}$$

so that for $\Theta \in [0, \frac{1}{4}]$

$$1 - r\,R(\Theta) = (1+r)\,R^{1/2}(\Theta)\,\frac{\sin 2\pi\Theta}{\cos 2\pi\Theta}\sqrt{\frac{1 - r\,R(\Theta)}{r - R(\Theta)}} \;. \tag{143}$$

When one uses (141) and (143) in the right-hand side of (137) one gets the first representation in (123) by setting $v = R(\Theta)$.

B. Consequences of the representations

We shall now give a number of consequences of the representations of $I_n(r)$ obtained in A. These consequences translate directly to properties of $h_{3,\alpha}$ since $h_{3,\alpha}(t+n) = I_n(r(t))$ for $t \in [0,1)$, $n = 0, 1, \ldots$ with $r(t)$ given in (114).

B.1. Behaviour of $h_{3,\alpha}$ at the integers and half-integers

It is clear that $h_{3,\alpha}$ is a smooth function away from the integers and the half-integers. From Prop. 4.1 (b) we know that h is continuous everywhere. To study the behaviour of $h_{3,\alpha}$ at the integers, we note that for $n = 1, 2, \ldots$ and $t \in [0,1)$ there holds

$$h_{3,\alpha}(n+t) = I_n(r) \;, \qquad h_{3,\alpha}(n-t) = I_{n-1}(1/r) \tag{144}$$

with $r = r(t)$, see (114). Thus we shall compare the behaviour of $I_n(r)$, $I_n(1/r)$ as $r \downarrow 0$ for $n = 1, 2, \ldots$. For $n = 0$ we have the special situation that $h_{3,\alpha}(t) = h_{3,\alpha}(-t)$, whence we only have to consider $I_0(r)$ as $r \downarrow 0$.

Proposition 6.1. We have

(a) $I_0(r)$ has a convergent power series in $r \in \mathbb{C}$ for $|r| < 1$ with only even powers of r, and $I_0(0) = 1$,

(b) $I_n(r)$ and $I_{n-1}(1/r)$ have for $n = 1, 2, ...$ a convergent power series in $r \in \mathbb{C}$ for $|r| < 1$, and there holds

$$I_n(r) = (-1)^n \binom{2n}{n} \left(\frac{r}{4}\right)^n + O(r^{n+1}) \ ,$$

$$I_{n-1}(1/r) = (-1)^{n-1} \frac{n}{2n-1} \binom{2n}{n} \left(\frac{r}{4}\right)^n + O(r^{n+1}) \qquad (145)$$

as $r \downarrow 0$.

Proof. We have for $0 \leq r < 1$ that

$$I_0(r) = 2 \int\limits_0^{1/2} \frac{1 + r \cos 2\pi\nu}{(1 + r^2 + 2r \cos 2\pi\nu)^{1/2}} \, d\nu \ . \qquad (146)$$

It is not difficult to see that for any $a \in [-1, 1]$ the image of $z \in \mathbb{C}$, $|z| < 1$ under the mapping $z \to 1 + z^2 + 2za$ and the set $(-\infty, 0]$ are disjoint. Hence the right-hand side of (146) extends to an analytic function of $r \in \mathbb{C}$, $|r| < 1$ when we take the principal square root for $(1 + r^2 + 2r \cos 2\pi\nu)^{1/2}$ where $|r| < 1$, $\nu \in [0, \frac{1}{2}]$. Thus $I_0(r)$ has a convergent power series in $r \in \mathbb{C}$, $|r| < 1$. Furthermore, we have

$$\int\limits_0^{1/2} \frac{1 - r \cos 2\pi\nu}{(1 + r^2 - 2r \cos 2\pi\nu)^{1/2}} \, d\nu = \int\limits_0^{1/2} \frac{1 + r \cos 2\pi\nu}{(1 + r^2 + 2r \cos 2\pi\nu)^{1/2}} \, d\nu \ , \qquad (147)$$

as we see by replacing ν by $\frac{1}{2} - \nu$ in the left-hand side integral in (147). It follows that the right-hand side of (146) is an even function of r, $|r| < 1$. Evidently, $I_0(0) = 1$ and this completes the proof of (a).

As to (b) we start by noting that

$$I_n(r) = \int\limits_{-1/2}^{1/2} \frac{1 + r \, e^{2\pi i\nu}}{(1 + r^2 + 2r \cos 2\pi\nu)^{1/2}} \, e^{2\pi i n\nu} \, d\nu \ , \qquad (148)$$

$$I_{n-1}(1/r) = \int\limits_{-1/2}^{1/2} \frac{1 + r \, e^{-2\pi i\nu}}{(1 + r^2 + 2r \cos 2\pi\nu)^{1/2}} \, e^{2\pi i n\nu} \, d\nu \ , \qquad (149)$$

and the statement about the convergent power series follows in a similar manner as in (a). The formulas in (145) are an easy consequence of the series representation in (122), also see (116).

Note. We have that

$$\lim_{r\downarrow 0} r^{-n}\, I_n(r) \neq (-1)^n \lim_{r\downarrow 0} r^{-n}\, I_{n-1}(1/r) \tag{150}$$

unless $n = 1$ in which case the two members of (150) equal $-1/2$.

As a consequence we have that $h_{3,\alpha}$ is analytic at $t = 0$, that $h_{3,\alpha}$ is differentiable at $t = n = 1$ (with derivative equal to $-\alpha/2 \sinh \alpha$), and that $h_{3,\alpha}$ has zeros of order n with non-zero, unequal left and right derivatives of order n at $t = n = 2, 3, \ldots$.

We next consider the behaviour of $h_{3,\alpha}$ at the half-integers $t = n + \frac{1}{2}$, $n = 0, 1, \ldots$. Since $r(\frac{1}{2}) = 1$ we thus need to consider I_n around $r = 1$.

Proposition 6.2. We have $(-1)^{n+1}\, I_n'(1) = +\infty$ for $n = 0, 1, \ldots$.

Proof. We consider the first representation of $I_n(r)$ in (123). It suffices for our purposes to study the functions

$$\int_0^r \sqrt{\frac{1-rv}{r-v}}\, dv \;, \qquad \int_0^s \sqrt{\frac{s-v}{1-sv}}\, dv \tag{151}$$

as $r \uparrow 1$ and $s = (1/r) \uparrow 1$, since the function $v \to v^{n-1/2}$ is smooth at $v = 1$. We compute, explicitly,

$$\int_0^r \sqrt{\frac{1-rv}{r-v}}\, dv = \frac{1}{2\sqrt{r}} \left\{2r + (1-r^2)\ln(1+r) - (1-r^2)\ln(1-r)\right\} \;, \tag{152}$$

$$\int_0^s \sqrt{\frac{s-v}{1-sv}}\, dv = \frac{1}{2s\sqrt{s}} \left\{2s - (1-s^2)\ln(1+s) + (1-s^2)\ln(1-s)\right\} \;. \tag{153}$$

Now it is evident that the terms $-(1-r^2)\ln(1-r)$ and $(1-s^2)\ln(1-s)$ cause the derivatives of the left-hand sides of (152) and (153) at $r = 1$ and $s = 1$ to be $-\infty$ and $+\infty$, respectively. This proves the result.

B.2. Special values

We have

$$K(k^2) = \tfrac{1}{2}\pi\, F(\tfrac{1}{2}, \tfrac{1}{2}\,;\, 1\,;\, k^2) \;, \qquad E(k^2) = \tfrac{1}{2}\pi\, F(-\tfrac{1}{2}, \tfrac{1}{2}\,;\, 1\,;\, k^2) \;, \tag{154}$$

with F a hypergeometric function, see [9], 17.3.9–10 on p. 591 and Ch. 15. A special value of K, E occurs when $k^2 = \frac{1}{2}$, i.e. $r = 3 \pm 2\sqrt{2}$, see [9], 15.1.25–26, cases $a = \frac{1}{2}$, $b = 1$ and $a = -\frac{1}{2}$, $b = \frac{1}{2}$, on p. 557. Thus

$$F(\tfrac{1}{2}, \tfrac{1}{2}\,;\,1\,;\,\tfrac{1}{2}) = \frac{\pi^{1/2}}{\Gamma^2(\frac{3}{4})}\,, \qquad F(-\tfrac{1}{2}, \tfrac{1}{2}\,;\,1\,;\,\tfrac{1}{2}) = 2\pi^{1/2}\Big(\frac{1}{4\Gamma^2(\frac{3}{4})} + \frac{1}{\Gamma^2(\frac{1}{4})}\Big)\,. \tag{155}$$

Then (119) yields

$$I_0(3 - 2\sqrt{2}) = \frac{(\pi/2)^{1/2}}{\Gamma^2(\frac{3}{4})} + \frac{\Gamma^2(\frac{3}{4})}{\pi^{3/2}}\,(2 - \sqrt{2}) = 0.992599562\,, \tag{156}$$

$$I_0(3 + 2\sqrt{2}) = -\,\frac{(\pi/2)^{1/2}}{\Gamma^2(\frac{3}{4})} + \frac{\Gamma^2(\frac{3}{4})}{\pi^{3/2}}\,(2 + \sqrt{2}) = 0.08610564\,. \tag{157}$$

B.3. Positivity and monotonicity of $(-1)^n\, I_n(r)$

We have the following result.

Theorem 6.1. There holds

(a) $I_0(r)$ is positive and decreasing in $r \geq 0$,

(b) $(-1)^n\, I_n(r)$ is positive for $r > 0$ and decreasing in $r \geq 1$.

Proof. (a) We have from (115) that

$$I_0(r) = 2\int_0^{1/2} \frac{J(r,\nu)}{(J^2(r,\nu) + K^2(r,\nu))^{1/2}}\, d\nu\,, \qquad r \geq 0\,, \tag{158}$$

where

$$J(r,\nu) = 1 + r\cos 2\pi\nu\,, \quad K(r,\nu) = r\sin 2\pi\nu\,, \qquad r \geq 0\,,\ \ 0 \leq \nu \leq \tfrac{1}{2}\,. \tag{159}$$

Fix $\nu \in (0, \frac{1}{2})$. Then $K(r,\nu) > 0$ for $r > 0$ while

$$\frac{J(r,\nu)}{K(r,\nu)} = \frac{1}{r\sin 2\pi\nu} + \operatorname{cotg} 2\pi\nu \tag{160}$$

decreases in $r > 0$. Then one easily sees that

$$\frac{J(r,\nu)}{(J^2(r,\nu) + K^2(r,\nu))^{1/2}} = \frac{\operatorname{sgn}(J(r,\nu))}{(1 + K^2(r,\nu)/J^2(r,\nu))^{1/2}} \tag{161}$$

decreases in $r > 0$. Since $I_0(0) = 1$, $I_0(\infty) = 0$, the proof of (a) is complete.

(b) That $(-1)^n I_n(r) > 0$ for $r > 0$ follows at once from either representation in (123). Next we have for $r \geq 1$ by the first representation in (123) that

$$
\begin{aligned}
(-1)^n I_n(r) &= \frac{1}{\pi} \int_0^{1/r} v^{n-1/2} \sqrt{\frac{1-rv}{r-v}}\, dv = \\
&= \frac{1}{\pi} \int_0^{\infty} v^{n-1/2} \sqrt{\frac{1-rv}{r-v}}\, \chi_{[0,1/r)}(v)\, dv \,. \qquad (162)
\end{aligned}
$$

One easily sees that the integrand in the integral on the second line of (162) decreases in $r \geq 1$ for any $v \geq 0$. This proves (b).

B.4. An inequality

We have the following counterpart of Prop. 4.3 for $h_{3,\alpha}$.

Proposition 6.3. With $R(r,\Theta) = R(\Theta)$ as defined through (124) we have

$$
I_0(r) + I_0(1/r) = \int_{-1/2}^{1/2} (1 + R(r,\Theta))\, d\Theta \geq 1 \,, \qquad 0 \leq r < \infty \,. \qquad (163)
$$

Proof. Taking $0 \leq r \leq 1$ we have from the second representation in (123) for $n = 0$ that

$$
\begin{aligned}
I_0(r) + I_0(1/r) &= \int_{-1/2}^{1/2} \frac{1 + R(r,\Theta)}{1+r} \, \frac{1 - r\,R(r,\Theta)}{1 - R(r,\Theta)}\, d\Theta \; + \\
&\quad + \int_{-1/2}^{1/2} \frac{1 + R(r,\Theta)}{1+1/r} \, \frac{1 - R(r,\Theta)/r}{1 - R(r,\Theta)}\, d\Theta \; = \\
&= \int_{-1/2}^{1/2} (1 + R(r,\Theta))\, d\Theta \,. \qquad (164)
\end{aligned}
$$

Since $R(r,\Theta) \geq 0$ we see that the right-hand side of (164) $\geq$ 1, and this completes the proof.

Improvements of the inequality in (163), as well as inequalities with $\leq$ instead of $\geq$, are obtained easily from the explicit forms (138), (139). We do not elaborate this point here.

B.5. Sharper form of Theorem 4.1.

We have the following sharper form of Theorem 4.1 for $h = h_{3,\alpha}$.

Theorem 6.2. We have

$$\sum_{n=0}^{\infty} |h_{3,\alpha}(t+n)| = O(\ln|t-\tfrac{1}{2}|^{-1}) \ , \qquad t \to \tfrac{1}{2} \ . \tag{165}$$

Proof. With $r = r(t)$, see (114), we have from Theorem 6.1 (a) and the second representation in (123) that

$$\sum_{n=0}^{\infty} |h_{3,\alpha}(t+n)| = \sum_{n=0}^{\infty} (-1)^n I_n(r) = \frac{4}{1+r} \int_0^{1/4} \frac{1+R(\Theta)}{1-R(\Theta)} \, \frac{1-r\,R(\Theta)}{1-R(\Theta)} \, d\Theta \ . \tag{166}$$

By (138), (139) we have that

$$4\int_0^{1/4} \frac{1+R(\Theta)}{1-R(\Theta)}\, d\Theta \ = \ 4\int_0^{1/4} \Big(\sin^2 2\pi\Theta + \Big(\frac{1-r}{1+r}\Big)^2 \cos^2 2\pi\Theta\Big)^{-1/2} d\Theta \ =$$

$$= \ \frac{2}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{2}{\pi}\, K(k^2) \ , \tag{167}$$

with k^2 as in (116). Also, by [9], 17.3.26 on p. 591 we have

$$K(k^2) = \ln 4 \Big|\frac{1+r}{1-r}\Big| + o(1) \ , \qquad r \to 1 \ . \tag{168}$$

Finally, by (125), we have that $(1-r\,R(\Theta))/(1-R(\Theta))$ is bounded between 0 and 2 for all $r \geq 0$ and all $\Theta \in [0, \frac{1}{4})$. Then the result follows on combining (166), (167), (168).

B.6. Inequality for $(-1)^n\, I_n(r)$

We have the following inequality for $(-1)^n\, I_n(r)$, $r \geq 0$, $n = 0, 1, \ldots$.

Proposition 6.4. There holds for $r \geq 0$ and $n = 0, 1, ...$ that

$$0 \ \leq \ (-1)^n \, I_n(r) \leq \frac{4}{\pi} \int\limits_0^{\pi/2} \exp(-2n \sqrt{1 - k^2 \cos^2 x}) \, dx \ \leq$$

$$\leq \ \exp\Bigl(-n \sqrt{2} \, \frac{1-r}{1+r}\Bigr) \ \min\Bigl(2, \frac{\sqrt{2}}{nk}\Bigr) \ , \tag{169}$$

where k^2 is given in (116).

Proof. We have from the second representation in (123) that

$$(-1)^n \, I_n(r) = 4 \int\limits_0^{1/4} R^n(\Theta) \, \frac{1 + R(\Theta)}{1 + r} \, \frac{1 - r\,R(\Theta)}{1 - R(\Theta)} \, d\Theta \leq 8 \int\limits_0^{1/4} R^n(\Theta) \, d\Theta \ , \tag{170}$$

since, see (125),

$$0 \leq \frac{1 + R(\Theta)}{1 + r} \leq 1 \ , \qquad 0 \leq \frac{1 - r\,R(\Theta)}{1 - R(\Theta)} \leq 2 \ . \tag{171}$$

Next by the inequality $(1 - x)(1 + x)^{-1} \leq \exp(-2x)$, $0 \leq x \leq 1$, we have from (138), (139) that

$$\begin{aligned} (-1)^n \, I_n(r) \ &\leq \ 8 \int\limits_0^{1/4} \exp\Bigl(-2n \sqrt{\sin^2 \, 2\pi\Theta + \Bigl(\frac{1-r}{1+r}\Bigr)^2 \, \cos^2 \, 2\pi\Theta}\Bigr) \, d\Theta \ = \\ &= \ \frac{4}{\pi} \int\limits_0^{\pi/2} \exp(-2n \sqrt{1 - k^2 + k^2 \, \sin^2 \, x}) \, dx \ , \end{aligned} \tag{172}$$

where we have used that $1 - k^2 = ((1 - r)/(1 + r))^2$. From this the first inequality in (169) follows.

Furthermore, by the inequality

$$\sqrt{a^2 + b^2} \geq \frac{1}{\sqrt{2}} \, (|a| + |b|) \ , \qquad a, b \in \mathbb{R} \ , \tag{173}$$

we get

$$(-1)^n \, I_n(r) \leq \frac{4}{\pi} \exp(-n \sqrt{2} \sqrt{1 - k^2}) \int\limits_0^{\pi/2} \exp(-kn \sqrt{2} \sin x) \, dx \ . \tag{174}$$

Using the inequality $\sin x \geq 2x/\pi$, $x \in [0, \pi/2]$ we can bound the integral at the right-hand side of (174) by

$$\int_0^{\pi/2} \exp(-kn\sqrt{2}\sin x)\,dx \leq \min\Big(\frac{\pi}{2}, \frac{\pi}{2kn\sqrt{2}}\Big) \ , \tag{175}$$

and the proof of the result is complete.

We note that, in particular, there is exponential decay of $I_n(r)$ as $n \to \infty$ away from $r = 1$, and that $I_n(r) = O(1/n)$ uniformly in $r \geq 0$.

B.7. Asymptotic behaviour of $(-1)^n\, I_n(r)$ as $n \to \infty$

There is some asymmetry in the behaviour or $I_n(r)$ on the respective ranges $0 \leq r \leq 1$, $r \geq 1$. One sees, for instance from the first representation in (123) that

$$(-1)^n\, I_n(r) \geq (-1)^n\, I_n(1/r) \ , \qquad n = 0, 1, \ldots,\ 0 \leq r \leq 1 \ . \tag{176}$$

This asymmetry is not reflected by Proposition 6.4. The next result on the asymptotic behaviour of $I_n(r)$ as $n \to \infty$ does show the asymmetry.

Proposition 6.5. We have

$$(-1)^n\, I_n(r) = \frac{r^n}{\sqrt{n\pi}}\,(1 - r^2)^{1/2}\,(1 + O(n^{-1})) \ , \qquad n \to \infty \ , \tag{177}$$

when $r \in [0, 1)$ is fixed, and

$$(-1)^n\, I_n(r) = \frac{r^{-n}}{2n\sqrt{n\pi}}\,(r^2 - 1)^{-1/2}\,(1 + O(n^{-1})) \ , \qquad n \to \infty \ , \tag{178}$$

when $r \in (1, \infty)$ is fixed.

Proof. We have from (138), (139) for fixed $r \in [0, 1)$ that

$$R(\Theta) = r\Big(1 - \frac{1+r}{1-r}\,4\pi^2\Theta^2 + O(\Theta^4)\Big) \ , \tag{179}$$

$$\frac{1 + R(\Theta)}{1 + r}\ \frac{1 - r\,R(\Theta)}{1 - R(\Theta)} = 1 + r + O(\Theta^2) \ . \tag{180}$$

Thus we get from the second representation in (123)

$$(-1)^n I_n(r) \;=$$

$$= \; 2 \int\limits_{-1/4}^{1/4} \Big[r\Big(1 - \frac{1+r}{1-r}\, 4\pi^2\Theta^2 + O(\Theta^4)\Big)\Big]^n \,(1 + r + O(\Theta^2))\, d\Theta \;=$$

$$= \; 2 \int\limits_{-1/4}^{1/4} r^n \exp\Big(-n\, \frac{1+r}{1-r}\, 4\pi^2\Theta^2 (1 + O(\Theta^4))\Big)\, (1 + r + O(\Theta^2))\, d\Theta \;=$$

$$= \; 2r^n (1+r) \Big(\pi / \Big(4\pi^2 n\, \frac{1+r}{1-r}\Big)\Big)^{1/2} \,(1 + O(n^{-1}))\;, \tag{181}$$

and this yields (177).

We have from (138), (139) for fixed $r \in (1, \infty)$ that

$$R(\Theta) = r^{-1}\Big(1 - \frac{r+1}{r-1}\, 4\pi^2\Theta^2 + O(\Theta^4)\Big)\;, \tag{182}$$

$$\frac{1 + R(\Theta)}{1 + r}\; \frac{1 - r\, R(\Theta)}{1 - R(\Theta)} = \frac{4\pi^2\Theta^2 (r+1)}{(r-1)^2} + O(\Theta^4)\;. \tag{183}$$

Thus, as above,

$$(-1)^n I_n(r) \;\; = \;\; 2 \int\limits_{-1/4}^{1/4} r^{-n} \exp\Big(-n\, \frac{r+1}{r-1}\, 4\pi^2\Theta^2 (1 + O(\Theta^4))\Big) \;\cdot$$

$$\cdot \Big(\frac{4\pi^2\Theta^2 (r+1)}{(r-1)^2} + O(\Theta^4)\Big)\, d\Theta \;=$$

$$= \;\; 2r^{-n}\, \frac{r+1}{(r-1)^2}\, 4\pi^2 \cdot \tfrac{1}{2}\, \sqrt{\pi}\, \Big(n\, \frac{r+1}{r-1}\, 4\pi^2\Big)(1 + O(n^{-1}))^{-3/2}, \tag{184}$$

and this yields (178).

Notes.

1. There is no uniform asymptotics in (177) or (178) as $r \uparrow 1$ or $r \downarrow 1$.

2. Consider the method of the proof for the case $r = 1$. Then

$$R(\Theta) = \frac{1 - |\sin\, 2\pi\Theta|}{1 + |\sin\, 2\pi\Theta|} = 1 - 4\pi\, |\Theta| + O(|\Theta|^3)\;,$$

$$\frac{1 + R(\Theta)}{1 + r}\; \frac{1 - r\, R(\Theta)}{1 - R(\Theta)} = 1 + O(|\Theta|)\;, \tag{185}$$

and we get

$$(-1)^n I_n(1) = 4 \int_0^{1/4} \exp(-4\pi n\Theta(1+O(\Theta)))(1+O(\Theta))\, d\Theta =$$

$$= \frac{1}{\pi n} + O\Big(\frac{1}{n^2}\Big) . \tag{186}$$

Note that Proposition 4.2 yields $(-1)^n I_n(1) = 1/\pi(n+\frac{1}{2})$.

7 The L^1-norm of $h_{3,0}$

In this section we consider the limit case $h_{3,0}$ that we obtain from $h_{3,\alpha}$ by letting $\alpha \downarrow 0$. We then get $r(t) = t(1-t)^{-1}$ in (114), and $h_{3,0}$ is given for $t \in [0,1)$, $n = 0, 1, ...$ by

$$h_{3,0}(t+n) = \int_{-1/2}^{1/2} \frac{1-t+t\, e^{2\pi i\nu}}{|1-t+t\, e^{2\pi i\nu}|}\, e^{2\pi i n\nu}\, d\nu . \tag{187}$$

This $h_{3,0}$ generates an orthonormal Gabor base for the parameters $a = b = 1$ since its Zak transform, see the integrand in (187), has unit modulus a.e. in the unit square. Note that for general $\alpha > 0$ we have

$$h_{3,\alpha}(s+n) = h_{3,0}(t+n) \; ; \qquad t = \frac{\sinh \alpha s}{\sinh \alpha s + \sinh \alpha(1-s)} \tag{188}$$

for $n = 0, 1, ..., s \in [0,1)$. Hence we only need a simple warping operation to express $h_{3,\alpha}$ in terms of $h_{3,0}$. We shall show the following result.

Theorem 7.1. We have $\|h_{3,0}\|_1 = \frac{1}{2}\pi$. More precisely there holds

$$\int_0^{1/2} |h_{3,0}(t+n)|\, dt = \Big(\frac{1}{2}+\frac{1}{\pi}\Big) H_n \; , \qquad \int_{1/2}^{1} |h_{3,0}(t+n)|\, dt = \Big(\frac{1}{2}-\frac{1}{\pi}\Big) H_n \; , \tag{189}$$

where

$$H_n = \int_0^1 |h_{3,0}(t+n)|\, dt = 2(-1)^n \Big(\frac{\pi}{4} - \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} - \frac{(-1)^n}{4n+2}\Big) \; , \tag{190}$$

and $n = 0, 1, \ldots$

Proof. By Theorem 6.1 (b) we have $(-1)^n\, h_{3,0}(t+n) \geq 0$ for $t \in [0,1)$ and $n = 0, 1, \ldots$. Hence when $I \subset [0,1)$ and $n = 0, 1, \ldots$ we get from (187)

$$\int\limits_I |h_{3,0}(t+n)|\, dt = (-1)^n \int\limits_I h_{3,0}(t+n)\, dt \ =$$

$$= \ (-1)^n \int\limits_{-1/2}^{1/2} e^{2\pi i n\nu} \left(\int\limits_I \frac{1 - t + t\, e^{2\pi i\nu}}{|1 - t + t\, e^{2\pi i\nu}|}\, dt \right) d\nu \ . \tag{191}$$

Writing

$$\frac{1 - t + t\, e^{2\pi i\nu}}{|1 - t + t\, e^{2\pi i\nu}|} = \frac{\frac{1}{2}\,(1 + e^{2\pi i\nu}) - (1 - e^{2\pi i\nu})(t - \frac{1}{2})}{A^{1/2}(\nu)\,\sqrt{(t - \frac{1}{2})^2 + A^{-1}(\nu) - \frac{1}{4}}} \ , \tag{192}$$

with $A(\nu) = 2(1 - \cos\, 2\pi\nu)$, we see that the inner integral on the second line of (191) allows expression in terms of elementary functions. Thus

$$\int \frac{1 - t + t\, e^{2\pi i\nu}}{|1 - t + t\, e^{2\pi i\nu}|}\, dt \ \ = \ \ \frac{1 + e^{2\pi i\nu}}{2A^{1/2}(\nu)} \ln(t - \tfrac{1}{2} + \sqrt{(t - \tfrac{1}{2})^2 + A^{-1}(\nu) - \tfrac{1}{4}}) \ +$$

$$- \ \frac{1 - e^{2\pi i\nu}}{A^{1/2}(\nu)} \sqrt{(t - \tfrac{1}{2})^2 + A^{-1}(\nu) - \tfrac{1}{4}} + C \ . \tag{193}$$

We take $I = [0,1), [0, \frac{1}{2}), [\frac{1}{2}, 1)$ in (191), and we get from (193) for $|\nu| \leq \frac{1}{2}$.

$$F(\nu) := \int\limits_0^1 \frac{1 - t + t\, e^{2\pi i\nu}}{|1 - t + t\, e^{2\pi i\nu}|}\, dt = \frac{1 + e^{2\pi i\nu}}{4 \sin \pi\nu} \ln\Big(\frac{1 + \sin\, \pi\nu}{1 - \sin\, \pi\nu}\Big) \ , \tag{194}$$

$$K(\nu) := \int\limits_0^{1/2} \frac{1 - t + t\, e^{2\pi i\nu}}{|1 - t + t\, e^{2\pi i\nu}|}\, dt = \tfrac{1}{2}\, F(\nu) + \frac{1}{4}\ \frac{1 - e^{2\pi i\nu}}{1 + \cos\, \pi\nu} \ , \tag{195}$$

$$L(\nu) := \int\limits_{1/2}^1 \frac{1 - t + t\, e^{2\pi i\nu}}{|1 - t + t\, e^{2\pi i\nu}|}\, dt = \tfrac{1}{2}\, F(\nu) - \frac{1}{4}\ \frac{1 - e^{2\pi i\nu}}{1 + \cos\, \pi\nu} \ . \tag{196}$$

To prove (189), (190) we must compute the Fourier coefficients F_n, K_n, L_n of F, K, L, where we note that by (191)

$$F_n, K_n, L_n = (-1)^n \int\limits_I h_{3,0}(t+n)\, dt \tag{197}$$

with $I = [0,1), [0,\frac{1}{2}), [\frac{1}{2},1)$, respectively. Thus we let

$$G(\nu) = \frac{1}{4\sin\pi\nu}\ln\Big(\frac{1+\sin\,\pi\nu}{1-\sin\,\pi\nu}\Big)\;. \tag{198}$$

Now by [12], 23.28–29 on p. 135

$$G(\nu+\tfrac{1}{2}) = \frac{1}{2\cos\pi\nu}\,[\ln|\cos\,\tfrac{1}{2}\,\pi\nu| - \ln|\sin\,\tfrac{1}{2}\,\pi\nu|] = \sum_{k=0}^{\infty}\frac{\cos(2k+1)\,\pi\nu}{(2k+1)\cos\pi\nu}\;, \tag{199}$$

whence

$$G(\nu) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\,\frac{\sin(2k+1)\,\pi\nu}{\sin\,\pi\nu}\;. \tag{200}$$

Since

$$\frac{\sin(2k+1)\,\pi\nu}{\sin\,\pi\nu} = \sum_{l=-k}^{k} e^{2\pi il\nu}\;, \tag{201}$$

we thus obtain

$$G(\nu) = \sum_{n=-\infty}^{\infty} G_n\,e^{2\pi in\nu} \tag{202}$$

with

$$G_n = \sum_{k=|n|}^{\infty}\frac{(-1)^k}{2k+1} = \frac{\pi}{4} - \sum_{k=0}^{|n|-1}\frac{(-1)^k}{2k+1}\;, \qquad n\in\mathbb{Z}\;. \tag{203}$$

Then we obtain for $n = 0, 1, \ldots$

$$F_n = G_n + G_{n+1} = 2\Big(\frac{\pi}{4} - \sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1} - \frac{(-1)^n}{2n+1}\Big)\;, \tag{204}$$

and this shows (190).

As to (189) we evaluate the Fourier coefficients of $(1-e^{2\pi i\nu})/(1+\cos\,\pi\nu)$, see the right-hand sides of (195), (196). Thus we get for $n = 0, 1, \ldots$

$$\int_{-1/2}^{1/2}\frac{1-e^{2\pi i\nu}}{1+\cos\,\pi\nu}\,e^{2\pi in\nu}\,d\nu = \int_{-1/2}^{1/2}\frac{\cos\,2\pi n\nu - \cos\,2\pi(n+1)\,\nu}{1+\cos\,\pi\nu}\,d\nu\;=$$

$$= \int_{-1/2}^{1/2}\frac{2\sin 2\pi(n+\frac{1}{2})\,\nu\sin\pi\nu}{1+\cos\,\pi\nu}\,d\nu\;=$$

$$= \int_{-1/2}^{1/2}\frac{2\sin 2\pi(n+\frac{1}{2})\,\nu}{\sin\,\pi\nu}\,(1-\cos\,\pi\nu)\,d\nu\;=$$

$$= \; 2 \int\limits_{-1/2}^{1/2} \sum_{k=-n}^{n} e^{2\pi i k \nu} (1 - \cos\, \pi\nu)\, d\nu \; =$$

$$= \; \frac{8}{\pi} \Big[\frac{\pi}{4} - \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} - \frac{(-1)^n}{4n+2} \Big] = \frac{4}{\pi}\, F_n \; . \tag{205}$$

From this we get (190) at once.

We finally show that $\|h_{3,0}\|_1 = \pi/2$. This can be done by using the explicit expression of F_n in (204) in

$$\|h_{3,0}\|_1 = 2 \sum_{n=0}^{\infty} (-1)^n\, F_n \; .$$

More directly, a formal computation shows that, see (194),

$$\|h_{3,0}\|_1 \;\; = \;\; 2 \sum_{n=0}^{\infty} (-1)^n \int\limits_{-1/2}^{1/2} e^{2\pi i n \nu}\, F(\nu)\, d\nu \; =$$

$$= \;\; 2 \int\limits_{-1/2}^{1/2} \frac{1}{1 + e^{2\pi i \nu}}\, F(\nu)\, d\nu \; =$$

$$= \;\; 2 \int\limits_{-1/2}^{1/2} \frac{1}{4 \sin \pi\nu} \ln\Big(\frac{1 + \sin\, \pi\nu}{1 - \sin\, \pi\nu}\Big)\, d\nu = 2G_0 = \frac{\pi}{2} \; , \tag{206}$$

with G_n from (203). This completes the proof.

Note. Since G_n of (203) with $n = 0, 1, \ldots$ is given as

$$G_n = (-1)^n \int\limits_0^1 \frac{t^{2n}}{1 + t^2}\, dt \; , \tag{207}$$

it follows that

$$F_n = G_n + G_{n+1} = (-1)^n \int\limits_0^1 t^{2n}\, \frac{1 - t^2}{1 + t^2}\, dt \; . \tag{208}$$

Hence H_n in (190) behaves asymptotically as

$$H_n = \int\limits_0^1 t^{2n}\, \frac{1 - t^2}{1 + t^2}\, dt \approx \int\limits_0^1 t^{2n} (1 - t)\, dt = \frac{1}{(2n+1)(2n+2)} \; . \tag{209}$$

In Fig. 6.a we have plotted $h_{3,0}$, and we have also (re)displayed (from Fig. 1) $h_{3,\alpha}$ with $\alpha = \sqrt{2\pi}$. In Fig. 6.b we show the Fourier transform of $h_{3,0}$ and of $h_{3,\alpha}$ with $\alpha = \sqrt{2\pi}$.

8 Interpolation between the box function and the sinc function

In this section we consider the choice $g_\gamma = g_{1,\gamma}$ or $g_{2,\gamma}$ and we study the behaviour of $h_\gamma = Z^{-1}(Zg_\gamma/|Zg_\gamma|)$ as the time constant $\gamma^{-1/2}$ tends to 0 or to ∞. As already said, $g_{1,\gamma}$ and $g_{2,\gamma}$ yield the same h, see [6].

Theorem 8.1. We have

$$\lim_{\gamma\to\infty} h_\gamma = \chi_{(-1/2,1/2)} \ , \qquad \lim_{\gamma\downarrow 0} h_\gamma = \text{sinc}(\pi\cdot) \ , \tag{210}$$

where the limits are in L^2-sense.

Proof. We take $g_\gamma = g_{\gamma,1}$, and the Zak transform of h_γ is given by

$$(Zh_\gamma)(t,\nu) = \frac{(Zg_\gamma)(t,\nu)}{|(Zg_\gamma)(t,\nu)|} = \frac{\sum_{k=-\infty}^{\infty} e^{-\pi\gamma(t-k)^2+2\pi ik\nu}}{\left|\sum_{k=-\infty}^{\infty} e^{-\pi\gamma(t-k)^2+2\pi ik\nu}\right|} \ . \tag{211}$$

Now when $(t,\nu) \in [0,1)^2$ we have

$$\lim_{\gamma\to\infty} (Zh_\gamma)(t,\nu) = \begin{cases} 1 \quad , & t\in[0,\frac{1}{2}) \, , \quad \nu\in[0,1) \ , \\ e^{2\pi i\nu} \ , & t\in(\frac{1}{2},1) \, , \quad \nu\in[0,1) \ . \end{cases} \tag{212}$$

This is so since for $t \in [0, \frac{1}{2})$, $k \in \mathbb{Z}$

$$e^{\pi\gamma t^2}\, e^{-\pi\gamma(t-k)^2} = O(\exp(-\pi\gamma\,|k|\,(|k|-2t))) \ , \tag{213}$$

and for $t \in (\frac{1}{2}, 1)$, $k \in \mathbb{Z}$

$$e^{\pi\gamma(t-1)^2}\, e^{-\pi\gamma(t-k)^2} = O(\exp(-\pi\gamma\,|k-1|\,(|k-1|-2(1-t))) \ , \tag{214}$$

showing that the relative decay of the terms in the two series at the right-hand side of (211) with $k \neq 0$ and $k \neq 1$, respectively for the upper case and lower case in (212), is exponentially fast. Since the Zak transform of $\chi_{(-1/2,1/2)}$ is a.e. identical to the right-hand side of (212) on $[0,1)^2$ while the convergence in (212) is certainly in $L^2([0,1)^2)$-sense, we easily get the first issue in (210) with $L^2(\mathbb{R})$-convergence.

Next we consider the second limit in (210). The Fourier transform of g_γ equals $g_{1/\gamma}$, whence, see the last paragraph in Sec. 1, the Fourier transform of h_γ equals $h_{1/\gamma}$. Now since the two right-hand side functions in (210) are Fourier pairs, the second limit relation in (210) follows from the first one. This completes the proof.

In Fig. 7 we have displayed h_γ for several values of γ (ranging from very small to very large).

Note. As the proof of Theorem 8.1 shows, one has similar limit relations as in (210) for a considerably wider class of windows g than Gaussians or hyperbolic secants. Denoting $g_\gamma = \gamma^{1/2}\, g(\gamma\cdot)$ and $h_\gamma = Z^{-1}(Zg_\gamma/|Zg_\gamma|)$, the first limit relation in (210) holds when $g(\gamma s)/g(\gamma t) \to 0$ sufficiently rapidly when $s > t \geq 0$ and $\gamma \to \infty$. Similarly, the second limit relation in (210) holds when $(\mathcal{F}g)(\gamma\mu)/(\mathcal{F}g)(\gamma\nu) \to 0$ sufficiently rapidly when $\mu > \nu \geq 0$ and $\gamma \to \infty$. Evidently, the two-sided exponential does satisfy the first condition but fails to satisfy the second one.

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Figure captions

Fig. 1. Tight frame generating window h associated to $g = g_{1,\gamma}$ and $g = g_{3,\alpha}$ of (9) and (10) according to (7) with $\gamma = 1$ and $\alpha = \sqrt{2\pi}$.

Fig. 2. Zak tight window $h_{3,\alpha}$ and frame tight window ${}^{a}h_{3,\alpha}$ associated to $g_{3,\alpha}$ in (10) for $\alpha = \sqrt{2\pi}$ and $a = b = \sqrt{0.9}$.

Fig. 3. Zak tight window $h_{3,\alpha}$ and frame tight window ${}^{a}h_{3,\alpha}$ associated to $g_{3,\alpha}$ in (10) for $\alpha = \sqrt{2\pi}$ and $a = b = \sqrt{0.99}$.

Fig. 4. The points $s_0, ..., s_6 \in [0, 1)$ as in (95) with $p = 4$, $a = 49/60$, $s_0 = 1/24$, together with the graphs of $r_2(s)$, $r_1(s)$ see (78), for $s \in [0, 1]$.

Fig. 5. The canonical dual $g^d_{3,\alpha}$ formally associated to $g_{3,\alpha}$ of (10) according to (109) and explicitly given by (110), (111) for $\alpha = \sqrt{2\pi}$.

Fig. 6. a. The limiting form of $h_{3,\alpha}$ as $\alpha \downarrow 0$ with $g_{3,\alpha}$ as in (10), given in integral form in (187), and $h_{3,\alpha}$ with $\alpha = \sqrt{2\pi}$.
b. The Fourier transform of $h_{3,\alpha}$ as $\alpha \downarrow 0$ with $g_{3,\alpha}$ as in (10), and the Fourier transform of $h_{3,\alpha}$ with $\alpha = \sqrt{2\pi}$.

Fig. 7. The tight frame generating window h_γ associated to $g_\gamma = g_{1,\gamma}$ of (9) for the values $\gamma = 0.1, 0.5, 1, 2, 10$.

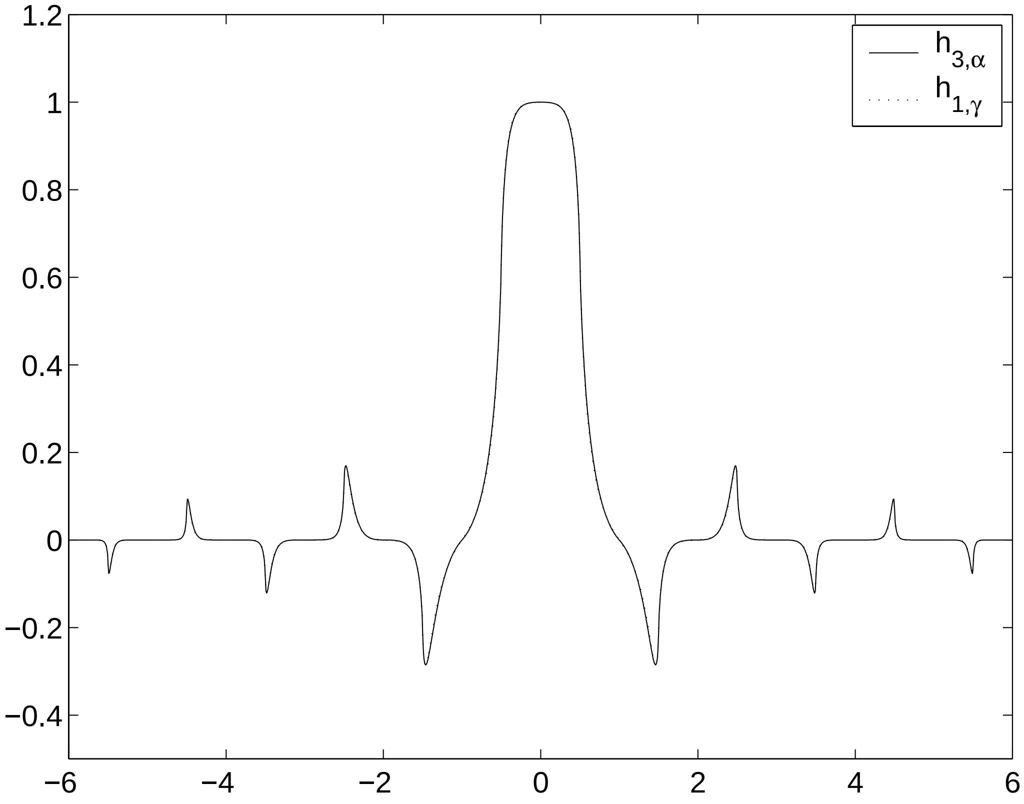

Figure 1.

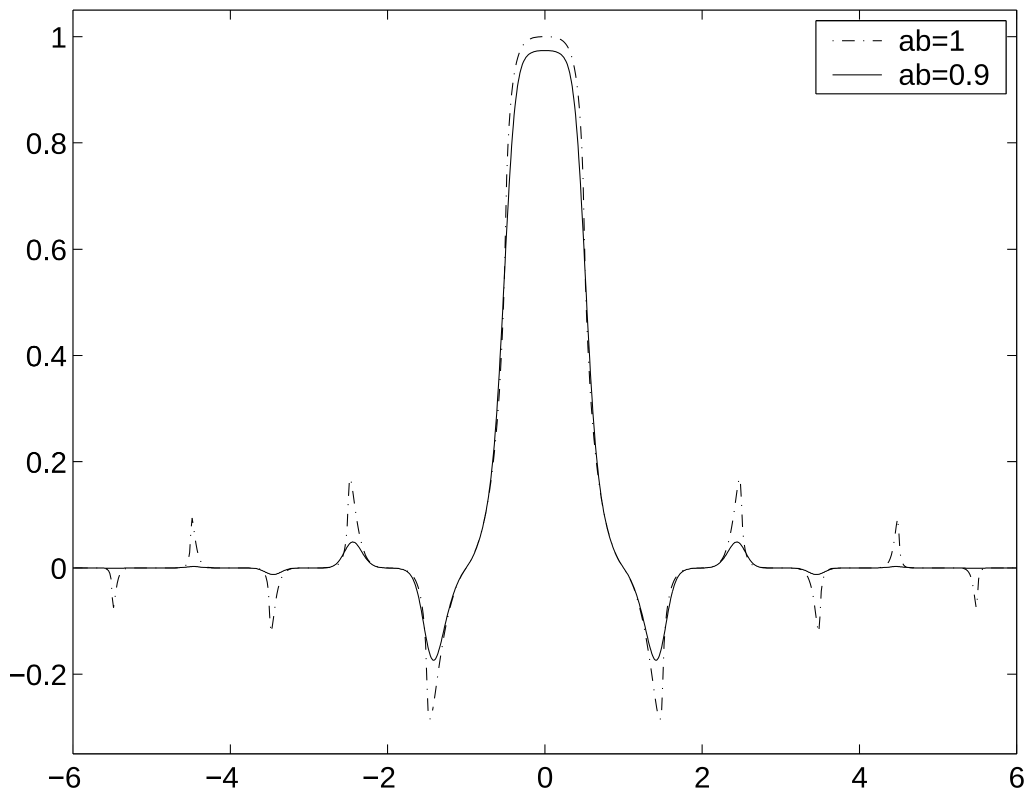

Figure 2.

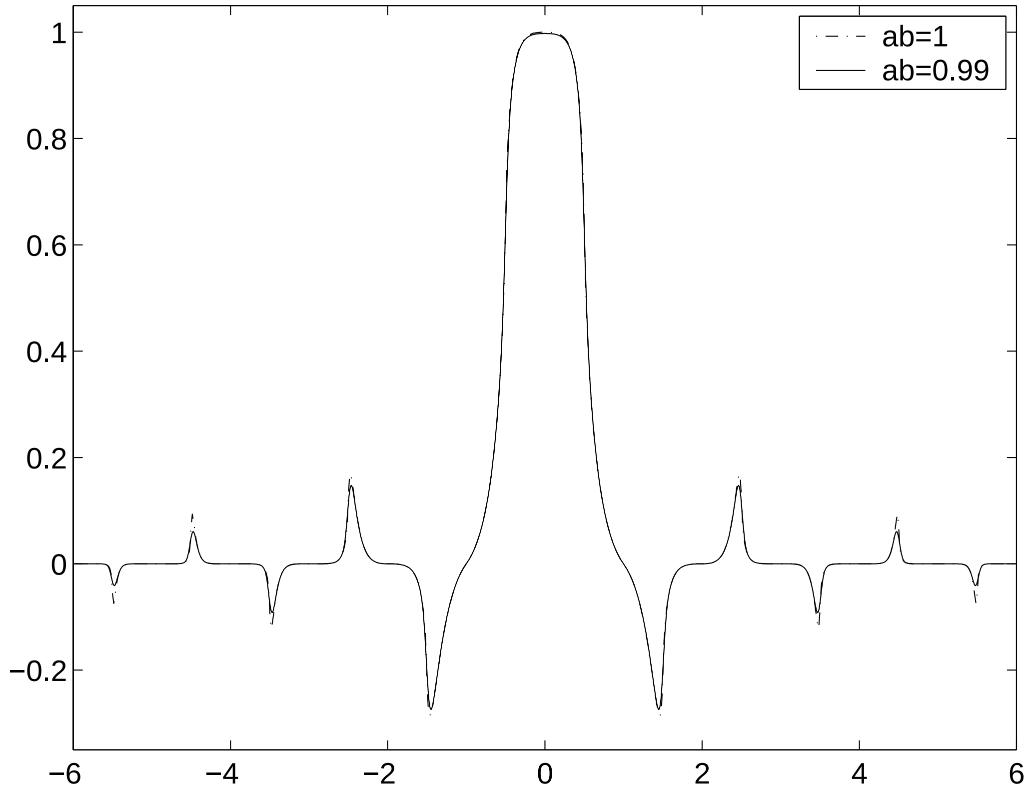

Figure 3.

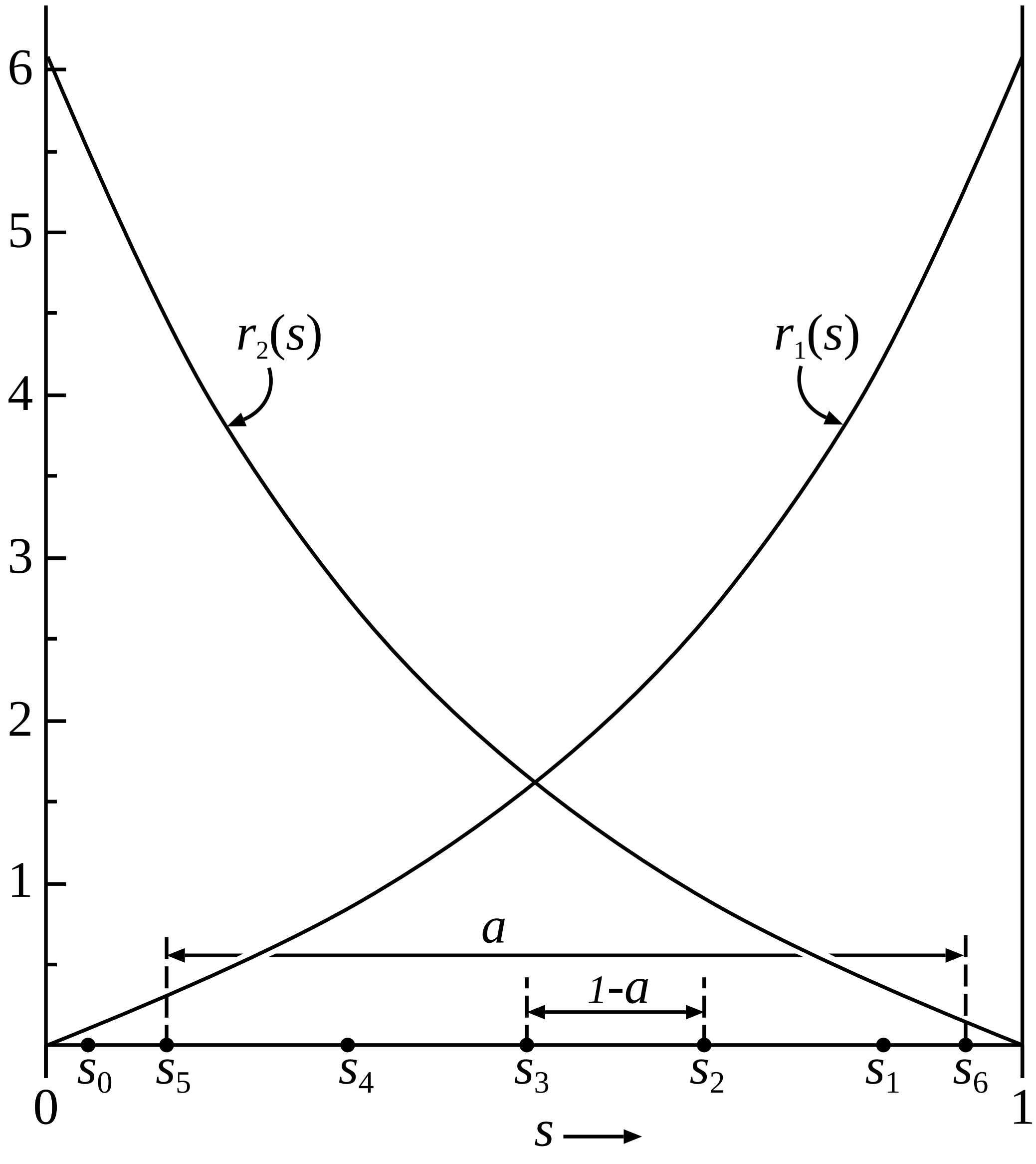

Figure 4.

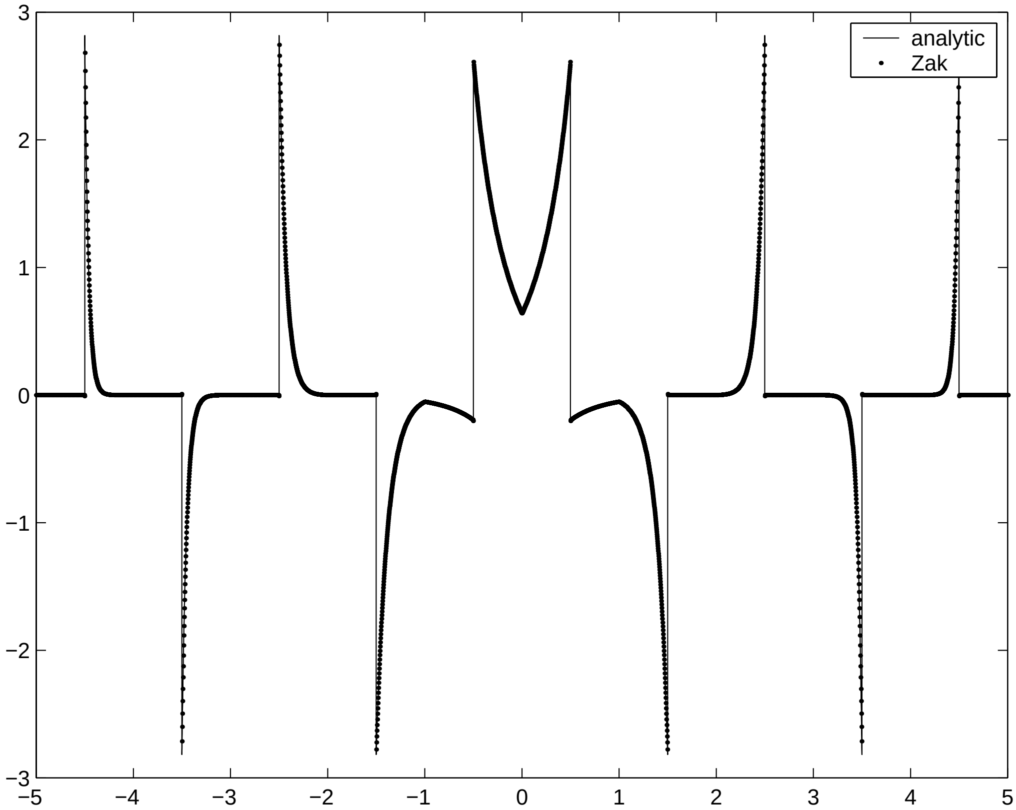

Figure 5.

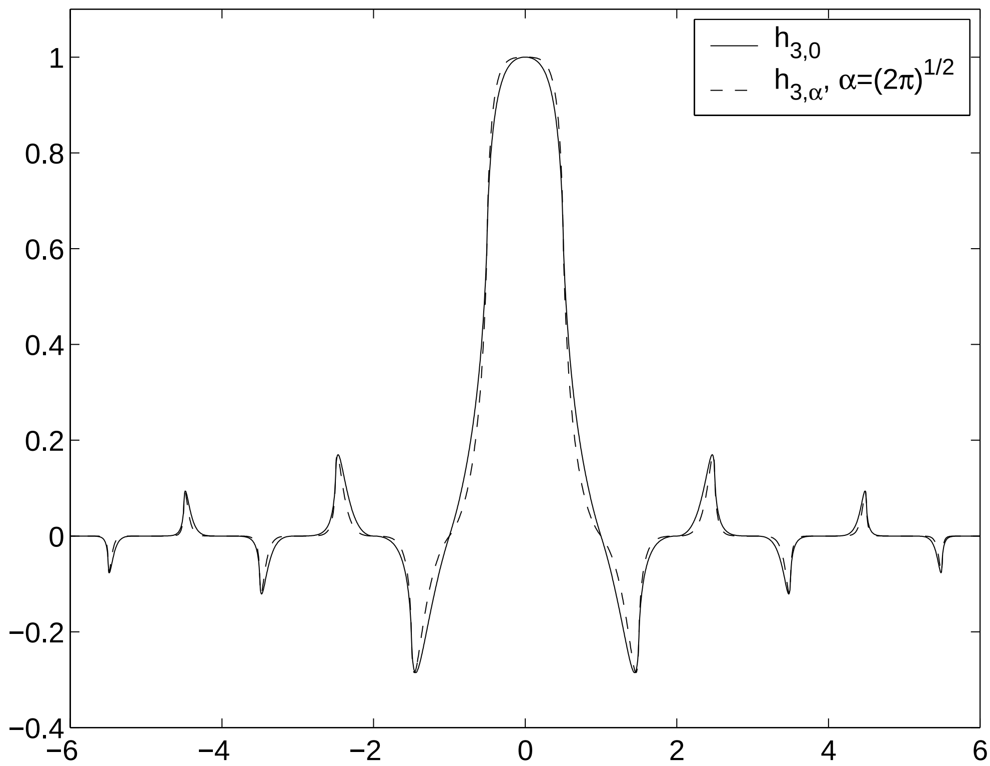

Figure 6a.

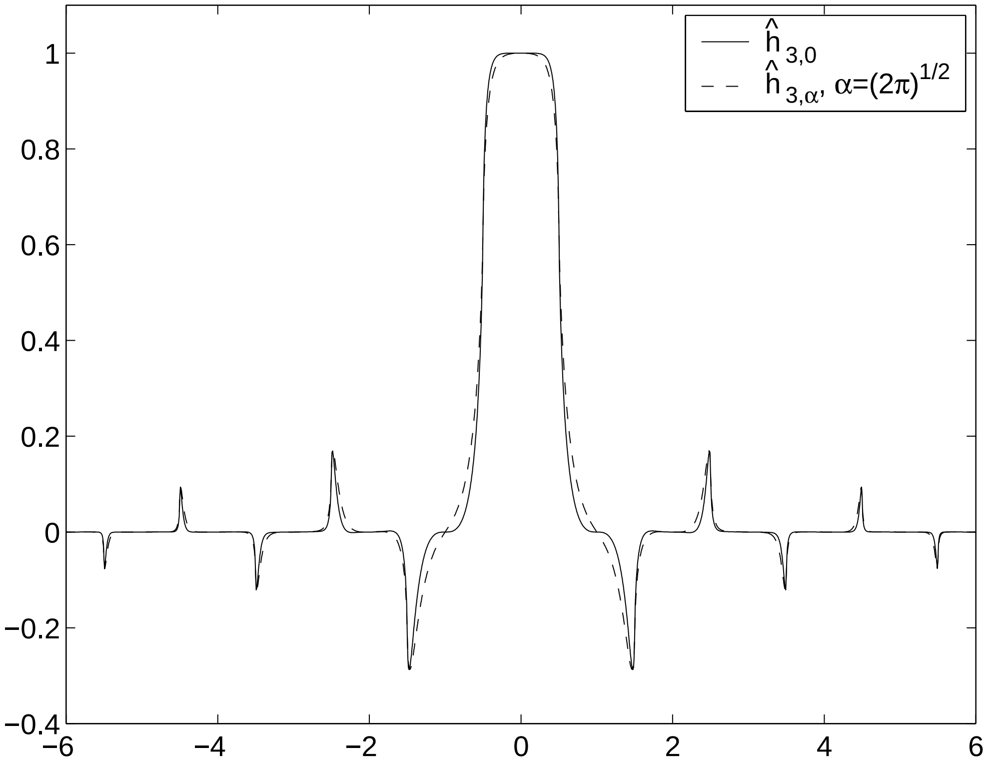

Figure 6b.

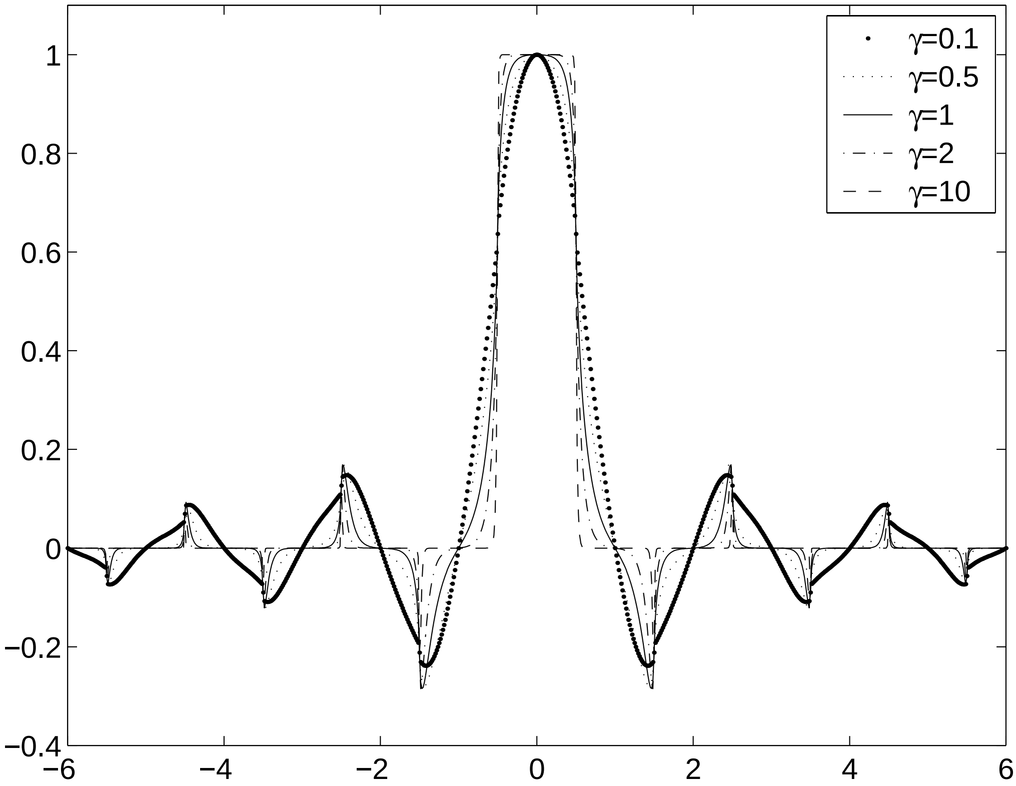

Figure 7.