Note on B-splines, wavelet scaling functions, and Gabor frames

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Abstract—**Let g be a continuous, compactly supported function on $\mathbb{R}$ such that the integer translates of g constitute a partition of unity. We show that the Gabor system (g, a, b), with window g and time-shift and frequency-shift parameters $a, b > 0$ has no lower frame bound larger than 0 if $b = 2, 3, \ldots$ and $a > 0$. In particular, (g, a, b) is not a Gabor frame if g is a continuous, compactly supported wavelet scaling function and if $b = 2, 3, \ldots$ and $a > 0$. We exemplify our result for the case that $g = B_1$, the triangle function supported by $[-1, 1]$, by showing pictures of the canonical dual corresponding to (g, a, b) when $ab = 1/4$ and b crosses the lines $N = 2, 3, \ldots$.**

Keywords—**B–splines, Gabor frame, partition of unity, Ron–Shen condition, wavelet scaling function.**

I. Introduction

LET $a > 0$, $b > 0$ and $g \in L^2(\mathbb{R})$. We call the function system

$$\left(g_{na,mb}\right)_{n,m\in\mathbb{Z}} \equiv (g, a, b) \tag{1}$$

a Gabor frame if there are $A > 0$, $B < \infty$ such that for all $f \in L^2(\mathbb{R})$

$$A\|f\|^2 \le \sum_{n,m} \left|\langle f, g_{na,mb}\rangle\right|^2 \le B\|f\|^2 . \tag{2}$$

Here $g_{x,y}$ denotes for $x, y \in \mathbb{R}$ the time-frequency shifted function

$$g_{x,y}(t) = e^{2\pi i y t} g(t - x), \quad t \in \mathbb{R} . \tag{3}$$

The numbers A and B that appear in (2) are called lower and upper frame bound respectively. It is well-known that (g, a, b) can be a Gabor frame only if $ab \le 1$; also, if $ab = 1$ and (g, a, b) is a Gabor frame, then g cannot be continuous and compactly supported. We refer for basic information about (Gabor) frames to [1, Sections 3.4, 3.5, 4.1, 4.2]; a comprehensive and recent treatment of Gabor systems and frames can be found in [2, Chapters 5-9, 11-13]. We shall use here the following criterion, due to Ron and Shen [3], for being a Gabor frame, see [2, p. 117, Proposition 6.3.4]; for convenience we restrict ourselves to continuous, compactly supported windows g. The system (g, a, b) is a Gabor frame with frame bounds $A > 0$, $B < \infty$ if and only if for all $c \in l^2(\mathbb{Z})$ and all $t \in \mathbb{R}$

$$A\|c\|^2 \le \frac{1}{b} \sum_{n=-\infty}^{\infty} \Big| \sum_{k=-\infty}^{\infty} g(t - na - k/b) c_k \Big|^2 \le B\|c\|^2 . \tag{4}$$

For $g \in L^2(\mathbb{R})$ we denote by F_g the set

$$F_g = \big\{ (a, b) \mid a > 0, b > 0, (g, a, b) \text{ is a Gabor frame} \big\} . \tag{5}$$

It is often quite hard to determine the set F_g for a given window $g \in L^2(\mathbb{R})$. In certain cases, for instance when one has restrictions on the supporting set of g, the Ron-Shen criterion can be of great help in telling whether (a, b) belongs to F_g. In [4, Section 3] a considerable effort has been made to determine F_g for the case that $g = B_0 = \chi_{[0,1)}$; the result is a complicated subset of $\big\{ (a, b) \mid a > 0, b > 0, ab \le 1 \big\}$ where (ir)rationality of ab plays a key role. Furthermore, only in a few cases of well-behaved windows g it has been shown that $F_g = \big\{ (a, b) \mid a > 0, b > 0, ab < 1 \big\}$; among these g are the Gaussians [5, 6], hyperbolic secants [7] and two sided exponentials [8, Section 5].

In this correspondence we ask the question whether for certain standard windows g from approximation theory and wavelet theory the set F_g consists of all (a, b) with $a > 0$, $b > 0$ and $ab < 1$ as well. Unlike the example $g = B_0$ given above, the windows of this type are smooth and well decaying, which implies that the sets F_g are open sets [9]. We shall show in Section II that for any continuous, compactly supported g satisfying the partition-of-unity identity

$$\sum_{k=-\infty}^{\infty} g(t - k) = 1 , \quad t \in \mathbb{R}, \tag{6}$$

no lower frame bound $A > 0$ for the Gabor system (g, a, b) exists when $a > 0$ and $b = 2, 3, \ldots$. Condition (6) can be shown to hold for large classes of windows, in particular it is satisfied for some commonly used windows in signal processing, such as the raised cosine

$$\mathrm{RC}(t) = \begin{cases} (1 + \cos \pi t)/2, & \text{when } |t| \le 1, \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

and the trapezoidal function, for $0 < \delta < 1/2$ defined as

$$\mathrm{T}(t) = \begin{cases} 1, & |t| \le 1/2 - \delta, \\ (2\delta)^{-1}(1/2 + \delta - |t|), & \big||t| - 1/2\big| < \delta, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

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Further, condition (6) on g is satisfied when g is a compactly supported, continuous scaling function from the theory of wavelets [1, Remark 5 on pp. 144–145 and Note 9 on p. 165], or when g is a B-spline with knots at the integers [10]. Hence, in these cases the set F_g consists apparently of countably many open sets, separated from one another by the horizontal lines $b = N = 2, 3, \ldots$.

This result came as a big surprise to us, because it is generally assumed that any "nice" window g and "reasonable" choice of $a, b > 0$ yield Gabor frames. Our observation demonstrates that even for window functions that are perfectly natural in approximation theory and wavelet theory, a and b have to be chosen extremely carefully.

In Section III we consider the linear B-spline or triangle function $g(t) = B_1(t) = \max(0, 1 - |t|)$ as an example. We show pictures of the canonical dual function corresponding to the Gabor system (g, a, b) with $ab = 1/4$ and b close to 2 and 3. It is conjectured that F_g consists of all (a, b) with $0 < a < 2$, $b > 0$, $ab < 1$ and $b \neq 2, 3, \ldots$. The pictures in Section III support this conjecture.

II. Proof of the main result

We assume that g is continuous and compactly supported (below we comment on weakening those conditions), and that g satisfies (6). Also, we let $a > 0$ and $b = 2, 3, \ldots$. We shall show that the Gabor system (g, a, b) has no positive lower frame bound (the Ron-Shen criterion (4) implies that (g, a, b) has a finite upper frame bound). To that end we shall display $c^K \in l^2(\mathbb{Z})$, $K = 1, 2, \ldots$, such that for all $t \in \mathbb{R}$

$$\frac{1}{\|c^K\|^2} \sum_{n=-\infty}^{\infty} \Big| \sum_{k=-\infty}^{\infty} g(t - na - k/N)\, c_k^K \Big|^2 \to 0, \quad K \to \infty\,. \tag{9}$$

Let $b = N = 2, 3, \ldots$ and set

$$c_k = e^{2\pi i k r/N}, \quad k \in \mathbb{Z}, \tag{10}$$

where $r = 1, \ldots, N-1$. Then for all $t \in \mathbb{R}$, $n \in \mathbb{Z}$, we have

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} g(t - na - k/N)\, c_k \\ &= \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} g\big(t - na - \tfrac{k}{N} - l\big) e^{2\pi i r (\frac{k}{N} + l)} \\ &= \sum_{k=0}^{N-1} e^{2\pi i k r/N} \ = 0\,. \end{aligned} \tag{11}$$

For $K = 1, 2, \ldots$ we define the sequence $c^K \in l^2(\mathbb{Z})$ by

$$c_k^K = \begin{cases} c_k\,, & |k| \le K\,, \\ 0\,, & |k| > K\,, \end{cases} \quad k \in \mathbb{Z}\,, \tag{12}$$

and the subsets of indices

$$\begin{aligned} V_k &= \big\{ l \in \mathbb{Z} \mid g(t - na - \tfrac{k}{N} - l) \neq 0 \big\}\,, \\ W_k &= \big\{ l \in \mathbb{Z} \mid c_{k+lN}^K \neq 0 \big\}\,. \end{aligned} \tag{13}$$

Then the equality $\sum_k g(t - na - \frac{k}{N})\, c_k^K = 0$ holds for all $t \in \mathbb{R}$, $n \in \mathbb{Z}$ whenever either

$$V_k \subset W_k\,, \quad k = 0, \ldots, N-1\,, \tag{14}$$

or

$$V_k \cap W_k = \varnothing\,, \quad k = 0, \ldots, N-1\,. \tag{15}$$

Conversely, for $t \in \mathbb{R}$, $n \in \mathbb{Z}$ both (14) and (15) fail to be true if and only if $W_k \setminus V_k \neq \varnothing$, i.e., if there are $j_1, j_2 \in \mathbb{Z}$ such that

$$g(t - na - \tfrac{j_1}{N}) \neq 0 \neq g(t - na - \tfrac{j_2}{N}), \quad c_{j_1}^K = 0 \neq c_{j_2}^K\,. \tag{16}$$

Let $|\mathrm{I}|$ be the length of a supporting interval I of g. Then the number of $n \in \mathbb{Z}$ such that both (14) and (15) fail to hold is at most $2(|\mathrm{I}|/a + 1)$ for any $t \in \mathbb{R}$. Hence

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} \Big| \sum_{k=-\infty}^{\infty} g(t - na - k/N) c_k^K \Big|^2 \\ &\le 2(|\mathrm{I}|/a + 1) \sum_{k=-\infty}^{\infty} |g(t - na - k/N)|^2 \\ &\le 2(|\mathrm{I}|/a + 1)(MN(|\mathrm{I}| + 1))^2 \end{aligned} \tag{17}$$

where M is an upper bound for $|g|$. Since (17) is independent of K and since $\|c^K\| = (2K+1)^{1/2}$, we obtain (9) for any $t \in \mathbb{R}$.

Remark. The property (11) with c_k given in (10) can be phrased as $(Zg)(s, r/N) = 0$ for $s \in \mathbb{R}$, $r = 1, \ldots, N-1$. Here

$$(Zg)(s, \nu) = N^{-1/2} \sum_{k=-\infty}^{\infty} g\Big(\frac{s-k}{N}\Big)\, e^{2\pi i k \nu}, \quad s, \nu \in \mathbb{R}\,, \tag{18}$$

is a Zak transform of g, see [11, Section 1.5]. Using Gabor frame operator theory in the Zak transform domain, one can get the main result under considerably weaker conditions on g than the ones made above.

III. Example

In this section we consider the choice

$$g(t) = B_1(t) = \max(0, 1 - |t|)\,, \quad t \in \mathbb{R}, \tag{19}$$

and we display the canonical dual $\gamma_{a,b} = S_{a,b}^{-1}\, g$, see [2, Section 7.6] for the role of the canonical dual in Gabor analysis, for some values of $a > 0$, $b > 0$, with $ab = 1/4$ and b near 2 or 3 (Fig. 1). Here $S_{a,b}$ is the frame operator corresponding to (g, a, b), which is invertible when (g, a, b) is a frame. One easily sees from the Ron-Shen criterion in (4) or [2, Thm. 6.4.1] that (g, a, b) is a frame when $a < 2 \le 1/b$. For the other cases that are shown in the figure, we have only numerical evidence that the lower frame bound A for (g, a, b) is positive. The dual windows were approximated by considering sampled Gabor systems for $l^2(\mathbb{Z})$ and their dual systems, see [12] for details, with sample rates taken so high that further increasing them produced no visible changes in the figure anymore. As we see, the $\gamma_{a,b}$ so obtained turns from a well-behaved function for the values $b = 1.5$, 2.5 into a quite irregularly behaved one when b approaches 2 or 3.

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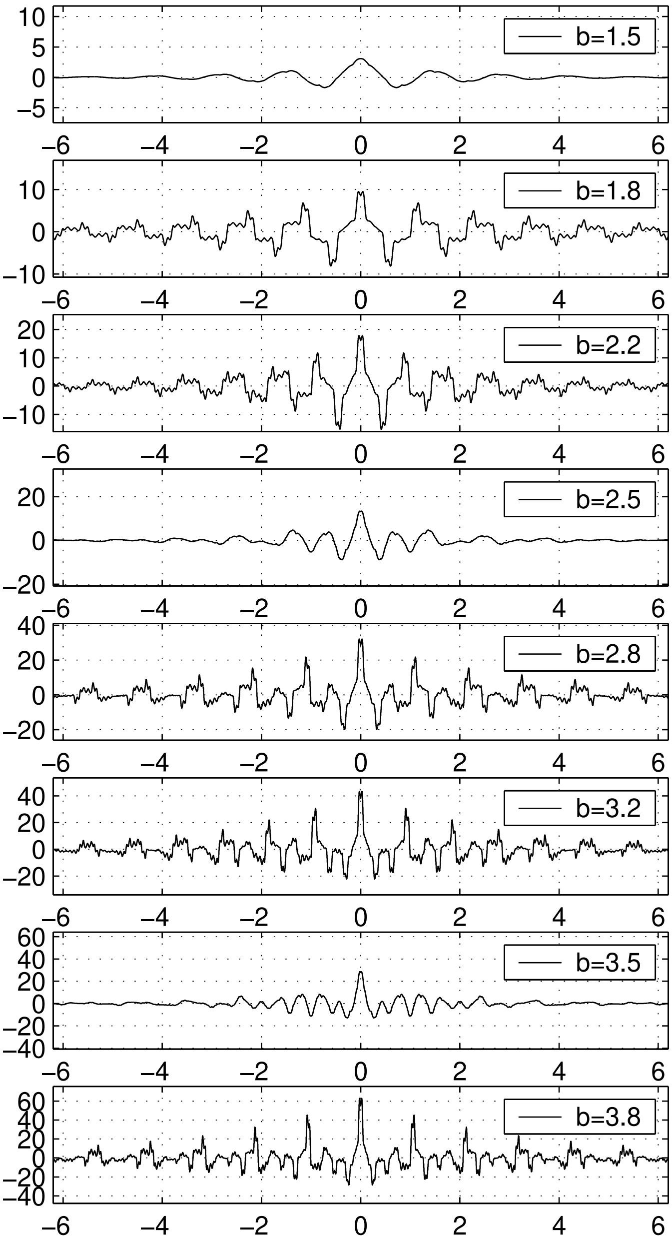

Fig. 1. The canonical Gabor dual $\gamma_{a,b} = S_{a,b}^{-1} g$ of the triangle function $g = B_1$, see (19), for $ab = 1/4$, $b = 1.5, 1.8, 2.2, 2.5, 2.8, 3.2, 3.5, 3.8$.