Analysis of some fast algorithms to compute canonical windows for Gabor frames

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Abstract.
We analyze some iterative algorithms for the computation of the canonical tight window g^t and the canonical dual window g^d associated with a Gabor frame (g, a, b). For these algorithms we present lower bounds on A/B, where $A > 0$ and $B < \infty$ are the best frame bounds of (g, a, b), such that convergence is guaranteed to the canonical window in question. As to computation of g^t we consider algorithms that do require inversion of intermediate frame operators as well as algorithms that do not require these inversions. As to computation of g^d we, naturally, consider algorithms where no frame operator inversions are required. Thus we propose for g^t an unconditionally converging algorithm with quadratic convergence using inversions, a conditionally ($A/B > \frac{1}{2}$) converging algorithm with quadratic convergence (without inversions), and a conditionally ($A/B > 3/7$) converging algorithm with cubic convergence (without inversions). For g^d we propose a conditionally ($A/B > \frac{1}{2}(\sqrt{5} - 1)$) converging algorithm with quadratic convergence, and a conditionally ($A/B > 0.5138...$) converging algorithm with cubic convergence. All these algorithms are iterative in nature. In the k^{th} iteration step, the $(k + 1)$-st window is expressed as a linear combination of two or three simple terms comprising γ_k, g and (the inverses of) the frame operator S and S_k corresponding to (g, a, b) and (γ_k, a, b), respectively. For the analysis of the algorithms with cubic convergence (that require three terms in the iteration steps), a sharp and possibly new form of Kantorovich's inequality is required. By considering the case that $a = b = 1$ (so that Zak transform techniques can be used), it is shown that this sharp Kantorovich inequality is best possible in the present context of Gabor frame operators. The analyses of the algorithms with quadratic convergence for g^t and g^d can be unified by considering coupled recursions comprising a parameter $\alpha \in [0, 1]$: the

choice $\alpha = \frac{1}{2}$ yields the algorithm for g^t and the choice $\alpha = 0$ or 1 yields the algorithm for g^d. Unfortunately, these coupled recursions do not yield an algorithm to compute $S^{-\alpha}g$, except in the cases ($\alpha = 0, \frac{1}{2}, 1$) that we already had. All analyses are based on an appropriate use of the spectral mapping theorem, relating the spectra of (frame) operators corresponding to γ_k and γ_{k+1}, and a fair amount of elementary but sometimes complicated considerations concerning extrema of low-degree polynomials on compact sets. These considerations also show that the above given convergence guaranteeing lower bounds on A/B are realistic for general Gabor frames but rather pessimistic for Gabor frames (g, a, b) with smooth, rapidly decaying window g.

Keywords: Gabor frame, canonical tight window, canonical dual window, iterative method, quadratic and cubic convergence, Kantorovich's inequality, coupled recursions, Zak transform.

1 Introduction and preview of results

We continue the developments in [1], Sec. 4, where we analyzed an algorithm, independently found by H. Feichtinger and T. Strohmer, for the computation of the tight window g^t canonically associated with a Gabor frame (g, a, b) for $L^2(\mathbb{R})$. We use the common conventions and notations in Gabor frame theory. Hence we have $a > 0$, $b > 0$, $ab \leq 1$ and $g \in L^2(\mathbb{R})$. The frame operator S corresponding to (g, a, b), defined by

$$f \in L^2(\mathbb{R}) \rightarrow Sf = \sum_{n,m} (f, g_{na,mb})\, g_{na,mb} \ , \tag{1}$$

is supposed to be a bounded, positive definite linear operator of $L^2(\mathbb{R})$. In (1) we have for $x, y \in \mathbb{R}$

$$g_{x,y}(t) = e^{2\pi iyt}\, g(t-x) \ , \qquad \text{a.e. } t \in \mathbb{R} \ . \tag{2}$$

We let $A = \min\, \sigma(S)$, $B = \max\, \sigma(S)$, with $\sigma(S)$ the spectrum of S, be the optimal frame bounds of (g, a, b). Furthermore, we denote by

$$g^t = S^{-1/2} g \ , \qquad g^d = S^{-1} g \tag{3}$$

the canonically associated tight frame generating window and dual window, respectively. For further information about Gabor frames and canonical windows we refer to [1], [2], Ch. 4, [3], and [4], Chs. 5–9. Before proceeding we want to point out that [1] contains a number of rather innocent but disturbing errors; these, with their corrections, can be found at the webpage

`http://www.math.ucdavis.edu/~strohmer/papers/2000/tight.html`

The algorithm analyzed in [1], Sec. 4 for the computation of g^t is as follows. Set

$$\gamma_0 = g\, ; \quad \gamma_{k+1} = \frac{1}{2}\, \frac{\gamma_k}{\|\gamma_k\|} + \frac{1}{2}\, \frac{S_k^{-1}\gamma_k}{\|S_k^{-1}\gamma_k\|} \ , \qquad k = 0, 1, \ldots \ , \tag{4}$$

where S_k is the frame operator corresponding to (γ_k, a, b). Denoting

$$A_k = \min\, \sigma(S_k) \ , \quad B_k = \max\, \sigma(S_k) \ , \quad Q_k = \frac{A_k}{B_k} \ , \qquad k = 0, 1, \ldots \ , \tag{5}$$

it can be shown that, see [1], (4.18),

$$1 \geq Q_{k+1} \geq \frac{4Q_k}{(1+Q_k)^2} = 1 - \Big(\frac{1-Q_k}{1+Q_k}\Big)^2 \ , \qquad k = 0, 1, \ldots \ . \tag{6}$$

Hence $Q_k \to 1$ at least quadratically. A supplementary argument (also see Sec. 3) then yields

$$\Big\| \frac{\gamma_k}{\|\gamma_k\|} - (ab)^{-1/2}\, g^t \Big\| \le (1 - Q_k^{1/4}) \sqrt{\frac{2}{1 + Q_k^{1/2}}} \ , \qquad k = 0, 1, \ldots \ . \tag{7}$$

This supplementary argument involves Kantorovich's inequality, see (19) below, and the observation that for all k

$$(ab)^{-1/2}\, g^t = \frac{S_k^{-1/2}\, \gamma_k}{\|S_k^{-1/2}\, \gamma_k\|} \ . \tag{8}$$

In this paper we present some more iterative algorithms, together with their convergence analysis, for the computation of g^t and g^d. All these algorithms have as a common feature that the iterates γ_k have the form $\gamma_k = \phi_k(S)\, g$ with ϕ_k a (possibly complicated) function analytic on $\sigma(S)$. As a consequence, all frame operators S_k corresponding to (γ_k, a, b) commute with S, and (8) is valid. We do not intend to compare these algorithms with one another or with other existing algorithms to compute canonical windows due to space limitations (see [1], Subsec. 4.2 for results of this type for algorithm (4)). The computation of g^t according to (4) requires the inversion of the frame operators S_k. In many cases these inversions are non-prohibitive from the point of view of computation time. This rests upon the following considerations. Assume that (γ, a, b) is a Gabor frame with frame operator T. When γ is sufficiently smooth and rapidly decaying there holds

$$T^{-1}\gamma = \sum_{i,j} \Big(\Big(\frac{1}{ab}\, GM \Big)^{-1} \Big)_{ij;oo} \gamma_{i/b, j/a} \ , \tag{9}$$

where GM is the Gram matrix

$$((\gamma_{i'/b, j'/a}, \gamma_{i/b, j/a}))_{i,j \in \mathbb{Z}; i',j' \in \mathbb{Z}} \tag{10}$$

of the Riesz system $(\gamma_{i/b,j/a})_{i,j\in\mathbb{Z}}$. This Gram matrix is rapidly decaying in $|i - i'|$, $|j - j'|$ when γ is well-behaved, and highly structured when ab is rational. This implies that the computation of $T^{-1}\gamma$ according to (9) is feasible in many cases. As a consequence the computation of $S_k^{-1}\, \gamma_k$ as required in (4) is feasible in many cases. There does exist a formula like (9) for $\gamma^t = S^{-1/2} g$, viz.

$$\gamma^t = \sum_{i,j} \Big(\Big(\frac{1}{ab}\, GM(g) \Big)^{-1/2} \Big)_{ij;oo} g_{i/b, j/a} \ , \tag{11}$$

with $GM(g)$ the Gram matrix of the Riesz system $(g_{i/b,j/a})_{i,j\in\mathbb{Z}}$. However, the computation of the inverse square root at the right-hand side of (11) presents much more problems than the computation of the inverse at the right-hand side of (9), even when g is well-behaved and ab is rational.

Although the inversion of frame operators does not present unsurmountable problems, one may ask whether the computation of g^t can be done with algorithms of simplicity comparable with that of (4) that do not require inversion of frame operators. This question can be answered in the positive. One such algorithm is the following one. Set

$$\gamma_0 = g\,; \quad \gamma_{k+1} = \frac{3}{2}\,\frac{\gamma_k}{\|\gamma_k\|} - \frac{1}{2}\,\frac{S_k\gamma_k}{\|S_k\gamma_k\|}\;, \qquad k = 0, 1, \ldots\;, \tag{12}$$

where S_k is the frame operator corresponding to (γ_k, a, b). We shall show for this algorithm results like (6), (7), provided that $A/B > \frac{1}{2}$ (here A, B are the best frame bounds of the frame (g, a, b)). Hence the advantage of not having to invert frame operators comes at the price that convergence is not always guaranteed.

An algorithm of a somewhat similar nature for the computation of the canonical dual g^d was communicated to the author by H. Feichtinger in December 2001. Here one sets

$$\gamma_0 = g\,; \quad \gamma_{k+1} = 2\,\frac{\gamma_k}{\|\gamma_k\|} - \frac{S_k g}{\|S_k g\|}\;, \qquad k = 0, 1, \ldots\;, \tag{13}$$

where S_k is the frame operator corresponding to (γ_k, a, b). We shall show that $\gamma_k \to g^d/\|g^d\|$ at least quadratically when $A/B > \frac{1}{2}(\sqrt{5}-1) = 0.6180\ldots$. Hence, as one might have expected, an even more stringent condition on the best frame bounds A, B of (g, a, b) is required to guarantee convergence.

The conditions $A/B > \frac{1}{2}$ and $A/B > \frac{1}{2}(\sqrt{5}-1)$, guaranteeing convergence of (12) and (13), respectively, can be weakened somewhat when one is willing to consider slightly more complicated recursion steps. Accordingly, we shall consider the algorithm

$$\gamma_0 = g\,; \quad \gamma_{k+1} = \frac{15}{8}\,\frac{\gamma_k}{\|\gamma_k\|} - \frac{5}{4}\,\frac{S_k\gamma_k}{\|S_k\gamma_k\|} + \frac{3}{8}\,\frac{S_k^2\gamma_k}{\|S_k^2\gamma_k\|}\;, \qquad k = 0, 1, \ldots\;, \tag{14}$$

where S_k is the frame operator corresponding to (γ_k, a, b). We shall show that $\gamma_k \to (ab)^{-1/2}\, g^t$ at least cubically when $A/B > \frac{3}{7}$. Hence, not only the range of allowed values of A/B is enlarged but also the convergence rate gets improved by including the term involving $S_k^2\gamma_k$. Similarly, we shall consider the algorithm

$$\gamma_0 = g\,; \quad \gamma_{k+1} = 3\,\frac{\gamma_k}{\|\gamma_k\|} - 3\,\frac{S_k g}{\|S_k g\|} + \frac{S\,S_k\,\gamma_k}{\|S\,S_k\,\gamma_k\|}\;, \qquad k = 0, 1, \ldots\;, \tag{15}$$

where S and S_k are the frame operators corresponding to (g, a, b) and (γ_k, a, b), respectively. We shall show that $\gamma_k \to g^d/\|g^d\|$ at least cubically when $A/B > Q_0 := 0.5138\ldots$. Here Q_0 is a root of a particular algebraic equation. Hence also in this case the range of allowed values of A/B is enlarged while the convergence rate is improved.

In the next section we give a more detailed overview of the results presented in this paper.

2 Paper outline

In this paper we shall analyze the algorithms described by (4), (12), (13), (14) and (15) with respect to their convergence behaviour and we shall establish conditions on A/B, the ratio of the best frame bounds of (g, a, b), under which convergence is guaranteed.

Up to now no motivation has been given for why one would write down any of the algorithms with a certain expectation regarding the convergence rate. In Section 3 we shall present a rationale for proposing algorithms of the type as in Section 1 with a desired convergence rate. This rationale leads readily to the algorithms (4), (12)–(15), and it is straightforward to see how one can get algorithms with a higher expected convergence rate.

In Section 4 we give the details for the algorithm described by (12) to compute g^t; that is, we shall show that $\gamma_k \to (ab)^{-1/2}\, g^t$ at least quadratically when $A/B > \frac{1}{2}$. By considering the case $a = b = 1$, so that we can examine the algorithm in the Zak transform domain, we shall show that the lower bound $\frac{1}{2}$ on A/B is close to being lowest possible when convergence to the correct tight window for general Gabor frames is required. For Gabor frames (g, a, b) with a well-behaved window g this lower bound $\frac{1}{2}$ seems somewhat pessimistic. In Section 5 we give the details for the algorithm described by (13) to compute g^d; that is, we show that $\gamma_k \to g^d/\|g^d\|$ at least quadratically when $A/B > \frac{1}{2}(\sqrt{5} - 1)$.

The analyses in Sections 4, 5 have several common features, and one could wonder whether they can be unified. In Section 6 we show that the algorithms (12), (13) can be considered as special instances of a coupled recursion. We take $\alpha \in [0, 1]$, and we consider the recursion

$$\begin{bmatrix} \gamma_0 \\ \phi_0 \end{bmatrix} = \begin{bmatrix} g \\ g \end{bmatrix} \; ; \quad \begin{bmatrix} \gamma_{k+1} \\ \phi_{k+1} \end{bmatrix} = \begin{bmatrix} (1+\alpha)\, \dfrac{\gamma_k}{\|\gamma_k\|} - \alpha\, \dfrac{S_k \phi_k}{\|S_k \phi_k\|} \\[2ex] (2-\alpha)\, \dfrac{\phi_k}{\|\phi_k\|} - (1-\alpha)\, \dfrac{T_k \gamma_k}{\|T_k \gamma_k\|} \end{bmatrix} \tag{16}$$

where S_k and T_k are the frame operators corresponding to (γ_k, a, b) and

(ϕ_k, a, b), respectively. For $\alpha = \frac{1}{2}$ we have that either component of the vectors in (16) coincide with the algorithm (12). For $\alpha = 0$ or 1, one component of the vectors in (16) reduces to the trivial recursion $g, g, \ldots$, while the other one coincides with algorithm (13). It can be shown that (16) converges at least quadratically when

$$(A/B)^3 + (A/B)^2 - (1 + \alpha - \alpha^2)\, A/B + \alpha - \alpha^2 > 0 \ , \tag{17}$$

but it is not so obvious, except in the cases $\alpha = 0, \frac{1}{2}, 1$, what the limiting windows actually are. It will be shown by an example with $a = b = 1$ in the Zak transform domain that these limiting windows are not simply of the form $\Phi_\alpha(S)\, g$ with Φ_α a g-independent function.

In Section 7 we present a sharpening of Kantorovich's inequality as a preparation of the analyses given in Sections 8, 9 of the algorithms (14), (15). Let T be a positive definite linear operator of a Hilbert space H and let $A = \min \sigma(T) > 0$, $B = \max \sigma(T) < \infty$. Then there holds for any $f \in H$

$$A \leq \frac{\|Tf\|}{\|f\|} \leq \frac{\|T^2 f\|}{\|Tf\|} \leq B \tag{18}$$

and

$$\frac{2AB}{A^2 + B^2} \leq \frac{\|Tf\|^2}{\|f\|\, \|T^2 f\|} \leq 1 \ . \tag{19}$$

The first inequality in (19) is Kantorovich's inequality. For a worst-case analysis of the algorithms (14), (15), a more precise type of inequalities is required: what values can be taken by $w = \|Tf\|/\|f\|$ under the condition that $\|T^2 f\|/\|Tf\|$ has a particular value $u \in [A, B]$. We shall show that

$$\frac{AB}{(A^2 + B^2 - u^2)^{1/2}} \leq w \leq u \tag{20}$$

and that the lower bound in (20) is best possible. Moreover, by taking $a = b = 1$ so that Zak transform techniques apply, we shall show that for any $u, w \in [A, B]$ satisfying (20) there is a Gabor frame(γ, a, b) with frame operator S and frame bounds A, B such that $\|S\gamma\|/\|\gamma\| = w$, $\|S^2\gamma\|/\|S\gamma\| = u$.

In Sections 8, 9 we present the analyses of the algorithms (14), (15) for the computation of g^t, g^d with cubic convergence. We shall show that (14) converges at least cubically, with limiting window $(ab)^{-1/2}\, g^t$, when $A/B > 3/7$. Furthermore, we shall show that (15) converges at least cubically, with limiting window $g^d/\|g^d\|$, when $A/B > Q_0 = 0.5138\ldots$. Here $Q_0 = \dfrac{3}{2W_0^3} -$

$\left(\frac{9}{4W_0^6} - 1\right)^{1/2}$ with W_0 the solution W larger but close to 1 of the equation

$$9W^2 - 16W^3 + 9W^4 + 6W^5 + 6W^7 + 2W^9 = 24 \ . \tag{21}$$

As with all algorithms in this paper, the analyses are based on an application of the spectral mapping theorem so that spectra of consecutive (frame) operators can be related. In the present cases this leads to the study of the ratio of the extrema of the functions

$$z\left[\frac{15}{8}\,v - \frac{5}{4}\,\frac{z^2}{u} + \frac{3}{8}\left(\frac{z^2}{u}\right)^2\right] , \qquad z\left[3v - 3\,\frac{z}{u} + \left(\frac{z}{u}\right)^2\right] , \tag{22}$$

respectively, where z ranges through a particular interval $[E, F] \subset (0, \infty)$ and u, v are arbitrary numbers satisfying

$$E \leq u \leq F \ , \qquad \frac{u^{-1}\,EF}{\sqrt{E^2 + F^2 - u^2}} \leq v \leq 1 \ . \tag{23}$$

Despite the simple appearance of the functions in (22) the analysis is quite involved.

In Section 10 we collect all the examples and illustrations from the previous sections for the case that $a = b = 1$. In this case we can consider the algorithms (4), (12)–(15) in a pointwise manner in the Zak transform domain. This is so since in the Zak transform domain frame operators with $a = b = 1$ are just multiplication operators. As a result the convergence behaviour of the algorithms is readily understood. Accordingly, it can be seen that the lower bound $\frac{1}{2}$ for A/B in algorithm (12) is close to being lowest possible, the pitfall of the coupled recursion (16) with $\alpha \neq 0, \frac{1}{2}, 1$ can be established explicitly, sharpness of the augmented Kantorovich inequality (20) in the context of Gabor frame operators can be shown, and some heuristics as to why the convergence guaranteeing lower bounds on A/B seems too pessimistic for well-behaved windows can be developed.

3 Rationale for proposing recursions

In this section we explain the mechanism that leads us to write down the recursions (4), (12)–(15) for the computation of g^t, with or without inversions, and of g^d. For this, and for all other developments in this paper, Lemma 2 in [1] is so basic that we repeat it here in extenso (somewhat modified).

Proposition.

Assume that (g, a, b) is a Gabor frame with frame operator S and best frame bounds $A = \min \sigma(S) > 0$, $B = \max \sigma(S) < \infty$. Let ϕ be a function analytic in an open neighourhood of $\sigma(S)$, and assume that $\phi(s) > 0$, $s \in \sigma(S)$. Then $(\phi(S)g, a, b)$ is a Gabor frame with frame operator $S\,\phi^2(S)$, and the best frame bounds are given by

$$\min_{s\in\sigma(S)}\ s\,\phi^2(s)\ , \qquad \max_{s\in\sigma(S)}\ s\,\phi^2(s)\ . \tag{24}$$

Furthermore, (g, a, b) and $(\phi(S)g, a, b)$ have the same canonically associated tight frame window, i.e.

$$g^t = S^{-1/2}g = T^{-1/2}h = h^t\ , \tag{25}$$

where $T = S\,\phi^2(S)$ is the frame operator corresponding to $(h = \phi(S)g, a, b)$.

By a repeated application of this Proposition it is easily seen that in all algorithms all γ_k have the form $\phi_k(S)g$ with ϕ_k a (possibly quite complicated) analytic function. As a consequence the corresponding frame operators $S\,\phi_k^2(S)$ commute with S and there holds

$$S_k^{-1/2}\gamma_k = S^{-1/2}g\ . \tag{26}$$

Of course, for the last thing to hold, it should be verified that ϕ_k is positive on $\sigma(S)$.

We start by developing a mechanism for proposing recursions for the computation of g^t without operator inversions. We note first that convergence of the sequence $\gamma_k/\|\gamma_k\|$ to $g^t/\|g^t\|$ is a consequence of convergence of the scaled frame operators $S_k/\|S_k\|$ to the identity I. Indeed, on account of (26) and the Kantorovich inequality (19) (applied to $T = S_k^{-1/4}$, $f = \gamma_k$ and $A = B_k^{-1/4}$, $B = A_k^{-1/4}$ where $A_k = \min \sigma(S_k) > 0$ and $B_k = \max \sigma(S_k) < \infty$), we have

$$\begin{aligned}
0 \ &\le\ \Big\|\frac{\gamma_k}{\|\gamma_k\|} - \frac{g^t}{\|g^t\|}\Big\|^2 = \Big\|\frac{\gamma_k}{\|\gamma_k\|} - \frac{S_k^{-1/2}\gamma_k}{\|S_k^{-1/2}\gamma_k\|}\Big\|^2 \ = \\
&=\ 2 - 2\,\frac{(S_k^{-1/2}\gamma_k, \gamma_k)}{\|\gamma_k\|\,\|S_k^{-1/2}\gamma_k\|} = 2\Big(1 - \frac{\|S_k^{-1/4}\gamma_k\|^2}{\|\gamma_k\|\,\|S_k^{-1/2}\gamma_k\|}\Big)\ \le \\
&\le\ 2\Big(1 - \frac{2B_k^{-1/4}A_k^{-1/4}}{B_k^{-1/2} + A_k^{-1/2}}\Big) = 2\,\frac{(B_k^{1/4} - A_k^{1/4})^2}{B_k^{1/2} + A_k^{1/2}}\ = \\
&=\ \frac{2(1 - Q_k^{1/4})^2}{1 + Q_k^{1/2}}\ ; \qquad Q_k := A_k/B_k\ .
\end{aligned} \tag{27}$$

Now assume we start with a Gabor frame (g, a, b) and that we construct γ_k, $k = 0, 1, \ldots$, according to a recursion

$$\gamma_0 = g\,; \quad \gamma_{k+1} = \phi(S_k)\gamma_k \ , \qquad k = 0, 1, \ldots \ , \tag{28}$$

where S_k is the frame operator corresponding to (γ_k, a, b) and ϕ is a smooth function. Suppose that for a certain $k = 0, 1, \ldots$ and some $c > 0$ we have that $S_k \approx cI$, and that we want

$$\|S_{k+1} - cI\| = O(\|S_k - cI\|^m) \tag{29}$$

for some $m = 2, 3, \ldots$. By the Proposition, applied to the Gabor frame (γ_k, a, b) and ϕ, we have that

$$S_{k+1} = S_k\, \phi^2(S_k) \ . \tag{30}$$

Hence we must choose ϕ such that

$$S_k^{1/2}\, \phi(S_k) = c^{1/2} I + O(\|S_k - cI\|^m) \ . \tag{31}$$

Now by Taylor expansion

$$\begin{aligned} S_k^{-1/2} &= c^{-1/2}(I - (I - c^{-1}S_k))^{-1/2} = \\ &= c^{-1/2} \sum_{j=0}^{m-1} (-1)^j \binom{-1/2}{j} (I - c^{-1}S_k)^{m-1} + O(\|S_k - cI\|^m) = \\ &= c^{-1/2} \sum_{j=0}^{m-1} a_{mj} (c^{-1}S_k)^j + O(\|S_k - cI\|^m) \end{aligned} \tag{32}$$

where the a_{mj} can be written down explicitly. It thus follows that

$$S_k^{1/2} \sum_{j=0}^{m-1} a_{mj} (c^{-1}S_k)^j = c^{1/2} I + O(\|S_k - cI\|^m) \ . \tag{33}$$

This suggests to take

$$\phi(s) = \sum_{j=0}^{m-1} a_{mj}\, c^{-j}\, s^j \ . \tag{34}$$

However, the number c and thus the numbers c^{-j} are not readily available. Fortunately, since $S_k \approx cI$ we can estimate the c^{-j} according to

$$c^{-j} \approx \frac{\|\gamma_k\|}{\|S_k^j \gamma_k\|} \ , \qquad j = 0, 1, \ldots, m-1 \ . \tag{35}$$

A division by $\|\gamma_k\|$ then finally leads to the proposal

$$\gamma_{k+1} = \sum_{j=0}^{m-1} a_{mj} \frac{S_k^j \gamma_k}{\|S_k^j \gamma_k\|} , \tag{36}$$

where we note that the $S_k^j \gamma_k / \|S_k^j \gamma_k\|$ all have unit norm while $\sum_{j=0}^{m-1} a_{mj} = 1$. The cases $m = 2, 3$ lead directly to the algorithms (12), (14).

We next develop a mechanism for generating recursions for the computation of the canonical dual g^d for a Gabor frame (g, a, b). Assume that we construct γ_k, $k = 0, 1, \ldots$, according to a recursion

$$\gamma_0 = g \, ; \quad \gamma_{k+1} = \psi(S, S_k)\gamma_k , \qquad k = 0, 1, \ldots , \tag{37}$$

where S and S_k are the frame operators corresponding to (g, a, b) and (γ_k, a, b), respectively, and ψ is an analytic function of two variables. There are smooth functions f_k, g_k such that

$$\gamma_k = f_k(S) g , \qquad S_k = g_k(S) , \tag{38}$$

and where $g_k(s) = s \, f_k^2(s)$. As a consequence we have that

$$\gamma_k = S^{-1/2} \, S_k^{1/2} \, g , \qquad g^d = S^{-1} g = (S S_k)^{-1/2} \gamma_k . \tag{39}$$

It thus follows as in (27) from

$$\Big\| \frac{\gamma_k}{\|\gamma_k\|} - \frac{g^d}{\|g^d\|} \Big\| = \Big\| \frac{\gamma_k}{\|\gamma_k\|} - \frac{(S S_k)^{-1/2} \gamma_k}{\|(S S_k)^{-1/2} \gamma_k\|} \Big\| \tag{40}$$

that is enough to show that $S S_k / \|S S_k\|$ converges to the identity operator I. Suppose that for a certain $k = 0, 1, \ldots$ and some $c > 0$ we have that $S S_k \approx cI$ and that we want

$$\|S S_{k+1} - cI\| = O(\|S S_k - cI\|^m) \tag{41}$$

for some $m = 2, 3, \ldots$. We have from (37), (38) that

$$\gamma_{k+1} = \psi(S, g_k(S)) \, f_k(S) \, g , \tag{42}$$

whence by the Proposition

$$S_{k+1} = S[\psi(S, g_k(S)) \, f_k(S)]^2 = S \, f_k^2(S) \, \psi^2(S, g_k(S)) = S_k \, \psi^2(S, S_k) . \tag{43}$$

Hence we must choose ψ such that

$$(S S_{k+1})^{1/2} = (S S_k)^{1/2} \, \psi(S, S_k) = c^{1/2} I + O(\|S S_k - cI\|^m) . \tag{44}$$

Now as in (32) we have

$$(SS_k)^{-1/2} = c^{-1/2}(I - (I - c^{-1/2}(SS_k)^{1/2})^{-1} =$$

$$= c^{-1/2} \sum_{j=0}^{m-1} b_{mj}\, c^{-\frac{1}{2}j} (SS_k)^{\frac{1}{2}j} + O(\|SS_k - cI\|^m) \;, \tag{45}$$

where the b_{mj} can be written down explicitly. Thus

$$(SS_k)^{1/2} \sum_{j=0}^{m-1} b_{mj}\, c^{-\frac{1}{2}j} (SS_k)^{\frac{1}{2}j} = c^{1/2} I + O(\|SS_k - cI\|^m) \;. \tag{46}$$

This suggests to take

$$\psi(S, S_k) = \sum_{j=0}^{m-1} b_{mj}\, c^{-\frac{1}{2}j} (SS_k)^{\frac{1}{2}j} \;. \tag{47}$$

We replace the numbers $c^{-\frac{1}{2}j}$ by their approximations

$$c^{-\frac{1}{2}j} \approx \Big(\frac{\|(SS_k)^{\frac{1}{2}j}\, \gamma_k\|}{\|\gamma_k\|} \Big)^{-1} \tag{48}$$

and dividing through by $\|\gamma_k\|$ we obtain the proposal

$$\gamma_{k+1} = \sum_{j=0}^{m-1} b_{mj}\, \frac{(SS_k)^{\frac{1}{2}j}\, \gamma_k}{\|(SS_k)^{\frac{1}{2}j}\, \gamma_k\|} \;. \tag{49}$$

Note that the $(SS_k)^{\frac{1}{2}j}\, \gamma_k / \|(SS_k)^{\frac{1}{2}j}\, \gamma_k\|$ all have unit norm and that $\sum_{j=0}^{m-1} b_{mj} = 1$. The terms at the right-hand side of (49) for even j are in a convenient form. As to the terms at the right-hand side of (49) with odd j we note that by (39) we have

$$(SS_k)^{1/2}\, \gamma_k = S_k g \;, \qquad (SS_k)^{3/2}\, \gamma_k = SS_k^2\, g, \ldots \;, \tag{50}$$

whence these terms are in a convenient form as well. The cases $m = 2, 3$ in (49) then directly lead to the recursions (13), (15).

In a similar fashion we can develop a mechanism for proposing recursions for the computation of g^t using inversions or mixed recursions (involving positive and negative powers of frame operators). This results, for instance, for $m = 2$ into the recursion (4) and for $m = 3$ to the recursions

$$\gamma_0 = g\,; \quad \gamma_{k+1} = \frac{3}{8}\, \frac{\gamma_k}{\|\gamma_k\|} + \frac{3}{4}\, \frac{S_k^{-1}\gamma_k}{\|S_k^{-1}\gamma_k\|} - \frac{1}{8}\, \frac{S_k^{-2}\gamma_k}{\|S_k^{-2}\gamma_k\|} \;, \qquad k = 0, 1, \ldots \;, \tag{51}$$

$$\gamma_0 = g\,; \quad \gamma_{k+1} = \frac{3}{8}\, \frac{S_k^{-1}\gamma_k}{\|S_k^{-1}\gamma_k\|} + \frac{3}{4}\, \frac{\gamma_k}{\|\gamma_k\|} - \frac{1}{8}\, \frac{S_k\gamma_k}{\|S_k\gamma_k\|} \;, \qquad k = 0, 1, \ldots \;, \tag{52}$$

etc., etc.

4 Algorithm for g^t with quadratic convergence and no inversions

In this section we analyze in detail the algorithm

$$\gamma_0 = g\,; \quad \gamma_{k+1} = \frac{3}{2}\,\frac{\gamma_k}{\|\gamma_k\|} - \frac{1}{2}\,\frac{S_k\gamma_k}{\|S_k\gamma_k\|}\,, \qquad k = 0, 1, \ldots\,, \tag{53}$$

where (g, a, b) is a Gabor frame with frame operator S and best frame bounds $A > 0$, $B < \infty$, and S_k is the frame operator corresponding to (γ_k, a, b) with best frame bounds A_k, B_k. In particular we have $A_0 = A$, $B_0 = B$. We shall show that $A/B > \frac{1}{2}$ is a sufficient condition for all (γ_k, a, b) to be a frame indeed and for at least quadratic convergence of $\gamma_k/\|\gamma_k\|$ to $(ab)^{-1/2}\, g^t = S^{-1/2}g/\|S^{-1/2}g\|$.

Let $k = 0, 1, \ldots$ and assume that (γ_k, a, b) is a frame. We let

$$\varepsilon_k := \|\gamma_k\|^{-1}\,, \quad \delta_k := \|S_k\gamma_k\|^{-1}\,; \qquad \phi(s) = \tfrac{3}{2}\,\varepsilon_k - \tfrac{1}{2}\,\delta_k s\,. \tag{54}$$

Then $\gamma_{k+1} = \phi(S)\gamma_k$. We want to apply the Proposition in Section 3, and to that end we must have that $\phi(s) > 0$, $s \in \sigma(S_k)$. Since $B_k \in \sigma(S_k) \subset [0, B_k]$ we thus require that

$$q_k := \frac{\varepsilon_k}{\delta_k} = \frac{\|S_k\gamma_k\|}{\|\gamma_k\|} > \tfrac{1}{3}\,B_k\,. \tag{55}$$

Since

$$\frac{\|S_k\gamma_k\|}{\|\gamma_k\|} \geq A_k\,, \tag{56}$$

the condition in (55) is certainly satisfied when $A_k/B_k > 1/3$. We shall thus assume that $A_k/B_k > 1/3$. By the Proposition in Section 3 we have that (γ_{k+1}, a, b) is a frame with frame operator

$$S_{k+1} = S_k\,\phi^2(S_k) = S_k(\tfrac{3}{2}\,\varepsilon_k I - \tfrac{1}{2}\,\delta_k S_k)^2\,, \tag{57}$$

and

$$g^t = S^{-1/2}g = S_k^{-1/2}g_k = S_{k+1}^{-1/2}g_{k+1}\,. \tag{58}$$

We now define

$$Z_k := S_k^{1/2}\,, \quad E_k := A_k^{1/2} = \min\sigma(Z_k)\,, \quad F_k := B_k^{1/2} = \max\sigma(Z_k)\,. \tag{59}$$

By (53) and (57) the Z_k satisfy the following recursion:

$$Z_0 = S^{1/2}\,; \quad Z_{k+1} = Z_k(\tfrac{3}{2}\,\varepsilon_k I - \tfrac{1}{2}\,\delta_k Z_k^2)\,, \qquad k = 0, 1, \ldots\,. \tag{60}$$

Since $\sigma(Z_k) \subset [E_k, F_k]$ there holds by the spectral mapping theorem that

$$E_{k+1} \geq \min_{z\in[E_k,F_k]} z(\tfrac{3}{2}\,\varepsilon_k - \tfrac{1}{2}\,\delta_k z^2) \ , \tag{61}$$

$$F_{k+1} \leq \max_{z\in[E_k,F_k]} z(\tfrac{3}{2}\,\varepsilon_k - \tfrac{1}{2}\,\delta_k z^2) \ . \tag{62}$$

We write

$$z(\tfrac{3}{2}\,\varepsilon_k - \tfrac{1}{2}\,\delta_k z^2) = \varepsilon_k^{3/2}\,\delta_k^{-1/2}\,P(z/r_k) \tag{63}$$

where

$$P(w) = w(\tfrac{3}{2} - \tfrac{1}{2}\,w^2) \ ; \qquad r_k = q_k^{1/2} = \Big(\frac{\varepsilon_k}{\delta_k}\Big)^{1/2} \ . \tag{64}$$

In Fig. 1 we have plotted $P(w)$ with particular attention for extreme values on the interval $[E/r, F/r]$ where $E = \frac{2}{3}$, $F = 1$ and $r \in [E, F]$. We note that P is positive on $(0, \sqrt{3})$, negative on $(\sqrt{3}, \infty)$ and that P has a unique maximum $P(1) = 1$ at $w = 1$. Hence, since $r_k \in [E_k, F_k]$ we have that the right-hand side of (62) equals 1. Therefore

$$\frac{E_{k+1}}{F_{k+1}} \geq \min\{P(E_k/r_k), P(F_k/r_k)\} =: V_k \ . \tag{65}$$

We shall determine the minimum value of E_k/F_k such that $V_k \geq E_k/F_k$ for any $r_k \in [E_k, F_k]$. It thus turns out that

$$\frac{E_k}{F_k} > (\tfrac{1}{2})^{1/2} \Rightarrow \frac{E_{k+1}}{F_{k+1}} > \frac{E_k}{F_k} \ , \tag{66}$$

and that therefore $(E_k/F_k)_{k=0,1,\dots}$ is an increasing sequence when $A/B = E_0^2/F_0^2 > \frac{1}{2}$. We note that the functions

$$r_k \to P(E_k/r_k), P(F_k/r_k) \tag{67}$$

are strictly decreasing, increasing in $r_k \in [E_k, F_k]$. Hence the minimum value of V_k under the constraint that $r_k \in [E_k, F_k]$ equals

$$\min\{P(E_k/F_k), P(F_k/E_k)\} \ . \tag{68}$$

The latter minimum equals $P(F_k/E_k)$ which easily follows from the explicit form of P in (64) and the inequality

$$\frac{1}{R}\Big(\frac{3}{2} - \frac{1}{2}\,\frac{1}{R^2}\Big) \leq R(\tfrac{3}{2} - \tfrac{1}{2}\,R^2) \ , \qquad 0 < R \leq 1 \ . \tag{69}$$

Hence

$$\frac{E_{k+1}}{F_{k+1}} \geq \frac{F_k}{E_k}\Big(\frac{3}{2} - \frac{1}{2}\Big(\frac{F_k}{E_k}\Big)^2\Big) \ , \tag{70}$$

and the right-hand side of (70) exceeds E_k/F_k when $E_k/F_k > (\frac{1}{2})^{1/2}$.

In terms of the frame bounds $A_k = E_k^2$, $B_k = F_k^2$ of S_k, we can write (70) as

$$Q_{k+1} \geq Q_k^{-1}(\tfrac{3}{2} - -\tfrac{1}{2}\,Q_k^{-1})^2 \ ; \qquad Q_k = \frac{A_k}{B_k} \ . \tag{71}$$

In Fig. 2 we have plotted the graph of

$$f_1(Q) = Q^{-1}(\tfrac{3}{2} - \tfrac{1}{2}\,Q^{-1})^2 \ , \qquad \tfrac{1}{3} \leq Q \leq 1 \ . \tag{72}$$

There holds

$$f_1(Q) = 1 - \frac{(1-Q)^2(Q-\frac{1}{4})}{Q^3} \ ; \qquad f_1(1) = 1 \ , \ \ f_1'(1) = 0 \ , \ \ f_1''(1) = -\tfrac{3}{2} \ , \tag{73}$$

and

$$f_1(\tfrac{1}{3}) = 0 \ ; \qquad f_1(Q) = Q \Leftrightarrow Q = \tfrac{1}{2}, 1 \ . \tag{74}$$

Hence we see that $Q_k \to 1$ at least quadratically when $Q_0 = A/B > \frac{1}{2}$. And since, see (27),

$$\Big\| \frac{\gamma_k}{\|\gamma_k\|} - \frac{g^t}{\|g^t\|} \Big\| \leq (1 - Q_k^{1/4}) \sqrt{\frac{2}{1 + Q_k^{1/2}}} \ , \tag{75}$$

we see that $\gamma_k/\|\gamma_k\| \to g^t/\|g^t\|$ at least quadratically.

We conclude this section with some notes. In (65) there is equality when $\sigma(Z_k) = [E_k, F_k]$. We have that $\sigma(Z_k) = [E_k, F_k]$ if and only if $\sigma(S) = [A, B]$. When $\sigma(S)$ is a proper subset of $[A, B]$, the inequality in (65) can be strict. Such an example, for the algorithm in (4) and with $\sigma(S) = \{A, B\}$, was given in [1], end of Subsec. 4.1.

The maximum value that V_k in (65) can take under the constraint that $r_k \in [E_k, F_k]$ occurs when $r_k \in [E_k, F_k]$ is such that

$$P(E_k/r_k) = P(F_k/r_k) \ . \tag{76}$$

This special r_k is given by (also see Fig. 1 (c))

$$\hat{r}_k = (\tfrac{1}{3}\,(E_k^2 + E_kF_k + F_k^2))^{1/2} \ , \tag{77}$$

and the corresponding value of V_k is given by

$$\hat{V}_k = \frac{3\sqrt{3}}{2}\ \frac{Q_k^{1/2} + Q_k}{(1 + Q_k^{1/2} + Q_k)^{3/2}} \ ; \qquad Q_k = \frac{A_k}{B_k} = \Big(\frac{E_k}{F_k}\Big)^2 \ . \tag{78}$$

Thus when $\sigma(S) = [A, B]$, we have that

$$Q_{k+1} \leq \frac{27}{4} \, \frac{(Q_k^{1/2} + Q_k)^2}{(1 + Q_k^{1/2} + Q_k)^3} =: f_2(Q_k) \ . \tag{79}$$

We have plotted this $f_2(Q)$, $0 \leq Q \leq 1$ also in Fig. 2.

We finally note that

$$\Big(\frac{r_k}{F_k}\Big)^2 = \frac{q_k}{B_k} > \frac{1}{2} \Rightarrow \frac{E_{k+1}}{F_{k+1}} > \frac{E_k}{F_k} \, , \, \frac{A_{k+1}}{B_{k+1}} > \frac{A_k}{B_k} \ , \tag{80}$$

no matter how small $E_k = A_k^{1/2}$ is. The condition

$$q_k = \frac{\|S_k \gamma_k\|}{\|\gamma_k\|} > \tfrac{1}{2} \max \sigma(S_k) = \tfrac{1}{2} \, B_k \tag{81}$$

is clearly weaker than the condition $A_k/B_k > \frac{1}{2}$, but we have not found a simple criterion in terms of $S_0 = S$, $A_0 = A$, $B_0 = B$ that guarantees (81) to hold for all k. In the cases of smooth, rapidly decaying windows γ_k, it is quite likely that (81) holds. Hence the algorithm (53) probably converges in many more cases than those restricted by $A/B > \frac{1}{2}$.

As an example we consider the standard Gaussian $g(t) = 2^{1/4} \exp(-\pi t^2)$ and $a = b = 1$. One can compute

$$A = 0 \ , \qquad B = \theta_3^2(0\,;\, e^{-\pi}) = 1.669253683 \ , \qquad \|g\| = 1 \ , \tag{82}$$

$$\begin{aligned} \|Sg\|^2 \ &= \ \Big\| \sum_{nm} (g, g_{nm}) \, g_{nm} \Big\|^2 \ = \\ &= \ \theta_3^2(0\,;\, e^{-\pi}) \, [\theta_3^2(0\,;\, e^{-3\pi}) + \sqrt{2} \, \theta_4(0\,;\, e^{-3\pi}) \, \theta_2(0\,;\, e^{-3\pi})] \ = \\ &= \ 1.5005272 \ , \end{aligned} \tag{83}$$

where

$$\theta_3(0\,;\, q) = \sum_n q^{n^2} \ , \qquad \theta_4(0\,;\, q) = \sum_n (-1)^n \, q^{n^2} \ , \qquad \theta_2(0\,;\, q) = \sum_n q^{(n+1/2)^2} \ . \tag{84}$$

It thus follows that

$$\frac{q_0}{B_0} = \frac{\|S_g\|}{\|g\|} / B = 0.73383698 > \tfrac{1}{2} \ . \tag{85}$$

When we apply algorithm (53) to this (g, a, b), we see rapid convergence of γ_k to the window h of [5], (7) with $g = g_{1,1}$ in (9) and Fig. 7 in [5], case $\gamma = 1$.

5 Algorithm for g^d with quadratic convergence

In this section we analyze in detail the algorithm

$$\gamma_0 = g\,; \quad \gamma_{k+1} = 2\,\frac{\gamma_k}{\|\gamma_k\|} - \frac{S_k g}{\|S_k g\|}\;, \qquad k = 0, 1, \ldots\;, \tag{86}$$

where (g, a, b) is a Gabor frame with frame operators S and best frame bounds $A > 0$, $B < \infty$, and S_k is the frame operator corresponding to (γ_k, a, b) with best frame bounds A_k, B_k. We shall show that $A/B > \frac{1}{2}(\sqrt{5} - 1)$ is a sufficient condition for all (γ_k, a, b) to be a frame indeed and for at least quadratic convergence of $\gamma_k/\|\gamma_k\|$ to $g^d/\|g^d\| = S^{-1}g/\|S^{-1}g\|$. Since the developments are reminiscent of those in Section 4, we shall be brief at some points.

Let $k = 0, 1, \ldots$ and assume that (γ_k, a, b) is a frame. We let

$$\varepsilon_k := \|\gamma_k\|^{-1}\;, \qquad \delta_k := \|S_k g\|^{-1}\;. \tag{87}$$

There is a smooth function f_k such that (see Section 3)

$$\gamma_k = f_k(S)\, g\;, \qquad S_k = S\, f_k^2(S)\;, \tag{88}$$

whence $f_k(S) = S^{-1/2} S_k^{1/2}$ and

$$\gamma_{k+1} = (2\varepsilon_k\, f_k(S) - \delta_k\, S\, f_k^2(S))\, g\;. \tag{89}$$

It follows that

$$\begin{aligned} S_{k+1} &= S(2\varepsilon_k\, f_k(S) - \delta_k\, S\, f_k^2(S))^2 = \\ &= S_k(2\varepsilon_k I - \delta_k (S S_k)^{1/2})^2\;. \end{aligned} \tag{90}$$

We consider

$$Z_k = (S S_k)^{1/2}\;; \qquad E_k = \min \sigma(Z_k)\;, \quad F_k = \max \sigma(Z_k)\;. \tag{91}$$

Under the condition that

$$q_k := \frac{\varepsilon_k}{\delta_k} > \tfrac{1}{2}\, F_k\;, \tag{92}$$

we can take a positive square root at the right-hand side of (90), and there results

$$Z_0 = S\;; \qquad Z_{k+1} = Z_k(2\varepsilon_k I - \delta_k Z_k)\;. \tag{93}$$

Since by the above

$$S_k g = S_k\,[S^{1/2}\,S_k^{-1/2}\,\gamma_k] = Z_k\gamma_k \tag{94}$$

we have that

$$\frac{\varepsilon_k}{\delta_k} = \frac{\|S_k g\|}{\|\gamma_k\|} = \frac{\|Z_k\gamma_k\|}{\|\gamma_k\|} \in [E_k, F_k]\ , \tag{95}$$

whence (92) is certainly satisfied when $E_k/F_k > \frac{1}{2}$.

As in Section 4 we get that

$$E_{k+1} \geq \min_{z\in[E_k,F_k]} z(2\varepsilon_k - \delta_k z)\ , \tag{96}$$

$$F_{k+1} \leq \max_{z\in[E_k,F_k]} z(2\varepsilon_k - \delta_k z)\ . \tag{97}$$

We write

$$z(2\varepsilon_k - \delta_k z) = \varepsilon_k^2\,\delta_k^{-1}\,P(z/q_k) \tag{98}$$

with q_k as in (92) and

$$P(w) = w(2-w)\ . \tag{99}$$

In Fig. 3 we have plotted $P(w)$ with particular attention for extreme values on the interval $[E/q, F/q]$ where $E = \frac{2}{3}$, $F = 1$ and $q \in [E, F]$. We note that P is positive on $(0,2)$, negative on $(2,\infty)$, and that P has a unique maximum $P(1) = 1$ at $w = 1$. Thus

$$\frac{E_{k+1}}{F_{k+1}} \geq \min\{P(E_k/q_k), P(F_k/q_k)\} =: W_k\ . \tag{100}$$

The lowest value W_k in (100) can assume under the restriction that $q_k \in [E_k, F_k]$ equals

$$\min\{P(E_k/F_k), P(F_k/E_k)\} = P(F_k/E_k)\ , \tag{101}$$

where the latter equality follows from the inequality $Q^{-1}(2-Q^{-1}) \leq Q(2-Q)$ which is valid for $0 < Q \leq 1$. It is concluded that

$$Q_{k+1} \geq Q_k^{-1}(2 - Q_k^{-1})\ ; \qquad Q_k := E_k/F_k\ . \tag{102}$$

In Fig. 4 we have plotted the graph of

$$f_1(Q) = Q^{-1}(2 - Q^{-1})\ , \qquad \tfrac{1}{2} \leq Q \leq 1\ . \tag{103}$$

There holds

$$f_1(Q) = 1 - \frac{(1-Q)^2}{Q^2}\ ; \qquad f_1(1) = 1\,,\quad f_1'(1) = 0\,,\quad f_1''(1) = -2\ , \tag{104}$$

and

$$f_1(\tfrac{1}{2}) = 0 \; ; \qquad f_1(Q) = Q \Leftrightarrow Q = \tfrac{1}{2}\,(\sqrt{5}-1), 1 \; . \tag{105}$$

Hence we see that $Q_k \to 1$ at least quadratically when $Q_0 = E_0/F_0 = A/B > \frac{1}{2}\,(\sqrt{5}-1)$.

We conclude now from $g^d = Z_k^{-1}\gamma_k$, see (39)–(40), and the Kantorovich inequality (19) (compare (27)) that

$$\Big\| \frac{\gamma_k}{\|\gamma_k\|} - \frac{g^d}{\|g^d\|} \Big\| = \Big\| \frac{\gamma_k}{\|\gamma_k\|} - \frac{Z_k^{-1}\gamma_k}{\|Z_k^{-1}\gamma_k\|} \Big\| \le (1 - Q_k^{1/2}) \sqrt{\frac{2}{1+Q_k}} \; . \tag{106}$$

Hence $\gamma_k/\|\gamma_k\| \to g^d/\|g^d\|$ at least quadratically.

We end this section by a best-case analysis. To this end we assume that $\sigma(S) = [A, B]$ so that $\sigma(Z_k) = [E_k, F_k]$ for all k. Then there is equality in (100). The maximum value of W_k in (100) under the restriction that $q_k \in [E_k, F_k]$ occurs for $\hat{q}_k = \frac{1}{2}\,(E_k + F_k)$ and equals

$$\hat{W}_k = \frac{4Q_k}{(1+Q_k)^2} =: f_2(Q_k) \; , \qquad Q_k = E_k/F_k \; . \tag{107}$$

Hence we have $Q_{k+1} \le f_2(Q_k)$ when $\sigma(S) = [A, B]$. We have plotted this $f_2(Q)$, $0 \le Q \le 1$ also in Fig. 4.

As a final comment we note that

$$\frac{q_k}{F_k} > \tfrac{1}{2}\,(\sqrt{5}-1) \Rightarrow \frac{E_{k+1}}{F_{k+1}} > \frac{E_k}{F_k} \tag{108}$$

so that (as in Section 4) the algorithm (86) probably converges in many more cases than those restricted by $A/B > \frac{1}{2}\,(\sqrt{5}-1)$. Note that for the example at the end of Section 4 we have

$$\frac{q_0}{F_0} = \frac{\|Zg\|}{\|g\|}/B = \frac{\|Sg\|}{\|g\|}/B = 0.73383698 > \tfrac{1}{2}\,(\sqrt{5}-1) \; . \tag{109}$$

Using algorithm (86) with this (g, a, b) we see rapid pointwise convergence of the γ_k outside the set of half-integers to the well-known singular function of Bastiaans.

6 Coupled recursions

We shall now indicate how the algorithms (53) and (86) can be analyzed in a unified manner. Since

$$g^t = S^{-1/2}g \; , \qquad g^d = S^{-1}g \tag{110}$$

it is natural to ask, more generally, for a recursive algorithm for the computation of $S^{-\alpha}g$ with $\alpha \in [0,1]$. The algorithms (53) and (86) suggest that one should consider a recursion

$$\gamma_0 = g\,; \quad \gamma_{k+1} = (1+\alpha)\,\frac{\gamma_k}{\|\gamma_k\|} - \alpha\,\frac{S_k\phi_k}{\|S_k\phi_k\|}\,, \qquad k = 0,1,\ldots\,, \tag{111}$$

with S_k the frame operator corresponding to (γ_k, a, b) and ϕ_k such that

$$\frac{\gamma_\infty}{\|\gamma_\infty\|} = \frac{S_\infty\phi_\infty}{\|S_\infty\phi_\infty\|}\,, \tag{112}$$

where γ_∞, S_∞, ϕ_∞ denotes the limit of γ_k, S_k, ϕ_k as $k \to \infty$. Since $\gamma_k/\|\gamma_k\|$ is supposed to converge to $S^{-\alpha}g/\|S^{-\alpha}g\|$, we should have that

$$S_\infty = \frac{\|\gamma_\infty\|^2}{\|S^{-\alpha}g\|^2}\,S^{1-2\alpha}\,. \tag{113}$$

This yields

$$\frac{\phi_\infty}{\|\phi_\infty\|} = \frac{S^{-(1-\alpha)}g}{\|S^{-(1-\alpha)}g\|}\,. \tag{114}$$

It thus seems quite natural to couple the recursion (111) with the corresponding recursion for the computation of $S^{-(1-\alpha)}g$. Now since $1-(1-\alpha) = \alpha$, it turns out that γ_k can be used in the latter recursion in a similar way as ϕ_k is used in recursion (111). Hence we couple (111) with

$$\phi_0 = g\,; \quad \phi_{k+1} = (2-\alpha)\,\frac{\phi_k}{\|\phi_k\|} - (1-\alpha)\,\frac{T_k\gamma_k}{\|T_k\gamma_k\|}\,, \qquad k = 0,1,\ldots\,, \tag{115}$$

where T_k is the frame operator corresponding to (ϕ_k, a, b).

Note that in case $\alpha = \frac{1}{2}$ the two recursions in (111) and (115) are identical and coincide with the recursion (53). Also, in the case that $\alpha = 0$ or 1, one of the two recursions in (111) and (115) is trivial (yielding $g, g, g, \ldots$) while the other one coincides with the recursion (86). We can show that the coupled recursions (111), (115) are at least quadratically convergent when the largest root $Q(\alpha)$ of

$$Q^3 + Q^2 - (1+\alpha-\alpha^2)\,Q + \alpha - \alpha^2 = 0 \tag{116}$$

in $(0,1)$ is less than A/B. We have $Q(\frac{1}{2}) = \frac{1}{2}$ and $Q(0) = Q(1) = \frac{1}{2}(\sqrt{5} - 1)$, in accordance with the results found in Secs. 4, 5. Now the surprising fact is that the limiting windows γ_∞, ϕ_∞ are in general NOT a multiple of $S^{-\alpha}g$, $S^{-(1-\alpha)}g$. We shall produce an example in Section 10 for the case that $a = b = 1$ so that we can consider the recursions in the Zak transform domain. Because of this pitfall, we shall not give an extensive analysis of the convergence behaviour of the coupled recursions (111), (115) but just briefly indicate the crucial steps. We let

$$Z_k = (S_k T_k)^{1/2} \ ; \qquad E_k = \min \sigma(Z_k) \, , \quad F_k = \max \sigma(Z_k) \ . \tag{117}$$

Noting that

$$\phi_k = S^{-1/2}\, T_k^{1/2}\, g \ , \qquad \gamma_k = S^{-1/2}\, S_k\, g \ , \tag{118}$$

we have

$$S_k \phi_k = Z_k \gamma_k \ , \qquad T_k \gamma_k = Z_k \phi_k \ . \tag{119}$$

Hence the recursions (111), (115) can be written as

$$\gamma_0 = g \ ; \qquad \gamma_{k+1} = (1+\alpha)\, \frac{\gamma_k}{\|\gamma_k\|} - \alpha\, \frac{Z_k \gamma_k}{\|Z_k \gamma_k\|} \ , \qquad k = 0, 1, \ldots \ , \tag{120}$$

$$\phi_0 = g \ ; \qquad \phi_{k+1} = (2-\alpha)\, \frac{\phi_k}{\|\phi_k\|} - (1-\alpha)\, \frac{Z_k \phi_k}{\|Z_k \phi_k\|} \ , \qquad k = 0, 1, \ldots \ . \tag{121}$$

By an appropriate use of the Proposition in Section 3 it can be shown that the Z_k satisfy the recursion

$$Z_0 = S \, ; \quad Z_{k+1} = Z_k((1+\alpha)\, \varepsilon_k I - \alpha\, \delta_k\, Z_k)((2-\alpha)\, \xi_k I - (1-\alpha)\, \eta_k\, Z_k) \ , \qquad k = 0, 1, \ldots \ , \tag{122}$$

where we have set

$$\varepsilon_k = \|\gamma_k\|^{-1} \, , \quad \delta_k = \|Z_k \gamma_k\|^{-1} \, , \quad \xi_k = \|\phi_k\|^{-1} \, , \quad \eta_k = \|Z_k \phi_k\|^{-1} \ . \tag{123}$$

Hence we must make the assumption $E_k/F_k \geq \max\{(1-\alpha)/(2-\alpha), \alpha/(1+\alpha)\}$ so that

$$((1+\alpha)\, \varepsilon_k - \alpha\, \delta_k\, z)((2-\alpha)\, \xi_k - (1-\alpha)\, \eta_k\, z) > 0 \ , \qquad z \in [E_k, F_k] \ . \tag{124}$$

Next we can apply the spectral mapping theorem to relate E_{k+1}, F_{k+1} and E_k, F_k via (122). For this a careful analysis of the extreme values of the polynomials

$$P(z \, ; \, q, r) = z(1+\alpha) - \alpha z/q)((2-\alpha) - \alpha z/q)((2-\alpha) - (1-\alpha)\, z/r) \ , \qquad z \in [E, F] \tag{125}$$

with $q, r \in [E, F]$ where E, F satisfy

$$0 < E \leq F \leq E \min\Big\{\frac{2-\alpha}{1-\alpha}, \frac{1+\alpha}{\alpha}\Big\} \ . \tag{126}$$

The final result is that $E_k/F_k \to 1$ at least quadratically when $Q = A/B > Q(\alpha)$ with $Q(\alpha)$ the above defined root of (116). It then follows easily from (120), (121) that γ_k, ϕ_k converge at least quadratically to limits γ_∞, ϕ_∞ of unit norm.

A further comment is that algorithm (4), which requires inversions of frame operators, can be considered in a coupled recursions setting as well. For $\alpha \in [0,1]$ we set

$$\begin{bmatrix} \gamma_0 \\ \phi_0 \end{bmatrix} = \begin{bmatrix} g \\ g \end{bmatrix} ; \quad \begin{bmatrix} \gamma_{k+1} \\ \phi_{k+1} \end{bmatrix} = \begin{bmatrix} (1-\alpha)\,\dfrac{\gamma_k}{||\gamma_k||} + \alpha\,\dfrac{T_k^{-1}\phi_k}{||T_k^{-1}\phi_k||} \\[2ex] \alpha\,\dfrac{\phi_k}{||\phi_k||} + (1-\alpha)\,\dfrac{S_k^{-1}\gamma_k}{||S_k^{-1}\gamma_k||} \end{bmatrix} , \quad k = 0,1,\ldots , \tag{127}$$

with S_k, T_k the frame operator corresponding to (γ_k, a, b), (ϕ_k, a, b), respectively. This coupled recursion always converges, but only in the cases $\alpha = 0, \frac{1}{2}, 1$ the limiting windows γ_∞, ϕ_∞ are equal to $S^{-\alpha}g/||S^{-\alpha}g||$, $S^{-(1-\alpha)}g/||S^{-(1-\alpha)}g||$, respectively. In Section 10 we shall consider examples.

7 A sharp Kantorovich inequality

The analysis of the algorithms (14), (15) requires insight into the set of points in $\mathbb{R}^2$ of the form

$$\Big(\frac{||Sf||}{||f||}, \frac{||S^2 f||}{||Sf||}\Big) \tag{128}$$

with f an element of a Hilbert space H and S a positive definite linear operator. As to (15) we note, see (50), that $S_k g = (SS_k)^{1/2}\gamma_k$. With $A = \min \sigma(S) > 0$, $B = \max \sigma(S) < \infty$ there is the inequality

$$A \le \frac{||Sf||}{||f||} \le \frac{||S^2 f||}{||Sf||} \le B \ , \tag{129}$$

and

$$\frac{2AB}{A^2+B^2} \le \frac{||Sf||^2}{||f||\,||S^2 f|} \le 1 \ . \tag{130}$$

While the inequalities in (129) are simple consequences of the Cauchy-Schwarz inequality and the fact that $\sigma(S) \subset [A,B]$, the first inequality in (130) is somewhat more subtle. The latter inequality, which is known as Kantorovich's inequality, is equivalent with

$$\frac{||Sf||}{||f||} \ge \frac{2AB}{A^2+B^2}\,\frac{||S^2 f||}{||Sf||} \ . \tag{131}$$

Obviously, (131) is not sharp when $\|Sf\|/\|f\|$ is close to A or B or when $\|S^2f\|/\|Sf\|$ is close to A or B: in these cases we have $\|Sf\|/\|f\| \approx \|S^2f\|/\|Sf\|$.

We shall prove the following sharpening of (131).

Proposition.
Let $f \in H$ and S as above.

(i) When $\|S^2f\|/\|Sf\| = u \in [A,B]$, we have

$$\frac{\|Sf\|}{\|f\|} \geq \frac{AB}{(A^2+B^2-u^2)^{1/2}} \ , \tag{132}$$

and this inequality is best possible.

(ii) When $\|Sf\|/\|f\| = t \in [A,B]$, we have

$$\frac{\|S^2f\|}{\|Sf\|} \leq (t^2A^2+t^2B^2-A^2B^2)^{1/2} \ , \tag{133}$$

and this inequality is best possible.

Proof. It suffices to prove (ii) only since (i) and (ii) are equivalent. We consider the case that H has a finite dimension $N+1$; the general case can be handled by using appropriate orthogonal projection operators of finite rank. We thus assume that we have an orthonormal base $f_0, ..., f_N \in \mathbb{C}^{N+1}$ of eigenvectors of S with eigenvalues $\lambda_0, ..., \lambda_N$ satisfying

$$0 < A = \lambda_0 \leq \lambda_1 \leq ... \leq \lambda_{N-1} \leq \lambda_N = B \ . \tag{134}$$

Take any $f \in \mathbb{C}^{N+1}$ where we assume that $\|f\| = 1$. Hence there are $a_0, ..., a_N \in \mathbb{C}$ such that

$$f = \sum_{n=0}^{N} a_n f_n \ ; \qquad \sum_{n=0}^{N} |a_n|^2 = 1 \ , \tag{135}$$

and

$$\|Sf\|^2 = \sum_{n=0}^{N} |a_n|^2 \, \lambda_n^2 \ ; \qquad \|S^2f\|^2 = \sum_{n=0}^{N} |a_n|^2 \, \lambda_n^4 \ . \tag{136}$$

We shall find the maximum of

$$\sum_{n=0}^{N} |a_n|^2 \, \lambda_n^4 \tag{137}$$

under the condition that

$$\sum_{n=0}^{N} |a_n|^2 = 1 \ , \qquad \sum_{n=0}^{N} |a_n|^2 \, \lambda_n^2 = t^2 \tag{138}$$

where $t \in [A, B]$. Letting $E = A^2$, $F = B^2$, $b = t^2 \in [E, F]$ and

$$x_n := |a_n|^2 \ , \quad b_n := \lambda_n^2 \in [E, F] \ , \qquad n = 0, ..., N \ , \tag{139}$$

we should find the maximum of

$$\sum_{n=0}^{N} b_n^2 \, x_n \tag{140}$$

under the condition that

$$x_n \geq 0 \ , \qquad \sum_{n=0}^{N} x_n = 1 \ , \qquad \sum_{n=0}^{N} b_n x_n = b \ . \tag{141}$$

Assume that there is an $m = 1, ..., N-1$ such that

$$x_m \neq 0 \ , \qquad E < b_m < F \ . \tag{142}$$

We shall show that (140) is not maximal. We write

$$x_m = y_m + z_m \ , \qquad b_m x_m = E y_m + F z_m \tag{143}$$

with positive numbers

$$y_m = \frac{F - b_m}{F - E} \, x_m \ , \qquad z_m = \frac{b_m - E}{F - E} \, x_m \ . \tag{144}$$

Then define

$$x'_0 = x_0 + y_m \ , \qquad x'_m = 0 \ , \qquad x'_N = x_N + z_m \tag{145}$$

and $x'_n = x_n$ for $n \neq 0, m, N$. We have (143) that

$$\sum_{n=0}^{N} x'_n = 1 \ , \qquad \sum_{n=0}^{N} b_n x'_n = b \ . \tag{146}$$

Also from (144)

$$\sum_{n=0}^{N} b_n^2 x'_n - \sum_{n=0}^{N} b_n^2 x_n = E^2 y_m + F^2 z_m - b_m^2 x_m \ =$$

$$= \ x_m \left\{ E^2 \, \frac{F - b_m}{F - E} + F^2 \, \frac{b_m - E}{F - E} - b_m^2 \right\} = x_m (F - b_m)(b_m - E) > 0 \ . \tag{147}$$

This proves the claim about non-maximality. We conclude that the maximum value of $\sum_{n=0}^{N} b_n^2 x_n$ can be found as the maximum of

$$E^2 x_0 + F^2 x_N \tag{148}$$

under the condition that

$$x_0 \geq 0 \ , \qquad x_N \geq 0 \ , \qquad x_0 + x_N = 1 \ , \qquad Ex_0 + Fx_N = b \ . \tag{149}$$

Clearly, this maximum value equals $b(E+F) - EF$, and now (133) follows on reinstalling the original variables $A = E^{1/2}$, $B = F^{1/2}$, $t = b^{1/2}$. We also note that there is equality for the particular f constructed in accordance with (149). In the general infinite-dimensional case, it may happen that equality does not occur in (133). Nevertheless, the right-hand side of (133) cannot be replaced by any smaller number. This completes the proof.

In Fig. 5 we have plotted the bounds (131), (132) for $\|Sf\|/\|f\|$ as a function of $u = \|S^2 f\|/\|Sf\| \in [A, B]$, where we have taken $A = \frac{1}{2}$, $B = 1$.

We observe that the two bounds (131), (132) agree at $u = (\frac{1}{2}\,(A^2+B^2))^{1/2}$, and that the bound (132) equals A when $u = A$ and equals B when $u = B$. Also,

$$\frac{2ABu}{A^2+B^2} \leq \frac{AB}{(A^2+B^2-u^2)^{1/2}} \leq u \ , \qquad u \in [A, B] \ , \tag{150}$$

since this is equivalent with

$$(A^2+B^2-2u^2)^2 \geq 0 \ , \qquad (u^2-A^2)(B^2-u^2) \geq 0 \ , \qquad u \in [A, B] \ . \tag{151}$$

In Sec. 10 we shall consider the case $a = b = 1$. Given four numbers A, B, u, t with

$$0 < A \leq \frac{AB}{(A^2+B^2-u^2)^{1/2}} \leq t \leq u \leq B < \infty \tag{152}$$

we shall display a Gabor frame (g, a, b) with frame operator S having best frame bounds A, B such that

$$\frac{\|Sg\|}{\|g\|} = t \ , \qquad \frac{\|S^2 g\|}{\|Sg\|} = u \ . \tag{153}$$

Example.
Let $g(t) = 2^{1/4}\exp(-\pi t^2)$, and let S be the frame operator of $(g, a = 2^{-1/2}, b = 2^{-1/2})$. One can compute, either analytically or numerically, that

$$A = 1.669253683\ , \qquad B = 2.360681197\ , \tag{154}$$

$$\frac{\|Sg\|}{\|g\|} = 2.022409392\ , \qquad \frac{\|S^2 g\|}{\|Sg\|} = 2.051916634\ , \tag{155}$$

$$\frac{AB}{(A^2+B^2-u^2)^{1/2}} = 1.934617914\ , \qquad \frac{2ABu}{A^2+B^2} = 1.934565555\ , \tag{156}$$

where $u = \|S^2 g\|/\|Sg\|$. It thus seems that for (very) well-behaved windows the new inequality is hardly sharper than the old one.

8 Algorithm for g^t with cubic convergence and no inversions

In this section we analyze in detail the algorithm

$$\gamma_0 = g\ ; \quad \gamma_{k+1} = \frac{15}{8}\,\frac{\gamma_k}{\|\gamma_k\|} - \frac{5}{4}\,\frac{S_k\gamma_k}{\|S_k\gamma_k\|} + \frac{3}{8}\,\frac{S_k^2\gamma_k}{\|S_k^2\gamma_k\|}\ , \qquad k = 0, 1, \ldots\ , \tag{157}$$

where (g, a, b) is a Gabor frame with best frame bounds $A > 0$, $B < \infty$, and S_k is the frame operator corresponding to (γ_k, a, b) with best frame bounds A_k, B_k. We shall show that the condition $A/B > \frac{3}{7}$ is sufficient for all (γ_k, a, b) to be a frame indeed and that then $\gamma_k/\|\gamma_k\|$ converges to $(ab)^{-1/2}\,g^t$ at least cubically.

We let

$$\varepsilon_{ki} = \|S_k^i\gamma_k\|^{-1}\ , \qquad i = 0, 1, 2\ ; \quad k = 0, 1, \ldots\ , \tag{158}$$

so that we can write $\gamma_{k+1} = \phi_k(S_k)\,\gamma_k$ with

$$\phi_k(s) = \tfrac{15}{8}\,\varepsilon_{k0} - \tfrac{5}{4}\,\varepsilon_{k1}\,s + \tfrac{3}{8}\,\varepsilon_{k2}\,s^2\ . \tag{159}$$

By the Proposition in Section 3 we thus have that

$$\begin{aligned} S_{k+1} &= S_k(\tfrac{15}{8}\,\varepsilon_{k0} - \tfrac{5}{4}\,\varepsilon_{k1}\,S_k + \tfrac{3}{8}\,\varepsilon_{k2}\,S_k^2)^2 = \\ &= w_k^2\,S_k\Big(\frac{15}{8}\,v_k - \frac{5}{4}\,\frac{S_k}{u_k} + \frac{3}{8}\,\frac{S_k^2}{u_k^2}\Big)^2\ , \end{aligned} \tag{160}$$

where $w_k = \varepsilon_{k1}^2/\varepsilon_{k2}$ and

$$u_k = \frac{\varepsilon_{k1}}{\varepsilon_{k2}} = \frac{\|S_k^2\gamma_k\|}{\|S_k\gamma_k\|}\ , \qquad v_k = \frac{\varepsilon_{k0}\varepsilon_{k2}}{\varepsilon_{k1}^2} = \frac{\|S_k\gamma_k\|^2}{\|\gamma_k\|\,\|S_k\gamma_k\|}\ . \tag{161}$$

We want to take a positive square root in (160), and to that end we must require that

$$\tfrac{15}{8}\,v_k - \tfrac{5}{4}\,x + \tfrac{3}{8}\,x^2 = \tfrac{3}{8}\,((x-\tfrac{5}{3})^2 + 5v_k - \tfrac{25}{9}) > 0\;, \qquad x > 0\;. \tag{162}$$

Hence we require that $v_k > \frac{5}{9}$. Given the form of v_k in (161) we see that Kantorovich's inequality (19) applies here. Therefore, $v_k > \frac{5}{9}$ surely when

$$\frac{A_k}{B_k} > \tfrac{1}{5}\,(9-\sqrt{56}) = 0.303337045\;. \tag{163}$$

Requiring (163) we now let

$$Z_k := S_k^{1/2}\;, \qquad E_k = A_k^{1/2} = \min\sigma(Z_k)\;, \qquad F_k = B_k^{1/2} = \max\sigma(Z_k)\;. \tag{164}$$

Then

$$Z_{k+1} = w_k\,Z_k\Big(\frac{15}{8}\,v_k I - \frac{5}{4}\,\frac{Z_k^2}{u_k} + \frac{3}{8}\,\frac{Z_k^4}{u_k^2}\Big)\;. \tag{165}$$

We define for arbitrary positive numbers u, v, $v \le 1$

$$P(z\,;\,u,v) = z\Big(\frac{15}{8}\,v - \frac{5}{4}\,\frac{z^2}{u} + \frac{3}{8}\,\frac{z^4}{u^2}\Big)\;. \tag{166}$$

Then by the spectral mapping theorem

$$\frac{E_{k+1}}{F_{k+1}} \ge \min_{z\in[E_k,F_k]} P(z\,;\,u_k,v_k)/\max_{z\in[E_k,F_k]} P(z\,;\,u_k,v_k)\;, \tag{167}$$

and we are interested in finding out the lowest value the right-hand side in (167) can take. The numbers u_k, v_k in (167) are restricted by, see Sec. 7,

$$A_k \le u_k \le B_k\;; \qquad \frac{u_k^{-1}A_kB_k}{(A_k^2+B_k^2-u_k^2)^{1/2}} \le v_k \le 1\;. \tag{168}$$

Thus we set ourselves the following problem: given numbers $A > 0$, $B < \infty$ such that $A/B > \frac{1}{5}\,(9-\sqrt{56})$, then with $E = A^{1/2}$, $F = B^{1/2}$, what is the lowest value of

$$\min_{z\in[E,F]} P(z\,;\,u,v)/\max_{z\in[E,F]} P(z\,;\,u,v) \tag{169}$$

under the constraint that

$$A \le u \le B\;; \qquad v_0(u) := \frac{u^{-1}AB}{(A^2+B^2-u^2)^{1/2}} \le v \le 1\;? \tag{170}$$

In particular we are interested to find out when the quantity in (169) exceeds E/F.

Our approach is as follows. For fixed $u \in [A, B]$ we shall show that the functional in (169) is a unimodal function of v, and this allows us to restrict attention to the extreme values of v in (170). Next, when $v = 1$ we have that $P(z\,;\,u, v = 1)$ increases in $z \in [E, F]$ and when $v = v_0(u)$ we have that $P(z\,;\,u, v = v_0(u))$ decreases in $z \in [E, F]$. Hence the worst case value of the functional in (156) equals for $u \in [A, B]$

$$\min\Bigl\{\frac{P(E\,;\,u,1)}{P(F\,;\,u,1)}\,,\,\frac{P(F\,;\,u,v_0(u))}{P(E\,;\,u,v_0(u))}\Bigr\}\ . \tag{171}$$

Finally we shall show that

$$\min_{A\leq u\leq B}\frac{P(E\,;\,u,1)}{P(F\,;\,u,1)} > \frac{E}{F} \Leftrightarrow \frac{A}{B} > \frac{3}{7}\ , \tag{172}$$

$$\min_{A\leq u\leq B}\frac{P(F\,;\,u,v_0(u))}{P(E\,;\,u,v_0(u))} > \frac{E}{F} \Leftrightarrow \frac{A}{B} > Q_2\ , \tag{173}$$

where $Q_2 = 0.401069994$ ($< \frac{3}{7} = 0.428571429$) is a root of a particular algebraic equation.

We shall now detail this approach.

Unimodality of (169) as a function of v

We fix $u \in [A, B]$ and we set

$$x = \frac{z}{\sqrt{u}} \in \Bigl[\Bigl(\frac{A}{u}\Bigr)^{1/2}, \Bigl(\frac{B}{u}\Bigr)^{1/2}\Bigr] \equiv [C, D] \supset \{1\}\ , \tag{174}$$

together with

$$S(x\,;\,v) = \tfrac{15}{8}\,xv - \tfrac{5}{4}\,x^3 + \tfrac{3}{8}\,x^5\ , \qquad x \in [C, D]\ . \tag{175}$$

In Figs. 6, 7, 8 we have plotted $S(x\,;\,v)$ as a function of x with particular attention to extreme values on the interval $[(A/u)^{1/2}, (B/u)^{1/2}]$ when $A = 0.36$, $B = 1$ and u and v as in (170). We show that

$$U(v) := \min_{x\in[C,D]} S(x\,;\,v) / \max_{x\in[C,D]} S(x\,;\,v) \tag{176}$$

is a unimodal function of $v \in [C^2D^2/(C^4 + D^4 - 1)^{1/2}, 1]$ when $C^2/D^2 = A/B > \frac{3}{7}$. Under the latter assumption we have that

$$\frac{S(C\,;\,1)}{S(D\,;\,1)} > \frac{C}{D} = \frac{E}{F} \tag{177}$$

no matter what value $u \in [A,B]$ has. This follows easily from the fact that $S(x\,;\,1)$ is strictly increasing in $x>0$ while the left-hand side of (177) is for given A, B minimal when $u=A$ with minimum value $(A/B)^{1/2}(\frac{15}{8}-\frac{5}{4}\,B/A+\frac{3}{8}\,(B/A)^2)^{-1}$ and this exceeds $(A/B)^{1/2}$ when $A/B>\frac{3}{7}$. See (205)–(210) for a more detailed argumentation. We may restrict attention to this range of C/D since we are interested only in the cases where (169) exceeds E/F.

We start by noting that

$$\frac{dS}{dx}\,(x\,;\,v)=\frac{15}{8}\,((1-x^2)^2-(1-v))\ . \tag{178}$$

Hence $\frac{dS}{dx}<0$ for $x>0$ if and only if

$$(1-(1-v)^{1/2})^{1/2}=:\psi(v)\leq x\leq \phi(v):=(1+(1-v)^{1/2})^{1/2}\ . \tag{179}$$

The reader is invited to sketch the graph of $S(x\,;\,v)$ as a function of x for various values of v, also see Figs. 6, 7, 8. It thus appears that the maximum value of $S(x\,;\,v)$, $x\in[C,D]$, is given by

$$\text{(i) } S(D\,;\,v) \quad \text{or} \quad \text{(ii) } S(\psi(v)\,;\,v) \quad \text{or} \quad \text{(iii) } S(c\,;\,v)\ , \tag{180}$$

while the minimum value of $S(x,v)$, $x\in[C,D]$ is given by

$$\text{(iv) } S(C\,;\,v) \quad \text{or} \quad \text{(v) } S(\phi(v)\,;\,v) \quad \text{or} \quad \text{(vi) } S(D\,;\,v)\ . \tag{181}$$

The cases in (180), (181) are ordered according to decreasing value of v. We note that the pairs (i),(vi) and (iii),(iv) cannot occur. We thus should consider

$$\text{(i),(iv)} \quad : \quad \frac{S(C\,;\,v)}{S(D\,;\,v)}\ , \tag{182}$$

$$\text{(i),(v)} \quad : \quad \frac{S(\phi(v)\,;\,v)}{S(D\,;\,v)}\ , \tag{183}$$

$$\text{(ii),(iv)} \quad : \quad \frac{S(C\,;\,v)}{S(\psi(v)\,;\,v)}\ , \tag{184}$$

$$\text{(ii),(v)} \quad : \quad \frac{S(\phi(v)\,;\,v)}{S(\psi(v)\,;\,v)}\ , \tag{185}$$

$$\text{(ii),(vi)} \quad : \quad \frac{S(D\,;\,v)}{S(\psi(v)\,;\,v)}\ , \tag{186}$$

$$\text{(iii),(v)} \quad : \quad \frac{S(\phi(v)\,;\,v)}{S(C\,;\,v)}\ , \tag{187}$$

$$\text{(iii),(vi)} \quad : \quad \frac{S(D\,;\,v)}{S(C\,;\,v)}\ . \tag{188}$$

Case (182)

We have, see (175),

$$U(v) = \frac{S(C\,;\,v)}{S(D\,;\,v)} = \frac{S(C\,;\,1) - \frac{C}{D}\,S(D\,;\,1)}{S(D\,;\,1) - \frac{15}{8}\,D(1-v)} + \frac{C}{D} \tag{189}$$

and this decreases in v on account of (177).

Case (183)

We have

$$U(v) = \frac{S(\phi(v)\,;\,v)}{S(D\,;\,v)}\ . \tag{190}$$

A computation shows that

$$\frac{d}{dv}\,S(\phi(v)\,;\,v) = \frac{15}{8}\,\phi(v)\ . \tag{191}$$

Hence the numerator $N(v)$ of $U'(v)$ equals

$$\begin{aligned} N(v) &= \tfrac{15}{8}\,\phi(v)(\tfrac{15}{8}\,Dv - \tfrac{5}{4}\,D^3 + \tfrac{3}{8}\,D^5)\ + \\ &\quad -\ \tfrac{15}{8}\,D(\tfrac{15}{8}\,v\phi(v) - \tfrac{5}{4}\,\phi^3(v) + \tfrac{3}{8}\,\phi^5(v))\ = \\ &= \tfrac{15}{64}\,D\phi(v)(D^2 - \phi^2(v))(3D^2 + 3\phi^2(v) - 10)\ . \end{aligned} \tag{192}$$

We have $\phi(1) = 1$, whence

$$N_3(1) = \tfrac{15}{64}\,D(D^2-1)(3D^2-7) \leq 0 \tag{193}$$

since $D^2 = B/u \in [1, \frac{7}{3}]$ by assumption. Since $\phi(v)$ decreases in v, there is at most one change of sign of $N_3(v)$ (as long as $\phi(v) \leq D$ which we may evidently assume). This change of sign does occur when $D^2 > \frac{5}{3}$. Thus for values of v such that $1 \leq \phi(v) \leq D$ we have that $U(v)$ decreases in v ($D^2 \leq \frac{5}{3}$) or is unimodal ($D^2 > \frac{5}{3}$) with maximum assumed when $\phi^2(v) = -D^2 + \frac{10}{3}$.

Case (184)

We have

$$U(v) = \frac{S(C\,;\,v)}{S(\psi(v)\,;\,v)}\ . \tag{194}$$

We have now $\frac{d}{dv}\,S(\psi(v)\,;\,v) = \frac{15}{8}\,\psi(v)$, and we compute the numerator $N(v)$ of $U'(v)$ as in (192) as

$$N(v) = -\,\tfrac{15}{64}\,C\,\psi(v)(C^2 - \psi^2(v))(3C^2 + 3\psi^2(v) - 10)\ . \tag{195}$$

Since $C^2 \le \psi^2(v) \le 1$ we see that $N(v) \le 0$, whence $U(v)$ decreases in v.

Case (185)

We have

$$U(v) = \frac{S(\phi(v)\,;\,v)}{S(\psi(v)\,;\,v)}\ . \tag{196}$$

We compute the numerator $N(v)$ of $U'(v)$ now as

$$\begin{aligned} N(v) &= \tfrac{15}{64}\,\phi(v)\psi(v)(\psi^2(v) - \phi^2(v))(3\psi^2(v) + 3\phi^2(v) - 10) = \\ &= \tfrac{15}{8}\,\phi(v)\psi(v)(1-v)^{1/2} > 0\ . \end{aligned} \tag{197}$$

Therefore $U(v)$ is increasing in v.

Case (186)

We have

$$U(v) = \frac{S(D\,;\,v)}{S(\psi(v)\,;\,v)}\ . \tag{198}$$

The numerator $N(v)$ of $U'(v)$ can be computed as

$$N(v) = -\,\tfrac{15}{64}\,D\,\psi(v)(D^2 - \psi^2(v))(3D^2 + 3\psi^2(v) - 10)\ . \tag{199}$$

Since $\psi^2(v) \le 1 \le D^2 \le \frac{7}{3}$ we see that $N(v) \ge 0$, whence $U(v)$ increases in v.

Case (187)

We have

$$U(v) = \frac{S(\phi(v)\,;\,v)}{S(C\,;\,v)}\ . \tag{200}$$

We compute the numerator $N(v)$ of $U'(v)$ now as

$$N(v) = \tfrac{15}{64}\,C\,\phi(v)(C^2 - \phi^2(v))(3C^2 + 3\phi^2(v) - 10)\ . \tag{201}$$

Since $C^2 \le \phi^2(v) \le D^2 \le \frac{7}{3}$ we see that $N(v) \ge 0$, whence $U(v)$ increases in v.

Case (188)

We have

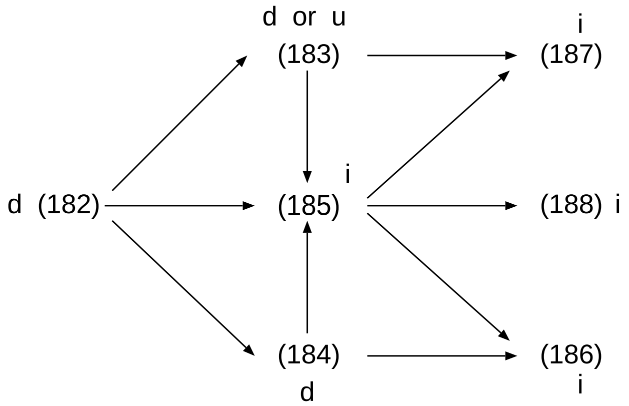
$$U(v) = \frac{S(D\,;\,v)}{S(C\,;\,v)}\ . \tag{202}$$

The numerator $N(v)$ of $U'(v)$ can now be computed as

$$N(v) = \tfrac{15}{8}\,D\,S(C\,;\,v) - \tfrac{15}{8}\,C\,S(D\,;\,v)\ . \tag{203}$$

Since $C \le D$, $S(D\,;\,v) \le S(C\,;\,v)$ we see that $N(v) \ge 0$, whence $U(v)$ increases in v.

Having established monotonicity of $U(v)$ in all cases that can occur, except where $U(v)$ is possibly unimodal with maximum value assumed at an interior point, the unimodality of U when v ranges between $v_0(u)$ and 1 follows on an inspection of the order in which the cases (182)–(188) occur. In the graph below we have indicated by arrows which one of the stages (182)–(188) can be reached from a particular one when v is decreased. Also the type of monotonicity of U as a function of (increasing) v at each of these stages is indicated by d (decreasing), i (increasing) or u (unimodal).



It is concluded that

$$\min_{v_0(u)\le v\le 1} U(v) = \min\{U(v_0(u)), U(1)\}\ . \tag{204}$$

We first analyze $U(1) = U(u\,;\,1)$ as a function of $u \in [A, B]$. By monotonicity of $S(x\,;\,1)$ we have

$$U(1) = \frac{S(C\,;\,1)}{S(D\,;\,1)} = \Big(\frac{A}{B}\Big)^{1/2}\,\frac{u^2 - \frac{2}{3}\,Au + \frac{1}{5}\,A^2}{u^2 - \frac{2}{3}\,Bu + \frac{1}{5}\,B^2} =: U(u\,;\,1)\ . \tag{205}$$

Next

$$\frac{d}{du}\,U(u\,;\,1)=-\,\frac{2}{3}\,\Big(\frac{A}{B}\Big)^{1/2}(B-A)\,\frac{u^2-\frac{3}{5}\,(A+B)\,u+\frac{1}{5}\,AB}{(u^2-\frac{2}{3}\,Bu+\frac{1}{5}\,B^2)^2} \tag{206}$$

from which one easily sees that the minimum of $U(u\,;\,1)$ when $u\in[A,B]$ equals $U(A\,;\,1)$ or $U(B\,;\,1)$. We compute

$$U(A\,;\,1)=\frac{8}{15}\,\frac{(A/B)^{5/2}}{(A/B)^2-\frac{2}{3}\,A/B+\frac{1}{5}}\ . \tag{207}$$

This $U(A\,;\,1)$ exceeds $(A/B)^{1/2}=E/F$ when $\frac{3}{7}<A/B$. In Fig. 9 we have plotted

$$H_1(Q)=\Big(\frac{8}{15}\,\frac{Q^{5/2}}{Q^2-\frac{2}{3}\,Q+\frac{1}{5}}\Big)^2\ ,\qquad Q=\frac{A}{B}\in[0,1]\ . \tag{208}$$

We furthermore compute

$$U(B\,;\,1)=\frac{15}{8}\,\Big(\frac{A}{B}\Big)^{1/2}(1-\tfrac{2}{3}\,A/B+\tfrac{1}{5}\,(A/B)^2)\ . \tag{209}$$

This $U(B\,;\,1)$ exceeds $(A/B)^{1/2}=E/F$ when $0<A/B<1$. In Fig. 9 we have also plotted

$$H_2(Q)=Q(\tfrac{15}{8}\,(1-\tfrac{2}{3}\,Q+\tfrac{1}{5}\,Q^2))^2\ ,\qquad Q=\frac{A}{B}\in[0,1]\ . \tag{210}$$

Thus the minimum of $U(u\,;\,1)$ when $u\in[A,B]$ equals $U(A\,;\,1)$ as given in (207) and exceeds $E/F=(A/B)^{1/2}$ when $A/B>\frac{3}{7}$.

We next consider $U(v_0(u))$. We shall assume[1] that u is sufficiently far away from A, B so that $\phi(v_0(u))\geq D$, $\psi(v_0(u))\leq C$, see (179). Then $S(x\,;\,v_0(u))$ is decreasing on $[C,D]$ and we compute, see (174)–(175), (170),

$$\begin{aligned} U(v_0(u)) &= \frac{S(D\,;\,v_0(u))}{S(C\,;\,v_0(u))}=\frac{\frac{15}{8}\,D\,v_0(u)-\frac{5}{4}\,D^3+\frac{3}{8}\,D^5}{\frac{15}{8}\,C\,v_0(u)-\frac{5}{4}\,C^3+\frac{3}{8}\,C^5}= \\ &= Q^{-1/2}\,\frac{15f(t,Q)-10t+3}{15f(t,Q)-10tQ+3Q^2}\equiv F(t,Q) \end{aligned} \tag{211}$$

where we have set

$$Q=\frac{A}{B}\ ,\qquad t=\frac{u}{B}\in[Q,1]\ , \tag{212}$$

[1] See Appendix at the end of this section.

and

$$f(t,Q)=\frac{tQ}{\sqrt{1+Q^2-t^2}}\ . \tag{213}$$

We consider the function

$$\hat{F}(Q)=\min_{t\in[Q,1]}\ F(t,Q)\ , \tag{214}$$

and we are particularly interested in the points Q_1, Q_2 where

$$\hat{F}(Q_1)=0\ ,\qquad \hat{F}(Q_2)=Q_2^{1/2}\ . \tag{215}$$

In Fig. 9 we have also included the graph of

$$\hat{H}(Q)=(\hat{F}(Q))^2\ ,\qquad Q\in[Q_1,1]\ . \tag{216}$$

The analysis of $\hat{F}(Q)$ is facilitated by the fact that

$$\hat{F}(Q)=F(\hat{t}(Q),Q)\ , \tag{217}$$

where $\hat{t}(Q)$ is the solution t near $\frac{1}{2}\,(1+Q)$ of

$$3(1+Q)(1+Q^2)-10t^3=2(1+Q^2-t^2)^{3/2}\ . \tag{218}$$

Furthermore, the points Q_1 and Q_2 can be found as follows. The point Q_1 and $\hat{t}(Q_1)$ follow from setting

$$15f(t,Q)-10t+3=\frac{\partial}{\partial t}\,[15f(t,Q)-10t+3]=0\ ; \tag{219}$$

this is so since $F(t,Q)$ should have a double zero at $t=\hat{t}(Q_1)$. A fairly technical but elementary analysis then gives that

$$Q_1=\tfrac{1}{5}\,S^{3/2}=0.294447771\ , \tag{220}$$

where S is the unique solution of

$$5S^3+27S^2+27S=91 \tag{221}$$

in $(0,\infty)$. The point Q_2 is found in a similar fashion as

$$Q_2=\frac{5V^{3/2}-1}{2}-\Big(\Big(\frac{5V^{3/2}-1}{2}\Big)^2-1\Big)^{1/2}=0.401069994 \tag{222}$$

where V is the unique solution of

$$91V^3-27V^2-27V=13 \tag{223}$$

in $[(\frac{3}{5})^{2/3}, \infty)$. Note that $Q_2 < \frac{3}{7}$.

We have now established (172), (173). Hence when $A/B > \frac{3}{7}$ we have that E_k/F_k increases to 1, and then convergence of $\gamma_k/\|\gamma_k\|$ to $(ab)^{-1/2}g^t$ is easily established as before by using that

$$\Big\|\frac{\gamma_k}{\|\gamma_k\|} - \frac{g^t}{\|g^t\|}\Big\|^2 = \Big\|\frac{\gamma_k}{\|\gamma_k\|} - \frac{S_k^{-1/2}\gamma_k}{\|S_k^{-1/2}\gamma_k\|}\Big\|^2 \tag{224}$$

and Kantorovich's inequality. As to the order of convergence it can be shows that it is at least cubic. Indeed, one has (neglecting higher orders)

$$1 - U(A\,;\,1) = \tfrac{5}{2}\,(1-R)^3\ , \qquad 1 - U(B\,;\,1) = \tfrac{5}{2}\,(1-R)^3 \tag{225}$$

and

$$1 - F(\alpha + (1-\alpha)\,Q, Q) = \tfrac{5}{2}\,(9\alpha - 9\alpha^2 - 1)(1-R)^3 \tag{226}$$

when $\alpha \in [\frac{1}{3}, \frac{2}{3}]$. Here we have set $R = Q^{1/2} = (A/B)^{1/2} = E/F$.

Appendix

We shall show that for the lowest value of $U(v_0(u))$ in the case that $\arg\max S(x\,;\,v_0(u)) \le \arg\min S(x\,;\,v_0(u))$ we may restrict attention to situations where case (188) is in effect. We start by noting that (generalization of the analysis for case (182))

$$\frac{S(x\,;\,v)}{S(y\,;\,v)} \ge \frac{S(C\,;\,1)}{S(D\,;\,1)}\ , \qquad v \le 1\,,\quad C \le x \le y \le D\ , \tag{227}$$

and, evidently, equality occurs for $v = 1$, $x = C$, $y = D$. The right-hand side equals $U(1)$, see (205). Therefore, the cases where we need to consider $U(v_0(u))$ are described by (185)–(188) in which we have $\arg\max S(x\,;\,v_0(u)) \le 1 \le \arg\min S(x\,;\,v_0(u))$.

We first show that there are indeed allowed u, v such that case (188) occurs. For this it is sufficient to show that

$$\psi(v_0(u)) \le C = \Big(\frac{A}{u}\Big)^{1/2}\ , \qquad \phi(v_0(u)) \ge D = \Big(\frac{B}{u}\Big)^{1/2} \tag{228}$$

where ψ, ϕ are given in (179) and $v_0(u)$ is given in (170). The function $v_0(u)$ has maximum value $2AB(A^2+B^2)^{-1}$ for $u = u_0 = (\frac{1}{2}\,(A^2+B^2))^{1/2}$, and it is easy to verify (228) for $u = u_0$. In (228) there is equality at $u = u_\psi = As_\psi$ and $u = u_\phi = Bt_\phi$, where $s = s_\psi$ and $t = t_\phi$ are the unique solutions $s > 1$ and $t < 1$ of the equations

$$\frac{R}{(1+R^2-s^2)^{1/2}} = 2 - \frac{1}{s}\ , \qquad \frac{Q}{(1+Q^2-t^2)^{1/2}} = 2 - \frac{1}{t}\ , \tag{229}$$

respectively. Here $Q = R^{-1} = A/B$ is assumed to be less than 1. Since there is inequality in either inequality in (228) for $u = u_0$ and $v_0(u) > v_0(u_0)$, $u \neq u_0$ one easily sees that $u_\phi < u_0 < u_\psi$.

Now assume that we increase u from u_0 onwards to the point u_ψ, a boundary case where case (188) holds. From the analysis for case (186) and the fact that $D = (B/u)^{1/2}$ decreases in u it is easily seen that

$$S(D\,;\, v_0(u))/S(\psi(v_0(u))\,;\, v_0(u)) \tag{230}$$

increases when we increase u beyond u_ψ. In a similar fashion it can be seen that

$$S(\phi(v_0(u))\,;\, v_0(u))/S(C\,;\, v_0(u)) \tag{231}$$

increases when we decrease u below u_ϕ, the other boundary case for (188). It is then concluded that it is sufficient to only consider the case (188) for $U(v_0(u))$.

9 Algorithm for g^d with cubic convergence

In this section we give the details for the analysis of the algorithm

$$\gamma_0 = g\,; \quad \gamma_{k+1} = 3\,\frac{\gamma_k}{\|\gamma_k\|} - 3\,\frac{S_k g}{\|S_k g\|} + \frac{S\, S_k\, \gamma_k}{\|S\, S_k\, \gamma_k\|}\ , \qquad k = 0, 1, \dots\ , \tag{232}$$

where (g, a, b) is a Gabor frame with best frame bounds $A > 0$, $B < \infty$ and S_k is the frame operator corresponding to (γ_k, a, b) with best frame bounds A_k, B_k. There holds, compare (39)

$$S_k g = Z_k \gamma_k\ , \qquad g^d = Z_k^{-1} \gamma_k \tag{233}$$

where we have set

$$Z_k = (S S_k)^{1/2}\ . \tag{234}$$

Hence we can write (232) as

$$\gamma_0 = g\,; \quad \gamma_{k+1} = 3\,\frac{\gamma_k}{\|\gamma_k\|} - 3\,\frac{Z_k \gamma_k}{\|Z_k \gamma_k\|} + \frac{Z_k^2 \gamma_k}{\|Z_k^2 \gamma_k\|}\ , \qquad k = 0, 1, \dots\ , \tag{235}$$

and by Kantorovich's inequality,

$$\Big\|\frac{\gamma_k}{\|\gamma_k\|} - \frac{g^d}{\|g^d\|}\Big\|^2 = \Big\|\frac{\gamma_k}{\|\gamma_k\|} - \frac{Z_k^{-1}\gamma_k}{\|Z_k^{-1}\gamma_k\|}\Big\|^2 \leq \frac{2(1 - Q_k^{1/2})^2}{1 + Q_k}\ , \tag{236}$$

where we have set

$$E_k = \min \sigma(Z_k)\,, \quad F_k = \max \sigma(Z_k)\ ; \qquad Q_k = E_k/F_k\ . \tag{237}$$

Hence for (rapid) convergence of $\gamma_k/\|\gamma_k\|$ to $g^d/\|g^d\|$ we should show that $Q_k \to 1$ (rapidly).

Let

$$\varepsilon_{ki} = \|Z_k^i \gamma_k\|^{-1}\ , \qquad i = 0,1,2\,; \quad k = 0,1,\ldots\ . \tag{238}$$

Then we have $Z_0 = S$ and, as before,

$$Z_{k+1} = Z_k(3\varepsilon_{k0} I - 3\varepsilon_{k1} Z_k + \varepsilon_{k2} Z_k^2) = w_k\, P(Z_k\,;\, u_k, v_k)\ , \tag{239}$$

where $w_k = \varepsilon_{k1}^2/\varepsilon_{k2}$ and

$$u_k = \frac{\varepsilon_{k2}}{\varepsilon_{k1}} = \frac{\|Z_k^2\gamma_k\|}{\|Z_k\gamma_k\|}\ , \qquad v_k = \frac{\varepsilon_{k0}\varepsilon_{k2}}{\varepsilon_{k1}^2} = \frac{\|Z_k\gamma_k\|^2}{\|\gamma_k\|\,\|Z_k^2\gamma_k\|}\ , \tag{240}$$

and where

$$P(z\,;\,u,v) = z\Big(3v - 3\,\frac{z}{u} + \Big(\frac{z}{u}\Big)^2\Big)\ . \tag{241}$$

We thus find ourselves in a situation that is quite similar to the one in Section 8. Accordingly, the analysis follows very much the same plan and we shall be somewhat brief about the details.

We require that $P(z\,;\,u,v) > 0$ for $z > 0$, and to that end we note that

$$3v - 3x + x^2 = (x - \tfrac{3}{2})^2 + 3(v - \tfrac{3}{4}) > 0 \tag{242}$$

for all $x > 0$ when $v > \frac{3}{4}$. By the Kantorovich inequality (19) and the definition of v_k in (240) we have that $v_k > 0$ when

$$\frac{E_k}{F_k} > \tfrac{1}{3}\,(4 - \sqrt{7}) = 0.451141623\ . \tag{243}$$

By the spectral mapping theorem we have that

$$\frac{E_{k+1}}{F_{k+1}} \geq \min_{z\in[E_k,F_k]} P(z\,;\,u_k,v_k)/\max_{z\in[E_k,F_k]} P(z\,;\,u_k,v_k)\ . \tag{244}$$

We thus consider the following problem. Let $0 < E \leq F < \infty$ with $E/F > \frac{1}{4}\,(4-\sqrt{7})$. What is the lowest value of

$$\min_{z\in[E,F]} P(z\,;\,u,v)/\max_{z\in[E,F]} P(z\,;\,u,v) \tag{245}$$

under the constraints that

$$E \le u \le F \; ; \qquad v_0(u) := \frac{u^{-1}EF}{\sqrt{E^2+F^2-u^2}} \le v \le 1 \; ? \tag{246}$$

We fix $u \in [E,F]$, and we shall show unimodality of the functional in (245) as a function of v. We let

$$x = \frac{z}{u} \in \Big[\frac{E}{u}\,,\,\frac{F}{u}\Big] \equiv [C,D] \supset \{1\} \tag{247}$$

and set

$$R(x\,;\,v) = x(3v - 3x + x^2) = 1 + 3(v-1)\,x - (1-x)^3 \;. \tag{248}$$

In Figs. 10, 11, 12 we have plotted $R(x\,;\,v)$ as a function of x with particular attention for extreme values on the interval $[E/u, F/u]$ for certain choices of E and F and where u and v are as in (246). We need to show that

$$W(v) := \min_{x\in[C,D]} R(x\,;\,v) / \max_{x\in[C,D]} R(x\,;\,v) \tag{249}$$

is a unimodal function of $v \in [CD(C^2+D^2-1)^{-1/2}, 1]$. By (248) we have that $\frac{dR}{dx} < 0$ (when $x > 0$) when

$$\Psi(v) := 1 - (1-v)^{1/2} \le x \le 1 + (1-v)^{1/2} =: \Phi(v) \;. \tag{250}$$

We restrict C, D such that

$$\frac{R(C\,;\,1)}{R(D\,;\,1)} > \frac{C}{D} = \frac{E}{F} \;. \tag{251}$$

The left-hand side of (251 equals the functional in (245) for the case that $u = E$, $v = 1$ so that the minimum over z is assumed at E while the maximum over z is assumed at F. Since we are particularly interested in the case that the functional in (245) exceeds E/F, the requirement in (251) is a logical one. We have that (251) holds for all allowed u, v if and only if

$$Q := E/F = C/D > \tfrac{1}{2} \;. \tag{252}$$

See (258)–(261) for a more detailed argumentation.

From the above definitions it appears that the maximum value of $R(x\,;\,v)$, $x \in [C,D]$ is given by

$$\text{(i) } R(D\,;\,v) \quad \text{or} \quad \text{(ii) } R(\Psi(v)\,;\,v) \quad \text{or} \quad \text{(iii) } R(C\,;\,v) \tag{253}$$

while the minimum values of $R(x\,;\,v)$, $x \in [C,D]$ is given by

$$\text{(iv) } R(C\,;\,v) \quad \text{or} \quad \text{(v) } R(\Phi(v)\,;\,v) \quad \text{or} \quad \text{(vi) } R(D\,;\,v)\ . \tag{254}$$

The cases in (253), (254) are ordered according to decreasing value of v. We note that (i),(vi) and (iii),(iv) do not occur. Just as in Section 8 we then have to consider the seven cases

$$\text{(i),(iv)}\,; \quad \text{(i),(v)}\,; \quad \text{(ii),(iv)}\,; \quad \text{(ii),(v)}\,; \quad \text{(ii),(vi)}\,; \quad \text{(iii),(v)}\,; \quad \text{(iii),(vi)}\ . \tag{255}$$

The first case in (255) is settled by the assumption in (252): decreeasing in v. The last case is easily settled and yields: increasing in v. To settle the other cases one can use the fact that

$$\frac{d}{dv}\,[R(\Phi(v)\,;\,v)] = 3\Phi(v)\ , \qquad \frac{d}{dv}\,[R(\Psi(v)\,;\,v)] = 3\Psi(v)\ . \tag{256}$$

Therefore, we get, for instance, that the numerator $N(v)$ of $(R(\Phi(v)\,;\,v)/R(D\,;\,v))'$ is given by

$$3D\,\Phi(v)(D-\Phi(v))(\Phi(v)+D-3)\ . \tag{257}$$

Hence $N(v)$ is negative as long as $\Phi(v) < 3-D$ and changes sign at most once. Etc., etc. It is concluded that $W(v)$ in (249) is unimodal as a function of v, whence $W(v)$ is minimal for $v=1$ or $v=v_0(u)$.

We consider first the case that $v=1$. Then

$$W(1) = \frac{R(C\,;\,1)}{R(D\,;\,1)} = \frac{E}{F}\,\frac{3u^2-3Eu+E^2}{3u^2-3Fu+F^2} =: W(u\,;\,1)\ , \tag{258}$$

and

$$\frac{dW(u\,;\,1)}{du} = -3\,\frac{E}{F}\,\frac{3u^2-2(F+E)\,u+EF}{(3u^2-3Fu+F^2)^2}\ . \tag{259}$$

It easily follows that the minimum value of $W(u\,;\,1)$ occurs at $u=E$ or $u=F$. We compute

$$W(E\,;\,1) = \frac{Q^3}{3Q^2-3Q+1} = 1-\frac{(1-Q)^3}{3Q^2-3Q+1} = G_1(Q) \tag{260}$$

where $Q=E/F=C/D$. Note that $W(E\,;\,1)\geq 0$ for $Q\in[0,1]$, and it is easily shown that $W(E\,;\,1)\geq Q$ if and only if $\frac{1}{2}\leq Q\leq 1$. We furthermore compute

$$W(F\,;\,1) = Q(3-3Q+Q^2) = 1-(1-Q)^3 = G_2(Q)\ , \tag{261}$$

and we note that $W(F\,;\,1) \geq Q$ for $Q \in [0,1]$. In Fig. 13 we have plotted the right-hand sides of (260), (261).

We next consider $W(v_0(u))$, and we shall assume that u is sufficiently far away from C, D so that $\Phi(v_0(u)) \geq D$, $\Psi(v_0(u)) \leq C$, see (250). It can be argued as in the Appendix in Section 8 that we can restrict attention to these values of u. Now $R(x\,;\,v_0(u))$ is decreasing in $x \in [C,D]$, and we compute

$$\begin{aligned} W(v_0(u)) &= \frac{R(D\,;\,v_0(u))}{R(C\,;\,v_0(u))} = \frac{3D\,v_0(u) - 3D^2 + D^3}{3C\,v_0(u) - 3C^2 + C^3} = \\ &= \frac{1}{Q}\,\frac{3f(t,Q) - 3t + 1}{3f(t,Q) - 3tQ + Q^2} \equiv G(t,Q)\ . \qquad (262) \end{aligned}$$

Here we have set $Q = E/F$, $t = u/F \in [Q,1]$ and f is as in (213) given by $tQ(1 + Q^2 - t^2)^{-1/2}$. We consider the function

$$\hat{G}(Q) = \min_{t \in [Q,1]} G(t,Q)\ , \qquad (263)$$

and we are particularly interested in the points Q_1, Q_2 for which

$$\hat{G}(Q_1) = 0\ , \qquad \hat{G}(Q_2) = Q_2\ . \qquad (264)$$

In Fig. 13 we have also included the graph of $\hat{G}(Q)$, $Q \in [Q_1,1]$. The analysis of $\hat{G}(Q)$ is facilitated by the fact that $\hat{G}(Q) = G(\hat{t}(Q),Q)$, where $\hat{t}(Q)$ is the solution t near $\frac{1}{2}(1+Q)$ of the equation

$$(1+Q)(1+Q^2) - 3t^2 = (1 + Q^2 - t^2)^{3/2}\ . \qquad (265)$$

The points Q_1, Q_2 can be found in a similar fashion as the points Q_1, Q_2 in Section 8, see (219). They are given as

$$Q_1 = \tfrac{1}{3}\,R^{3/2} = 0.442471025\,, \qquad Q_2 = \frac{3}{2W^3} - \Big(\Big(\frac{3}{2W^3}\Big)^2 - 1\Big)^{1/2} = 0.513829766\,, \qquad (266)$$

where R, W are the unique solutions in $(0,\infty)$ and $(0,(\frac{3}{2})^{1/2})$ of

$$3R^2 + 3R = 8\ , \qquad 9W^2 - 16W^3 + 9W^4 + 6W^5 + 6W^7 + 2W^9 = 24\ , \ (267)$$

respectively. We observe that $Q_2 > \frac{1}{2}$. In Fig. 11 we have plotted $R(x\,;\,v)$ as a function of x on the interval $[E/u, F/u]$ with $E = Q_2$, $F = 1$, $u = \hat{t}(Q_2)$, $v = v_0(u)$, so that $R(D\,;\,v)/R(C\,;\,v) = C/D = E/F$.

We have shown now that the quantity in (245) exceeds E/F when $Q = E/F > Q_2$ with Q_2 given in (266). Hence when $E_0/F_0 = A/B > Q_2$ we have that E_k/F_k increases to 1 and that, see (236), $\gamma_k/\|\gamma_k\|$ converges to $g^d/\|g^d\|$. As to the order of convergence, it can be shown that it is at least cubic. Indeed one has (neglecting higher orders)

$$1 - W(E\,;\,1) = (1-Q)^3\;, \qquad 1 - W(F\,;\,1) = (1-Q)^3 \tag{268}$$

and

$$1 - G(\alpha + (1-\alpha)\,Q, Q) = (9\alpha - 9\alpha^2 - 1)(1-Q)^3 \tag{269}$$

when $\alpha \in [\frac{1}{3}, \frac{2}{3}]$.

10 Consideration of the algorithms for $a = b = 1$ in the Zak transform domain

In this section we consider the case that $a = b = 1$. As is well known, Gabor analysis can be done in this case conveniently in the Zak transform domain, see [2], Sec. 4.1, [6] and [7], Sec. 1.5. For $f \in L^2(\mathbb{R})$ we define Zf by

$$(Zf)(t,\nu) = \sum_{l=-\infty}^{\infty} f(t-l)\, e^{2\pi i l \nu}\;, \qquad \text{a.e. } t,\nu \in \mathbb{R}\;. \tag{270}$$

By quasi-periodicity it is sufficient to consider Zf on $[0,1)^2$. Doing so, the mapping $f \to Zf$ maps $L^2(\mathbb{R})$ onto $L^2([0,1)^2)$ unitarily. Furthermore, when $g \in L^2(\mathbb{R})$ then $(g, a=1, b=1)$ is a Gabor frame if and only if

$$\text{ess inf}\, |Zg|^2 > 0\;, \qquad \text{ess sup}\, |Zg|^2 < \infty \tag{271}$$

while the two numbers in (271) are the best frame bounds. Finally, when $(g,1,1)$ is a Gabor frame with frame operator S and best frame bounds A, B and ϕ is a continuous function on $[A,B]$, then

$$Z(\phi(S)f)(t,\nu) = \phi(|(Zg)(t,\nu)|^2)(Zf)(t,\nu)\;, \qquad \text{a.e. } t,\nu\;. \tag{272}$$

In particular, by taking $\phi(s) = s$ we see that $Z(Sf) = |Zg|^2\, Zf$. Accordingly, $\sigma(S)$ is the spectrum of the multiplication operator $F \in L^2([0,1)^2) \to |Zg|^2\, F \in L^2([0,1)^2)$.

In the example in [1], end of Subsec. 4.1, we have a g such that $\sigma(S) = \{A, B\}$. As a consequence, we get γ_k with $\sigma(S_k) = \{A_k, B_k\}$ as well. Accordingly, the expression for the lower frame bound A_1 in [1], (4.12) is too low in general. This is exemplified by [1], (4.42) where the choice $t = \frac{1}{2}$ leads to $A_1/B_1 = 1$ while $A/B < 1$. Similar examples can be constructed for the examples in the present paper. A slight modification of the example in [1], end of Subsec. 4.1 yields an example of an S such that $\sigma(S) = [A, B]$. To this end we only need to take care that $|Zg|^2$ is a continuous function on $[0,1)^2$ that assumes the two values A, B on two sets N, M whose union differs from $[0,1)^2$ by a set of arbitrarily small measure.

The algorithm in [1] and those of the present paper can be considered conveniently in the Zak transform domain. For instance, denoting $G = Zg$ and $\Gamma_k = Z\gamma_k$, the algorithm in (4) takes the form

$$\begin{aligned}\Gamma_0 = G\,;\quad \Gamma_{k+1} &= \frac{1}{2}\,\frac{\Gamma_k}{\|\Gamma_k\|} + \frac{1}{2}\,\frac{\Gamma_k/|\Gamma_k|^2}{\|\Gamma_k/|\Gamma_k|^2\|} \\ &= \frac{1}{2}\,\frac{\Gamma_k}{\|\Gamma_k\|} + \frac{1}{2}\,\frac{1/\Gamma_k^*}{\|1/\Gamma_k^*\|}\,, \qquad k = 0, \ldots\,. \end{aligned} \tag{273}$$

Here (272) has been used with $\phi(s) = s^{-1}$.

We shall show in this section that the lower bounds on A/B that ensure convergence of the various algorithms are realistic in the sense that for general windows they cannot be lowered by much. For this we shall treat the algorithm of Section 4 in detail. We shall also demonstrate the pitfall of the coupled recursions of Section 6 when $\alpha \neq 0, \frac{1}{2}, 1$; this we do in detail for the algorithm in (4) whose Zak transform representation is given in (273). Furthermore, we shall present an example of a Gabor frame $(\gamma, a = 1, b = 1)$ with frame operator S such that the quantities $\|S\gamma\|/\|\gamma\|$, $\|S^2\gamma\|/\|S\gamma\|$ assume any value as allowed by the sharp Kantorovich inequality given in Section 7. Finally, we present some heuristics as to why one should expect the algorithms to converge under considerably less stringent conditions on A/B when the initial window g is well-behaved.

10.1 Sharpness of lower bounds

The algorithm of Section 4 for the computation of the canonical tight window g^t for the Gabor frame $(g, a = 1, b = 1)$ with quadratic convergence without inversions assumes in the Zak transform domain the following form:

$$\Gamma_0 = G\,;\quad \Gamma_{k+1} = \frac{3}{2}\,\frac{\Gamma_k}{\|\Gamma_k\|} - \frac{1}{2}\,\frac{|\Gamma_k|^2\,\Gamma_k}{\|\Gamma_k^3\|}\,, \qquad k = 0, 1, \ldots\,. \tag{274}$$

Here $G = Zg$ and $\Gamma_k = Z\gamma_k$. We show that the condition $A/B > \frac{1}{2}$, that has been shown to be sufficient for all $(\gamma_k, a = 1, b = 1)$ to be a Gabor frame with quadratic convergence of $\gamma_k/\|\gamma_k\|$ to $(ab)^{-1/2}\,g^t$, is not much more restrictive than necessary. To that end we take g such that $|Zg|^2 = A$ on a set N with measure nearly equal to 1 and $|Zg|^2 = B$ on the complementary set M in

$[0,1)^2$. Then the γ_k have $|\Gamma_k|^2 = C_k$ on N and $|\Gamma_k|^2 = D_k$ on M for certain constants C_k, D_k. Also, one has

$$||G|| = ||\Gamma_0|| \approx A^{1/2} \ , \qquad ||G^3|| = ||\Gamma_0^3|| \approx A^{3/2} \ . \tag{275}$$

Let us also assume that G is real and positive on $[0,1)^2$. Then one computes the Zak transform G^t of g^t as

$$G^t = Zg^t = Z(S^{-1/2}g) = \frac{Zg}{|Zg|} \equiv 1 \ . \tag{276}$$

The values of Γ_1 on N and M are given approximately as

$$\frac{3}{2}\ \frac{A^{1/2}}{A^{1/2}} - \frac{1}{2}\ \frac{A^{3/2}}{A^{3/2}} = 1 \ , \tag{277}$$

$$\frac{3}{2}\ \frac{B^{1/2}}{A^{1/2}} - \frac{1}{2}\ \frac{B^{3/2}}{A^{3/2}} = \frac{1}{2}\Big(\frac{B}{A}\Big)^{1/2}\Big(3 - \frac{B}{A}\Big) \ . \tag{278}$$

Note that (278) is negative when $A/B < \frac{1}{3}$ and that one must expect chaotic behaviour of the sequence of values of Γ_k on M. In particular, it may happen that Γ_k does not converge at all, or that it converges to a Γ that takes the value -1 on M. Furthermore, when $A/B > \frac{1}{3}$, the ratio of the squares of the quantities in (278) and (277) is larger than A/B if and only if $A/B \in (\frac{1}{2}, 1)$. Hence, when $\frac{1}{2} < A/B < \frac{1}{3}$, the frame bound ratio A_1/B_1 of $(\gamma_1, a=1, b=1)$ is, under the approximations in (277) and (278), less than $A_0/B_0 = A/B$. However, as one easily sees, the subsequent frames $(\gamma_k, a=1, b=1)$, $k = 1,2,...$ have frame bound ratios A_k/B_k that rapidly increase to 1 (when $\frac{1}{2} < A/B < \frac{1}{3}$).

10.2 Pitfall of coupled recursions for $\alpha \neq 0, \frac{1}{2}, 1$

We consider now the coupled recursions as we had them in Section 6 in the Zak transform domain ($a = b = 1$), and for demonstration of the pitfall for $\alpha \neq 0, \frac{1}{2}, 1$ we focus on the case (127). We then have $\Gamma_0 = \Phi_0 = G$ and for $k = 0,1,...$

$$\Gamma_{k+1} = (1-\alpha)\,\frac{\Gamma_k}{||\Gamma_k||} + \alpha\,\frac{1/\Phi_k^*}{||1/\Phi_k^*||} \ , \tag{279}$$

$$\Phi_{k+1} = \alpha\,\frac{\Phi_k}{||\Phi_k||} + (1-\alpha)\,\frac{1/\Gamma_k^*}{||1/\Gamma_k^*||} \ . \tag{280}$$

Here we have $G = Zg$, $\Gamma_k = Z\gamma_k$, $\Phi_k = Z\phi_k$ as before. Note that it follows from $g^t = \gamma_k^t = \phi_k^t$ that

$$\frac{G}{|G|} = \frac{\Gamma_k}{|\Gamma_k|} = \frac{\Phi_k}{|\Phi_k|} \ , \qquad k = 0,1,... \ . \tag{281}$$

We consider the question whether or not γ_k, ϕ_k converge to $S^{-\alpha}g/\|S^{-\alpha}g\|$, $S^{-(1-\alpha)}g/\|S^{-(1-\alpha)}g\|$. In the Zak transform domain this means that we ask whether

$$\Gamma_k \to |G|^{-2\alpha}G/\|G^{-2\alpha+1}\| \ , \qquad \Phi_k \to |G|^{-2(1-\alpha)}G/\|G^{2\alpha-1}\| \ . \tag{282}$$

As an example we take $\alpha = \frac{1}{2}, \frac{1}{4}$ and G of the form

$$G = c\chi_M + d\chi_N \ , \tag{283}$$

where $c > 0$, $d > 0$ and M, N are pairwise disjoint sets in $[0,1)^2$ with positive measures m, n such that $m + n = 1$. We then get for $k = 0, 1, ...$

$$\Gamma_k = c_k\chi_M + d_k\chi_N \ , \qquad \Phi_k = e_k\chi_M + f_k\chi_N \tag{284}$$

with the c_k, d_k, e_k, f_k given recursively by

$$c_0 = e_0 = c \ , \qquad d_0 = f_0 = d \ , \tag{285}$$

and

$$\begin{bmatrix} c_{k+1} \\ d_{k+1} \\ e_{k+1} \\ f_{k+1} \end{bmatrix} = \begin{bmatrix} c_k & e_k^{-1} & 0 & 0 \\ d_k & f_k^{-1} & 0 & 0 \\ 0 & 0 & e_k & c_k^{-1} \\ 0 & 0 & f_k & d_k^{-1} \end{bmatrix} \begin{bmatrix} (1-\alpha)/\|\Gamma_k\| \\ \alpha/\|\Phi_k^{-1}\| \\ \alpha/\|\Phi_k\| \\ (1-\alpha)/\|\Gamma_k^{-1}\| \end{bmatrix} . \tag{286}$$

Denote

$$\begin{bmatrix} \hat{c} \\ \hat{d} \\ \hat{e} \\ \hat{f} \end{bmatrix} = \begin{bmatrix} c^{1-2\alpha}/\|G^{-2\alpha+1}\| \\ d^{1-2\alpha}/\|G^{-2\alpha+1}\| \\ c^{2\alpha-1}/\|G^{2\alpha-1}\| \\ d^{2\alpha-1}/\|G^{2\alpha-1}\| \end{bmatrix} , \qquad \begin{bmatrix} c_\infty \\ d_\infty \\ e_\infty \\ f_\infty \end{bmatrix} = \lim_{k\to\infty} \begin{bmatrix} c_k \\ d_k \\ e_k \\ f_k \end{bmatrix} . \tag{287}$$

Then, with accuracy at least $0.5 \cdot 10^{-4}$ there are the following tables of results.

Table I: $\alpha = \frac{1}{2}$

c	d	n	$c_\infty - \hat{c}$	$d_\infty - \hat{d}$	$e_\infty - \hat{e}$	$f_\infty - \hat{f}$
1	2	$\frac{1}{2}$	0.0000	0.0000	0.0000	0.0000
1	10^3	$\frac{1}{2}$	0.0000	0.0000	0.0000	0.0000
1	2	10^{-3}	0.0000	0.0000	0.0000	0.0000
1	10^3	10^{-3}	0.0000	0.0000	0.0000	0.0000

Table II: $\alpha = \frac{1}{4}$

c	d	n	$\frac{c_\infty - \hat{c}}{\hat{c}}$	$\frac{d_\infty - \hat{d}}{\hat{d}}$	$\frac{e_\infty - \hat{e}}{\hat{e}}$	$\frac{f_\infty - \hat{f}}{\hat{f}}$
1	2	$\frac{1}{2}$	0.0055	−0.0039	−0.0039	0.0055
			0.8165	1.1547	1.1547	0.8165
1	10^3	$\frac{1}{2}$	0.4036	−0.0722	−0.0722	0.4036
			0.0447	1.4135	1.4135	0.0447
1	2	10^{-3}	0.0980	−0.0001	−0.1718	0.0002
			0.7073	1.0003	1.4135	0.9995
1	10^3	10^{-3}	3.4207	−0.0060	−22.0781	0.2932
			0.0316	1.0050	22.3663	0.7073

Note that the limiting windows $\gamma_\infty = \lim_{k\to\infty} \gamma_k$ and $\phi_\infty = \lim_{k\to\infty} \phi_k$ are significantly off in the case that $\alpha = \frac{1}{4}$. Also, it is definitely not so that

$$\gamma_\infty = \Psi(S)\, g / \|\Psi(S)\, g\| \ , \qquad \phi_\infty = \chi(S)/\|\chi(S)\| \tag{288}$$

with Ψ and χ functions independent of g. Indeed, in that case the ratios c_∞/d_∞ and e_∞/f_∞ should be independent of n, and they are not.

10.3 Realization of the sharp Kantorovich inequality

We shall show now that the sharp Kantorovich inequality involving $\|Sf\|/\|f\|$ and $\|S^2 f\|/\|Sf\|$, see Proposition in Section 7, can be realized with $f = \gamma \in L^2(\mathbb{R})$ and S the frame operator corresponding to $(\gamma, a = 1, b = 1)$. We let $0 < E \le F < \infty$ and we choose numbers $u, z \in [E, F]$ such that

$$\frac{EF}{(E^2 + F^2 - u^2)^{1/2}} \le z \le u \ . \tag{289}$$

We want to find γ such that the frame operator S corresponding to $(\gamma, 1, 1)$ has best frame bounds E, F while

$$\frac{\|S\gamma\|}{\|\gamma\|} = z \ , \qquad \frac{\|S^2\gamma\|}{\|S\gamma\|} = u \ . \tag{290}$$

To that end we consider $\Gamma = Z\gamma$ of the form

$$\Gamma = E^{1/2}\chi_M + F^{1/2}\chi_N + G^{1/2}\chi_P \ , \tag{291}$$

where M, N, P are pairwise disjoint sets $\subset [0,1)^2$ with measures $m = \mu(M)$, $n = \mu(N)$, $p = \mu(P)$ such that $m + n + p = 1$. By taking $G \in [E, F]$ we assure that $(\gamma, 1, 1)$ has best frame bounds E, F. Furthermore, we have

$$Z(S^j\gamma) = |Z\gamma|^{2j} Z\gamma = E^{j+1/2}\chi_M + F^{j+1/2}\chi_N + G^{j+1/2}\chi_P \ , \tag{292}$$

so that

$$\|S^j\gamma\|^2 = m\,E^{2j+1} + n\,F^{2j+1} + p\,G^{2j+1}\ . \tag{293}$$

Since the cases $z = u$ and $z = F$ are fairly easy to deal with we shall assume that $z < u$. We must choose $G \in [E, F]$ and $m, n, p \geq 0$ with $m + n + p = 1$ such that (290) holds. We choose any $G \in [E, F]$ such that

$$G^2 \in \Big[\frac{F^2 - u^2}{F^2 - z^2}\,z^2, \frac{u^2 - E^2}{z^2 - E^2}\,z^2\Big]\ ; \tag{294}$$

it follows from (289) and $u, z \in [E, F]$ that the interval in (294) contains z^2 and is contained in $[E^2, F^2]$. This choice implies that

$$z^2F^2 + z^2G^2 - F^2G^2 - u^2z^2 \leq 0\ , \qquad z^2G^2 + z^2E^2 - E^2G^2 - u^2z^2 \leq 0\ . \tag{295}$$

In addition we have from the first inequality in (289) that

$$z^2E^2 + z^2F^2 - E^2F^2 - u^2z^2 \geq 0\ . \tag{296}$$

We then take

$$m = \frac{\alpha_E}{\alpha_E + \alpha_F + \alpha_G}\ , \qquad n = \frac{\alpha_F}{\alpha_E + \alpha_F + \alpha_G}\ , \qquad p = \frac{\alpha_G}{\alpha_E + \alpha_F + \alpha_G}\ , \tag{297}$$

where

$$\alpha_E = FG(G^2 - F^2)(z^2F^2 + z^2G^2 - F^2G^2 - u^2z^2) \geq 0\ , \tag{298}$$

$$\alpha_F = GE(E^2 - G^2)(z^2G^2 + z^2E^2 - E^2G^2 - u^2z^2) \geq 0\ , \tag{299}$$

$$\alpha_G = EF(F^2 - E^2)(z^2E^2 + z^2F^2 - E^2F^2 - u^2z^2) \geq 0\ . \tag{300}$$

When there is equality in (296) the interval in (294) equals $[E^2, F^2]$ and thus allows us to take G in the interior of that interval so that we have inequality signs in (295). That is, we can take care that at least one of the α's is positive.

It can be shown that this choice of m, n, p, G is such that (290). In fact, these choices are found by requiring the three conditions on m, n, p

$$m + n + p = 1\ , \qquad \|S\gamma\|^2 = z^2\,\|\gamma\|^2\ , \qquad \|S^2\gamma\|^2 = u^2\,\|S\gamma\|^2 \tag{301}$$

while using (293), where the m, n, p are constrained by $m \geq 0$, $n \geq 0$, $p \geq 0$.

10.4 Performance for well-behaved windows

We make some heuristic comments on the performance of the algorithms for well-behaved (i.e. smooth and rapidly decaying) initial windows g. These windows have a smooth Zak transform with at least one zero in $[0,1)^2$, whence the windows γ considered in 10.3 are certainly not well-behaved.

In the algorithms of Sections 4, 5, 8, 9 we have generically in the k^{th} step a positive definite operator Z_k with $E_k = \min\sigma(Z_k)$, $F_k = \max\sigma(Z_k)$ and quantities q_k or u_k, v_k given as

$$q_k = \frac{\|Z_k\gamma_k\|}{\|\gamma_k\|}\ ; \qquad u_k = \frac{\|Z_k^2\gamma_k\|}{\|Z_k\gamma_k\|}\ , \quad v_k = \frac{\|Z_k\gamma_k\|^2}{\|\gamma_k\|\,\|Z_k^2\gamma_k\|}\ . \tag{302}$$

We have spent a considerable effort in finding a lower bound L such that $E_k/F_k > L$ implies that $E_{k+1}/F_{k+1} > E_k/F_k$. In the algorithms of Sections 4, 5 we have seen, see (81) and (108), that $E_{k+1}/F_{k+1} > E_k/F_k$ is already implied by the condition that $q_k/F_k > L$. Hence relatively large values of q_k yield good convergence behaviour of these algorithms even when we start with low values of E_k/F_k. In the algorithms of Sections 8, 9 a consideration of the relevant polynomials $P(z\,;\,u_k,v_k)$ shows that the ratio E_{k+1}/F_{k+1} is close to being largest when u_k is about halfway between E_k and F_k and v_k is of the order $\frac{1}{10}\,(1-E_k/F_k)^2$. Such u_k, v_k give rise to a $P(z\,;\,u_k,v_k)$ with values at $z = E_k, F_k, \frac{1}{2}\,(E_k+F_k)$ that are about equal while the extrema occur around $\frac{1}{2}\,(E_k+F_k) \pm \frac{1}{4}\,(F_k-E_k)$.

When we consider the Gaussian $2^{1/4}\exp(-\pi t^2)$ and we let S be the frame operator corresponding to (g,a,b), we see that $q_0 = \|Sg\|/\|g\|$ is indeed relatively large for $a=b=1$. Also, we see from the Example at the end of Section 7 that $u_0 = \|S^2g\|/\|Sg\|$ is indeed somewhere halfway between $E_0 = A$, $F_0 = B$, and that $v_0 = \|Sg\|^2/\|g\|\,\|S^2g\|$ is indeed of the order $\frac{1}{10}\,(1-E_0/F_0)^2$. Hence the Gaussian window is a very good initial window for all algorithms considered here, and we expect this to hold for more general well-behaved windows. All this is of course very speculative; extensive numerical experiments with the algorithms should give a better insight into the actual state of affairs. Some insight for the case $a=b=1$ may also be obtained from the fact that the quantities $\|Sg\|/\|g\|$, $\|S^2g\|/\|Sg\|$ can be expressed in terms of the Zak transform $G = Zg$ as

$$\left(\frac{\iint |G(t,\nu)|^6\,dt d\nu}{\iint |G(t,\nu)|^2\,dt d\nu}\right)^{1/2}, \qquad \left(\frac{\iint |G(t,\nu)|^{10}\,dt d\nu}{\iint |G(t,\nu)|^6\,dt d\nu}\right)^{1/2}, \tag{303}$$

respectively (integrations over a unit square).

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Figure captions

Fig. 1. Plot of $P(w) = w(\frac{3}{2} - \frac{1}{2}\,w^2)$ on the interval $[E/r, F/r]$ where $E = \frac{2}{3}$, $F = 1$, and (a) $r = 1$, (b) $r = \frac{2}{3}$, (c) $r = (19/27)^{1/2}$ so that $P(C = E/r) = P(D = F/r)$.

Fig. 2. Plot of $f_1(Q) = Q^{-1}(\frac{3}{2} - \frac{1}{2}\,Q^{-1})^2$ for $\frac{1}{3} \leq Q \leq 1$ and of $f_2(Q) = \frac{27}{4}\,(Q^{1/2} + Q)^2/(1 + Q^{1/2} + Q)^3$ for $0 \leq Q \leq 1$.

Fig. 3. Plot of $P(w) = w(2 - w)$ on the interval $[E/q, F/q]$ where $E = \frac{2}{3}$, $F = 1$, and (a) $q = 1$, (b) $q = \frac{2}{3}$, (c) $q = \frac{5}{6}$ so that $P(C = E/q) = P(D = E/q)$.

Fig. 4. Plot of $f_1(Q) = (2Q - 1)\,Q^{-2}$ for $\frac{1}{2} \leq Q \leq 1$ and of $f_2(Q) = 4Q(1 + Q)^{-2}$ for $0 \leq Q \leq 1$.

Fig. 5. Illustration of the inequalities (150) with $A = \frac{1}{2}$, $B = 1$.

Fig. 6. Plot of $S(x\,;\, v = 1) = \frac{15}{8}\,x - \frac{5}{4}\,x^3 + \frac{3}{8}\,x^5$ on the interval $[(A/u)^{1/2}, (B/u)^{1/2}]$ where $A = 0.36$, $B = 1$ and (a) $u = 1$, (b) $u = 0.36$.

Fig. 7. Plot of $S(x\,;\, v = 0.65) = \frac{15}{8}\,xv - \frac{5}{4}\,x^3 + \frac{3}{8}\,x^5$ on the interval $[C, D] = [(A/u)^{1/2}, (B/u)^{1/2}]$ where $A = 0.36$, $B = 1$ and $u = 0.70$.

Fig. 8. Plot of $S(x\,;\, v = 0.9159) = \frac{15}{8}\,xv - \frac{5}{4}\,x^3 + \frac{3}{8}\,x^5$ on the interval $[C, D] = [(A/u)^{1/2}, (B/u)^{1/2}]$ where $A = 0.36$, $B = 1$ and $u = 0.64$.

Fig. 9. Plot of $H_1(Q)$ and $H_2(Q)$ of (208) and (210), respectively, for $Q = A/B \in [0, 1]$, and of $\hat{H}(Q)$ of (216) and $Q \in [Q_1, 1]$ with Q_1 given in (220).

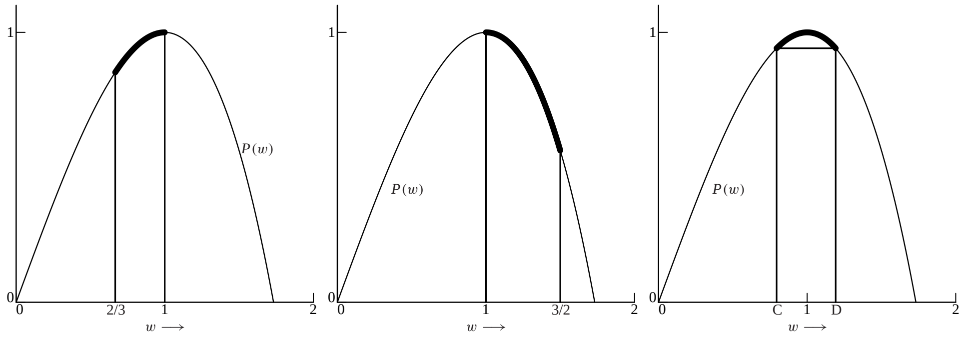
Fig. 10. Plot of $R(x\,;\, v = 1) = 3x - 3x^2 + x^3$ on the interval $[E/u, F/u]$ where $E = \frac{1}{2}$, $F = 1$ and (a) $u = 1$, (b) $u = \frac{1}{2}$.

Fig. 11. Plot of $R(x\,;\, v) = 3xv - 3x^2 + x^3$ on the interval $[C, D] = [E/u, F/u]$ where $E = 0.513829766$, $F = 1$, $u = 0.768286186$, $v = 0.814787712$.

Fig. 12. Plot of $R(x\,;\, v) = 3xv - 3x^2 + x^3$ on the interval $[C, D] = [E/u, F/u]$ where $E = \frac{1}{2}$, $F = 1$, $u = \frac{3}{4}$, $v = \frac{26}{27}$.

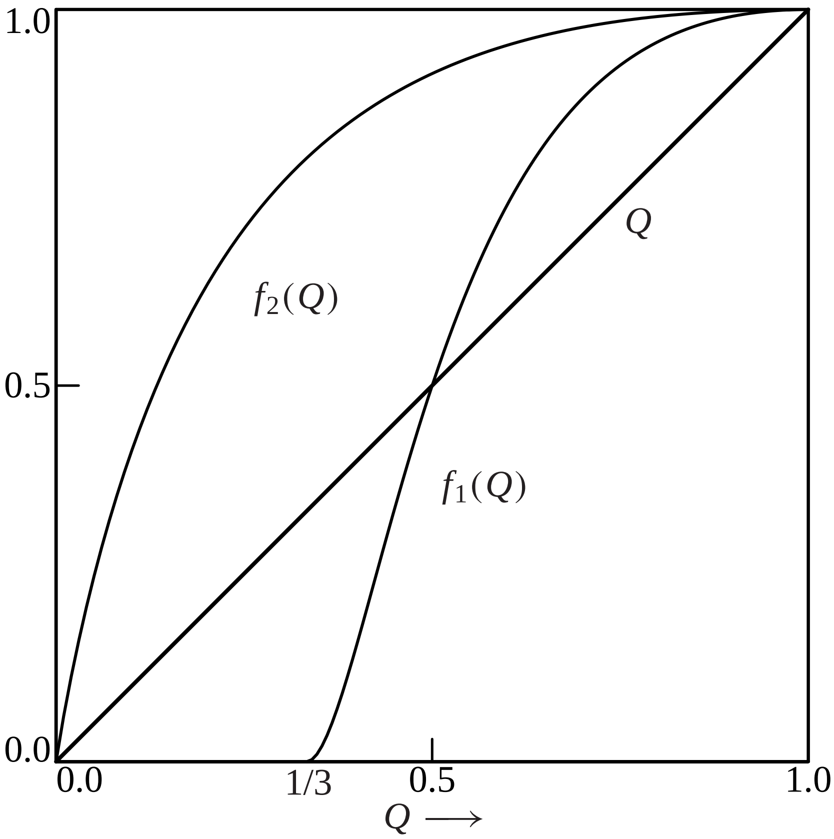
Fig. 13. Plot of $G_1(Q)$ and $G_2(Q)$ of (260) and (261), respectively, for $Q = E/F \in [0, 1]$, and of $\hat{G}(Q)$ with $\hat{G}$ given in (263) and $Q \in [Q_1, 1]$ with Q_1 given in (266).

figure 1



(a) (b) (c)

figure 2



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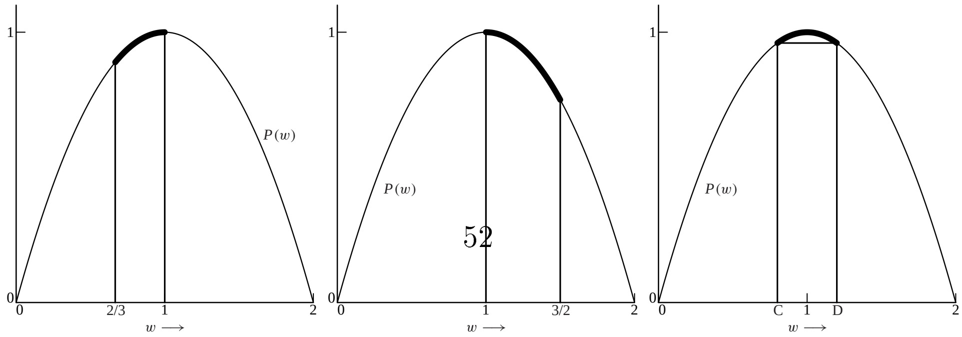
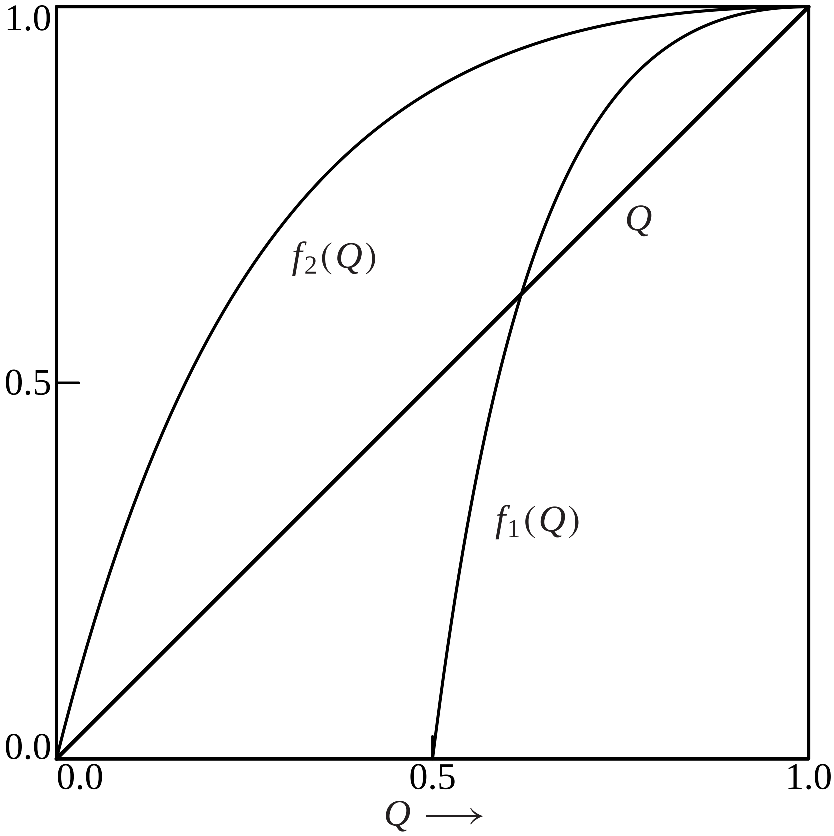


(a) (b) (c)

figure 4



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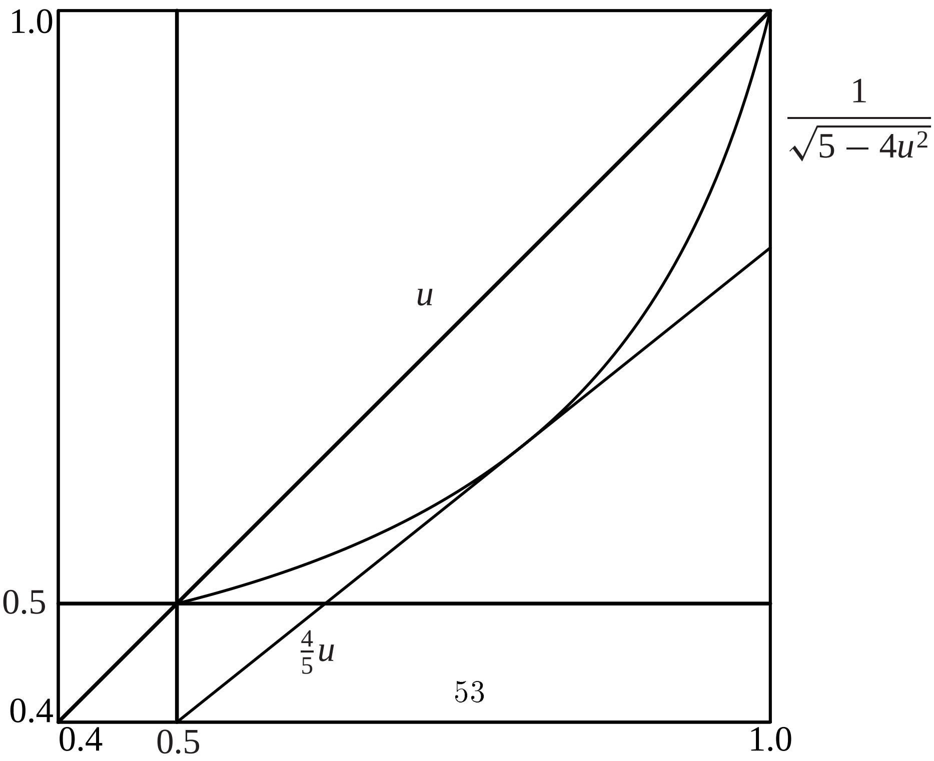

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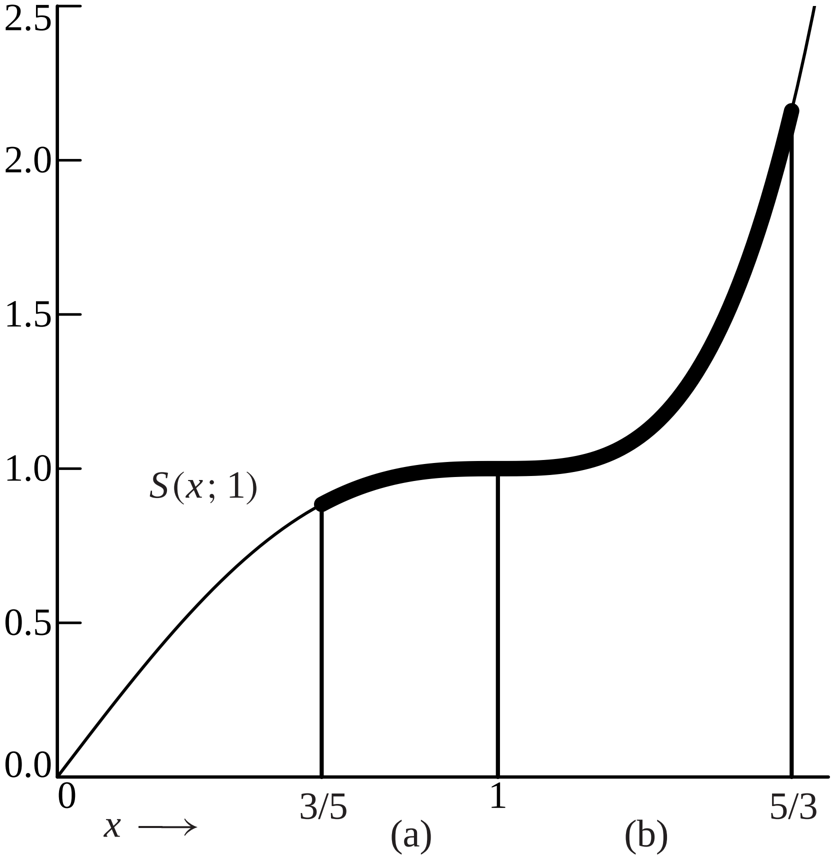

figure 7

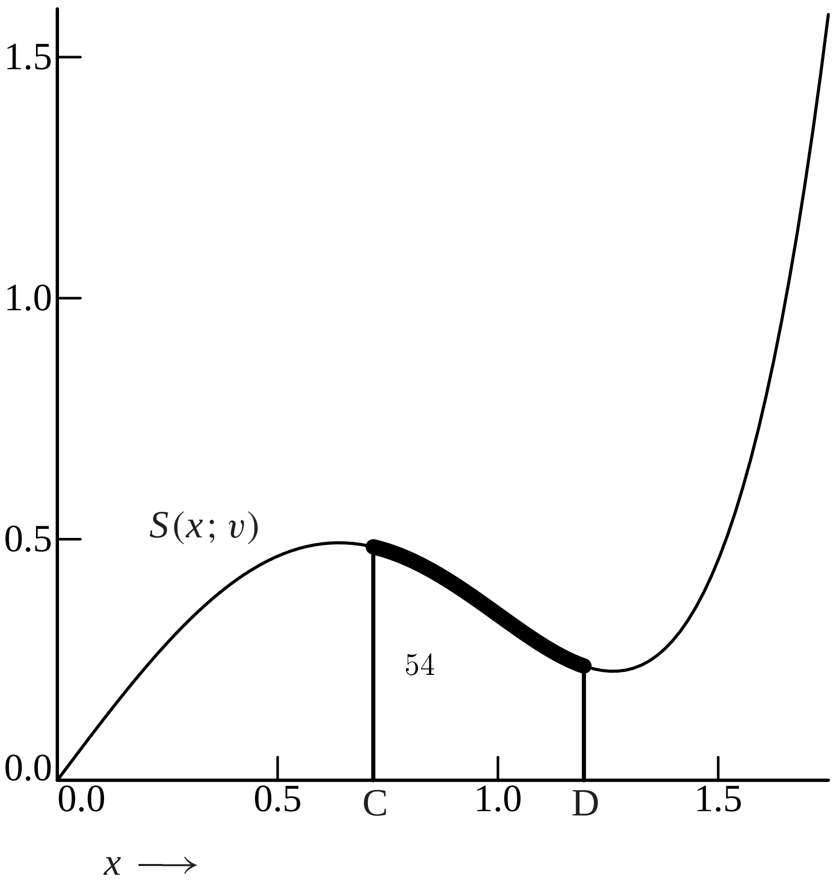

figure 8

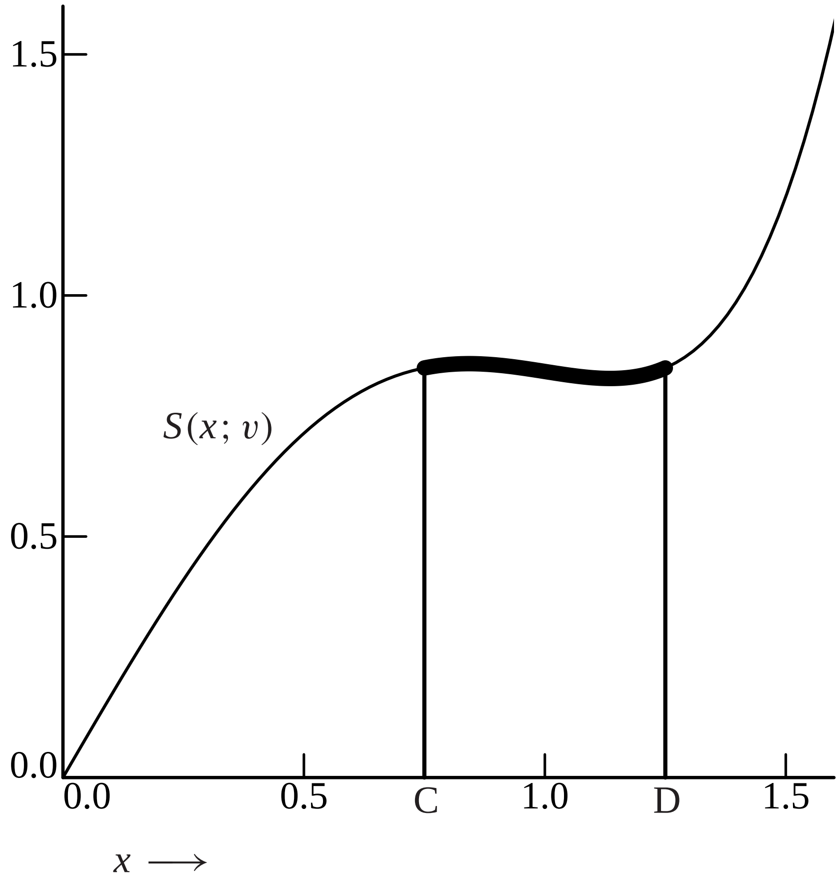

figure 9

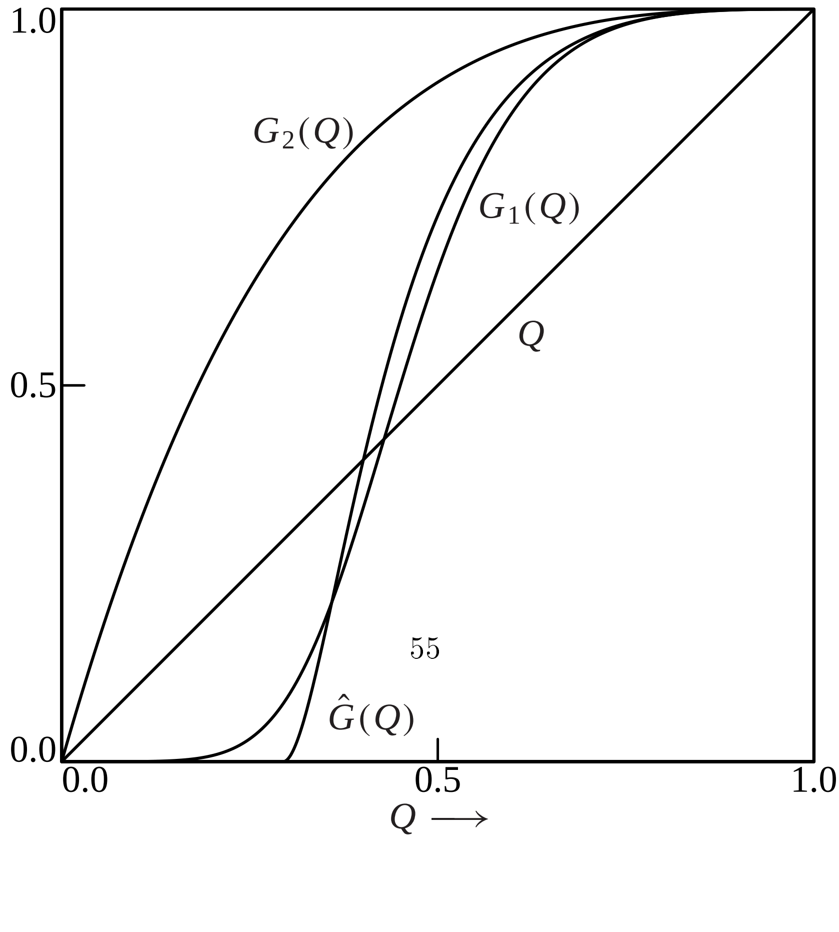

figure 10

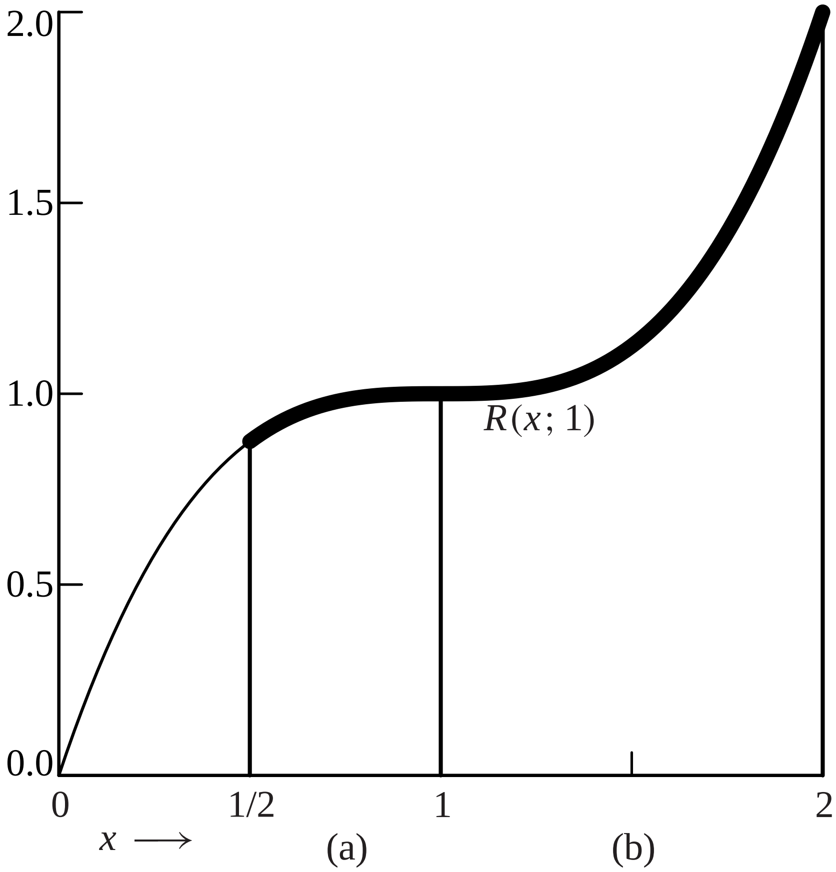

figure 11

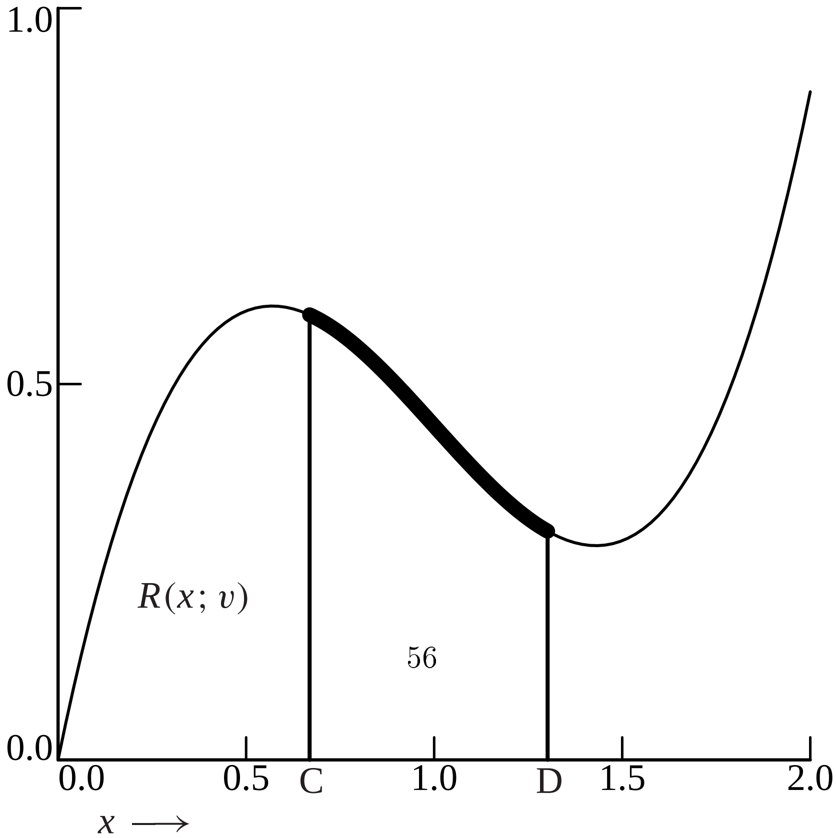

figure 12

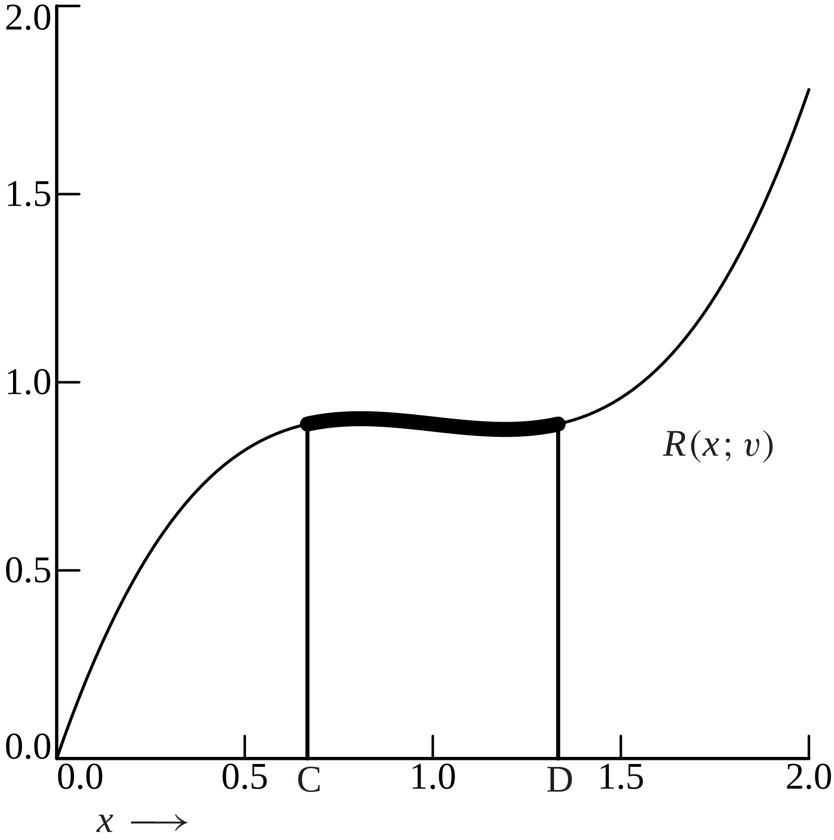

figure 13