Moment inequalities for the discrete-time bulk service queue

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Abstract

For the discrete-time bulk service queueing model, the mean and variance of the steady-state queue length can be expressed in terms of moments of the arrival distribution and series of the zeros of a characteristic equation. In this paper we investigate the behaviour of these series. In particular, we derive bounds on the series, from which bounds on the mean and variance of the queue length follow. We pay considerable attention to the case in which the arrivals follow a Poisson distribution. For this case, additional properties of the series are proved leading to even sharper bounds. The Poisson case serves as a pilot study for a broader range of distributions.

keywords: bulk service queue, discrete-time, zeros, moment inequalities

1 Introduction and motivation

We consider a discrete-time queueing model with bulk service as defined by the recursion

$$X_{n+1} = \max\{X_n - s, 0\} + A_n. \tag{1}$$

Here, time is assumed to be slotted, X_n denotes the queue length at the beginning of slot n, A_n denotes the number of newly arriving customers during slot n, and s denotes the fixed number of customers that can be served during one slot. The numbers of new customers arriving per slot are assumed to be i.i.d. according to a discrete random variable A with $a_j = P(A = j)$, and probability generating function (pgf)

$$A(z) = \sum_{j=0}^{\infty} a_j z^j, \tag{2}$$

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that we assume to be analytic in an open set containing the closed unit disk $|z| \leq 1$. In particular, the random variable A has finite moments of all order. The model described by (1) has a wide range of applications, including ATM switching elements [3], data transmission over satellites [20], high performance serial busses [16], and cable access networks [8].

Let X denote the random variable following the stationary distribution of the Markov chain defined by the recursion (1), with

$$x_j = P(X = j) = \lim_{n\to\infty} P(X_n = j), \quad j = 0, 1, 2, \ldots, \tag{3}$$

that exists under the assumption that $E(A) < s$. It follows that the pgf of X is given by (see e.g. [3])

$$X(z) = \frac{A(z)\sum_{j=0}^{s-1} x_j(z^s - z^j)}{z^s - A(z)}, \tag{4}$$

as an analytic function in an open set containing the closed unit disk $|z| \leq 1$. The expression (4) is of indeterminate form, but the s unknowns $x_0, \ldots, x_{s-1}$ can be determined by consideration of the zeros of the denominator in (4) that lie in the closed unit disk (see e.g. [2, 21]). By applying Rouché's theorem on a curve $|z| = 1+\epsilon$ where $\epsilon > 0$, and using $E(A) < s$, it can be shown that there are exactly s of these zeros. Thus, by analyticity, the numerator of $X(z)$ should vanish at each of the zeros, yielding s equations. One of the zeros equals 1, and leads to a trivial equation. However, the normalization condition $X(1) = 1$ provides an additional equation. Using l'Hôpital's rule, this condition is found to be

$$s - E(A) = \sum_{j=0}^{s-1} x_j(s - j), \tag{5}$$

which equates two expressions for the mean unused service capacity.

The s roots of $A(z) = z^s$ in $|z| \leq 1$ are denoted by $z_0 = 1, z_1, \ldots, z_{s-1}$. By writing $\sum_{j=0}^{s-1} x_j(z^s - z^j)$ as $c(z-1)\prod_{k=1}^{s-1}(z - z_k)$ with c a constant, and using (5) to derive the value of c, it follows that

$$\prod_{k=1}^{s-1} \frac{z - z_k}{1 - z_k} = \frac{1}{s - \mu_A}\sum_{j=0}^{s-1} x_j \frac{z^s - z^j}{z - 1}, \tag{6}$$

so that (4) can be written as

$$X(z) = \frac{A(z)(s - \mu_A)}{z^s - A(z)}(z - 1)\prod_{k=1}^{s-1}\frac{z - z_k}{1 - z_k}, \quad |z| \leq 1. \tag{7}$$

Expectations and variances are denoted throughout by appending the involved random variable to μ and σ^2, respectively. Accordingly,

$$E(A) = \mu_A = A'(1); \quad \sigma_A^2 = A''(1) + A'(1) - (A'(1))^2, \tag{8}$$

and similarly for X. In cases where the random variable is a complicated expression, we use the E-notation rather that the μ-notation. Explicit expressions for the mean and variance of the steady-state queue length can be obtained by evaluating derivatives of $X(z)$ at $z = 1$.

There holds (see e.g. [14])

$$\begin{aligned} \mu_X &= \frac{\sigma_A^2}{2(s-\mu_A)} + \frac{1}{2}\mu_A - \frac{1}{2}(s-1) + \sum_{k=1}^{s-1} \frac{1}{1-z_k}, && (9) \\ \sigma_X^2 &= \sigma_A^2 + \frac{A'''(1) - s(s-1)(s-2)}{3(s-\mu_A)} + \frac{A''(1) - s(s-1)}{2(s-\mu_A)} \\ &\quad + \left(\frac{A''(1)-s(s-1)}{2(s-\mu_A)}\right)^2 - \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2}. && (10) \end{aligned}$$

The series

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k}, \quad \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2}, \tag{11}$$

will be called the μ-series and σ^2-series, respectively. Evidently, both series are real since the zeros z_k are either real or come in conjugate pairs. For $s = 1$ both series equal zero and one obtains the exact expressions for μ_X and σ_X^2 from (9) and (10), respectively. We will therefore consider $s \geq 2$.

In [8] the bounds

$$\frac{\sigma_A^2}{2(s-\mu_A)} + \frac{\mu_A}{2} \leq \mu_X \leq \frac{\sigma_A^2}{2(s-\mu_A)} + \frac{\mu_A}{2} + \frac{\min(\mu_A, s-1)}{2} \tag{12}$$

have been shown to hold for the μ-series. The proof of these bounds was based on the representation

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} = \frac{1}{2}(s-1) + \sum_{j=0}^{s-1} x_j \frac{j(s-j)}{2(s-\mu_A)}, \tag{13}$$

and identity (5). In this paper we give further bounds on μ_X and extend the methods of [8] to provide bounds for σ_X^2.

There is no real need for deriving bounds when the distribution of A is fully specified. In fact, for many cases the roots of $z^s = A(z)$ can be easily determined by some numerical procedure (see Chaudhry et al. [4]), so that one can find the mean and variance of X explicitly through (9) and (10). Moreover, it is also possible to give explicit, analytic expressions of the Spitzer type (that is, involving the power series coefficients of $A^l(z)$ for $l = 1, 2, \ldots$, see [1], formulas (8)-(9)) for both μ_X and σ_X^2 and for the boundary probabilities $x_j,\ j = 0, 1, \ldots$, see [10]. Furthermore, for a wide class of allowed distributions, among which the Poisson case of Sec. 6, we present in [10] an explicit Fourier series representation for the roots $z_k,\ k = 0, 1, \ldots, s$. However, the bounds have added value as compared to these exact approaches in that they give more intuitive insight in the behavior of the performance characteristics and some bounds can be used for back-of-the-envelope computations. Furthermore, when one has only knowledge of the first two or three moments of A, the exact approaches cannot be applied while the presented bounds retain their value. Additionally, we identify the distributions of A for which the bounds are attained, which gives additional insight into the behaviour of the estimated values.

The model defined by (1) fits into the framework of the $G/G/1$ queue. That is, one can think of X_n as being the sojourn time of the n-th customer, with A_{n-1} its service requirement, and s the deterministic and integer-valued interarrival time between customer n and $n+1$. This model is also referred to as the discrete-time $D/G/1$ queue (see e.g. Servi [18]). Ever since the publication of Kingman [11], a vast literature on bounding waiting time characteristics for the $G/G/1$ queue has been developed. Daley et al. [6] give a comprehensive treatment of most of this research. Simple bounds for the mean and variance of the waiting time can be constructed by observing that the deterministic interarrival times s belong to the class of Increasing Failure Rate (see e.g Daley et al. [6], p. 200). For the mean, we then obtain a lower and an upper bound (see Kleinrock [12], (2.51), and Kingman [11], respectively) that, translated to the current setting, read

$$\frac{\sigma_A^2}{2(s-\mu_A)} + \frac{\mu_A}{2} \le \mu_X \le \frac{\sigma_A^2}{2(s-\mu_A)} + \mu_A. \tag{14}$$

The right-hand side of (14) is known as Kingman's upper bound. For μ_X the bound in (12) is slightly sharper. We will show that it is relatively easy to further sharpen the Kingman upper bound, although the gain is marginal.

Largely paralleling the approach used for the mean, bounds for the variance were derived as well. The lower bound in Daley et al. [6] and the upper bound derived by Fainberg [9] yield for the $D/G/1$ queue

$$-\frac{s^2}{4} + \frac{1}{12} \le \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \le -\frac{1}{12}(s-\mu_A)^2 + \frac{1}{12}, \tag{15}$$

where, for reasons of brevity, we have given the bounds on the σ^2-series. Together with (10), they yield bounds on σ_X^2. We strengthen these bounds on σ_X^2 and further derive some new bounds on σ_X^2 that will be shown to be very sharp.

In this paper we extend and complete the approach adopted in [8] and derive relatively simple bounds for the μ-series and the σ^2-series. Here simple means bounds that require knowledge on the arrival distribution by at most the first three moments. We do so by representing the moment series (11) in terms of random variables related to the idle time of the system. One of the features of this study is the fact that we extend the bounding techniques to a discrete setting. In doing so, we obtain simple bounds on the mean and variance of the steady-state queue length that are sharper then comparably simple bounds (14) and (15). Finally, we show that the bounds can be further strengthened by combining them with specific properties of the moment series (11) leading to even sharper bounds. This is done for the Poisson distribution, which serves as a sort of pilot study for other distributions. It is noteworthy to mention that for the $E_k/G/1$ queue, Daley [5] also proves properties of roots on a particular curve in order to derive bounds.

In Sec. 2 we give a detailed account of the main results, a comparison with the bounds (14) and (15), along with an overview of the paper.

2 Overview and results

For the discrete-time $D/G/1$ queue, the stationary distribution of the length of the idle periods, I, is completely determined by the probabilities $x_0, \ldots, x_s$. That is, once a customer has a sojourn that is less than s, the slots remaining until the arrival of the next customer remain idle, i.e.

$$P(I=j) = \frac{x_{s-j}}{\sum_{i=0}^{s} x_i}, \quad j = 0, 1, \ldots, s. \tag{16}$$

For a convenient presentation of our results we now define two auxiliary random variables Y and Z that are closely related to I and take values in $\{0, 1, \ldots, s\}$ according to

$$P(Y = j) = \frac{x_j}{\sum_{i=0}^{s} x_i}, \quad P(Z = j) = \frac{(s - j)x_j}{s - \mu_A}, \quad j = 0, 1, \ldots, s, \tag{17}$$

and $P(Y = j) = P(Z = j) = 0$, $j = s + 1, s + 2, \ldots$. Note that Y represents both X conditional on $X \leq s$ and $s - I$. Further note that the kth moment of Z can be expressed in terms of the first $k + 1$ moments of I. For example, $\mu_Z = s - E(I^2)/\mu_I$. The random variables Y and Z are studied in detail in Sec. 3. There holds, in particular,

$$\mu_Y \leq \mu_A; \quad 0 \leq \mu_Z \leq s - 1, \tag{18}$$

with equality in the first inequality if and only if A is concentrated on $\{0, 1, \ldots, s\}$. In Sec. 3 we also present representations for the μ-series and σ^2-series in terms of Y and Z (Equations (32-34) and (35-37), respectively). From these representations, one can obtain various inequalities, as well as insight into the matter when equality occurs in these.

We show the following bounds on the μ-series in Sec. 4.

Theorem 2.1. *(i) We have*

$$\sum_{k=1}^{s-1} \frac{1}{1 - z_k} \geq \frac{1}{2}(s - 1) + \frac{1}{2}\mu_A - \frac{\sigma_A^2}{2(s - \mu_A)}, \tag{19}$$

and there is equality if and only if A is concentrated on $\{0, 1, \ldots, s\}$.
(ii) Define $f : [0, s) \to [0, \infty)$ by

$$f(\mu) = \frac{1}{2}(s - 1) + \frac{1}{2}\mu - \frac{\langle\mu\rangle - \langle\mu\rangle^2}{2(s - \mu)}, \tag{20}$$

where we have defined $\langle\mu\rangle = \mu - \lfloor\mu\rfloor$ and $\lfloor\mu\rfloor$ = largest integer $\leq \mu$. Then we have

$$\sum_{k=1}^{s-1} \frac{1}{1 - z_k} \leq f(\mu_A), \tag{21}$$

and there is equality if and only if A is concentrated on $\{j, j + 1\}$ with $j = 0, 1, \ldots, s - 2$ or A is concentrated on $\{s - 1, s, s + 1, \ldots\}$.

In Sec. 4 we present somewhat sharper forms of Thm. 2.1 that explicitly involve μ_Y and σ_Y^2. The result in Thm. 2.1(i) presents a sharpening of the first inequality in (12) in case that $\sigma_A^2 \leq \mu_A(s - \mu_A)$. The inequality in Thm. 2.1(ii) is a refinement of the second inequality in (12) in which the discrete nature of the involved random variables is taken into account. In Fig. 1, we have plotted the graphs of both $f(\mu)$ and $\mu \to \frac{1}{2}(s - 1) + \frac{1}{2}\min\{\mu, s - 1\}$ for $s = 5$. As one sees, the graph of f hangs down from the second graph as a sort of guirlande with nodes at all integers $\mu = 0, 1, \ldots, s - 1$.

In Sec. 5 we show the following result.

Theorem 2.2. *We have*

$$\frac{-s^2}{3(4 - \mu_A/s)} + \frac{1}{12} \leq \sum_{k=1}^{s-1} \frac{z_k}{(1 - z_k)^2} \leq -\frac{1}{12}(s - \mu_A)^2 + \frac{1}{12}. \tag{22}$$

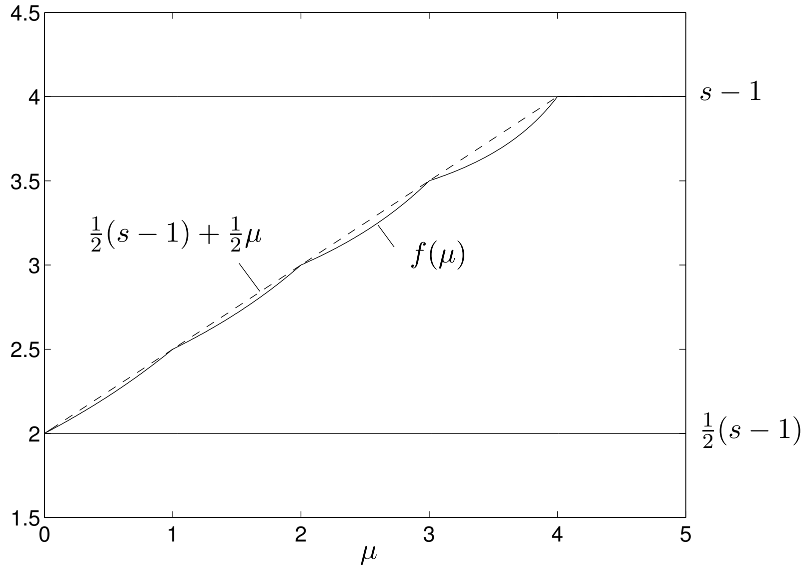

Figure 1: Universal bounds for the μ-series, $s = 5$.

Theorem 2.2 should be considered as a counterpart of the bounds in (12) for μ_X. The lower bound in (22) is far sharper than the one in (15).

In Sec. 5 we present a more precise and sharper result in which the σ^2-series is bounded in terms of μ_Y and σ_Y^2, and from which one can infer the cases of equality in (22). This requires a result, communicated to us by E. Verbitskiy, on the extreme values of the third central moment of a random variable taking all real values between 0 and s, whose mean and variance are prescribed. The bounds in Thm. 2.2 disregard the discrete nature of the involved random variable, and, indeed, there is again a guirlande phenomenon that is detailed in Sec. 5. The bounds in (22) can be sharpened somewhat by using (37). We then have

$$-\frac{1}{9}(s-\frac{1}{2})^2 \le \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \le 0, \tag{23}$$

and this improves the bounds in (22) when $\mu_A \uparrow s$.

Theorem 2.3. *(i) We have*

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \le \frac{A'''(1) - s(s-1)(s-2)}{3(s-\mu_A)} + \frac{A''(1) - s(s-1)}{2(s-\mu_A)} + \left(\frac{A''(1) - s(s-1)}{2(s-\mu_A)}\right)^2, \tag{24}$$

and there is equality if and only if A is concentrated on $\{0, 1, \ldots, s\}$.
(ii) Defining $h : [0, s) \to [0, \infty)$ by

$$h(\mu) = \begin{cases} 0, & 0 \le \mu \le 2, \\ \mu(\mu-1)(\mu-2), & \mu > 2, \end{cases} \tag{25}$$

there holds

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \ge \frac{h(\mu_A) - h(s)}{3(s-\mu_A)} + \frac{A''(1) - s(s-1)}{2(s-\mu_A)} + \left(\frac{A''(1) - s(s-1)}{2(s-\mu_A)}\right)^2. \tag{26}$$

Here σ_A^2 and μ_A must be constrained according to

$$\sigma_A^2 \leq (s-\mu_A)(\mu_A+2s-4). \tag{27}$$

There is equality in (26) if and only if A is concentrated on $\{0,1,2\}$ or on $\{j\}$ with $j = 2,\ldots,s-1$.

The proof of this result uses the representation (10) together with $\sigma_X^2 \geq \sigma_A^2$ for Thm. 2.3(i), and representation (35) in conjunction with Jensen's inequality and $\mu_Y \leq \mu_A$ for Thm. 2.3(ii).

In Sec. 6 we study in considerable detail the case that A is distributed according to the Poisson distribution. Among other things, it is shown that both the μ-series and σ^2-series increase in $\mu_A \in [0,s)$ in the Poisson case, which can be exploited to derive the following theorems:

Theorem 2.4. *For A distributed according to the Poisson distribution, i.e. $A(z) = e^{\lambda(z-1)}$, that satisfies $\lambda < s$, the corresponding μ-series can be bounded as*

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} \quad \geq \quad \frac{1}{2}(s-1) + m_1(\lambda), \tag{28}$$

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} \quad \leq \quad \frac{1}{2}(s-1) + \frac{1}{2}\lambda - \frac{\langle\lambda\rangle - \langle\lambda\rangle^2}{2(s-\lambda)}, \tag{29}$$

where $m_1(\lambda) = \max\{\frac{\tau}{2} + \frac{\tau}{2(s-\tau)} \mid 0 \leq \tau \leq \lambda\}$.

Theorem 2.5. *For A distributed according to the Poisson distribution, i.e. $A(z) = e^{\lambda(z-1)}$, that satisfies $\lambda < s$, and when Cond. (27) holds, the corresponding σ^2-series can be bounded as*

$$\sum_{k=0}^{s-1} \frac{z_k}{(1-z_k)^2} \quad \geq \quad m_2(\lambda), \tag{30}$$

$$\sum_{k=0}^{s-1} \frac{z_k}{(1-z_k)^2} \quad \leq \quad -\frac{1}{12}(s-\lambda)^2 - \frac{1}{2}\lambda + \frac{s(s+2\lambda)}{12(s-\lambda)^2}, \tag{31}$$

where $m_2(\lambda) = \max\{-\frac{1}{12}(s-\tau)^2 - \frac{1}{2}\tau + \frac{s(s+2\tau)}{12(s-\tau)^2} - \frac{\tau}{s-\tau}(\tau - \frac{2}{3}) \mid 0 \leq \tau \leq \lambda\}$.

Note that the functions $m_1(\lambda)$ and $m_2(\lambda)$ are strictly increasing for $\lambda \in [0, s-\sqrt{s}]$ and $\lambda \in [0, \lambda_2(s)]$, respectively, where $\lambda_2(s)$ is a point close to $s - (6(s^2 - \frac{1}{2}s))^{1/3}$.

In Sec. 7 we present examples of distributions A to illustrate the bounds on the μ-series and σ^2-series. For the Poisson case, we use the bounds in Thms. 2.4 and 2.5. For other distributions, we employ for the μ-series the bounds in Thm. 2.1 together with $\frac{1}{2}(s-1)$ as an overall lower bound. For the σ^2-series we employ the bounds in Thm. 2.3, where the lower bound (26) is only used when condition (27) is satisfied. If not, we use the overall lower bound $-\frac{1}{9}(s-\frac{1}{2})^2$, and the overall upper bound 0.

The bounds on the μ-series and σ^2-series provide more insight in the behaviour of the model. However, we are primarily interested in bounds on μ_X and σ_X^2. In Sec. 8 we present the bounds on μ_X and σ_X^2 for the same distributions as in Sec. 7. These bounds will be shown to be sharp, both for the low and high load situations.

3 Representations of the μ-series and σ^2-series

In this section we take a closer look at the random variables Y and Z as defined by (17), and we show that they give rise to the representations

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} = \frac{1}{2}(s-1) + \frac{1}{2}\mu_Y - \frac{\sigma_Y^2}{2(s-\mu_Y)} \tag{32}$$

$$= \frac{s(s-1) - Y''(1)}{2(s-\mu_Y)} = \frac{s^2 - E(Y^2)}{2(s-\mu_Y)} - \frac{1}{2} \tag{33}$$

$$= \frac{1}{2}(s-1) + \frac{1}{2}\mu_Z, \tag{34}$$

for the μ-series, and

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} = \frac{Y'''(1) - s(s-1)(s-2)}{3(s-\mu_Y)} + \frac{Y''(1) - s(s-1)}{2(s-\mu_Y)} + \left(\frac{Y''(1) - s(s-1)}{2(s-\mu_Y)}\right)^2 \tag{35}$$

$$= \frac{1}{4}\left(\frac{s^2 - E(Y^2)}{s-\mu_Y}\right)^2 - \frac{1}{3}\frac{s^3 - E(Y^3)}{s-\mu_Y} + \frac{1}{12} \tag{36}$$

$$= -\frac{1}{12}(s-\mu_Z)^2 - \frac{1}{3}\sigma_Z^2 + \frac{1}{12}, \tag{37}$$

for the σ^2-series.

We note that $Y(z)$ has degree s and that the roots of $Y(z) = z^s$ are precisely $z_0 = 1, z_1, \ldots, z_{s-1}$. The latter statement follows from the fact that the numerator $A(z)\sum_{j=0}^{s} x_j(z^s - z^j)$ at the right-hand side of (4) has to cancel the s zeros of the denominator $z^s - A(z)$ within the closed unit disk $|z| \leq 1$ (when $A(0) = 0$ some trivial modifications are required). As a consequence, the random variables Y and A give rise to the same μ-series and σ^2-series while $P(Y > s) = 0$. It follows from (5) that

$$s - \mu_A = (s - \mu_Y)P(X \leq s), \tag{38}$$

and thus $\mu_Y \leq \mu_A$ with equality if and only if $P(X > s) = 0$. From the process definition we see furthermore that

$$A = X = Y \quad \Leftrightarrow \quad P(A > s) = 0. \tag{39}$$

We now derive the representations (32-34) and (35-37). The representations (32), (35) follow from the observation that A and Y yield the same μ-series and σ^2-series, and the fact that $P(Y > s) = 0$, so that (32), (35) result from consideration of the process definition and application of (9), (10) with Y instead of A. The derivation of (33) and (36) follows from straightforward rewriting.

Finally, we show the representations (34), (37). The former follows from

$$\frac{s^2 - E(Y^2)}{s-\mu_Y} = \frac{1}{s-\mu_Y}\sum_{j=0}^{s}(s^2 - j^2)P(Y=j) = \frac{1}{(s-\mu_Y)P(X \leq s)}\sum_{j=0}^{s}(s+j)(s-j)x_j$$

$$= \frac{s-\mu_A}{(s-\mu_Y)P(X \leq s)}E(s+Z) = s + \mu_Z, \tag{40}$$

where we have used the definitions of Y and Z together with (38). Similarly, we have

$$\frac{s^3 - E(Y^3)}{s - \mu_Y} = E(s^2 + sZ + Z^2) = s^2 + s\mu_Z + E(Z^2), \tag{41}$$

and (37) follows after some administration.

We shall now be concerned with the question how certain concentration properties of Y (and Z) are reflected by corresponding properties of A. The result given below is vital in Secs. 4, 5 for settling cases of equality in our theorems.

Definition 3.1. *Let B be a random variable with values in $\{0, 1, \ldots\}$ and let $\mathcal{S}$ be a subset of $\{0, 1, \ldots\}$. We say that B is concentrated on $\mathcal{S}$ when $P(B \notin \mathcal{S}) = 0$.*

According to this definition Y is concentrated on $\{0, 1, \ldots, s\}$ while Z is concentrated on $\{0, 1, \ldots, s-1\}$. Moreover, we have the following result.

Lemma 3.2. *(i) Let $j = 0, 1, \ldots, s-1$. Then Y concentrated on $\{j\}$ $\Leftrightarrow$ A concentrated on $\{j\}$.*

(ii) Let $j = 0, 1, \ldots, s-2$. Then Y concentrated on $\{j, j+1\}$ $\Leftrightarrow$ A concentrated on $\{j, j+1\}$.

(iii) Y concentrated on $\{s-1, s\}$ $\Leftrightarrow$ A concentrated on $\{s-1, s, s+1, \ldots\}$.

(iv) Y concentrated on $\{0, s\}$ $\Leftrightarrow$ Z concentrated on $\{0\}$ $\Leftrightarrow$ A concentrated on $\{0, s, 2s, \ldots\}$.

For reasons of brevity we omit the proof of Lemma 3.2. It follows by a careful analysis from the process definition.

4 Bounds for the μ-series

In this section we prove (the claims associated with) Thm. 2.1. From the process definition in (1) we see that $\mu_X \geq \mu_A$. So from (9) it follows that

$$\sum_{k=1}^{s-1} \frac{1}{1 - z_k} \geq \frac{1}{2}(s-1) + \frac{1}{2}\mu_A - \frac{\sigma_A^2}{2(s - \mu_A)}, \tag{42}$$

with equality if and only if A is concentrated on $\{0, \ldots, s\}$. We further see from representation (34) that

$$\sum_{k=1}^{s-1} \frac{1}{1 - z_k} \geq \frac{1}{2}(s-1), \tag{43}$$

and there is equality if and only if A is concentrated on $\{0, s, 2s, \ldots\}$. Next we consider the representation (32) in which the μ-series is expressed in terms of the mean and variance of Y. Observe that for any random variable B concentrated on $\{0, \ldots, s\}$ with mean μ the smallest value of σ_B^2 is given by $\langle\mu\rangle - \langle\mu\rangle^2$ (as defined in Thm. 2.1), and is assumed when

$$P(B = \lfloor\mu\rfloor) = 1 - \langle\mu\rangle, \quad P(B = \lfloor\mu\rfloor + 1) = \langle\mu\rangle. \tag{44}$$

The function f as defined by (20) is strictly increasing in $\mu \in [0, s-1]$, and constant, $s-1$, for $\mu \in [s-1, s)$. We thus have

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} \leq f(\mu_Y) \leq f(\mu_A) \leq \frac{1}{2}(s-1) + \frac{1}{2}\min\{\mu_A, s-1\}. \tag{45}$$

In the first inequality there is equality if and only if $\mu_Y = 0, 1, \ldots, s-1$ and Y is concentrated on $\{\mu_Y\}$, or μ_Y is non-integer and Y is concentrated on $\{\lfloor \mu_Y \rfloor, \lfloor \mu_Y \rfloor + 1\}$. In the second inequality there is equality if and only if $\mu_Y < s-1$ and $\mu_A = \mu_Y$, or $s-1 \leq \mu_Y < s$. In the third inequality there is equality if and only if $\mu_A = 0, 1, \ldots, s-2$ or $\mu_A \geq s-1$. The inequalities (42-43) together with the second inequality in (45) prove Thm. 2.1. And also the case of equality in the third inequality in (45) is settled now: it holds if and only if A is concentrated on $\{j\}$ with $j = 0, 1, \ldots, s-2$ or A is concentrated on $\{s-1, s, s+1, \ldots\}$.

5 Bounds for the σ^2-series

In this section we prove Thms. 2.2-2.3. We first derive bounds for the σ^2-series that depend on the mean and the variance of Y, from which we derive bounds that depend on μ_A. We consider the representation (36) in which the σ^2-series is expressed in terms of μ_Y, σ_Y^2 and $E(Y^3)$. We are interested in the smallest and largest value of (36) under the condition that μ_Y and σ_Y^2 take prescribed values. For convenience we assume Y takes, not necessarily integer, values between 0 and s, and that $0 < \mu_Y < s$. Under these assumptions, we have

$$0 < \theta := \frac{\mu_Y}{s} < 1, \quad 0 \leq \omega := \frac{\sigma_Y^2}{\mu_Y(s-\mu_Y)} \leq 1, \tag{46}$$

and equality in the last inequality occurs if and only if Y is concentrated on $\{0, s\}$. We start by presenting a lemma.

Lemma 5.1. *Let D be a random variable with values in $[-c, d]$, where $c \geq 0$, $d \geq 0$, and assume that $\mu_D = 0$, $\sigma_D^2 = \sigma^2$ is fixed. Then the minimum and maximum value of $E(D^3)$ are given by*

$$\frac{\sigma^4}{c} - c\sigma^2, \quad d\sigma^2 - \frac{\sigma^4}{d}, \tag{47}$$

respectively. The minimum and maximum value occur when D is concentrated on $\{-c, \sigma^2/c\}$ and $\{-\sigma^2/d, d\}$, respectively.

The proof of this result follows from Thm. 2.4 in [13], as was kindly communicated to us by E. Verbitskiy.

We next present three results from which Thm. 2.2 follows. In Thms. 5.2-5.4 the random variable Y is allowed to take non-integer values in $[0, s]$ and θ, ω are as in (46).

Theorem 5.2. *We have*

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \quad \geq \quad -\frac{1}{12}s^2(1-\theta+\theta\omega)^2 + \frac{1}{12} - \frac{1}{3}s^2(1-\omega)\theta\omega, \tag{48}$$

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \quad \leq \quad -\frac{1}{12}s^2(1-\theta+\theta\omega)^2 + \frac{1}{12}. \tag{49}$$

The lower bound is assumed if and only if Y is concentrated on

$$\{0, \mu_Y + \frac{\sigma_Y^2}{\mu_Y}\} = \{0, s\omega + s(1-\omega)\theta\}, \tag{50}$$

and the upper bound is assumed if and only if Y is concentrated on

$$\{\mu_Y - \frac{\sigma_Y^2}{s-\mu_Y}, s\} = \{s(1-\omega)\theta, s\}. \tag{51}$$

Theorem 5.3. *We have*

$$-\frac{s^2}{3(4-\theta)} + \frac{1}{12} \le \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \le -\frac{1}{12}s^2(1-\theta)^2 + \frac{1}{12}. \tag{52}$$

The lower bound is assumed if and only if Y is concentrated on the set in (50) with $\omega = (3-\theta)/(4-\theta)$, and the upper bound is assumed if and only if Y is concentrated on the set in (51) with $\omega = 0$.

Theorem 5.4. *We have*

$$-\frac{1}{9}s^2 + \frac{1}{12} \le \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \le \frac{1}{12}. \tag{53}$$

The lower bound is assumed if and only if Y is concentrated on the set in (50) with $\omega = (3-\theta)/(4-\theta) \to \frac{2}{3}$ and $\theta \uparrow 1$, and the upper bound is assumed if and only if Y is concentrated on the set in (51) with $\omega = 0$ and $\theta \uparrow 1$.

Proofs. It is convenient to combine the proofs of the above results. We rewrite representation (36) using

$$E(Y^2) = \sigma_Y^2 + \mu_Y^2, \quad E(Y^3) = m_Y^3 + 3\mu_Y\sigma_Y^2 + \mu_Y^3, \tag{54}$$

where $m_Y^3 = E((Y-\mu_Y)^3)$. This yields

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} = -\frac{1}{12}(s-\mu_Y)^2 - \frac{1}{2}\sigma_Y^2 + \left(\frac{\sigma_Y^2}{2(s-\mu_Y)}\right)^2 + \frac{m_Y^3}{3(s-\mu_Y)} + \frac{1}{12}. \tag{55}$$

We then use Lemma 5.1 with $D = Y - \mu_Y$, $c = \mu_Y$, $d = s - \mu_Y$ and some administration, to see that

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \ge -\frac{1}{12}\left(s-\mu_Y+\frac{\sigma_Y^2}{s-\mu_Y}\right)^2 + \frac{1}{12} - \frac{s\sigma_Y^2}{3(s-\mu_Y)}\left(1-\frac{\sigma_Y^2}{\mu_Y(s-\mu_Y)}\right), \tag{56}$$

$$\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \le -\frac{1}{12}\left(s-\mu_Y+\frac{\sigma_Y^2}{s-\mu_Y}\right)^2 + \frac{1}{12}, \tag{57}$$

with equality if and only if Y is concentrated on $\{0, \mu_Y + \sigma_Y^2/\mu_Y\}$ and on $\{\mu_Y - \sigma_Y^2/(s-\mu_Y), s\}$, respectively. The inequalities in (56) and (57) can be written succinctly, in terms of θ, ω as (48) and (49), respectively, and this shows Thm. 5.2.

For fixed $\theta \in (0,1)$, the minimum of (48) equals $-s^2/(4(3-\theta)) + 1/12$ and occurs uniquely at $\omega = (3-\theta)/(4-\theta)$. The maximum of (49) equals $-s^2(1-\theta)^2/12 + 1/12$ and occurs uniquely at $\omega = 0$. This shows Thm. 5.3.

Finally, the minimum of the first member of (52) equals $-\frac{1}{9}s^2 + 1/12$ and occurs uniquely when $\omega = (3-\theta)/(4-\theta) \to 2/3$ and $\theta \uparrow 1$, while the maximum of the third member of (52) equals 1/12 and occurs uniquely when $\omega = 0$ and $\theta \uparrow 1$. This then also shows Thm. 5.4. □

The bounds in Thm. 2.2 are in terms of μ_A. They can be obtained straightforwardly from Thm. 5.3 by noting that $\mu_Y \le \mu_A$ and the fact that the first member in (52) is decreasing in θ while the third member in (52) is increasing in θ. A corresponding result for the inequalities in (48) and (49) is unlikely to hold since the relation between σ_Y^2 and σ_A^2 seems much more awkward. Note once more that $Y = A$ when A is concentrated on $\{0, 1, \ldots, s\}$, and then Thms. 5.2-5.4 hold with Y replaced by A.

In Thms. 5.2-5.4 the discrete nature of the random variables has been disregarded. Accordingly, the two bounds in (48) and (49) are achieved by some integer-valued Y if and only if

$$\mu_Y + \frac{\sigma_Y^2}{\mu_Y} = s\omega + s(1-\omega)\theta \in \mathbb{Z}, \tag{58}$$

$$\mu_Y - \frac{\sigma_Y^2}{s-\mu_Y} = s(1-\omega)\theta \in \mathbb{Z}, \tag{59}$$

respectively. In general, when these integrality conditions are not met, slight improvement of the bounds in Thm. 5.2 can be achieved by invoking an appropriate discrete version of Lemma 5.1 in Formula (55). This then gives rise to two guirlanded (μ, σ)- or (θ, ω)-surfaces, with contact curves described by (58) and (59), just as we had a guirlanded graph in Thm. 2.1 for the upper bound for the μ-series (since the lower bound is constant and achievable by Y concentrated on $\{0, s\}$, no guirlande phenomenon occurs for the lower bound of the μ-series).

A slight improvement of the upper bound in (52) can be obtained by observing that $\sigma_Y^2 \ge \langle \mu_Y \rangle - \langle \mu_Y \rangle^2$ when Y is integer-valued. Thus we find, see (57), in a similar fashion as in Sec. 4 for the μ-series

$$\begin{aligned}\sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} &\le -\frac{1}{12}\left(s - \mu_Y + \frac{\langle \mu_Y \rangle - \langle \mu_Y \rangle^2}{s-\mu_Y}\right)^2 + \frac{1}{12} \\ &= -\frac{1}{12}(2s-1-2f(\mu_Y))^2 + \frac{1}{12} \\ &\le -\frac{1}{12}(2s-1-2f(\mu_A))^2 + \frac{1}{12} \ =: \ g(\mu_A) \ \le \ 0,\end{aligned} \tag{60}$$

with f as in Thm. 2.1.

We may also observe the bounds

$$-\frac{1}{9}(s-\frac{1}{2})^2 \le \sum_{k=1}^{s-1} \frac{z_k}{(1-z_k)^2} \le 0, \tag{61}$$

and their simple proofs from the representation (37) in terms of W. Indeed, consider an arbitrary random variable C concentrated on $\{0, 1, \ldots, s-1\}$ with mean μ and variance σ^2.

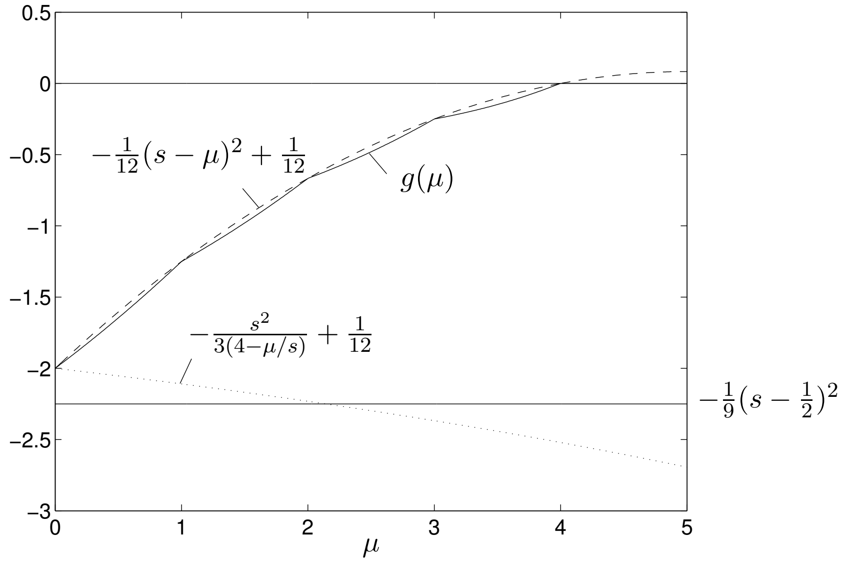

Figure 2: Universal bounds for the σ^2-series, $s = 5$.

When μ is fixed, the minimum value of

$$-\frac{1}{12}(s-\mu)^2 - \frac{1}{3}\sigma^2 + \frac{1}{12} \tag{62}$$

occurs when C is concentrated on $\{0, s-1\}$ and equals

$$-\frac{1}{9}(s-\frac{1}{2})^2 + \frac{1}{4}(\mu - \frac{1}{3}(s-2))^2 \geq -\frac{1}{9}(s-\frac{1}{2})^2. \tag{63}$$

Similarly, the maximum value of (62) occurs when C is concentrated on $\{\mu\}$ or on $\{\lfloor\mu\rfloor, \lfloor\mu\rfloor+1\}$ (μ non-integer) and equals

$$-\frac{1}{12}(s-\mu)^2 - \frac{1}{3}(\langle\mu\rangle - \langle\mu\rangle^2) + \frac{1}{12} \leq 0, \tag{64}$$

with equality if and only if $\mu = s-1$.

In Fig. 2 we have plotted the bounds in (52), (60) and (61) for $s = 5$ and $0 \leq \mu_A < s$. Observe that the graph of g hangs down from $-\frac{1}{12}(s-\mu)^2 + \frac{1}{12}$ as a guirlande with nodes at all integers $\mu = 0, 1, \ldots, s-1$.

We conclude this section by proving Thm. 2.3. Theorem 2.3(i) follows at once from (10) and the fact that $\sigma_X^2 \geq \sigma_A^2$, with equality if and only if A is concentrated on $\{0, 1, \ldots, s\}$. As to Thm. 2.3(ii) we start from the representation (35) in which we write

$$Y'''(1) = E(Y(Y-1)(Y-2)) = E(h(Y)), \tag{65}$$

with h given in (25). In (65) the last identity follows from the fact that Y is integer-valued. The function h is convex on $[0,\infty)$ and strictly convex on $[2,\infty)$, whence by Jensen's inequality there holds

$$E(h(Y)) \geq h(E(Y)) = h(\mu_Y), \tag{66}$$

with equality if and only if Y is concentrated on $\{0,1,2\}$ or Y is concentrated on $\{j\}$ with $j = 2,3,\ldots,s-1$. Next we observe from convexity of h that the function $(h(\mu)-h(s))/(s-\mu)$ is strictly decreasing in $\mu \in [0,s)$. Hence, as $\mu_Y \le \mu_A$, we have

$$\frac{Y'''(1) - s(s-1)(s-2)}{3(s-\mu_Y)} \ge \frac{h(\mu_Y) - h(s)}{3(s-\mu_Y)} \ge \frac{h(\mu_A) - h(s)}{3(s-\mu_A)}, \tag{67}$$

with equality if and only $\mu_A = \mu_Y$. We next turn to the quantity

$$\left(\frac{s(s-1) - Y''(1)}{2(s-\mu_Y)}\right)^2 - \frac{s(s-1) - Y''(1)}{2(s-\mu_Y)}, \tag{68}$$

that occurs at the right-hand side of (35). We note from (33) that

$$\frac{s(s-1) - Y''(1)}{2(s-\mu_Y)} \ge \frac{1}{2}(s-1). \tag{69}$$

Furthermore, we have from (33) and Thm. 2.1(i) that

$$\frac{s(s-1) - Y''(1)}{2(s-\mu_Y)} \ge \frac{1}{2}(s-1) + \frac{1}{2}\mu_A - \frac{\sigma_A^2}{2(s-\mu_A)} = \frac{s(s-1) - A''(1)}{2(s-\mu_A)}. \tag{70}$$

Denoting the far left-hand side of (70) by x_Y and the far right-hand side of (70) by x_A we have $x_Y \ge \frac{1}{2}(s-1)$ and $x_A \ge \frac{1}{2}(s-1)$, whence

$$(x_Y^2 - x_Y) - (x_A^2 - x_A) = (x_Y - x_A)(x_Y + x_A - 1) \ge 0, \tag{71}$$

whenever $x_A \ge -\frac{1}{2}(s-1)+1$. This latter condition can be worked out to yield constraint (27). Hence, under this constraint, (24) follows. The cases with equality easily follow from what has been said in connection with occurrence of equality in (66) and (67).

6 Special results for the Poisson distribution

In case one has, or wants to use, more knowledge on the distribution of A, sharper bounds can be derived. For example, the Kingman upper bound in case of the discrete-time $D/G/1$ queue (14) can be sharpened by using the quantity $P(A < s)$ to give (see [6], (3.11))

$$\sum_{k=1}^{s-1} \frac{1}{1-z_k} \le \frac{1}{2}(s-1) + \frac{1}{2}\mu_A - \frac{1}{2}(P(A<s)^{-1} - 1)(s-\mu_A). \tag{72}$$

In this section we show for the case that A is distributed according to a Poisson distribution, i.e.

$$a_j = e^{-\lambda}\frac{\lambda^j}{j!}, \quad j = 0,1,\ldots; \qquad A(z) = e^{\lambda(z-1)}, \tag{73}$$

that the μ-series and σ^2-series are monotone functions of μ_A, which facilitates a sharpening of the lower bounds for both series. For that, we consider the curve on which the roots of $A(z) = z^s$ lie, and prove some properties for all points on this curve.

We have

$$\mu_A = \sigma_A^2 = \lambda; \qquad A^{(k)}(1) = \lambda^k, \tag{74}$$

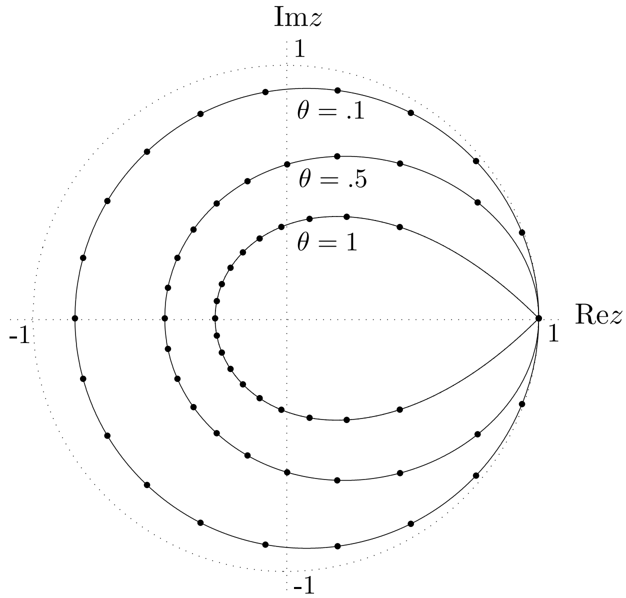

Figure 3: $\mathcal{S}_\theta$ for $\theta = .1, .5, 1$. The roots $z_0, \ldots, z_{19}$ ($s = 20$) are indicated as dots.

with $A^{(k)}(1)$ the k-th derivative of $A(z)$ evaluated at $z = 1$. The roots $z_0, z_1, \ldots, z_{s-1}$ now occur on, what we have called, the generalized Szegö curve

$$\mathcal{S}_\theta = \{z \in \mathbb{C} \mid |z| \leq 1,\ |z| = |e^{\theta(z-1)}|\}, \quad \theta := \lambda/s, \tag{75}$$

see [10, 19]. In Fig. 3 some examples of $\mathcal{S}_\theta$ are plotted.

We now introduce two useful parametrizations of $\mathcal{S}_\theta$. First, we represent a point z on $\mathcal{S}_\theta$ as

$$z = r_\theta(\varphi)e^{i\varphi}, \quad 0 \leq \varphi \leq 2\pi, \tag{76}$$

where $0 \leq r_\theta(\varphi) \leq 1$. In (75) and (76) we allow $\theta = 1$, i.e. $\lambda = s$. There holds

$$r_\theta(\varphi) = \exp\{\theta(r_\theta(\varphi)\cos\varphi - 1)\}, \quad 0 \leq \varphi \leq 2\pi. \tag{77}$$

A second parametrization of $\mathcal{S}_\theta$ is obtained by solving for $\alpha \in [0, 2\pi]$ the equation

$$ze^{\theta(1-z)} = e^{i\alpha}. \tag{78}$$

Denoting the solution of (78) by $z_\theta(\alpha)$, we have the following Fourier series representation, see [10] where this is done for more general A as well,

$$z_\theta(\alpha) = \sum_{l=1}^{\infty} e^{-l\theta}\frac{(\theta l)^{l-1}}{l!}e^{il\alpha}, \quad \alpha \in [0, 2\pi]. \tag{79}$$

This allows convenient computation of all z_k's, since

$$z_k = z_{k,\theta} = z_\theta(2\pi k/s), \quad k = 0, 1, \ldots, s-1. \tag{80}$$

Using the parametrizations of $\mathcal{S}_\theta$, we derive the following results (for the proofs we refer to Appendix A):

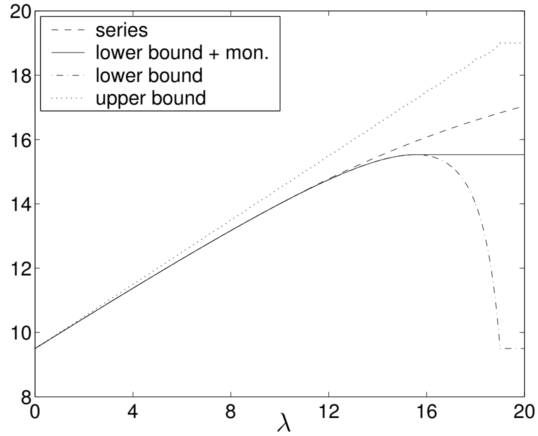

Figure 4: The μ-series, Poisson, $s = 20$.

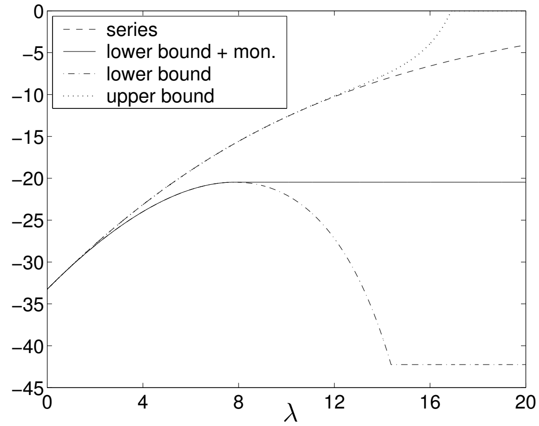

Figure 5: The σ^2-series, Poisson, $s = 20$.

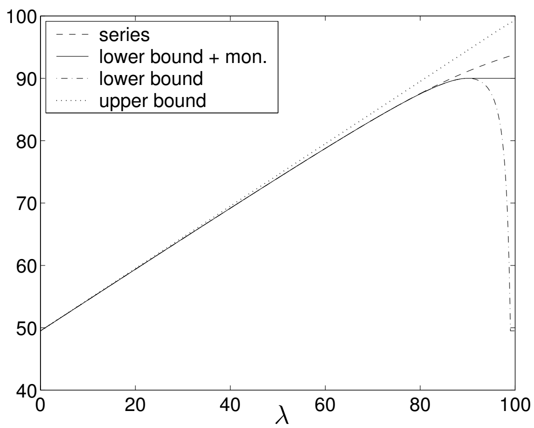

Figure 6: The μ-series, Poisson, $s = 100$.

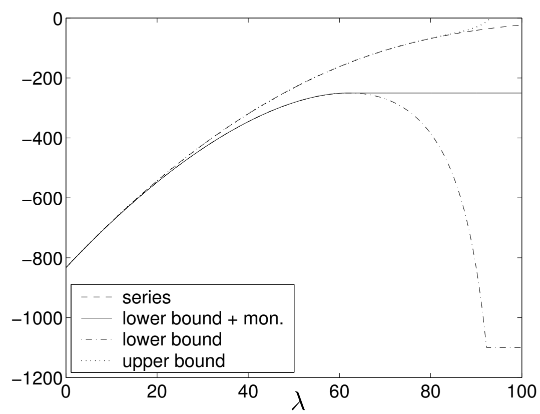

Figure 7: The σ^2-series, Poisson, $s = 100$.

Lemma 6.1. *For any z on the generalized Szegö curve $\mathcal{S}_\theta$, it holds that*

$$Re\left[\frac{z}{(1-z)(1-\theta z)}\right] \leq 0, \tag{81}$$

with equality if and only if $z \to 1$.

Lemma 6.2. *The μ-series in case of $A(z) = e^{\theta s(z-1)}$ is increasing in $\theta \in [0, 1)$.*

Lemma 6.3. *The σ^2-series in case of $A(z) = e^{\theta s(z-1)}$ is increasing in $\theta \in [0, 1)$.*

Combining the monotonicity of the μ-series and σ^2-series, as proven in Lemma 6.2 and Lemma 6.3, and the bounds in Thms. 2.1 and 2.3 yield the proofs of Thms. 2.4 and 2.5.

Fig. 4 and Fig. 6 display the μ-series and the bounds in Thm. 2.4 for $s = 20$ and $s = 100$, respectively, with $\frac{1}{2}(s-1)$ as an overall lower bound. The more general lower bound arising from Thm. 2.1 is also plotted. Fig. 5 and Fig. 7 display the σ^2-series and the bounds in Thm. 2.5 for $s = 20$ and $s = 100$, respectively, where $-\frac{1}{9}(s-\frac{1}{2})^2$ holds as an overall lower bound and as the lower bound when condition (27), i.e. $\lambda \leq 19.64$ for $s = 20$ and $\lambda \leq 99.66$ for $s = 100$, is not met. The more general lower bound arising from Thm. 2.3 is also plotted.

In Figs. 4-7 it is nicely demonstrated that the lower bound is sharpened substantially when monotonicity can be proven. We conjecture that monotonicity of the μ-series and σ^2-series can be shown for distributions of A other than Poisson, e.g. the binomial and geometric distribution.

7 Numerical examples of bounds on the μ-series and σ^2-series

In this section we first present some more examples of distributions of A to illustrate the behaviour of the μ-series and σ^2-series and the sharpness of the bounds in Thms. 2.1 and 2.3. In [7] we present some more examples that emphasize particular properties of the μ-series and σ^2-series. Moreover, we study in [7] the geometric properties of the series.

The μ-series and σ^2-series can be computed numerically by finding the roots $z_1, \ldots, z_{s-1}$, which is feasible in the cases below. We display the μ-series and σ^2-series, with corresponding lower and upper bounds, for a number of parametrically given A in which μ_A covers the whole range of permitted values below $s = 5$. For these cases we also exhibit explicitly the quantities μ_A, σ_A^2 and $A''(1)$, $A'''(1)$, as required in the various bounds.

For the μ-series we employ the bounds in Thm. 2.1 together with $\frac{1}{2}(s-1)$ as an overall lower bound. For the σ^2-series we employ the bounds in Thm. 2.3, where the lower bound (26) is only used when condition (27) is satisfied. If not, we use the overall lower bound $-\frac{1}{9}(s-\frac{1}{2})^2$, and the overall upper bound 0.

Example 7.1. Let A be uniformly distributed on $\{0, 1, \ldots, n-1\}$ so that

$$A(z) = \frac{1}{n}(1 + z + \ldots + z^{n-1}) = \frac{1}{n}\frac{z^n - 1}{z - 1}. \tag{82}$$

We have

$$\mu_A = \frac{1}{2}(n-1), \quad \sigma_A^2 = \frac{1}{12}(n^2 - 1), \tag{83}$$

and for $k = 2, 3, \ldots$

$$A^{(k)}(1) = \frac{1}{k+1}(n-1)(n-2) \cdot \ldots \cdot (n-k). \tag{84}$$

Fig. 8 and Fig. 9 display the μ-series and σ^2-series for $s = 10$, $\mu_A \in [0, s - \frac{1}{2})$, i.e. $1 \leq n \leq 2s$. As a curiosity we mention that the values of the μ-series and σ^2-series at $n = s, s+1$ are identical, viz. $\frac{2}{3}(s-1)$ and $-\frac{1}{18}(s-1)(s+2)$, respectively. Condition (27) is satisfied for $\mu_A \leq 8.83$.

Example 7.2. Take a symmetric binomially distributed A,

$$a_j = \frac{1}{2^{n-1}}\binom{n-1}{j}, \quad j = 0, 1, \ldots, n-1; \quad a_j = 0, \quad j = n, n+1, \ldots, \tag{85}$$

so that

$$A(z) = \left(\frac{1+z}{2}\right)^{n-1}. \tag{86}$$

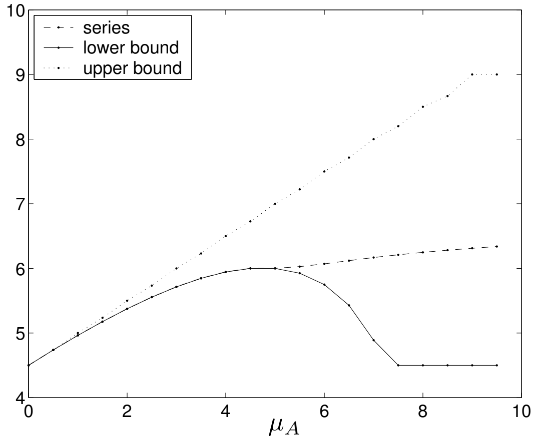

Figure 8: The μ-series, Ex. 7.1, $s = 10$.

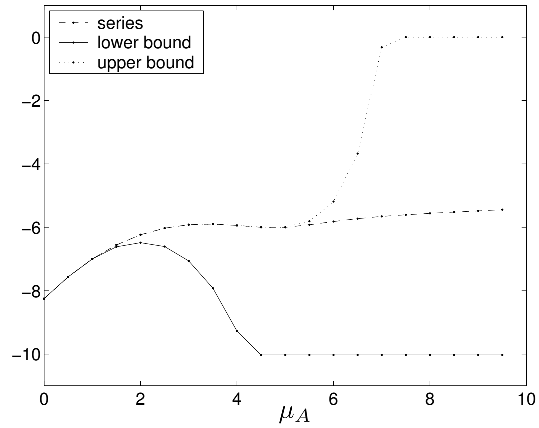

Figure 9: The σ^2-series, Ex. 7.1, $s = 10$.

We now have

$$\mu_A = \frac{1}{2}(n-1), \quad \sigma_A^2 = \frac{1}{4}(n-1), \tag{87}$$

and for $k = 2, 3, \ldots$

$$A^{(k)}(1) = \left(\frac{1}{2}\right)^k (n-1)(n-2) \cdot \ldots \cdot (n-k). \tag{88}$$

Fig. 10 and Fig. 11 display the μ-series and σ^2-series for $s = 10$, $\mu_A \in [0, s - \frac{1}{2}]$, i.e. for $1 \le n \le 2s$, and we observe a qualitatively similar behaviour for the two series as in the Poisson case, see Sec. 6. Condition (27) is satisfied for $\mu_A \le 9.81$.

Example 7.3. Take $a_0 = 1/2$, $a_{n-1} = 1/2$ where $n \in [1, 2s]$, so that

$$A(z) = \frac{1}{2} + \frac{1}{2} z^{n-1}. \tag{89}$$

We have

$$\mu_A = \frac{1}{2}(n-1), \quad \sigma_A^2 = \frac{1}{4}(n-1)^2, \tag{90}$$

and for $k = 2, 3, \ldots$

$$A^{(k)}(1) = \frac{1}{2}(n-1)(n-2) \cdot \ldots \cdot (n-k). \tag{91}$$

Fig. 12 and Fig. 13 display the μ-series and σ^2-series for $s = 10$, $\mu_A \in [0, s - \frac{1}{2}]$, i.e. for $1 \le n \le 2s$. Note that the μ-series starts decreasing as a function of $n-1$ around $n - 1 = s(2 - \sqrt{2})$, which is well before $n - 1 = s$. Condition (27) is satisfied for $\mu_A \le 7.57$.

8 Numerical examples of bounds on μ_X and σ_X^2

We now present bounds on μ_X and σ_X^2 for the Poisson case and the examples given in Sec. 7, to see how well these bounds perform as approximations to the actual values. We denote the lower and upper bound on μ_X by $\check{\mu}_X$ and $\hat{\mu}_X$, respectively, and similarly for σ_X^2. These bounds are simply the addition of the bounds on the μ-series and σ^2-series as described at the beginning of Sec. 7. For comparison we also display the known bounds given by Expressions

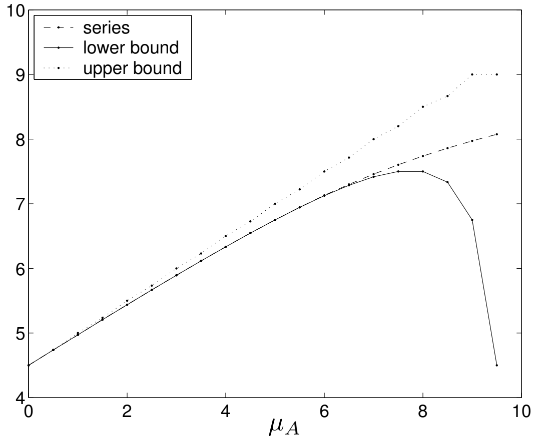

Figure 10: The μ-series, Ex. 7.2, $s = 10$.

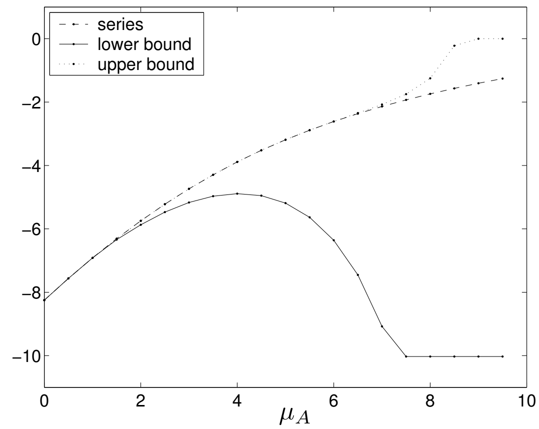

Figure 11: The σ^2-series, Ex. 7.2, $s = 10$.

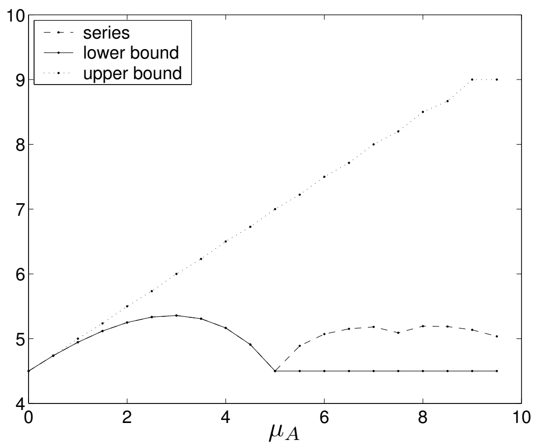

Figure 12: The μ-series, Ex. 7.3, $s = 10$.

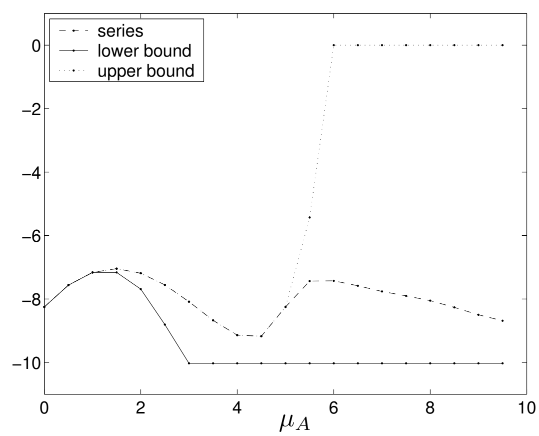

Figure 13: The σ^2-series, Ex. 7.3, $s = 10$.

(14) and (15). The actual values μ_X and σ_X^2 are computed numerically by finding the roots $z_1, \ldots, z_{s-1}$, which is feasible in the cases below.

Table 1 displays the bounds on μ_X for various load values $\theta := \mu_A/s$ and $s = 5, 10$. For $s = 5$, the bounds are sharp, irrespective of the load values θ. For low load values the lower bound on the μ-series is extremely sharp, as we have seen in Sec. 7, leading to sharp bounds on μ_X as well. For higher load values, the bounds on the μ-series tend to be less sharp. However, comparing with the μ-series, the other terms in (9) containing the first two moments of A get more dominant for an increasing load θ, which gradually diminishes the influence of the μ-series. This makes that the bounds on μ_X are sharp, even asymptotically for $\theta \uparrow 1$. We further note that while the series in Sec. 7 have similar order of magnitudes for all four examples, the mean and variance of X show large differences, in particular for high load values. Again, this is due to $\sigma_A^2/(2(s - \mu_A))$ being dominant. Finally, we observe that the bounds derived in this paper slightly improve the known bounds.

Although for $s = 10$ the values are somewhat larger than those for $s = 5$, the bounds remain sharp. In particular, the lower bound $\breve{\mu}_X$ substantially improves the known lower bound.

Table 1: Bounds on the mean queue length μ_X. Lower bound $\check{\mu}_X$ and upper bound $\hat{\mu}_X$ following from the addition of the bounds on the μ-series as described at the beginning of Sec. 7. Lower bound lb(14) and upper bound ub(14) from expression (14). Various traffic intensities $\theta = \mu_A/s$ for $s = 5$ and 10.

$s = 5$	Poisson					Example 7.1				
θ	lb(14)	$\check{\mu}_X$	μ_X	$\hat{\mu}_X$	ub(14)	lb(14)	$\check{\mu}_X$	μ_X	$\hat{\mu}_X$	ub(14)
0.2	0.63	0.97	0.97	1.06	1.13	0.58	1.00	1.00	1.08	1.08
0.4	1.33	1.98	2.00	2.27	2.33	1.33	2.00	2.00	2.33	2.33
0.6	2.25	3.01	3.17	3.68	3.75	2.50	3.00	3.21	4.00	4.00
0.8	4.00	4.74	5.14	5.91	6.00	5.33	5.33	6.13	7.33	7.33
0.9	6.75	7.51	7.98	8.75	9.00	10.50	10.50	11.32	12.40	12.75
$s = 5$	Example 7.2					Example 7.3				
θ	lb(14)	$\check{\mu}_X$	μ_X	$\hat{\mu}_X$	ub(14)	lb(14)	$\check{\mu}_X$	μ_X	$\hat{\mu}_X$	ub(14)
0.2	0.56	1.00	1.00	1.06	1.06	0.63	1.00	1.00	1.13	1.13
0.4	1.17	2.00	2.00	2.17	2.17	1.67	2.00	2.00	2.67	2.67
0.6	1.88	3.00	3.02	3.38	3.38	3.75	3.75	4.04	5.25	5.25
0.8	3.00	4.00	4.37	5.00	5.00	10.00	10.00	10.35	12.00	12.00
0.9	4.50	4.50	4.96	6.50	6.75	22.50	22.50	22.82	24.50	24.75
$s = 10$	Poisson					Example 7.1				
θ	lb(14)	$\check{\mu}_X$	μ_X	$\hat{\mu}_X$	ub(14)	lb(14)	$\check{\mu}_X$	μ_X	$\hat{\mu}_X$	ub(14)
0.2	1.13	1.96	1.96	2.07	2.13	1.13	2.00	2.00	2.13	2.13
0.4	2.33	3.96	3.97	4.28	4.33	2.56	4.00	4.00	4.56	4.56
0.6	3.75	5.98	6.06	6.69	6.75	4.75	6.00	6.32	7.75	7.75
0.8	6.00	8.07	8.85	9.93	10.00	10.00	10.00	11.75	14.00	14.00
0.9	9.00	9.41	12.08	13.41	13.50	19.50	19.50	21.31	24.00	24.00
$s = 10$	Example 7.2					Example 7.3				
θ	lb(14)	$\check{\mu}_X$	μ_X	$\hat{\mu}_X$	ub(14)	lb(14)	$\check{\mu}_X$	μ_X	$\hat{\mu}_X$	ub(14)
0.2	1.06	2.00	2.00	2.06	2.06	1.25	2.00	2.00	2.25	2.25
0.4	2.17	4.00	4.00	4.17	4.17	3.33	4.00	4.00	5.33	5.33
0.6	3.38	6.00	6.00	6.38	6.38	7.50	7.50	8.07	10.50	10.50
0.8	5.00	8.00	8.24	9.00	9.00	20.00	20.00	20.69	24.00	24.00
0.9	6.75	9.00	10.22	11.25	11.25	45.00	45.00	45.67	49.50	49.50

Table 2 displays the bounds on σ_X for various load values and $s = 5$ and $s = 10$. As for the bounds on μ_X, the bounds are sharp, both in the lower and higher load regime. The improvement of the upper bound $\hat{\sigma}_X^2$ in comparison with the known bound is considerable.

9 Conclusions

For the discrete-time bulk service queueing model we have derived bounds for the mean and variance of the stationary queue length distribution. These bounds are simple, in the sense that they involve at most the first three moments of the arrival distribution. For various settings we have shown that the bounds are sharp indeed, and that they improve the bounds known in the literature. In particular, the bounds on the variance of the stationary queue length are sharpened substantially.

The bounds can be easily generalized to more complicated structures such as the delay in cable networks (see [8]). Furthermore, the bounds in this paper also apply in more general settings, see e.g. [17]. It is our belief that using similar techniques bounds for the discrete-time bulk service queue with correlated arrivals can be derived as well. This is subject of future research.

Table 2: Bounds on the mean queue length σ_X^2. Lower bound $\breve{\sigma}_X^2$ and upper bound $\hat{\sigma}_X^2$ following from the addition of the bounds on the σ^2-series as described at the beginning of Sec. 7. Lower bound lb(15) and upper bound ub(15) from expression (15). Various traffic intensities $\theta = \sigma_A/s$ for $s = 5$ and 10.

$s=5$	Poisson					Example 7.1				
θ	lb(15)	$\breve{\sigma}_X^2$	σ_X^2	$\hat{\sigma}_X^2$	ub(15)	lb(15)	$\breve{\sigma}_X^2$	σ_X^2	$\hat{\sigma}_X^2$	ub(15)
0.2	0.59	1.03	1.03	1.08	5.52	0.34	0.67	0.67	0.67	5.26
0.4	1.33	2.04	2.07	2.79	6.83	1.11	2.00	2.00	2.67	6.61
0.6	2.56	3.11	3.48	4.56	8.48	3.00	4.00	4.25	5.00	8.92
0.8	7.33	7.33	8.29	9.58	13.50	14.44	14.44	15.88	16.69	20.61
0.9	25.50	25.56	26.43	27.81	31.73	72.19	72.25	73.67	74.50	78.42
$s=5$	Example 7.2					Example 7.3				
θ	lb(15)	$\breve{\sigma}_X^2$	σ_X^2	$\hat{\sigma}_X^2$	ub(15)	lb(15)	$\breve{\sigma}_X^2$	σ_X^2	$\hat{\sigma}_X^2$	ub(15)
0.2	0.25	0.50	0.50	0.50	5.17	0.52	1.00	1.00	1.00	5.43
0.4	0.53	1.00	1.00	1.33	6.03	2.44	4.00	4.00	4.02	7.94
0.6	0.89	1.50	1.52	2.89	6.81	9.31	9.31	11.12	11.56	15.48
0.8	2.00	2.00	2.67	4.24	8.17	72.00	72.00	73.95	74.25	78.17
0.9	6.19	6.25	6.83	8.50	12.42	420.18	420.25	422.31	422.50	426.42
$s=10$	Poisson					Example 7.1				
θ	lb(15)	$\breve{\sigma}_X^2$	σ_X^2	$\hat{\sigma}_X^2$	ub(15)	lb(15)	$\breve{\sigma}_X^2$	σ_X^2	$\hat{\sigma}_X^2$	ub(15)
0.2	1.10	2.09	2.09	2.38	20.76	1.02	2.00	2.00	2.25	20.68
0.4	2.33	4.07	4.08	6.14	24.33	3.64	6.67	6.67	10.00	25.64
0.6	4.06	6.08	6.29	12.84	27.73	10.06	14.00	14.63	18.84	33.73
0.8	9.08	9.08	11.65	19.11	34.00	47.75	47.75	53.31	57.78	72.67
0.9	27.75	27.75	29.94	37.78	52.67	240.00	240.00	245.48	250.03	264.92
$s=10$	Example 7.2					Example 7.3				
θ	lb(15)	$\breve{\sigma}_X^2$	σ_X^2	$\hat{\sigma}_X^2$	ub(15)	lb(15)	$\breve{\sigma}_X^2$	σ_X^2	$\hat{\sigma}_X^2$	ub(15)
0.2	0.50	1.00	1.00	1.13	20.17	2.06	4.00	4.00	4.50	21.72
0.4	1.03	2.00	2.00	3.00	23.02	9.77	16.00	16.00	16.89	31.77
0.6	1.64	3.00	3.00	10.42	25.30	37.00	37.00	44.42	47.03	61.92
0.8	3.00	4.00	4.49	12.78	27.66	287.75	287.75	295.79	297.77	312.66
0.9	7.31	7.31	8.72	17.34	32.23	1680.75	1680.75	1688.81	1690.77	1705.66

A Proofs Sec. 6

Proof Lemma 6.1
With $z = re^{i\varphi}$, we get

$$\begin{aligned}\mathrm{Re}\left[\frac{z}{(1-z)(1-\theta z)}\right] &= \frac{r}{|1-z|^2|1-\theta z|^2}\mathrm{Re}[e^{i\varphi}(1-re^{-i\varphi})(1-\theta re^{-i\varphi})] \\ &= \frac{r}{|1-z|^2|1-\theta z|^2}(\cos\varphi - (1+\theta)r + \theta r^2\cos\varphi),\end{aligned}$$

and it suffices to show that, omitting the subindex θ in r_θ for notational convenience,

$$g(\varphi) := (1+\theta r^2(\varphi))\cos\varphi - (1+\theta)r(\varphi) \le 0, \tag{92}$$

with equality if and only if $\varphi = 0$. Here it is evidently sufficient to consider the case that $\cos\varphi > 0,\ \varphi \ge 0$, i.e. $\varphi \in [0, \frac{1}{2}\pi)$. There is indeed equality in (92) when $\varphi = 0$ since $r(0) = 1$. It follows from (77) that

$$r'(\varphi) = \frac{-\theta r^2(\varphi)\sin\varphi}{1-\theta r(\varphi)\cos\varphi}, \tag{93}$$

and hence

$$\begin{aligned} g'(\varphi) &= -(1+\theta r^2(\varphi))\sin\varphi + (2\theta r(\varphi)\cos\varphi - 1 - \theta) r'(\varphi)) \\ &= -(1+\theta r^2(\varphi))\sin\varphi - \frac{(2\theta r(\varphi)\cos\varphi - 1-\theta)\theta r^2(\varphi)\sin\varphi}{1-\theta r(\varphi)\cos\varphi} \\ &= \frac{-\sin\varphi}{1-\theta r(\varphi)\cos\varphi}[(1+\theta r^2(\varphi))(1-\theta r(\varphi)\cos\varphi) + (2\theta r(\varphi)\cos\varphi - 1-\theta)\theta r^2(\varphi))] \\ &= \frac{-\sin\varphi}{1-\theta r(\varphi)\cos\varphi}[1-\theta r(\varphi)\cos\varphi - \theta^2 r^2(\varphi)(1-r(\varphi)\cos\varphi)]. \end{aligned} \tag{94}$$

Now, as $\cos\varphi > 0$ and $\theta \leq 1$,

$$\begin{aligned} 1-\theta r(\varphi)\cos\varphi - \theta^2 r^2(\varphi)(1-r(\varphi)\cos\varphi) &\geq 1-r(\varphi)\cos\varphi - \theta^2 r^2(\varphi)(1-r(\varphi)\cos\varphi) \\ &= (1-r(\varphi)\cos\varphi)(1-\theta^2 r^2(\varphi)) \geq 0, \end{aligned} \tag{95}$$

with equality in the last inequality if and only if $\varphi = 0$. Thus $g'(\varphi) < 0$ for $\varphi > 0$, and it follows that (92) is less than or equal to 0 with equality if and only if $\varphi = 0$. □

Proof Lemma 6.2
From

$$z_\theta(\alpha) = e^{\theta(z_\theta(\alpha)-1)}, \quad \frac{dz_\theta(\alpha)}{d\theta} = \frac{z_\theta(\alpha)(z_\theta(\alpha)-1)}{1-\theta z_\theta(\alpha)}, \tag{96}$$

we obtain

$$\frac{d}{d\theta}(1-z_\theta(\alpha))^{-1} = \frac{1}{(1-z_\theta(\alpha))^2}\frac{dz_\theta(\alpha)}{d\theta} = \frac{-z_\theta(\alpha)}{(1-z_\theta(\alpha))(1-\theta z_\theta(\alpha))}. \tag{97}$$

Applying Lemma 6.1 then shows that the real part of (97) is greater than or equal to 0 for each point on $\mathcal{S}_\theta$, and thus for all roots $z_1, \ldots, z_{s-1}$. □

Proof Lemma 6.3
It is readily seen that

$$\frac{d}{d\theta}\left(\frac{z_\theta(\alpha)}{(1-z_\theta(\alpha))^2}\right) = \frac{-z_\theta(\alpha)}{(1-z_\theta(\alpha))(1-\theta z_\theta(\alpha))}\frac{1+z_\theta(\alpha)}{1-z_\theta(\alpha)}, \tag{98}$$

and thus

$$\begin{aligned} \mathrm{Re}\left[\frac{d}{d\theta}\left(\frac{z_\theta(\alpha)}{(1-z_\theta(\alpha))^2}\right)\right] &= \mathrm{Re}\left[\frac{-z_\theta(\alpha)}{(1-z_\theta(\alpha))(1-\theta z_\theta(\alpha))}\right]\mathrm{Re}\left[\frac{1+z_\theta(\alpha)}{1-z_\theta(\alpha)}\right] \\ &- \mathrm{Im}\left[\frac{-z_\theta(\alpha)}{(1-z_\theta(\alpha))(1-\theta z_\theta(\alpha))}\right]\mathrm{Im}\left[\frac{1+z_\theta(\alpha)}{1-z_\theta(\alpha)}\right]. \end{aligned} \tag{99}$$

First note that with $z = re^{i\varphi}$

$$\begin{aligned} \mathrm{Im}\left[\frac{z}{(1-z)(1-\theta z)}\right] &= \frac{r}{|1-z|^2|1-\theta z|^2}\mathrm{Im}[e^{i\varphi}(1-re^{-i\varphi})(1-\theta re^{-i\varphi})] \\ &= \frac{r(1-\theta r^2)}{|1-z|^2|1-\theta z|^2}\sin\varphi. \end{aligned} \tag{100}$$

Furthermore, we have

$$\frac{1+z}{1-z} = \frac{1}{|1-z|^2}(1 - r^2 + 2ir\sin\varphi), \tag{101}$$

whence

$$\mathrm{Re}\left[\frac{1+z}{1-z}\right] = \frac{1-r^2}{|1-z|^2}, \quad \mathrm{Im}\left[\frac{1+z}{1-z}\right] = \frac{2r}{|1-z|^2}\sin\varphi. \tag{102}$$

Altogether, this shows that both members at the right-hand side of (99) are greater than or equal to 0, and thus the real part of (98) is greater than or equal to 0 for each point on $\mathcal{S}_\theta$, including all roots $z_1, \ldots, z_{s-1}$. □

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