A NEW CRITERION FOR GABOR FRAMES

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Abstract. We give a new criterion for the invertibility of the Gabor frame operator in the irrational case. It is sufficient to verify the invertibility of a single (infinite) matrix.

We give a new criterion for the invertibility of the Gabor frame operator in the irrational case. Usually this is the difficult case because Zak transform methods do not apply and thus explicit calculations are no longer applicable. However, in the irrational case it is sufficient to verify the invertibility of a single infinite matrix. This fact is not so surprising for the operator algebraist because the associated operator algebras, the so-called irrational rotation algebras or non-commutative tori, are simple whereas in the rational case they have a different structure [1, Ch. 6] or [11]. Also for the applied mathematician the main result of this note is not entirely unexpected and can be made plausible by the Zibulski-Zeevi representation of the Gabor frame operator in the rational case. Indeed, in dimension $d = 1$ when $\alpha\beta = p/q$ with $p, q \in \mathbb{N}$ relatively prime, it suffices to inspect the spectra of the relevant Zibulski-Zeevi matrices in any rectangle in the Zak transform domain with side lengths $1/q$ and $1/p$ [4, 14]. When $p, q \to \infty$, as would be the case when we were to approximate an irrational number $\alpha\beta$ by rationals p/q, then the area of the rectangle to be checked tends to 0. A similar reduction of the checking problem was noted in [6, Section 3.6] where an algorithm based on the matrices of type (3) below was proposed to decide for which lattice parameters the characteristic function $\chi_{[0,c]}$ forms a Gabor frame.

We denote time-frequency shifts by $M_\omega T_x f(t) := e^{2\pi i\omega\cdot t} f(t-x)$ for $t, x, \omega \in \mathbb{R}^d$. For a given window function $g \in L^2(\mathbb{R}^d)$ and lattice parameters $\alpha, \beta > 0$, we consider the Gabor system $\mathcal{G}(g, \alpha, \beta) = \{M_{\beta n} T_{\alpha k} g : k, n \in \mathbb{Z}^d\}$ and the associated Gabor frame operator S defined by

$$Sf = \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k} g \tag{1}$$

If S is invertible on $L^2(\mathbb{R}^d)$, then $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame or Weyl-Heisenberg frame. Following the established literature we will use the following practical class of windows: we say that g belongs to the Wiener amalgam space $W := W(C, \ell^1)$, if g is continuous and if the norm

$$\|g\|_W = \sum_{k\in\mathbb{Z}^d} \sup_{t\in[0,1]^d} |g(t+k)| \tag{2}$$

is finite.

Key words and phrases. Gabor frames, duality principle, irrational case.

The literature contains a number of sufficient conditions and a few necessary and sufficient conditions for the invertibility of S (see, e.g., [3], in particular Prop. 6.3.4 and Thm. 7.4.3). In this note we work with the representation of S in terms of Ron-Shen matrices that are defined as follows. Set

$$G_{jl}(x) = \sum_{k\in\mathbb{Z}^d} \bar{g}(x - \frac{l}{\beta} - \alpha k)\, g(x - \frac{j}{\beta} - \alpha k) \tag{3}$$

and for each x in the cube $[0,\alpha]^d$ let $G(x)$ be the operator (matrix) acting on $c = (c_k)_{k\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ with finite support by matrix multiplication

$$(G(x)c)_j = \sum_{l\in\mathbb{Z}^d} G_{jl}(x)c_l \tag{4}$$

Note that each $G(x)$ is a positive operator in the sense that $\langle G(x)c, c\rangle \geq 0$ for all $c \in \ell^2(\mathbb{Z}^d)$ with finite support.

If $\alpha\beta$ is irrational and g reasonably nice, then we have the following criterion for $\mathcal{G}(g,\alpha,\beta)$ to be a Gabor frame.

Theorem 1. *Assume that $g \in W(C,\ell^1)$. If $\alpha\beta \notin \mathbb{Q}$, then the following are equivalent:*

(i) S is invertible on $L^2(\mathbb{R}^d)$.

(ii) There exist constants $a, b > 0$ such that for all $x \in [0,\alpha]^d$

$$a\,\mathrm{Id}_{\ell^2} \leq G(x) \leq b\,\mathrm{Id}_{\ell^2} \tag{5}$$

(iii) There exists $a > 0$ such that for all $x \in [0,\alpha]^d$

$$\sum_{k\in\mathbb{Z}^d} |\sum_{l\in\mathbb{Z}^d} g(x - \frac{l}{\beta} - \alpha k)\, c_l|^2 \geq a\|c\|_2^2 \qquad c \in \ell^2(\mathbb{Z}^d)\,. \tag{6}$$

(iv) There exists a single $\xi \in [0,\alpha]^d$ such that $G(\xi)$ is invertible on $\ell^2(\mathbb{Z}^d)$.

If any of these conditions is satisfied, then all operators $G(x)$ have the same spectrum, in short $\sigma(G(x)) = \sigma(G(\xi))$ for all $x \in \mathbb{R}^d$.

The theorem is a summary of known results, see [3, Ch. 6 and 7] for details and precise references. The novelty is the new condition (iv).

The equivalence (i) $\Longleftrightarrow$ (ii) is well known and was proved in [12] and [2, Thm. 3.2] (however, for $g \in S_0$ only). It holds without any restriction on $\alpha\beta < 1$ and for arbitrary $g \in L^2(\mathbb{R}^d)$ (in this case (5) needs to hold for almost all x). We remark that criterion (iii) is particularly useful to verify that certain Gabor systems are Gabor frames, see [6, Section 3] and [5,7].

The equivalence of (ii) and (iii) is a consequence of the following calculation

$$
\begin{aligned}
\langle G(x)c, c\rangle &= \sum_{j\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} G_{jl}(x)c_l\bar{c}_j \\
&= \sum_{j\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} \bar{g}(x-\frac{l}{\beta}-\alpha k)\, g(x-\frac{j}{\beta}-\alpha k)c_l\bar{c}_j \\
&= \sum_{k\in\mathbb{Z}^d} |\sum_{l\in\mathbb{Z}^d} g(x-\frac{l}{\beta}-\alpha k)\,\bar{c}_l|^2
\end{aligned}
$$

Here the interchange of the order of summation is perfectly justified because we may assume that c has finite support.

The implication (iii) $\Longrightarrow$ (iv) is obvious.

In the remainder of this note we prove the decisive implication (iv) $\Longrightarrow$ (ii). We need the following lemma which can be traced back to Walnut [13], see also [3], p. 116, Step 2.

Lemma 1. *Let $g, \gamma \in W$ and define the modified matrices $G(x) = G^{g,\gamma}(x)$ by their entries*

$$G_{jl}(x) = \sum_{k\in\mathbb{Z}^d} \bar{g}(x-\frac{l}{\beta}-\alpha k)\,\gamma(x-\frac{j}{\beta}-\alpha k) \tag{7}$$

Then

$$\sum_{l\in\mathbb{Z}^d}|G_{jl}(x)| \le (\beta+1)^d\,(\tfrac{1}{\alpha}+1)^d\,\|g\|_W\,\|\gamma\|_W \tag{8}$$

for all $x \in \mathbb{R}^d$, and the same estimate holds for the sum $\sum_{j\in\mathbb{Z}^d}|G_{jl}(x)|$.

By Schur's test (e.g., [3, Lemma 6.2.1]) the operator norm of $G(x)$ acting on $\ell^2(\mathbb{Z}^d)$ is bounded by

$$\|G(x)\|_{op} \le (\beta+1)^d\,(\tfrac{1}{\alpha}+1)^d\,\|g\|_W\,\|\gamma\|_W$$

Consequently we can choose the bound $b = (\beta+1)^d\,(\frac{1}{\alpha}+1)^d\,\|g\|_W\,\|\gamma\|_W$ in (5) and the upper estimate is always satisfied. Thus we need only take care of the lower bound.

To show the existence of a uniform lower bound $a > 0$, we proceed in two steps.

Step 1. Let $\tau_n, n \in \mathbb{Z}^d$, be the translation operator on $\ell^2(\mathbb{Z}^d)$ defined by

$$(\tau_n c)_l = c_{l-n} \quad \text{for } c \in \ell^2(\mathbb{Z}^d) \tag{9}$$

The definition of $G(x)$ implies that for all $n \in \mathbb{Z}^d$ we have

$$G_{jl}(x-\frac{n}{\beta}) = G_{j+n,l+n}(x)\,. \tag{10}$$

Now observe that for all finitely supported $c, d \in \ell^2(\mathbb{Z}^d)$ we have

$$\begin{aligned}\langle G(x-\frac{n}{\beta})c, d\rangle &= \sum_{j,l\in\mathbb{Z}^d} G_{jl}(x-\frac{n}{\beta})c_l\bar{d}_j \\ &= \sum_{j,l\in\mathbb{Z}^d} G_{j+n,l+n}(x)c_l\bar{d}_j \\ &= \sum_{j,l\in\mathbb{Z}^d} G_{jl}(x)c_{l-n}\bar{d}_{j-n} \\ &= \langle G(x)\tau_n c, \tau_n d\rangle\end{aligned}$$

This property implies that

$$G(x-\frac{n}{\beta}) = \tau_{-n}G(x)\tau_n \tag{11}$$

Since τ_n is unitary on $\ell^2(\mathbb{Z}^d)$, conjugation by τ_n does not change the spectrum and thus all operators $G(x-\frac{n}{\beta})$ for $n \in \mathbb{Z}^d$ possess the same spectrum $\sigma(G(x))$.

Since we have assumed the existence of a $\xi \in \mathbb{R}^d$ such that

$$a\,\mathrm{Id}_{\ell^2} \le G(\xi) \le b\,\mathrm{Id}_{\ell^2}\,, \tag{12}$$

we conclude that

$$a\,\mathrm{Id}_{\ell^2} \le G(\xi-\frac{n}{\beta}) \le b\,\mathrm{Id}_{\ell^2} \qquad \text{for } \textit{all } \; n\in\mathbb{Z}^d\,.$$

Now $G(x) = G(x+\alpha)$ for all $x \in \mathbb{R}^d$ and $\alpha\beta$ is irrational. By Kronecker's theorem [9] the set $\{x = \xi - \frac{n}{\beta} (\mod \alpha) : n \in \mathbb{Z}^d\}$ is therefore dense in the cube $[0,\alpha]^d$. So far we have shown that (5) holds for a dense subset of $[0,\alpha]^d$.

Step 2. To show that (5) holds for all $x \in [0,\alpha]^d$, we use an approximation argument. We will show that the map $x \mapsto G(x)$ is uniformly continuous in the operator norm. In other words, we show that for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|G(x) - G(x')\|_{op} < \epsilon \quad \text{ for } |x-x'| < \delta\,. \tag{13}$$

If in this case we choose $\epsilon = a/2$ where a is the lower bound in (12), then for $|x-x'| < \delta$ we obtain

$$\begin{aligned}\langle G(x')c, c\rangle &= \langle G(x)c, c\rangle - \langle (G(x)-G(x'))c, c\rangle \\ &\ge a\|c\|_2^2 - \|G(x)-G(x')\|_{op}\|c\|_2^2 \\ &\ge \frac{a}{2}\|c\|_2^2\,, p\end{aligned}$$

or equivalently $G(x') \ge \frac{a}{2}\,\mathrm{Id}_{\ell^2}$ for $|x-x'| < \delta$. Since we know from Step 1 that $G(x) \ge a\,\mathrm{Id}_{\ell^2}$ on a dense subset of $[0,\alpha]^d$, we are done.

Since the map $T \mapsto \sigma(T)$ from bounded *selfadjoint* operators endowed with the operator topology into compact sets of $\mathbb{R}$ endowed with the Hausdorff topology is continuous, (13) implies that the spectrum $\sigma(G(x))$ is independent of x. More precisely, let $d(A,B) = \max\big(\sup_{a\in A}\inf_{b\in B}|a-b|\,;\,\sup_{b\in B}\inf_{a\in A}|a-b|\big)$ be the Hausdorff distance between the closed sets $A, B \subseteq \mathbb{R}$. If T_1, T_2 are bounded selfadjoint operators on some Hilbert space, then the perturbation result [8, Thm. 4.10,p. 291] states that

$$d(\sigma(T_1), \sigma(T_2)) \le \|T_1 - T_2\|_{op}\,.$$

Given $x \in \mathbb{R}^d$, there exist $\ell, n \in \mathbb{N}^d$ such that $|x - (\xi - n/\beta - \ell\alpha)| < \delta$ and (13) implies that $\|G(x) - G(\xi - n/\beta - \ell\alpha)\|_{op} < \epsilon$. Consequently

$$d\big(\sigma(G(x)), \sigma(G(\xi))\big) = d\big(\sigma(G(x)), \sigma(G(\xi - n/\beta - \ell\alpha))\big) < \epsilon\,.$$

Since $\epsilon > 0$ was arbitrary, we obtain that $d\big(\sigma(G(x)), \sigma(G(\xi))\big) = 0$, and the equality of the spectra $\sigma(G(x)) = G(\xi))$ follows.

So it remains to prove the following lemma.

Lemma 2. *If $g \in W$, then the map $x \mapsto G(x)$ from $\mathbb{R}^d$ into the bounded operators on $\ell^2(\mathbb{Z}^d)$ is uniformly continuous in the operator norm.*

Proof. For $x' = x + h$ set $\gamma(t) = T_{-h}g(t) = g(x+h)$. Then we have

$$G^{g,g}_{jl}(x+h) = \sum_{k\in\mathbb{Z}^d} \bar{g}(x+h-\frac{l}{\beta}-\alpha k)\, g(x+h-\frac{j}{\beta}-\alpha k) = G^{\gamma,\gamma}_{jl}(x)\,.$$

By the usual product trick we can write the difference $G(x+h) - G(x)$ as

$$G^{g,g}(x+h) - G^{g,g}(x) = G^{\gamma,\gamma}(x) - G^{g,g}(x) = G^{\gamma,\gamma-g}(x) + G^{\gamma-g,g}(x)\,.$$

Now Lemma 1 implies that

$$\begin{aligned} \|G^{g,g}(x+h) - G^{g,g}(x)\|_{op} &\le \|G^{\gamma,\gamma-g}(x)\|_{op} + \|G^{\gamma-g,g}(x)\|_{op} \\ &\le (\beta+1)^d(\frac{1}{\alpha}+1)^d\Big(2\|\gamma\|_W\,\|\gamma-g\|_W\Big) \end{aligned} \tag{14}$$

We know that translation is uniformly bounded and strongly continuous on W, i.e., $\|T_x g\|_W \le C\|g\|_W$ for all $x \in \mathbb{R}^d$ and $\|T_h g - g\|_W \to 0$ as $|h| \to 0$ [10, Ex. 1.5.3]. Since $\gamma = T_{-h}g$, (14) implies that

$$\lim_{h\to 0} \|G^{g,g}(x+h) - G^{g,g}(x)\|_{op} = 0\,, \tag{15}$$

which is the statement of the lemma. ∎

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